1 Question 1

Show that, for any vector norm $\|:\|$ for $\mathbb{R}^n$ and for any $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$,

$$\Big|\|x\| - \|y\|\Big| \leq \|x-y\|$$

Let $\|:\|$ be any norm in $\mathbb{R}^n$.
Let $x, y \in \mathbb{R}^n$

$$\begin{aligned}
\|x-y\| &= \sqrt{\langle x-y, x-y\rangle} && \text{by property} \|x\| = \sqrt{\langle x,x\rangle} \text{where } \langle,\rangle \text{ is the corresponding inner product} \\
&= \sqrt{\langle x,x\rangle - 2\langle x,y\rangle + \langle y,y\rangle} && \text{by property} \langle a+b,c\rangle = \langle a,c\rangle + \langle b,c\rangle \text{ and } \langle a,b\rangle = \langle b,a\rangle \\
&= \sqrt{\|x\|^2 - 2\langle x,y\rangle + \|y\|^2}
\end{aligned}$$

if $\langle x,y\rangle > 0$

$$\begin{aligned}
&= \sqrt{\|x\|^2 - 2|\langle x,y\rangle| + \|y\|^2} \\
&\geq \sqrt{\|x\|^2 - 2\|x\|\|y\| + \|y\|^2} && \text{by Cauchy–Schwarz inequality, } |\langle x,y\rangle| \leq \|x\|\|y\| \text{ and } \sqrt{..} \text{ is incr func} \\
&= \sqrt{(\|x\| - \|y\|)^2} \\
&= \Big|\|x\| - \|y\|\Big|
\end{aligned}$$

if $\langle x,y\rangle \leq 0$

$$\begin{aligned}
&= \sqrt{\|x\|^2 - 2\langle x,y\rangle + \|y\|^2} \\
&\geq \sqrt{\|x\|^2 + \|y\|^2} && \text{since } \langle x,y\rangle \leq 0 \text{, then } -2\langle x,y\rangle \geq 0, \sqrt{..} \text{ is an increasing function} \\
&= \sqrt{\|x\|^2 + \|y\|^2 - 2\|x\|\|y\| + 2\|x\|\|y\|} \\
&= \sqrt{(\|x\| - \|y\|)^2 + 2\|x\|\|y\|} \\
&\geq \sqrt{(\|x\| - \|y\|)^2} && \text{since } \|x\| \geq 0, \|y\| \geq 0 \text{, then } 2\|x\|\|y\| \geq 0, \sqrt{..} \text{ is an increasing function} \\
&= \Big|\|x\| - \|y\|\Big|
\end{aligned}$$

2 Question 2

Show that

$$\max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\|$$

Can write the right hand side,

$$\max_{\|x\|=1} \|Ax\| = \max_{\|x\|=1} \frac{\|Ax\|}{\|x\|}$$

Clearly,

$$\max_{x \neq 0} \frac{\|Ax\|}{\|x\|} \geq \max_{\|x\|=1} \frac{\|Ax\|}{\|x\|} \qquad (1.1)$$

because the set of x over which the left max is taken, completely contains the set of x on the right max (for same function).
The set of x's s.t $\|x\| = 1$ is contained in the set of x's s.t $x \neq 0$ or equivalently $\|x\| \neq 0$

Let $y \neq 0 \in \mathbb{R}^n$ be some vector from the set on the left hand side max.
There exists a vector x with $\|x\| = 1$ such that it achieves the same for value for $\frac{\|Ax\|}{\|x\|}$ as y.

Let $x = \frac{y}{\|y\|}$, so $y = \|y\| \, x$ and clearly $\|x\| = 1$

$$\frac{\|Ay\|}{\|y\|} = \frac{\left\| A(\|y\| \, x) \right\|}{\left\| \|y\| \, x \right\|}$$

$$= \frac{\left\| \|y\| \, (Ax) \right\|}{\left\| \|y\| \, x \right\|}$$

$\|y\|$ is just a scalar that we can pull out of by properties of norms (for both $\mathbb{R}^m$ on the top and $\mathbb{R}^n$ on the bottom)

$$= \frac{\|y\| \|Ax\|}{\|y\| \|x\|}$$

$$= \frac{\|Ax\|}{\|x\|}$$

So for any vector from the set on the left ($x \neq 0$), there is a vector from the right set ($\|x\| = 1$) that achieves the same value for the function $\frac{\|Ax\|}{\|x\|}$

Therefore, the max of the right side is at least as large as the max of the left.

$$\max_{x \neq 0} \frac{\|Ax\|}{\|x\|} \leq \max_{\|x\|=1} \frac{\|Ax\|}{\|x\|} \qquad (2.1)$$

