# 1 Question 1

**Q** Explain why $\phi$ is a quintic spline with respect to knots 0, 1, 2, 3, 4, 5 and 6. (Write down all the relations that $\phi(x)$ has to satisfy in order to be a quintic spline, and verify they hold.)
To be a quintic spline with respect to knots 0, 1, 2, 3, 4, 5 and 6, $\phi(x)$ has to be continuous on those knots and have continuous derivatives up to order 4 on the knots.
Labeling the 7 branches of $\phi(x)$ as $\phi_1(x)$, $\phi_2(x)$, .. $\phi_7(x)$ and the knots $x_0 = 0$, $x_1 = 1$, .. $x_6 = 6$. We have the following conditions it must satisfy. NOTE: notice the structure of $\phi_i$'s on the right.

$$
\begin{aligned}
\phi_i(x_1) &= \phi_{i+1}(x_i) \ , \ i = 1, 2, ..5 & \qquad \phi_2(x) &= \phi_1(x) - 6(x-1)^5 \\
\phi_i^{(1)}(x_1) &= \phi_{i+1}^{(1)}(x_i) \ , \ i = 1, 2, ..5 & \phi_3(x) &= \phi_2(x) + 15(x-2)^5 \\
\phi_i^{(2)}(x_1) &= \phi_{i+1}^{(2)}(x_i) \ , \ i = 1, 2, ..5 & \phi_4(x) &= \phi_3(x) - 20(x-3)^5 \\
\phi_i^{(3)}(x_1) &= \phi_{i+1}^{(3)}(x_i) \ , \ i = 1, 2, ..5 & \phi_5(x) &= \phi_4(x) + 15(x-4)^5 \\
\phi_i^{(4)}(x_1) &= \phi_{i+1}^{(4)}(x_i) \ , \ i = 1, 2, ..5 & \phi_6(x) &= \phi_5(x) - 6(x-5)^5
\end{aligned}
$$

$$
\begin{aligned}
\phi_2(1) &= \phi_1(1) - 6(1-1)^5 = \phi_1(1) & \phi_2^{(1)}(1) &= \phi_1^{(1)}(1) - 30(1-1)^4 = \phi_1^{(1)}(1) & \phi_2^{(2)}(1) &= \phi_1^{(2)}(1) - 120(1-1)^3 = \phi_1^{(2)}(1) \\
\phi_3(2) &= \phi_2(2) + 15(2-2)^5 = \phi_2(2) & \phi_3^{(1)}(2) &= \phi_2^{(1)}(2) + 75(2-2)^4 = \phi_2^{(1)}(2) & \phi_3^{(2)}(2) &= \phi_2^{(2)}(2) + 300(2-2)^3 = \phi_2^{(2)}(2) \\
\phi_4(3) &= \phi_3(3) - 20(3-3)^5 = \phi_3(3) & \phi_4^{(1)}(3) &= \phi_3^{(1)}(3) - 100(3-3)^4 = \phi_3^{(1)}(3) & \phi_4^{(2)}(3) &= \phi_3^{(2)}(3) - 400(3-3)^3 = \phi_3^{(2)}(3) \\
\phi_5(4) &= \phi_4(4) + 15(4-4)^5 = \phi_4(4) & \phi_5^{(1)}(4) &= \phi_4^{(1)}(4) + 75(4-4)^4 = \phi_4^{(1)}(4) & \phi_5^{(2)}(4) &= \phi_4^{(2)}(4) + 300(4-4)^3 = \phi_4^{(2)}(4) \\
\phi_6(5) &= \phi_5(5) - 6(5-5)^5 = \phi_5(5) & \phi_6^{(1)}(5) &= \phi_5^{(1)}(5) - 30(5-5)^4 = \phi_5^{(1)}(5) & \phi_6^{(2)}(5) &= \phi_5^{(2)}(5) - 120(5-5)^3 = \phi_5^{(2)}(5)
\end{aligned}
$$

