# Week 2: Multiple Regression, Training Exercises

Coursera/Erasumus U., Econometric Methods and Applications

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## Training Exercise 2.4.2

### Notes:

- For this exercise, you need the formulas for the F-test and t-test as discussed in Lecture 2.4.2.

### Questions

Consider the unrestricted multiple regression model $y = X_1\beta_1 + X_2\beta_2 + \varepsilon$. If we impose the null hypothesis that $\beta_2 = 0$, we get the restricted model $y = X_1\beta_1 + \varepsilon$.

**(a) Suppose that both the restricted and the unrestricted model contain a constant term. Then prove that:**

$$F = \frac{(R_1^2 - R_0'2)/g}{(1 - R_1^2)/(n-k)}$$

where $R_0^2$ and $R_1^2$ are, respectively, the R-squared of of the restricted and unrestricted model.

**(b) Suppose that we test for a single restriction$\boldsymbol{H_0 : \beta_j = 0}$, so that $\boldsymbol{g = 1}$. Then prove that $\boldsymbol{F = t^2}$.**

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## Answers

Consider the unrestricted multiple regression model $y = X_1\beta_1 + X_2\beta_2 + \varepsilon$. If we impose the null hypothesis that $\beta_2 = 0$, we get the restricted model $y = X_1\beta_1 + \varepsilon$.

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The F-test is expressed in terms of $e'e$, and we need an expression of the F-test in terms of $R^2$.

$R^2$ can be expressed in terms of $e'e$ for models with a constant term, such that:

$$\begin{aligned}
R^2 &= 1 - \frac{e'e}{SST} && \text{where } SST = \sum_{i=1}^{n}(y_i - \bar{y})2 \\
e'e &= SST(1 - R^2) && \text{re-written} \\
e_0'e_0 &= SST(1 - R_0^2) && \text{restricted} \\
e_1'e_1 &= SST(1 - R_1'2) && \text{unrestricted}
\end{aligned}$$

Finally, we can show that the F-statistic is equal to:

$$\begin{aligned}
F &= \frac{(e_0'e_0 - e_1'e_1)/g}{e_1'e_1/(n-u)} = \frac{(SST(1 - R_0^2) - SST(1 - R_1'2))/g}{SST(1 - R_1^2)/(n-u)} \\
&= \frac{(R_1^2 - R_0^2)/g}{(1 - R_1^2)/(n-u)}
\end{aligned}$$

**(b) Suppose that we test for a single restriction$H_0 : \beta_j = 0$, so that $g = 1$. Then prove that $F = t^2$.**

The t-statistic is defined as:

$$t = \frac{b_j}{s\sqrt{a_j j}} \quad \text{where } a_{jj} \text{ is the } (j,j)^{th} \text{ element of } (X'X)^{-1}$$

In the derivation of the F-test, the null hypothesis, $H_0 : R\beta = r$ can be tested using:

$$F = \frac{1}{s^2}(R\beta - r)'V^{-1}(R\beta - r)/g \quad \text{with } V = R(X'X)^{-1}R'$$

Here we test the null hypothesis $H0 : \beta_j = 0$, with $g = 1$ restriction, and with $R = (0 \dots 0\ 1\ 0\ \dots 0)$ (the $j^{th}$ unit vector), and $r = 0$.

Then:

$$V = R(X'X)^{-1} = (0 \ldots 0\ 1\ 0\ \ldots 0)(X'X)^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = a_{jj}$$

If we substitute this into the definition of the F-test, then we find that:

$$F = \frac{1}{s^2}(b_j - 0)' \cdot \frac{1}{a_{jj}}(b_j - 0) = \frac{b_j^2}{s^2 a_{jj}} = t^2$$