1 Question 1

$$y' = \frac{y^2}{t} + \frac{y}{t} \qquad t \geq 1, y(1) = -2$$

a) Is the above ODE stable? Justify your answer.
We know that if $f_y < 0$ then the ODE is stable. Here $f(t,y) = \frac{y^2}{t} + \frac{y}{t}$

$$f_y = \frac{2y}{t} + \frac{1}{t} < \frac{-2}{t} + \frac{1}{t} = \frac{-1}{t} < 0$$

Here the first $<$ is using the given fact that $y(t) < -1$ for $t \geq 1$ and the second is from the fact that the ODE is defined on $t \geq 1$, t always positive.

b) Apply (by hand) two steps of forward Euler's method with h = 0.2
$t_0 = 1$, $y_0 = -2$, $h = 0.2$
$t_1 = t_0 + h = 1 + 0.2 = 1.2$
$y_1 = y_0 + hf(t_0, y_0) = -2 + 0.2(\frac{(-2)^2}{1} + \frac{-2}{1}) = -2 + 0.2(2) = -1.6$
$t_2 = t_1 + h = 1.2 + 0.2 = 1.4$
$y_2 = y_1 + hf(t_1, y_1) = -1.6 + 0.2(\frac{(-1.6)^2}{1.2} + \frac{-1.6}{1.2})$

c) Under what conditions for h is forward Euler's method (numerically) stable for the above IVP? What is the maximum h you could use at the first step so that forward Euler's method would be stable? Justify your answers.
$f_y = J = \frac{2y+1}{t}$, $J_i = \frac{2\tau_i+1}{t_i}$ where $\tau_i \in ospr\{y_i, y(t_i)\}$
To be stable must have $h < \frac{-2}{J_i}$
At the fist step $J_i = \frac{2(-2)+1}{1} = -3$ so $h < -2/-3 = 0.6667$
But this might not be most accurate, notice that $y' = \frac{y}{t}(y+1)$ so there is an equilibrium solution for this $y = -1$.
It is then easy to see that the flow field is towards $y = 1$ for $0 < y < -1$ ($y' < 0$, y decreasing) and towards $y = 1$ for $-1 < y$ ($y' > 0$, y increasing).
So we want max h s.t $y_1 < -1$ (also part of given fact for function $y(t)$)
$y_1 = -2 + h(2) < -1 \Rightarrow 2h < 1 \Rightarrow h < 0.5$

d) Apply (by hand) one step of backward Euler's method with h = 0.5. Show the approximate value of y computed at t1.
$t_1 = t_0 + h = 1 + 0.5 = 1.5$
$y_1 = y_0 + hf(t_1, y_1) \Rightarrow y_1 = -2 + 0.5(\frac{y_1^2+y_1}{1.5}) \Rightarrow 0 = -2 + \frac{1}{3}y_1^2 + \frac{1}{3}y_1 - y_1 \Rightarrow 0 = \frac{1}{3}y_1^2 - \frac{2}{3}y_1 - 2 \Rightarrow 0 = y_1^2 - 2y_1 - 6$
$y_1 = \frac{2 \pm \sqrt{4-4(1)(-6)}}{2} = \frac{2 \pm \sqrt{28}}{2} = \frac{2 \pm 2\sqrt{7}}{2} = 1 \pm \sqrt{7}$
We know to take the negative solution since $y < -1$

2 Question 2

a) Write the ODE (1) as a system of first order ODEs of the form $u' = f(t, u)$. Indicate the components of u and f
$u_1(t) = \theta(t) \qquad u_1'(t) = u_2(t)$
$u_2(t) = \theta'(t) \qquad u_2'(t) = \frac{A}{ml^2}cos(\omega t) - \frac{g}{l}sin(u_1(t)) - \frac{\gamma}{ml^2}u_2(t) \qquad (\theta'' = \frac{A}{ml^2}cos(\omega t) - \frac{g}{l}sin(\theta) - \frac{\gamma}{ml^2}\theta')$

$$\bar{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \bar{u'} = \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} \bar{f}(t, \bar{u}) = \begin{bmatrix} u_2 \\ \frac{A}{ml^2}cos(\omega t) - \frac{g}{l}sin(u_1) - \frac{\gamma}{ml^2}u_2 \end{bmatrix} \tag{1}$$

