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ON SOME PROPERTIES OF KOROBOV POLYNOMIALS

We represent Korobov polynomials as paradeterminants of triangular matrices and prove some of their properties.

Key words and phrases: Korobov polynomial, triangular matrix, paradeterminant, partition polynomial.

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INTRODUCTION

Korobov in [2] introduces polynomials of a special form, which are discrete analogs of Bernoulli polynomials. These polynomials are used to derive some interpolation formulas of many variables and a discrete analog of the Euler summation formula [3]. Therefore, it is topical to conduct further research of their properties.

1 OVERVIEW ON TRIANGULAR MATRICES AND THEIR PARADETERMINANTS

Definition 1 ([4]). *A triangular table of numbers from some field K*

$$A = \begin{pmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}_n \tag{1}$$

is called a triangular matrix, and a number n — its order.

Note, that in our understanding a triangular matrix is not a matrix in its usual sense, it is a triangular but not rectangular table of numbers.

To every elements a_{ij} of the matrix (1) we correspond the $(i-j+1)$ elements a_{ik}, $k = j, \ldots, i$, which are called the *derived elements* of the matrix, generated by the *key element* a_{ij}.

The product of all derived elements generated by the element a_{ij} is denoted by $\{a_{ij}\}$ and is called the *factorial product* of the key element a_{ij}, i.e.

$$\{a_{ij}\} = \prod_{k=j}^{i} a_{ik}.$$

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Definition 2. *The paradeterminant and the parapermanent of the triangular matrix*

$$A = \begin{pmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \dots & \ddots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

are, respectively, the functions

$$\mathrm{ddet}(A) = \sum_{r=1}^{n} \sum_{p_1+\ldots+p_r=n} (-1)^{n-r} \prod_{s=1}^{r} \{a_{p_1+\ldots+p_s, p_1+\ldots+p_{s-1}+1}\},$$

$$\mathrm{pper(A)} = \sum_{r=1}^{n} \sum_{p_1+\ldots+p_r=n} \prod_{s=1}^{r} \{a_{p_1+\ldots+p_s, p_1+\ldots+p_{s-1}+1}\}.$$

To every element a_{ij} of the triangular matrix (1) we correspond the triangular matrix with this element in the bottom left corner, which is called a *corner* of the triangular matrix and denoted by $R_{ij}(A)$. It is obvious that the corner $R_{ij}(A)$ is the triangular matrix of the $(i-j+1)$-th order. The corner $R_{ij}(A)$ comprises only those elements a_{rs} of the triangular matrix (1), the indexes of which satisfy the relations $j \leqslant s \leqslant r \leqslant i$.

The parafunctions of triangular matrices can be decomposed by the elements of their last row:

$$ddet\,(A) = \sum_{s=1}^{n} (-1)^{n+s} \{a_{ns}\} \cdot ddet\,(R_{s-1,1}),$$

$$pper\,(A) = \sum_{s=1}^{n} \{a_{ns}\} \cdot pper(R_{s-1,1}).$$

Proposition 1. *The following is true:*

$$\mathrm{pper}\,(A) := \begin{bmatrix} a_1 & & & & & & \\ a_2 & a_1 & & & & & \\ \vdots & \dots & \ddots & & & & \\ a_n & a_{n-1} & \dots & a_1 & & & \\ 0 & a_n & \dots & a_2 & a_1 & & \\ \vdots & \dots & \dots & \dots & \dots & \ddots & \\ 0 & 0 & \dots & a_n & a_{n-1} & \dots & a_1 \end{bmatrix}_m$$

$$= \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=m} \frac{k!}{\lambda_1!\lambda_2! \cdot \ldots \cdot \lambda_n!} a_1^k a_2^{\lambda_2+\ldots+\lambda_n} \cdot \ldots \cdot a_n^{\lambda_n},$$

and

$$\mathrm{ddet}\,(A) := \left\langle \begin{matrix} a_1 & & & & & & \\ a_2 & a_1 & & & & & \\ \vdots & \dots & \ddots & & & & \\ a_n & a_{n-1} & \dots & a_1 & & & \\ 0 & a_n & \dots & a_2 & a_1 & & \\ \vdots & \dots & \dots & \dots & \dots & \ddots & \\ 0 & 0 & \dots & a_n & a_{n-1} & \dots & a_1 \end{matrix} \right\rangle_m \tag{2}$$

$$= \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=m} (-1)^{n-k} \frac{k!}{\lambda_1!\lambda_2! \cdot \ldots \cdot \lambda_n!} a_1^k a_2^{\lambda_2+\ldots+\lambda_n} \cdot \ldots \cdot a_n^{\lambda_n},$$

where $k = \lambda_1 + \lambda_2 + \ldots + \lambda_n$.

