Please write your family and given names and **underline** your family name on the front page of your paper.

1. [*10 points*] Consider the function

$$\phi(x) = \frac{1}{120}\begin{cases} x^5 & \text{for } 0 \le x \le 1 \\ x^5 - 6(x-1)^5 & \text{for } 1 \le x \le 2 \\ x^5 - 6(x-1)^5 + 15(x-2)^5 & \text{for } 2 \le x \le 3 \\ x^5 - 6(x-1)^5 + 15(x-2)^5 - 20(x-3)^5 & \text{for } 3 \le x \le 4 \\ x^5 - 6(x-1)^5 + 15(x-2)^5 - 20(x-3)^5 + 15(x-4)^5 & \text{for } 4 \le x \le 5 \\ x^5 - 6(x-1)^5 + 15(x-2)^5 - 20(x-3)^5 + 15(x-4)^5 - 6(x-5)^5 & \text{for } 5 \le x \le 6 \\ 0 & \text{elsewhere} \end{cases} \tag{1}$$

Explain why ϕ is a quintic spline with respect to knots 0, 1, 2, 3, 4, 5 and 6. (Write down all the relations that $\phi(x)$ has to satisfy in order to be a quintic spline, and verify they hold.)
What is the value of $\phi(x)$ at each of the knots 0, 1, 2, 3, 4, 5 and 6?
What is the value of $\phi(x)$ at each of the midpoints 0.5, 1.5, 2.5, 3.5, 4.5, and 5.5?
Give the form of the first derivative of $\phi(x)$.
What is the value of $\phi'(x)$ at each of the knots 0, 1, 2, 3, 4, 5 and 6?

In calculating values, it is more convenient to give them in fractional format.

Background: Consider the set of knots $\Delta = \{x_i, i = 0, \cdots, n\}$, with $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. For later convenience, let $\tau_i = (x_{i-1} + x_i)/2$, $i = 1, \cdots, n$, be the midpoints of the subintervals. It is known that a quintic spline defined with respect to $n+1$ knots has $n+5$ degrees of freedom (free parameters). Based on $\phi(x)$ of (1) and using appropriate mappings we can define a set of $n+5$ quintic spline basis functions $\{\phi_i(x), i = -2, \cdots, n+2\}$ with respect to the knots $\{x_i, i = 0, \cdots, n\}$. That is, each $\phi_i(x)$ is obtained from $\phi(x)$ by mapping the intervals [0, 1], [1, 2], [2, 3], [3, 4], [4, 5] and [5, 6] to $[x_{i-3}, x_{i-2}]$, $[x_{i-2}, x_{i-1}]$, $[x_{i-1}, x_i]$, $[x_i, x_{i+1}]$, $[x_{i+1}, x_{i+2}]$, and $[x_{i+2}, x_{i+3}]$, respectively.

For simplicity, assume that the knots are equidistant with stepsize $h = \dfrac{b-a}{n}$. Now consider mapping the interval [0, 6] to $[x_{i-3}, x_{i+3}]$, for $i = -2, \cdots, n+2$, and the respective mappings of the function $\phi(x)$ to

$$\phi_i(x) = \phi\left(\frac{x-a}{h} - i + 3\right), \; i = -2, \cdots, n+2. \tag{2}$$

(When $[x_{i-3}, x_{i+3}]$ goes out of $[a, b]$, consider only the part of $\phi_i(x)$ which is inside $[a, b]$.) It can be shown that the set of functions $\phi_i(x)$, $i = -2, \cdots, n+2$, forms a basis for the space of quintic splines with respect to the given knots. That is, any quintic spline $s(x)$ with respect to the knots of Δ can be written as a linear combination of the basis functions

$$s(x) = \sum_{i=-2}^{n+2} c_i \phi_i(x) \tag{3}$$

for some coefficients (degrees of freedom or parameters) c_i, $i = -2, \cdots, n+2$. In order to uniquely determine a quintic spline, the coefficients c_i need to be determined.

