Week 2: Multiple Regression, Training Exercises

Coursera/Erasumus U., Econometric Methods and Applications

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Training Exercise 2.4.1

Questions

By solving the questions of this exercise, you provide a proof of the Gauss-Markov theorem. We use the following notation:

- The OLS estimator is $b = A_0 y$, where $A_0 = (X'X)^{-1}X'$.
- Let $\hat{\beta} = Ay$ be linear, unbiased, with $A(k \text{ x } n)$ matrix.
- Define the difference matrix $D = A - A_0$.

(a) Prove the following three results:

(i) $var(\hat{\beta}) = \sigma^2 AA'$.

(ii) $\hat{\beta}$ unbiased implies $AX = I$ and $DX = 0$.

(iii) Part (ii) implies $AA' = DD' + (X'X)^{-1}$.

(b) Prove that part (a-iii) implies $var(\hat{\beta}) = \sigma^2 DD'$.

(c) Prove that part (b) implies $var(\hat{\beta}) - var(b)$ is positive semidefinite (Gauss-Markov).

(d) Prove that $var(\hat{\beta}) \geq var(b_j)$ for every $j = 1, \ldots, k$.

Answers

(a) Prove the following three results:

(i) $var(\hat{\beta}) = \sigma^2 AA'$.

Given that $y = X\beta + \varepsilon$, then we find that:

$$\begin{aligned}\hat{\beta} &= Ay \\ &= A(X\beta + \varepsilon) \\ &= AX\beta + A\varepsilon\end{aligned}$$

Now given that A, X, and β are fixed and $E(\varepsilon) = 0$, we find that:

$$\begin{aligned}E(\hat{\beta}) &= AX\beta + AE(\varepsilon) \\ &= AX\beta \\ var(\hat{\beta}) &= E((\hat{\beta} - E(\hat{\beta}))(\hat{\beta} - E(\hat{\beta}))') \\ &= E(A\varepsilon(A\varepsilon)') \\ &= E(A\varepsilon\varepsilon' A') \\ &= AE(\varepsilon\varepsilon')A' \\ &= A\sigma^2 IA' \\ &= \sigma^2 AA'\end{aligned}$$

(ii) $\hat{\beta}$ unbiased implies $AX = I$ and $DX = 0$.

$E(\hat{\beta})$ is unbiased if $AX\beta = \beta$ for all β, or:

$$\begin{aligned}DX &= (A - A_0)X \\ &= AX - A_0X \\ &= I - (x'x)^{-1}X'X \\ &= I - I \\ &= 0\end{aligned}$$

(iii) Part (ii) implies $AA' = DD' + (X'X)^{-1}$.

$$
\begin{aligned}
AA' &= (D + A_0)(D + A_0)' \\
&= DD' + A_0 D' + DA_0' + A_0 A_0'
\end{aligned}
$$

$$DA_0' = DX(X'X)^{-1} = 0 \cdot (X'X)^{-1} = 0$$

$$A_0 D' = (DA_0')^{-1} = 0$$

$$
\begin{aligned}
AA' &= DD' + A_0 A_0' \\
&= DD' + (X'X)^{-1}X'X(X'X)^{-1} \\
&= DD' + (X'X)^{-1}
\end{aligned}
$$

(b) Prove that part (a-iii) implies $var(\hat{\beta}) = \sigma^2 DD'$.

Given that $var(b) = \sigma^2(X'X)^{-1}$, we find that:

$$
\begin{aligned}
var(\hat{\beta}) &= \sigma^2 AA' \\
&= \sigma^2(DD' + (X'X)^{-1}) \\
&= \sigma^2 DD' + \sigma^2(X'X)^{-1} \\
&= \sigma^2 DD' + var(b)
\end{aligned}
$$

(c) Prove that part (b) implies $var(\hat{\beta}) - var(b)$ is positive semidefinite (Gauss-Markov).

From (b) we've seen that $var(\hat{\beta}) - var(b) = \sigma^2 DD'$. Given that $\sigma^2 > 0$, then it remains to be shown that DD' is positive semidefinite (p.s.d.):

D is a (k x n) matrix
D' is a (n x n) matrix
c is a (n x 1) vector
$d = D'c$ is a (n x 1) vector with components $d_1, \ldots, d_n$

$$
\begin{aligned}
\therefore c'DD'c &= (D'c)'D'c \\
&= d'd \\
&= \sum_{i=1}^{n} d_i^2 \geq 0 \quad \text{proving that } DD' \text{ is p.s.d.}
\end{aligned}
$$

(d) Prove that $var(\hat{\beta}) \geq var(b_j)$ for every $j = 1, \ldots, k$.

Let c_j be the (k x 1) j^{th} unit vector, which means c_j can be re-written as:

$$c_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{with the single element 1 at the } j^{th} \text{ entry}$$

from c we obtain that:

$$c_j'(var(\hat{\beta}) - var(b))c_j = \sigma^2 c_j' DD' c_j \geq 0$$

hence:

$$c_j' var(\hat{\beta}) c_j \geq c_j' var(b) c_j$$

or equivalently:

$$var(c_j'\hat{\beta}) \geq var(c_j' b)$$

As $c_j'\hat{\beta} = \hat{\beta}_j$ and $c_j' b = b_j$, it follow that:

$$var(\hat{\beta}_j) \geq var(b_j) \quad \text{for every } j = 1, \ldots, n$$