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CALCULATION ALGORITHM OF RATIONAL ESTIMATIONS OF RECURRENCE PERIODICAL FOURTH ORDER FRACTION

Recurrence fourth order fractions are studied. Connection with algebraic fourth order equations is established. Calculation algorithms of rational contractions of such fractions are built.

Key words and phrases: periodical recurrence fraction, triangular matrix, parapermanent, parade-terminant, rational approximation.

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INTRODUCTION

Continued fractions are generalized by quite a few Ukrainian [12, 13] and foreign mathematicians [1]–[11], [14].

The important conditions for generalization of continued fractions are following:

- construction of an easy-to-use algebraic object, the form of which would be similar to the form of continued fractions, would make it possible to naturally introduce the notion of their order and to single out the class of periodic objects generalizing periodic continued fractions;
- the algorithm for calculating the value of rational contractions of mathematical objects is to be simple in realization and efficient;
- by analogy with periodic chain fractions, random periodic algebraic objects of higher orders are to be of the forms of some algebraic irrationalities of higher orders.

In [15] it is suggested new generalization of continued fractions — recurrence fractions, which satisfy the above-mentioned conditions. In addition, the connection between singly periodic recurrence fractions of order n and algebraic equations of order n has been established. Recurrence fractions of order three have been studied in [16].

This article focuses on recurrence fractions of order four, proves their connection with corresponding algebraic equations of order four and determines algorithms for constructing rational approximations of order four.

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1 PERIODIC RECURRENCE FRACTIONS OF ORDER 4

A recurrence fraction of order four takes the form

$$\left[\begin{array}{c|cccccccc} q_1 & & & & & & & \\ \frac{p_2}{q_2} & q_2 & & & & & & \\ \frac{r_3}{p_3} & \frac{p_3}{q_3} & q_3 & & & & & \\ \frac{s_4}{r_4} & \frac{r_4}{p_4} & \frac{p_4}{q_4} & q_4 & & & & \\ 0 & \frac{s_5}{r_5} & \frac{r_5}{p_5} & \frac{p_5}{q_5} & q_5 & & & \\ \vdots & \dots & \dots & \dots & \dots & \ddots & & \\ 0 & 0 & \dots & \frac{s_m}{r_m} & \frac{r_m}{p_m} & \frac{p_m}{q_m} & q_m & \\ \vdots & \dots & \dots & \dots & \dots & \dots & \dots & \ddots \end{array} \right]_{\infty} \cdot \quad (1)$$

Its rational contractions

$$\frac{P_n}{Q_n} = \left[\begin{array}{c|cccccccc} q_1 & & & & & & & \\ \frac{p_2}{q_2} & q_2 & & & & & & \\ \frac{r_3}{p_3} & \frac{p_3}{q_3} & q_3 & & & & & \\ \frac{s_4}{r_4} & \frac{r_4}{p_4} & \frac{p_4}{q_4} & q_4 & & & & \\ 0 & \frac{s_5}{r_5} & \frac{r_5}{p_5} & \frac{p_5}{q_5} & q_5 & & & \\ \vdots & \dots & \dots & \dots & \dots & \ddots & & \\ 0 & 0 & \dots & \frac{s_n}{r_n} & \frac{r_n}{p_n} & \frac{p_n}{q_n} & q_n & \end{array} \right]_n$$

satisfy the recurrence equations

$$\begin{aligned} P_n &= q_n P_{n-1} + p_n P_{n-2} + r_n P_{n-3} + s_n P_{n-4}, & n &= 1, 2, 3, \dots, \\ Q_n &= q_n Q_{n-1} + p_n Q_{n-2} + r_n Q_{n-3} + s_n Q_{n-4}, & n &= 2, 3, 4, \dots \end{aligned} \quad (2)$$

with the initial conditions

$$P_0 = 1, \quad P_{i<0} = 0,$$

$$Q_1 = 1, \quad Q_{i<1} = 0.$$

Definition 1. The recurrence fraction (1) of order 4, the elements of which satisfy the conditions

$$p_{rk+m} = p_m, \quad q_{rk+m} = q_m, \quad r_{rk+m} = r_m, \quad s_{rk+m} = s_m, \quad m = 1, 2, \dots, k, \quad r = 0, 1, 2, \dots \quad (3)$$

is a periodic recurrence fraction of order 4 with the period k .

We shall determine the connections between periodic recurrence fractions of order four and real positive roots of quartic equations.

1. Consider a singly periodic recurrence fraction of order four. Let us decompose the permanent of the numerator of the rational contraction

$$\frac{P_n}{Q_n} = \left[\begin{array}{c|cccccc} q & & & & & \\ \frac{p}{q} & q & & & & \\ \frac{r}{p} & \frac{p}{q} & q & & & \\ \frac{s}{r} & \frac{r}{p} & \frac{p}{q} & q & & \\ 0 & \frac{s}{r} & \frac{r}{p} & \frac{p}{q} & q & \\ \vdots & \dots & \dots & \dots & \dots & \ddots \\ 0 & 0 & 0 & 0 & 0 & \dots & q \\ 0 & 0 & 0 & 0 & 0 & \dots & \frac{p}{q} & q \\ 0 & 0 & 0 & 0 & 0 & \dots & \frac{r}{p} & \frac{p}{q} & q \\ 0 & 0 & 0 & 0 & 0 & \dots & \frac{s}{r} & \frac{r}{p} & \frac{p}{q} & q \end{array} \right]_{n+1}$$

by the elements of the first column. We get the equality

$$\frac{P_n}{Q_n} = q + \frac{p}{\frac{P_{n-1}}{Q_{n-1}}} + \frac{r}{\frac{P_{n-1}}{Q_{n-2}}} + \frac{s}{\frac{P_{n-1}}{Q_{n-3}}} = q + \frac{p}{\frac{P_{n-1}}{Q_{n-1}}} + \frac{r}{\frac{P_{n-1}}{Q_{n-1}} \cdot \frac{P_{n-2}}{Q_{n-2}}} + \frac{s}{\frac{P_{n-1}}{Q_{n-1}} \cdot \frac{P_{n-2}}{Q_{n-2}} \cdot \frac{P_{n-3}}{Q_{n-3}}}. \quad (4)$$