Notice that if x belongs to (x_{j-1}, x_j), then (3) simplifies to

$$s(x) = \sum_{i=j-3}^{j+2} c_i \phi_i(x). \tag{4}$$

If $x = x_j$, then (3) simplifies to

$$s(x) = \sum_{i=j-2}^{j+2} c_i \phi_i(x) = \frac{1}{120}(c_{j-2} + 26c_{j-1} + 66c_j + 26c_{j+1} + c_{j+2}). \tag{5}$$

If $x = \tau_j$, i.e. $x = (x_j + x_{j-1})/2$, then (3) simplifies to

$$s(x) = \sum_{i=j-3}^{j+2} c_i \phi_i(x) = \frac{1}{3840}(c_{j-3} + 237c_{j-2} + 1682c_{j-1} + 1682c_j + 237c_{j+1} + c_{j+2}). \tag{6}$$

Differentiating relation (3), we get

$$s'(x) = \sum_{i=-2}^{n+2} c_i \phi_i'(x). \tag{7}$$

If $x = x_j$, then (7) simplifies to

$$s'(x) = \sum_{i=j-2}^{j+2} c_i \phi_i'(x) = \frac{1}{24h}\left(-c_{j-2} - 10c_{j-1} + 10c_{j+1} + c_{j+2}\right). \tag{8}$$

Consider interpolating a function $u(x)$ by a quintic spline $s(x)$ in some interval $I = [a, b]$. Assume that the values of u at the knots x_i, $u = 0, \cdots, n$, and those at the midpoints τ_i, $i = 1, 2, n-1, n$, are given. We can then write down $n+5$ interpolating conditions. Thus, to compute the quintic spline interpolant s of u, we need to find the coefficients so that $s(x)$ satisfies the interpolation relations

$$\begin{gathered} s(x_0) = u(x_0),\ \ s(\tau_1) = u(\tau_1),\ \ s(x_1) = u(x_1),\ \ s(\tau_2) = u(\tau_2), \\ s(x_i) = u(x_i),\ i = 2, \cdots, n-1, \\ s(\tau_{n-1}) = u(\tau_{n-1}),\ \ s(x_{n-1}) = u(x_{n-1}),\ \ s(\tau_n) = u(\tau_n),\ \ s(x_n) = u(x_n). \end{gathered} \tag{9}$$

The interpolation relations (9) result to the linear system

$$\frac{1}{3840}\left|\begin{array}{ccccccccccccc}
32 & 832 & 2112 & 832 & 32 & & & & & & & & \\
1 & 237 & 1682 & 1682 & 237 & 1 & & & & & & & \\
 & 32 & 832 & 2112 & 832 & 32 & & & & & & & \\
 & 1 & 237 & 1682 & 1682 & 237 & 1 & & & & & & \\
 & & 32 & 832 & 2112 & 832 & 32 & & & & & & \\
 & & & 32 & 832 & 2112 & 832 & 32 & & & & & \\
 & & & & . & . & . & . & . & & & & \\
 & & & & & & 32 & 832 & 2112 & 832 & 32 & & \\
 & & & & & & 1 & 237 & 1682 & 1682 & 237 & 1 & \\
 & & & & & & & 32 & 832 & 2112 & 832 & 32 & \\
 & & & & & & & 1 & 237 & 1682 & 1682 & 237 & 1 \\
 & & & & & & & & 32 & 832 & 2112 & 832 & 32
\end{array}\right|
\left|\begin{array}{c} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ c_3 \\ . \\ c_{n-2} \\ c_{n-1} \\ c_n \\ c_{n+1} \\ c_{n+2} \end{array}\right|
=
\left|\begin{array}{c} u(x_0) \\ u(\tau_1) \\ u(x_1) \\ u(\tau_2) \\ u(x_2) \\ u(x_3) \\ \\ u(x_{n-2}) \\ u(\tau_{n-1}) \\ u(x_{n-1}) \\ u(\tau_n) \\ u(x_n) \end{array}\right| \tag{10}$$

By solving the system (10), the degrees of freedom are computed and the quintic spline interpolant can be evaluated using (4) or, selectively, (5) or (6).