b)Some observations about the results
α) For the first two experiments of part a) we have that for the $\gamma = 2$ and A=0 case the pendulum does come to a stop before t_f ($trst = 33.36$) . This result makes sense since there is a damping force but no driving force to cancel it out. For the $\gamma = 2$ and A=2 case we have that the pendulum doesn't stop oscillating ($trst = t_f = 40$). Naturally since the pendulum doesn't stop oscillating in the second case the values of number of steps and number of function evaluations are higher.
β) Combining and running different combinations of γ and A.
For the cases where A=0, the higher the value of γ goes, the fewer oscillations there are before the pendulum comes to a stop. Interestingly for all the cases where the pendulum does not stop oscillating, the number of steps taken is about 650 and number of function evaluations is about 1000 for all of them.

angvel or θ'_{last} is very small for all. We should be concerned if it wasn't since this value is the value of θ at the last event i (pendulum at it's highest right/up position ie at rest). When looking at the runs for any of the γ's as A is increasing, we see that θ_{last} is increasing. This makes some sense as the driving force grows from one run to the next and the pendulum's assymptotic maximum angle increases. (sort of suggestion on how to do part c) of this question)

Some observations about the plots

thetha and thetha' vs time t

The graph of the first experiment $\gamma = 2$ and A=0 shows that the oscillations (the values of thetha) very quickly approach 0, this makes sense because there is only the damping force acting on the pendulum and no driving.

For the second experiment we see something different, the pendulum quickly goes into periodic motion repeating the same motion again and again. The driving and damping force reach a sort of equilibrium.

The stepsize vs time graph:

From these we see that for both runs in α) the step sizes at the very begining are very small: the first 4 are $6.9247e-07$, the next four are $3.4624e-06$ they stay in the e-5 to e-4 range until step size 21 where its 0.0022 followed by 0.01. After that they seem to become stable, alternating between about 0.06 and 0.07 for both.

Let's look into at one of these low points of the stepsize in the alternating region, for $\gamma = 2$ and A=2 at

t(108:112) = 5.3220, 5.3810, 5.4495, 5.5180, 5.5865

h(105:110) = 0.0590, 0.0590, 0.0590, 0.0590, 0.0685, 0.0685

Examining the event vector we see that at te(6) = 5.4768 and ie(6) = 2, around this t event 2 occurs i.e. the pendulum approaches its top left position before turning to swing back.

It makes sense that the step size decreases at around this point as this is a sort of critical point in where the behaviour of the system changes.

At the directly nearest region where step sizes are large

t(116:120) = 5.8297, 5.8880, 5.9596, 6.0313, 6.1029

h(116:120) = 0.0583, 0.0716, 0.0716, 0.0716, 0.0716

y(116:120, :) =

-0.1631	0.9158
-0.1075	0.9885
-0.0347	1.0344
0.0396	1.0298
0.1116	0.9749

From this we see that at this point the pendulum is going through the rest point (x=0, y=-1) because θ (first column) goes through 0. There is no huge change in the system so the step sizes chosen are slightly larger.

The xy plots:

The second experiment show the pendulum reaching higher to the upper left corner (before stopping and turning back, ie event2 and we stop graphing) then the fist experiment. This is again accounted to the fact that the first experiment has no driving force.

c) Develop a technique (an algorithm) to find the appropriate amplitude

The algorithm/technique is the following. Given the relevant parameters and gamma we find the amplitude A such that the maximum angle is asymptotically θ_{given} at each oscillation by:

0) Sort of a base case if $\gamma = 0$ return $A = 0$ (ideal case, no damping so no force required)

1) Start with relatively small initial $A_0 = 0.25$ such that for this A_0 we have the maximum angle is smaller then θ_{given}

2) Keep doubling A_0 until we have that the maximum angle is larger then θ_{given}. Save the two A_{und}, A_{ovr}, one is over and the other is under

3) Assume that the true A lies somewhere between the two A's from 2). Perform a sort of bisection search for this A