So by 1.1 and 2.1 we get

$$\max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\|$$

Let $B = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ and let $f : B \to \mathbb{R}$ defined by $f(x) = \|Ax\|$ Will show B is compact and that $f(x)$ is continuous. Then once we have that, by theorem covered in class we can conclude that there does exist a vector y that achieves the maximum value for $f(x)$ on B, ie $\|A\| = \|Ay\|$

• B is closed $\iff$ for any sequence $\{x_k\} \subset B$ that converges to x^*, ($x_k \to x^*$) we have that $x^* \in B$
Like we showed in class,
Let $\{x_k\} \subset B$ be some sequence that converges to x^*, $x_k \to x^*$ ($\|x_k\| = 1$)
So $\forall \epsilon > 0 \exists N > 0 \forall k \geq N \, \|x_k - x^*\| < \epsilon$
Let $\epsilon > 0$ be arbitrary, so $\exists N > 0$ s.t $\|x_N - x^*\| < \epsilon$ $\qquad$ ($x_N \in B$ so $\|x_N\| = 1$)
$\big| \, \|x^*\| - 1 \, \big| = \big| \, \|x^*\| - \|x_N\| \, \big| \leq \|x^* - x_N\| < \epsilon$ $\qquad$ the $\leq$ is by Question 1
So $\forall \epsilon > 0$ we have $\big| \, \|x^*\| - 1 \, \big| < \epsilon$
Therefore $\big| \, \|x^*\| - 1 \, \big| = 0$, so $\|x^*\| = 1$ and hence $x^* \in B$
Proving B is a closed set.

• B is bounded $\iff$ $\exists M > 0$ s.t. $\|x\| \leq M$ for all $x \in B$
Clearly there exists such an M, set $M = 1$, by def of B $x \in B \to \|x\| = 1 \leq M$
Proving B is bounded
So since B is closed and bounded, it is compact

• $f(x) = \|Ax\|$ is continuous on B if $\forall x \in B \forall \epsilon > 0 \exists \delta > 0 \forall y \in B \quad \left\| y - x \right\| < \delta \implies \left| f(y) - f(x) \right| < \epsilon$
Let $x \in B$, $A \in \mathbb{R}^{mxn}$ and $\epsilon > 0$. Write A in terms of its columns $A = [a_1, a_2, ..a_n]$, $a_i \in \mathbb{R}^m$
$\|\|$ is the norm in $\mathbb{R}^n$ we are using. By norm equivalences $\exists 0 < c_1 \leq c_2 \in \mathbb{R}$ such that $\forall z \in \mathbb{R}^n$ $c_1 \|z\| \leq \|z\|_1 \leq c_2 \|z\|$
Let

$$\delta = \frac{\epsilon}{\|a_t\| \, c_2} \qquad \text{where } \|a_t\| \text{ is the max of } \|a_1\|, \|a_2\|, ..\|a_n\|$$

Let $y \in B$ s.t $\|y - x\| < \delta$
By norm equivalences $c_1\|y - x\| \leq \|y - x\|_1 \leq c_2\|y - x\|$

$$
\begin{aligned}
|f(y) - f(x)| &= \Big| \, \|Ay\| - \|Ax\| \, \Big| \\
&\leq \|Ay - Ax\| && \text{by Question 1} \\
&= \|A(y - x)\| \\
&= \|A[(y_1 - x_1)e_1 + (y_2 - x_2)e_2 + ... + (y_n - x_n)e_n]\| && \text{wrote x - y in terms of the standard basis, } x_i, y_i \in \mathbb{R} \\
&= \|(y_1 - x_1)Ae_1 + (y_2 - x_2)Ae_2 + ... + (y_n - x_n)Ae_n\| \\
&= \|(y_1 - x_1)a_1 + (y_2 - x_2)a_2 + ... + (y_n - x_n)a_n\| \\
&\leq \|(y_1 - x_1)a_1\| + \|(y_2 - x_2)a_2\| + ... + \|(y_n - x_n)a_n\| && \text{by triangle inequality} \\
&= |y_1 - x_1|\|a_1\| + |y_2 - x_2|\|a_2\| + ... + |y_n - x_n|\|a_n\| && \text{by prop of norm} \\
&\leq |y_1 - x_1|\|a_t\| + |y_2 - x_2|\|a_t\| + ... + |y_n - x_n|\|a_t\| && a_t \text{ is the max of } \|a_1\|, \|a_2\|, ..\|a_n\| \\
&= \|a_t\| \left(|y_1 - x_1| + |y_2 - x_2| + ... + |y_n - x_n|\right) \\
&= \|a_t\|\|y - x\|_1 && \text{by def of norm 1} \\
&\leq \|a_t\| c_2\|y - x\| && \text{by above norm equivalence} \\
&< \|a_t\| c_2\delta \\
&= \epsilon
\end{aligned}
$$

