8.- Para el caso de estimar la media muestral con $\bar{x}$, derivar el valor esperado del estimado jackknife de la varianza y el sesgo.

Sea $\hat{\theta} = \frac{1}{n}\sum x_i = \bar{x}$ el estimador de la media muestral, la replica i-ésima Jackknife es $\hat{\theta}_{-i} = \frac{1}{n-1}\sum_{j\neq i} x_j \Rightarrow sesgo_{jack} = (n-1)\left(\frac{\sum_{i=1}^{n}\hat{\theta}_{-i}}{n} - \hat{\theta}\right) = \frac{n-1}{n}\sum_{i=1}^{n}\frac{1}{n-1}\sum_{j\neq i}^{n} x_j - \frac{n-1}{n}\sum_{i=1}^{n} x_i = \frac{1}{n}\sum_{i=1}^{n}(n-1)x_i - \frac{n-1}{n}\sum x_i = 0 \therefore \mathbb{E}[sesgo_{jack}] = \mathbb{E}[0] = 0$. Ahora si tenemos el estimador JK del error estándar $\hat{se}_{jack} \Rightarrow$ el estimado de la varianza sería $\hat{\sigma}^2_{jack} = (\sqrt{n}\,\hat{se}_{jack})^2$ entonces

$$\hat{\sigma}^2_{jack} = \left(\sqrt{n}\left[\frac{n-1}{n}\sum\left(\hat{\theta}_{-i} - \frac{1}{n}\sum\hat{\theta}_{-i}\right)^2\right]^{1/2}\right)^2 = n-1\sum_{i=1}^{n}\left(\hat{\theta}_{-i} - \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_{-i}\right)^2 = (n-1)\sum\left(\hat{\theta}_{-i}^2 - \frac{2}{n}\hat{\theta}_{-i}\sum\hat{\theta}_{-i} + \frac{1}{n^2}\left(\sum\hat{\theta}_{-i}\right)^2\right)$$

$$= (n-1)\sum\left(\frac{1}{(n-1)^2}\left(\sum_{j\neq i}x_j\right)^2 - \frac{2}{n(n-1)}\left(\sum_{j\neq i}x_j\right)\left(\sum x_i\right) + \frac{1}{n^2}\left(\sum x_i\right)^2\right) = (n-1)\sum\left(\hat{\theta}_{-i}^2 - 2\hat{\theta}_{-i}\bar{x} + \bar{x}^2\right)$$

$$= (n-1)\sum\left(\hat{\theta}_{-i} - \bar{x}\right)^2 \therefore \mathbb{E}[\hat{\sigma}^2_{jack}] = (n-1)\sum\left(\hat{\theta}_{-i} - \bar{x}\right)^2$$ pues $\hat{\sigma}^2_{jack}$ es un valor definido