$$
\begin{aligned}
\phi_2^{(3)}(1) &= \phi_1^{(3)}(1) - 360(1-1)^2 = \phi_1^{(3)}(1) & \phi_2^{(4)}(1) &= \phi_1^{(4)}(1) - 720(1-1)^1 = \phi_1^{(4)}(1) \\
\phi_3^{(3)}(2) &= \phi_2^{(3)}(2) + 900(2-2)^2 = \phi_2^{(3)}(2) & \phi_3^{(4)}(2) &= \phi_2^{(4)}(2) + 1800(2-2)^1 = \phi_2^{(4)}(2) \\
\phi_4^{(3)}(3) &= \phi_3^{(3)}(3) - 1200(3-3)^2 = \phi_3^{(3)}(3) & \phi_4^{(4)}(3) &= \phi_3^{(4)}(3) - 2400(3-3)^1 = \phi_3^{(4)}(3) \\
\phi_5^{(3)}(4) &= \phi_4^{(3)}(4) + 900(4-4)^2 = \phi_4^{(3)}(4) & \phi_5^{(4)}(4) &= \phi_4^{(4)}(4) + 1800(4-4)^1 = \phi_4^{(4)}(4) \\
\phi_6^{(3)}(5) &= \phi_5^{(3)}(5) - 360(5-5)^2 = \phi_5^{(3)}(5) & \phi_6^{(4)}(5) &= \phi_5^{(4)}(5) - 360(5-5)^1 = \phi_5^{(4)}(5)
\end{aligned}
$$

Since $\phi$ satisfies all the relations, it is a quintic spline.

**Q** What is the value of $\phi(x)$ at each of the knots 0, 1, 2, 3, 4, 5 and 6?
$\phi(0) = \phi_1(0) = (1/120)(0^5) = 0$
$\phi(1) = \phi_1(1) = (1/120)(1^5) = \frac{1}{120}$
$\phi(2) = \phi_2(2) = (1/120)[2^5 - 6(2-1)^5] = \frac{26}{120}$
$\phi(3) = \phi_3(3) = (1/120)[3^5 - 6(3-1)^5 + 15(3-2)^5] = \frac{66}{120}$
$\phi(4) = \phi_4(4) = (1/120)[4^5 - 6(4-1)^5 + 15(4-2)^5 - 20(4-3)^5] = \frac{26}{120}$
$\phi(5) = \phi_5(5) = (1/120)[5^5 - 6(5-1)^5 + 15(5-2)^5 - 20(5-3)^5 + 15(5-4)^5] = \frac{1}{120}$
$\phi(6) = \phi_6(6) = (1/120)[6^5 - 6(6-1)^5 + 15(6-2)^5 - 20(6-3)^5 + 15(6-4)^5 - 6(6-5)^5] = (1/120)(0) = 0$

**Q** What is the value of $\phi(x)$ at each of the midpoints 0.5, 1.5, 2.5, 3.5, 4.5, and 5.5?
$\phi(0.5) = \phi_1(0.5) = 0.03125/120$ (or $1/3840$)
$\phi(1.5) = \phi_2(1.5) = 7.40625/120$ (or $237/3840$)
$\phi(2.5) = \phi_3(2.5) = 52.5625/120$ (or $1682/3840$)
$\phi(3.5) = \phi_4(3.5) = 52.5625/120$ (or $1682/3840$)
$\phi(4.5) = \phi_5(4.5) = 7.40625/120$ (or $237/3840$)
$\phi(5.5) = \phi_6(5.5) = 0.03125/120$ (or $1/3840$)