2. [*5 points*] What is the solution to system (10), if the data of the right side vector correspond to the function $u(x) = 1$? Explain.

3. [*50 points*] Implement quintic spline interpolation as above. Write a MATLAB functions, `function yv = qntspline(x, y, xv)` that takes as input the $n+1$ (equidistant) knots in vector `x`, the appropriate y-values in vector `y` (the values $u(x_0)$, $u(\tau_1)$, $u(x_1)$, $u(\tau_2)$, $u(x_i)$, $i = 2, \cdots, n-2$, $u(\tau_{n-1})$, $u(x_{n-1})$, $u(\tau_n)$, $u(x_n)$), and the abscissae of the points of evaluation in vector `xv`. The output is vector `yv` which gives the values of the respective interpolant at the points `xv`. Note that `y` has four more components than `x`.

Carry out experiments for $n = 8, 16, 32, 64, 128, 256$ and assuming the data come from the functions
(α) $u(x) = x^5$ in [0, 1]
(β) $u(x) = 1/(1+25x^2)$ in $[-1, 1]$ and
(γ) $u(x) = x^{11/2}$ in [0, 1].
For each value of n and each function (α), (β) and (γ), compute and output the maximum (in abs. value) error at the midpoints $\max_{i=3}^{n-2} |s(\tau_i) - u(\tau_i)|$, the maximum (in abs. value) error at 1000 (equidistant) evaluation points in $[a, b]$, which approximates $\max_{a \le x \le b} |u(x) - s(x)|$ and the maximum (in abs. value) error of the derivative at the knots $\max_{i=0}^{n} |s'(x_i) - u'(x_i)|$.

Compare the errors obtained by quintic spline interpolation with the respective errors of the cubic spline interpolant (*not-a-knot cubic spline interpolant*) obtained by calling the MATLAB function `yv = spline(x, y, xv)`, where `y` is a vector of the same length as `x` holding the respective y-values (at the knots `x`). For convenience, present the results in a form of a **table** so that the errors for each n, function and interpolant are easily distinguished.

Discuss the quality of the results. Do the error results from quintic spline interpolation for the test function $u(x) = x^5$ agree with those expected? Based on the errors obtained experimentally, come up with a formula that gives a bound for the error of the form

$$\max_{a \le x \le b} |u(x) - s(x)| \approx \kappa_s h^{\rho_s} \qquad (12)$$

$$\max_{a \le x \le b} |u(x) - C(x)| \approx \kappa_c h^{\rho_c} \qquad (13)$$

$$\max_{i=0,\ldots,n} |u'(x_i) - s'(x_i)| \approx \kappa'_s h^{\rho'_s} \qquad (14)$$

$$\max_{i=0,\ldots,n} |u'(x_i) - C'(x_i)| \approx \kappa'_c h^{\rho'_c} \qquad (14)$$

where $C(x)$ is the cubic spline interpolant. Determine ρ_s and ρ'_s (experimentally) by using the pairs of results corresponding to grid sizes n and $2n$, from each of the functions (β) and (γ). Do the same for ρ_c and ρ'_c. Note that ρ_s, ρ'_s, ρ_c and ρ'_c may not be the same for the two functions (β) and (γ) and for the various pairs of grid sizes. Finally, note that $\max_{a \le x \le b}$ is the maximum over all points in $[a, b]$, but you are going to approximate it by the maximum over the 1000 evaluation points.

Do the error results from the numerical experiments on the test functions (β) and (γ) agree with those expected from theory?

Notes: Use sparse matrices and sparse MATLAB functions (e.g. `spdiags`) whenever a (two-dimensional) sparse matrix needs to be constructed and/or stored. Use the \ division operator for solving the linear systems in MATLAB. Use `format compact` throughout the MATLAB program or format your output by other statements (e.g. `sprintf`) so that it is compact. Use `format short e` or a reasonable exponential format for outputting the error. Please do not output more than requested. Hand in a copy of your source code, together with the output, and discussion.