For more detailed information on triangular matrices and their paradeterminants, the reader is referred to [4], [5].

2 KOROBOV POLYNOMIALS AND PARADETERMINANTS

In [2] the Korobov numbers P_n and polynomials $P_n(x)$ are defined by the equalities

$$P_0 = 1,\ \binom{p}{1} P_n + \ldots + \binom{p}{n+1} P_0 = 0,\ n \geqslant 1; \tag{3}$$

$$P_0(x) = 1,\ P_n(x) = P_0 \binom{x}{n} + \ldots + P_{n-1} \binom{x}{1} + P_n,\ n \geqslant 1.$$

We shall write the Korobov numbers as the paradeterminant of the triangular matrix.

Theorem 1. *The following is true*

$$P_n = (-1)^n \left\langle \begin{array}{cccc} \frac{p-1}{2} & & & \\ \frac{p-2}{3} & \frac{p-1}{2} & & \\ \vdots & \cdots & \ddots & \\ \frac{p-n}{n+1} & \frac{p-n+1}{n} & \cdots & \frac{p-1}{2} \end{array} \right\rangle_n . \tag{4}$$

Proof. Let us divide the second equality (3) by $\binom{p}{1}$, and we get the recurrence equality

$$P_n + a_1 P_{n-1} + a_2 P_{n-2} + \ldots + a_{n-1} P_1 + a_n P_0 = 0,$$

where

$$a_i = \frac{(p-1)^{\underline{i}}}{(i+1)!}.$$

The last equality, according to [1], has the solution

$$P_n = (-1)^n \left\langle \begin{array}{cccc} a_1 & & & \\ \frac{a_2}{a_1} & a_1 & & \\ \vdots & \cdots & \ddots & \\ \frac{a_n}{a_{n-1}} & \frac{a_{n-1}}{a_{n-2}} & \cdots & a_1 \end{array} \right\rangle_n . \tag{5}$$

That is why, in virtue of the equality

$$\frac{a_i}{a_{i-1}} = \frac{p-i}{i+1},$$

the equality (4) is true. □

It should be noted that due to the connection between the paradeterminants of triangular matrices and the parapermanents of some triangular matrices, the Korobov numbers can also be written as the parapermanent of a triangular matrix.

By now there are several presentations of some algebraic objects as partition polynomials (e.g., Waring's formula presenting power sums through elementary symmetric polynomials). The following theorem obviously presents the Korobov numbers with the help of the partition polynomials.

Theorem 2. *The following is true:*

$$P_n = \sum_{\lambda_1+\ldots+n\lambda_n=n} (-1)^k \frac{k!}{\lambda_1!\lambda_2!2!^{\lambda_1}\ldots\lambda_n!(n+1)!^{\lambda_n}}(p-1)^k(p-2)^{k-\lambda_1}\ldots(p-n)^{\lambda_n}, \quad n=1,2,\ldots. \tag{6}$$

Proof. Considering the equality (5) and the identity (2), after some simplifications, we get the presentation of the Korobov numbers as partition polynomials (6). □

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Пилипів В.М., Малярчук А.Р. *Про деякі властивості многочленів Коробова* // Карпатські матем. публ. — 2014. — Т.6, №1. — С. 130–133.

Стаття присвячена представленню многочленів Коробова у вигляді парадетермінантів трикутних матриць і встановленню деяких їхніх властивостей.

Ключові слова і фрази: многочлен Коробова, трикутна матриця, парадетермінант, многочлен розбиттів.

Пылыпив В.М., Малярчук А.Р. *О некоторых свойствах многочленов Коробова* // Карпатские матем. публ. — 2014. — Т.6, №1. — С. 130–133.

Заметка посвящена представлению многочленов Коробова в виде парадетерминантов треугольных матриц и установлению некоторых их свойств.

Ключевые слова и фразы: многочлен Коробова, треугольная матрица, парадетерминант, многочлен разбиений.