**Q** Give the form of the first derivative of $\phi(x)$

$$\phi^{'}(x) = \frac{1}{120}\begin{cases} 5x^4 & \text{for } 0 \le x \le 1 \\ 5x^4 - 30(x-1)^4 & \text{for } 1 \le x \le 2 \\ 5x^4 - 30(x-1)^4 + 75(x-2)^4 & \text{for } 2 \le x \le 3 \\ 5x^4 - 30(x-1)^4 + 75(x-2)^4 - 100(x-3)^4 & \text{for } 3 \le x \le 4 \\ 5x^4 - 30(x-1)^4 + 75(x-2)^4 - 100(x-3)^4 + 75(x-4)^4 & \text{for } 4 \le x \le 5 \\ 5x^4 - 30(x-1)^4 + 75(x-2)^4 - 100(x-3)^4 + 75(x-4)^4 - 30(x-5)^4 & \text{for } 5 \le x \le 6 \\ 0 & \text{elsewhere} \end{cases}$$

**Q** What is the value of $\phi^{'}(x)$ at each of the knots 0, 1, 2, 3, 4, 5 and 6?
$\phi^{'}(0) = \phi_1^{'}(0) = 0$
$\phi^{'}(1) = \phi_1^{'}(1) = 5/120$ (or $1/24$)
$\phi^{'}(2) = \phi_2^{'}(2) = 50/120$ (or $10/24$)
$\phi^{'}(3) = \phi_3^{'}(3) = 0$
$\phi^{'}(4) = \phi_4^{'}(4) = -50/120$ (or $-10/24$)
$\phi^{'}(5) = \phi_5^{'}(5) = -5/120$ (or $-1/24$)
$\phi^{'}(6) = \phi_6^{'}(6) = 0$

## 2 Question 2

**Q** What is the solution to system (10), if the data of the right side vector correspond to the function u(x) = 1? Explain.
We can build u(x) exactly using the basis functions in this case. So the solution to the system will always be some ci's such that S(x) = 1 for all x in the inteval a, b. In this case the ci's that work, which is easily checked, are ci = 1 for all i. Every row of the matrix is either 1/3840 |32 832 2112 832 32| or 1/3840 |1 237 1682 1682 237 1|. So each row evaluates to $(1/3840)(32 + 832 + 2112 + 832 + 32) = 3840/3840 = 1$ and $(1/3840)(1 + 237 + 1682 + 1682 + 237 + 1) = 3840/3840 = 1$

## 3 Question 3

**OUTPUT**

```
>> a2q3script
u(x) = x^5 in [0, 1]
     N     QuinticMidErr     QuinticIntrErr     QuinticDervErr     CubicMidErr     CubicIntrErr     CubicDervErr
   ___     _____________     ______________     ______________     ___________     ____________     ____________

     8       2.2204e-16        2.8866e-15         2.8422e-14       0.00065796       0.00072132        0.036984
    16       1.1102e-16        1.0436e-14         2.4869e-14       4.4234e-05       4.8409e-05       0.0049419
    32       1.1102e-16        1.1324e-14          1.279e-13       2.8617e-06       3.1276e-06      0.00063765
    64       2.2204e-16        1.3323e-14         1.6165e-13       1.8189e-07       1.9845e-07      8.0951e-05
   128       2.2204e-16        3.1752e-14         4.1922e-13       1.1463e-08       1.2505e-08      1.0197e-05
   256       2.2204e-16        2.2871e-14         3.6238e-13       7.1941e-10       7.3159e-10      1.2794e-06

u(x) = 1/(1 + 25x^2) in [-1, 1]
     N     QuinticMidErr     QuinticIntrErr     QuinticDervErr     CubicMidErr     CubicIntrErr     CubicDervErr
   ___     _____________     ______________     ______________     ___________     ____________     ____________

     8       3.5057e-11        4.3843e-11          2.405e-09       1.0925e-07       1.1875e-07      2.9689e-06
    16       4.6354e-13        4.6378e-13         4.8619e-11       5.8713e-09       6.3952e-09      3.2184e-07
    32       7.1748e-15        7.6952e-15         1.1419e-12       3.3528e-10       3.6539e-10      3.6977e-08
    64       1.1796e-16        1.5057e-15         2.6975e-14       1.9946e-11       2.1736e-11      4.4152e-09
   128       2.0817e-17        1.5959e-15         1.9358e-14       1.2148e-12       1.3244e-12      5.3886e-10
   256       2.0817e-17        1.4017e-15         4.0834e-14       7.4933e-14       7.6147e-14      6.6539e-11

u(x) = x^(11/2) in [0, 1]
     N     QuinticMidErr     QuinticIntrErr     QuinticDervErr     CubicMidErr     CubicIntrErr     CubicDervErr
   ___     _____________     ______________     ______________     ___________     ____________     ____________

     8       1.0751e-07        3.0836e-07         3.1501e-05         0.001107        0.0012155        0.062602
    16       2.4077e-09        6.7896e-09         1.3868e-06       7.7166e-05       8.4508e-05       0.0086451
    32       5.3204e-11        1.4957e-10         6.1287e-08       5.0791e-06       5.5531e-06       0.0011332
    64       1.1756e-12        3.3051e-12         2.7085e-09       3.2555e-07       3.5524e-07      0.00014498
   128       2.5978e-14        6.8476e-14          1.197e-10       2.0602e-08       2.2476e-08      1.8332e-05
   256       5.7405e-16        2.2204e-14         5.2901e-12       1.2956e-09       1.3177e-09      2.3046e-06
```