$A_{cur} = (A_{und} + A_{ovr})/2$, if for A_{cur} we are under set $A_{und} = A_{cur}$ else over, set $A_{ovr} = A_{cur}$

It should be noted that the 'about' the maximum angle should be formulated somewhat loose. For example when $\gamma = 0$, the solution should be $A = 0$ for any initial θ_0 ($\theta_{given} = \theta_0$), however simulating this (for $\theta_0 = 1$) we get that the values of thetha at those maximums are

-0.9992, -0.9983, -0.9968, -0.9958, -0.9945, -0.9930, -0.9922, -0.9905, -0.9897, -0.9882, -0.9872, -0.9860, -0.9846, -0.9837, -0.9821, -0.9813, -0.9798, -0.9790, -0.9776

And

1.0000, 0.9988, 0.9973, 0.9965, 0.9948, 0.9940, 0.9925, 0.9915, 0.9902, 0.9888, 0.9879, 0.9863, 0.9855, 0.9840, 0.9831, 0.9818, 0.9805, 0.9795, 0.9780

The numbers are close to 1 but their abs. difference with 1 should be taken into account. So lets say the tolerance will be set to being 0.05 within θ_{given}. Also let's ignore the first say 5 maximum angles, perhaps the pendulum needs some incorrect initial oscillations before swinging forever asymptotically to θ_{given} from starting at θ_0

3 Question 3

Comment on the evolution of the CO2 concentrations with the different models
With the 10-year variable only ranging from 0 to 80, both the quadratic and cubic model Seem to be very similar, follow the same curve and look to be evolving the same way. If the plot is extended to t = 300, the domination of the t^3 term in the cubic model can already be seen.
As for the linear model, it evolves much slower and even visually is seen to be the worst model fitting the given data. (not as close to the given points as the other two models).

4 Question 4

Q) Let A be a n n (row) diagonally dominant matrix with nonzero diagonal entries, and D be the diagonal matrix arising by extracting the diagonal of A (i.e. $D_{ii} = A_{ii}$, i = 1, .. n). Show that no eigenvalue of the matrix $G = D^{-1}(DA)$ is greater than 1 in magnitude.

First because D is a diagonal matrix, then D^{-1} is also a diagonal matrix with entries $\frac{1}{a_{ii}}$
The matrix $D - A$ is almost matrix -A but the only difference is that the diagonal entries of it are 0.
The product $G = D^{-1}(DA)$ is the matrix (DA) with each row scaled by the matrix D^{-1}. So the diagonal entries of G are still 0 ($\frac{0}{a_{ii}} = 0$) and the i,j entry is $\frac{-a_{i,j}}{a_{ii}}$
Now will apply Gerschgorin's theorem for the rows. Take arbitrary $i \in \{1, ..n\}$

$$r_i = \sum_{j=1,j\neq i}^{n} |a_{i,j}| = \sum_{j=1,j\neq i}^{n} |\frac{-a_{i,j}}{a_{ii}}| = \frac{1}{|a_{ii}|} \sum_{j=1,j\neq i}^{n} |a_{i,j}| \leq \frac{1}{|a_{ii}|}|a_{ii}| = 1$$

The $\leq$ comes from the fact that A is (row) diagonally dominant matrix, $|a_{i,i}| \geq \sum_{j=1,j\neq i}^{n} |a_{i,j}|$
Now since all the diagonal entries of G are 0, the disk R_i is centred at the origin and has radius at most 1.
This is shown for an arbitrary row i, meaning all the disks R_i are like this.

Therefore the eigenvalues must lie in the union of all these disks, union of disks radius at most 1 centred on the origin is again at most radius 1.
Hence the magnitude of all the eigenvalues is at most 1. $\lambda = a + bi$, and $a^2 + b^2 \leq 1$ (in circle rad at most 1), then $|\lambda| = \sqrt{a^2 + b^2} \leq 1$

5 Question 5

All the asked indicated values are on the attached a3q5scriptOutput.txt
Found approx. $\lambda^{(1)}$ to λ_3 using $\mu^{(1)}$ by the fact that evals of A^{-1} are reciprocals of evals of A
The smallest eval of A, is the largest eval of A^{-1} since (1/ ..)