So $f(x) = \|Ax\|$ is continuous on B
Finally we can conclude by the theorem seen in class that there does exists a vector y that maximizes f(x) on B.

$$f(y) = \max_{\|x\|=1} f(x) = \max_{\|x\|=1} \|Ax\| = \max_{\|x\|=1} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|\neq 0} \frac{\|Ax\|}{\|x\|} = \|A\|$$

3 Question 3

Show that, for any matrix $A \in \mathbb{R}^{mxn}$ and any vector $x \in \mathbb{R}^n$,

$$\|Ax\| \leq \|A\|\|x\|$$

Let $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{mxn}$. Also let $y \neq 0 \in \mathbb{R}^n$ be the vector that maximizes the quantity $\frac{\|Ax\|}{\|x\|}$, ie $\|A\| = \frac{\|Ay\|}{\|y\|}$
If $x = 0$
then $Ax = 0$ and by properties of norms $\|Ax\| = 0$ as well as $\|x\| = 0$
We have $\|Ax\| = \|A\|\|x\|$ so $\|Ax\| \leq \|A\|\|x\|$

Else (if $x \neq 0$)
then by prop. of norms $\|x\| \neq 0$ and so we can write
$\|Ax\| = \frac{\|Ax\|}{\|x\|}\|x\| \leq \frac{\|Ay\|}{\|y\|}\|x\| = \|A\|\|x\|$

4 Question 4

Show that there is always an $x^* \neq 0$ such that

$$\|Ax^*\| = \|A\|\|x^*\|$$

Let $A \in \mathbb{R}^{mxn}$ be some matrix. And let $y \neq 0 \in \mathbb{R}^n$ be the vector that maximizes the quantity $\frac{\|Ax\|}{\|x\|}$, ie $\|A\| = \frac{\|Ay\|}{\|y\|}$
Let $x^* = y$
then since $y \neq 0$ and so $x^* \neq 0$, we have $\|x^*\| \neq 0$
$\|Ax^*\| = \frac{\|Ax^*\|}{\|x^*\|}\|x^*\| = \frac{\|Ay\|}{\|y\|}\|x^*\| = \|A\|\|x^*\|$

Moreover, show that, if x^* satisfies $\|Ax^*\| = \|A\|\|x^*\|$, then $\hat{x} = \alpha x^*$, for all $\alpha \in \mathbb{R}$, satisfies

$$\|A\hat{x}\| = \|A\|\|\hat{x}\|$$

Let $x^* \in \mathbb{R}^n$ satisfy $\|Ax^*\| = \|A\|\|x^*\|$
Let $\alpha \in \mathbb{R}$ and $\hat{x} = \alpha x^*$
$\|A\hat{x}\| = \left\|A(\alpha x^*)\right\| = \left\|\alpha(Ax^*)\right\| = |\alpha|\|Ax^*\| = |\alpha|\|A\|\|x^*\| = \|A\|\|\alpha x^*\| = \|A\|\|\hat{x}\|$
by prop of vector scalar mult, prop of norms and by assumption on x^*

5 Question 5

Show that an isolated local minimizer of a function $f : \mathbb{R}^n \to \mathbb{R}$ is a strict local minimizer of f

Let $f : \mathbb{R}^n \to \mathbb{R}$ be some function.
Assume x^* is an isolated local minimizer of f.
So there is a neighborhood N_1 of x^* such that x^* is the only local minimizer in N_1.
In other words $\exists r_1 > 0$ s.t x^* is the only local minimizer in $B(x^*, r_1)$.
WTP that x^* is a strict local minimizer of f.
That there exists a neighborhood N_2 of x^* s.t. $f(x^*) < f(x)$ for all $x \in N_2$ with $x \neq x^*$
In other words $\exists r_2 > 0$ s.t $f(x^*) < f(x)$ for all $x \in B(x^*, r_2)$ with $x \neq x^*$.

Let $r_2 = r_1$, (x^* is a strict minimizer in the same neigborhood)
Claim: $\forall x \in B(x^*, r_1),\ x \neq x^* \implies f(x^*) < f(x)$
Suppose for a contradiction that $\exists y \in B(x^*, r_1)$, $y \neq x^*$ and $f(x^*) \geq f(y)$
Since x^* is a local minimizer in $B(x^*, r_1)$, $\forall x \in B(x^*, r_1)$, $f(x^*) \leq f(x)$ and it follows that
$\forall x \in B(x^*, r_1)$, $f(y) \leq f(x)$ since $f(x^*) \geq f(y)$
This implies that y is a local minimizer in $B(x^*, r_1)$
But since $y \neq x^*$ this contradicts that fact that x^* is the **only** local minimizer in $B(x^*, r_1)$.
Therefore the claim is true and so x^* is a strict local minimizer of f.