**Q** Discuss the quality of the results. Do the error results from quintic spline interpolation for the test function $u(x) = x^5$ agree with those expected?
If u happens to be a polynomial of degree up to 5, we expect (from our study of linear and cubic splines) the interpolant to be the function itself, regardless of the size of the interval size h. We expect the error formula to be something like $|f(x) - L(x)| = C \max_{x\in[a,b]} |f^{(6)}(x)|$ The numerical results do agree with this, the max errors for all n are on the order of machine epsilon, almost 0.

**Q** Based on the errors obtained experimentally, come up with a formula that gives a bound for the error of the form.

$$\max_{a\leq x\leq b} |u(x) - s(x)| \approx k_s h^{p_s}$$

$$\max_{a\leq x\leq b} |u(x) - C(x)| \approx k_c h^{p_c}$$

$$\max_{a\leq x\leq b} |u'(x) - s'(x)| \approx k'_s h^{p'_s}$$

$$\max_{a\leq x\leq b} |u'(x) - C'(x)| \approx k'_s h^{p'_c}$$

```
Estimating ps
Function B:      6.5627     5.9133     2.3535    -0.0839     0.1873
Function Y:      5.5051     5.5044     5.5000     5.5929     1.6247

Estimating pc
Function B:      4.2148     4.1295     4.0713     4.0367     4.1204
Function Y:      3.8463     3.9277     3.9664     3.9823     4.0923

Estimating ps'
Function B:      5.6284     5.4120     5.4037     0.4787    -1.0768
Function Y:      4.5056     4.5000     4.5000     4.5000     4.5000

Estimating pc'
Function B:      3.2055     3.1216     3.0661     3.0345     3.0176
Function Y:      2.8563     2.9315     2.9665     2.9834     2.9918
```

Now determining ps and p's (experimentally) by using the pairs of results corresponding to grid sizes n and 2n, from each of the functions (B) and (Y).
We have an approximation to $\max_{a\leq x\leq b} |u(x) - s(x)| =$ r for n and $\max_{a\leq x\leq b} |u(x) - s(x)| =$ w for 2n.
So we have $r \approx k_s h^{p_s}$ and $w \approx k_s(h/2)^{p_s}$. Then $r/w \approx h^{p_s}/(h/2)^{p_s} = 2^{p_s}$ and $p_s \approx log(r/w)$ is an approximation. The same is done for all 4p's using all pairs.