To obtain the value of the quintic spline and its derivative at the midpoints and grid points (knots), respectively, you do not need to hardcode the basis functions ϕ_i. (Use (6) and (8).) However, to obtain the value of the quintic spline at the 1000 evaluation points, and to code a general version of `qntspline` you need to hardcode ϕ_i. If you have that coded correctly, you can obtain the value of the quintic spline and its derivatives up to 5th order at any point of the domain.

Also, to obtain the value of the cubic spline at any point is simple. However, obtaining the value of the derivative of the cubic spline requires extra care. See `help spline, help ppval, help mkpp` (also `doc spline, doc ppval, doc mkpp`), and, if you need more help, ask me early enough!

4. [*15 points*]

(a) Determine the constants w_1 and w_2 so that the rule of the form

$$\int_0^1 f(x)dx \approx w_1 f(\frac{1}{4}) + w_2 f(\frac{3}{4})$$

is exact for all polynomials of degree ≤ 1.

(b) Use transformation of variables in order to transform the rule constructed in (a) into one which is appropriate for approximating $\int_a^b g(t)dt$ in a general interval (a, b). Show the transformation you have used.

(c) What is the polynomial degree (degree of accuracy) of the rule constructed in (b)? Explain.

(d) Based on the rule constructed in (b) obtain a composite (compound) quadrature rule for approximating $\int_a^b f(x)dx$ in a general interval (a, b) with n panels.

(e) Consider now the trapezoidal rule

$$\int_a^b f(x)dx \approx \frac{b-a}{2}(f(a) + f(b))$$

Which rule has higher polynomial degree, the trapezoidal rule or the rule constructed in (b)? Explain.
Which rule is preferable to use in a composite form and why?

Note: This question is to be done by hand calculations.

5. [*20 points*] Calculate the exact value of $I = \int_0^1 f(x)dx$ for $f(x) = x^{\frac{i}{2}}$, $i = 5, 7, 9, 11, 13, 15$, for $f(x) = \max\{x - \frac{1}{3}, 0\}$, and for $f(x) = \begin{cases} 0 & \text{for } 0 \le x < 1/3 \\ 1 & \text{for } 1/3 \le x \le 1 \end{cases}$.

Write a MATLAB script that approximates $I = \int_0^1 f(x)dx$ using the quadrature rules

(α) midpoint
(β) Gauss based on 2 points
(γ) Gauss based on 4 points
with 1, 2, 4, 8, 16 and 32 panels.

For each function, and for each of the quadrature rules, with the number of panels indicated, compute and output the error (including its sign). Also output the experimentally observed order of convergence for each function and rule and function, and for each pair of numbers of panels (1 and 2, 2 and 4, etc). Comment on the results.

Notes: The weights and points for the Gauss rules are given in the course website, hence there can be no excuses for calculation errors.
For each of the rules write a MATLAB function that implements the (composite version of the) rule, takes as input arguments the function name, endpoints of interval of integration and number of panels, and outputs the integral approximation.
The exact value of each integral can be calculated by hand, and hard-coded.
It would be convenient to output the three approximations to each integral for a certain number of panels together with the respective errors in one row, and the orders of convergence in another row. For example,

```
for i = 1:6
    n = 2^(i-1);
    % ... fill the code
    fprintf('%3d %13.11f %13.11f %13.11f %9.2e %9.2e %9.2e\n', ...
             n, Qm(i), Qg2(i), Qg4(i), Q-Qm(i), Q-Qg2(i), Q-Qg4(i));
end
for i = 1:5
    n = 2^(i-1);
    % ... fill the code
    fprintf('(%3d, %3d)    %13.2f %13.2f %13.2f\n', ...
            n, 2*n, convm(i), convg2(i), convg4(i));
end
```

General note: For all programming questions, hand in a hard-copy of your source code and output, together with any hand-written or typed comments and/or discussion.