**Q** Do the error results from the numerical experiments on the test functions (B) and (Y) agree with those expected from theory?
$p_c$: For both the functions B and Y, $p_c$ experimentally turns out to be around 4 for all the pairs sizes n and 2n. These results do agree with those expected from theory since we know that the error of cubic spline interpolants is of fourth order $O(h^4)$

$p'_c$: For both the functions B and Y, $p'_c$ experimentally turns out to be around 3 for all the pairs sizes n and 2n. This can be looked at as fitting a quadratic spline (derivative of cubis) to a function, There is no theory to compare these results to, but it makes sense that the order of the error of the quadratic interpolant (derivatie here) is lower than that of cubic spline interpolants.

$p_s$: For the function B, the value of $p_s$ is hard to estimate. Only the values 6.5627 and 5.9133 could be considered since the after that the error between pairs barely goes down because it's already at about 0 (close ish to machine epsilon). For the function Y, the value of $p_s$ experimentally seems to be about 5.5. Now since linear spline interpolant error is of second order and cubic spline interpolant error is of fourth order, one would naturally expect quintic spline interpolant error to be of sixth order. The experimental values of 5.5 for function Y and 6 for function B somewhat support this.

$p'_s$: For the functions B, $p'_s$ experimentally turns out to be around 5.5 (don't count the last 2). For the functions Y, $p'_s$ experimentally turns out to be around 4.5. Again there is no theory to compare these results to, but it makes sense that the order of the error of the derivative interpolant is lower than that of the interpolant itself.

# 4 Question 4

**a)** Determine the constants $w_1$ and $w_2$ so that the rule of the form is exact for all polynomials of degree $\leq 1$

$$\int_0^1 f(x)dx = w_1 f(\frac{1}{4}) + w_2 f(\frac{3}{4})$$

Has to be exact for $f(x) = 1$

$$\int_0^1 1dx = w_1 + w_2 => 1 = w_1 + w_2$$

Has to be exact for $f(x) = x$

$$\int_0^1 xdx = w_1(\frac{1}{4}) + w_2(\frac{3}{4}) => \frac{1}{2} = \frac{1}{4}w_1 + \frac{3}{4}w_2$$

We have the following system of equations

$$\begin{bmatrix} 1 & 1 \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} => \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} => \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} => \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} => \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \tag{1}$$

So the constants are $w_1 = \frac{1}{2}$ and $w_2 = \frac{1}{2}$

**b)** The transformation used is $x = (b-a)t + a$. The general linear transformation from $[c, d]$ to $[a, b]$, $x = \frac{(b-a)t+ad-bc}{d-c}$ $(b-a)\frac{1}{4} + a$ and $(b-a)\frac{3}{4} + a$. The weights are scaled by $(b-a)$. So the rule is

$$\int_a^b g(t)dt = \frac{(b-a)}{2}[g((b-a)\frac{1}{4} + a) + g((b-a)\frac{3}{4} + a)]$$

**c)** We shown the rule has polynomial degree at least 1, it is actually exactly 1. Consider $f(x) = x^2$

$$\int_0^1 f(x)dx = \int_0^1 x^2 dx = \frac{1}{3}x^3|_{x=0}^1 = \frac{1}{3} \text{ exact value}$$

$$\int_0^1 f(x)dx = \frac{1}{2}f(\frac{1}{4}) + \frac{1}{2}f(\frac{3}{4}) = \frac{1}{2}(\frac{1}{4})^2 + \frac{1}{2}(\frac{3}{4})^2 = \frac{5}{16} \neq \frac{1}{3}$$

**d)** Composite (compound) quadrature rule

$$\int_a^b f(x)dx = \frac{h}{2}\sum_{i=1}^{n} f(a + (i-1+h)\frac{1}{4}) + f(a + (i-1+h)\frac{3}{4})$$

**e)** Which rule has higher polynomial degree, the trapezoidal rule or the rule constructed in (b)? Explain.
We know from lecture that the trapezoid rule has polynomial degree 1, and it was shown that this rule also has degree 1 so both rules have the same polynomial degree.
**Q** Which rule is preferable to use in a composite form and why?
The trapezoid rule is more preferable because it uses a lot less function evaluations. In the trapezoid rule about one function evaluation is done per panel (except for first and last) and multiplied by 2, while as in this rule there are always 2 function evaluations per panel.
On the other hand, this rule is an open rules which can be used to compute integrals with singularities on the boundaries (endpoints) where as the trapezoid, a closed rule, can not be.

# 5 Question 5

**Q** Calculate the exact value of $I = \int_0^1 f(x)dx$ for $f(x) = x^{i/2}$, i = 5, 7, 9, 11, 13, 15
$I = \int_0^1 x^{5/2}dx = (2/7)x^{7/2}|_{x=0}^1 = \frac{2}{7}$
$I = \int_0^1 x^{7/2}dx = (2/9)x^{9/2}|_{x=0}^1 = \frac{2}{9}$
$I = \int_0^1 x^{9/2}dx = (2/11)x^{11/2}|_{x=0}^1 = \frac{2}{11}$
$I = \int_0^1 x^{11/2}dx = (2/13)x^{13/2}|_{x=0}^1 = \frac{2}{13}$
$I = \int_0^1 x^{13/2}dx = (2/15)x^{15/2}|_{x=0}^1 = \frac{2}{15}$
$I = \int_0^1 x^{15/2}dx = (2/17)x^{17/2}|_{x=0}^1 = \frac{2}{17}$
Calculate the exact value of $I = \int_0^1 f(x)dx$ for $f(x) = max\{x - \frac{1}{3}, 0\}$
$I = \int_0^1 max\{x - \frac{1}{3}, 0\}dx = \int_0^{\frac{1}{3}} 0dx + \int_{\frac{1}{3}}^1 (x - \frac{1}{3})dx = (1/2)x^2 - (1/3)x|_{x=\frac{1}{3}}^1 = 1/2 - 1/3 - [(1/2)(1/3)^2 - (1/3)(1/3)] = \frac{2}{9}$
Calculate the exact value of $I = \int_0^1 f(x)dx$ for

$$f(x) = \begin{cases} 0 & \text{for } 0 \leq x < 1/3 \\ 1 & \text{for } 1/3 \leq x \leq 1 \end{cases}$$

$I = \int_0^{\frac{1}{3}} 0dx + \int_{\frac{1}{3}}^1 1dx = x|_{x=\frac{1}{3}}^1 = 1 - 1/3 = \frac{2}{3}$
Getting the experimentally observed order of convergence for each function and rule. We have the error for n $Err_n \approx kh^p$ and the error for 2n $Err_{2n} \approx k(h/2)^p$.
Then $Err_n/Err_{2n} \approx h^p/(h/2)^p = 2^p$ and $p \approx log(Err_n/Err_{2n})$ is an approximation for the order.

We know from theory that the midpoint rule is of order 2 and that a gauss rule with points $i = 0, 1, ..n$ is of order d+1 where d = 2n + 1
For all most all the functions and pairs of n and 2n, the order of convergence of the midpoint is at around 2, which agrees with theory.
We expect the 2 point gauss rule to have order $2(1) + 1 + 1 = 4$ which is about what we see.
We expect the 4 point gauss rule to have order $2(3)+1+1 = 8$ but for most of the functions we don't see this. The reason for this is because we can't apply the error formula, which has the d+1 derivative of the function. For the first type of function the d+1 derivative becomes unbounded at x=0 so the $f^{d+1}(n)$ term becomes unbounded.
There is the exception with the last function where all the rules don't do very well (order 1), the resaon for this is that the function we are integrating isn't continuous, so the error formulas can't be applied.
With the second last function the gauss rules don't do very well either, the reason for this is because the derivative of the function isn't continous. These two last functions aren't smooth.