Let us take the limit

$$\lim_{n \rightarrow \infty} \frac{P_n}{Q_n} = \lim_{n \rightarrow \infty} \frac{P_n}{P_{n-1}} = x,$$

then the equality (4) is written as $x = q + \frac{p}{x} + \frac{r}{x^2} + \frac{s}{x^3}$, or $x^4 = qx^3 + px^2 + rx + s$. One of the roots of this equation is

$$x = \frac{q}{4} + \frac{\sqrt{\alpha + 2y} + \sqrt{-\left(3\alpha + 2y + \frac{2\beta}{\sqrt{\alpha + 2y}}\right)}}{2},$$

where

$$y = -\frac{5}{6}\alpha + \sqrt[3]{-\frac{Q}{2} + \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}} - \frac{P}{3 \cdot \sqrt[3]{-\frac{Q}{2} + \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}}}, \quad \alpha = -\frac{3}{8}q^2 - p,$$

$$\beta = -\frac{q^3}{8} - \frac{qp}{2} - r, \quad \gamma = -\frac{3q^4}{256} - \frac{pq^2}{16} - \frac{qr}{4} - s, \quad P = -\frac{\alpha^2}{12} - \gamma, \quad Q = -\frac{\alpha^3}{108} - \frac{\alpha\gamma}{3} - \frac{\beta^2}{8}.$$

It is easy to establish that if $q = 4$, $p = -6$, $r = 4$, the singly periodic fraction will represent the irrationality $\sqrt[4]{1+s}$.

Example 1. If

$$q = 4, \quad p = -6, \quad r = 4, \quad s = 2,$$

then the recurrence fraction is written as

$$\left[\begin{array}{c|cccccc} 4 & & & & & \\ -\frac{3}{2} & 4 & & & & \\ -\frac{2}{3} & -\frac{3}{2} & 4 & & & \\ \frac{1}{2} & -\frac{2}{3} & -\frac{3}{2} & 4 & & \\ 0 & \frac{1}{2} & -\frac{2}{3} & -\frac{3}{2} & 4 & \\ 0 & 0 & \frac{1}{2} & -\frac{2}{3} & -\frac{3}{2} & 4 \\ \vdots & \dots & \dots & \dots & \dots & \ddots \end{array} \right]_{\infty}.$$

The relevant algebraic equation of order four is of the form $x^4 = 4x^3 - 6x^2 + 4x + 2$. The rational approximations to the maximum modulo root $x = 1 + \sqrt[4]{3} \approx 2,31607401295249246$ of this equation can be found with the help of the linear recurrence relations of order four

$$P_n = 4P_{n-1} - 6P_{n-2} + 4P_{n-3} + 2P_{n-4}, \quad P_0 = 1,$$

while $x = \lim_{m \rightarrow \infty} \frac{P_m}{P_{m-1}}$.

Here are the first 35 rational approximations to this root:

$$\begin{array}{lllll} u_1 = 4, & u_8 = 1164, & u_{15} = 404736, & u_{22} = 144235520, & u_{29} = 51545829376, \\ u_2 = 10, & u_9 = 2704, & u_{16} = 937104, & u_{23} = 334031360, & u_{30} = 119382376448, \\ u_3 = 20, & u_{10} = 6136, & u_{17} = 2165568, & u_{24} = 773463744, & u_{31} = 276492099584, \\ u_4 = 38, & u_{11} = 13936, & u_{18} = 5006752, & u_{25} = 1791122688, & u_{32} = 640367841536, \\ u_5 = 80, & u_{12} = 32072, & u_{19} = 11591488, & u_{26} = 4148304768, & u_{33} = 1483139933184, \\ u_6 = 192, & u_{13} = 74624, & u_{20} = 26861920, & u_{27} = 9608400640, & u_{34} = 3435085834752, \\ u_7 = 480, & u_{14} = 174080, & u_{21} = 62256896, & u_{28} = 22255192192, & u_{35} = 7955959305216, \end{array}$$

while $\frac{u_{35}}{u_{34}} = \frac{7955959305216}{3435085834752} \approx 2.3160874$.

2. Consider a doubly periodic recurrence fraction of order four

$$\left[\begin{array}{c|cccccc} q_1 & & & & & & \\ p_2 & & & & & & \\ q_2 & q_2 & & & & & \\ r_1 & p_1 & q_1 & & & & \\ p_1 & r_2 & p_2 & q_2 & & & \\ s_2 & p_2 & q_2 & q_2 & & & \\ r_2 & s_1 & r_1 & p_1 & q_1 & & \\ 0 & r_1 & p_1 & q_1 & q_1 & & \\ 0 & 0 & s_2 & r_2 & p_2 & q_2 & \\ \vdots & \dots & \dots & \dots & \dots & \dots & \ddots \end{array} \right]_{\infty},$$

where q_i, p_i, r_i, s_i are some rational positive numbers.

Let us decompose the parapermanent of the numerator of the rational contraction by the elements of the first column

$$\begin{aligned} \frac{[q_1]_n}{[q_2]_{n-1}} &= \frac{q_1[q_2]_{n-1} + p_2[q_1]_{n-2} + r_1[q_2]_{n-3} + s_2[q_1]_{n-4}}{[q_2]_{n-1}} \\ &= q_1 + \frac{p_2}{\frac{[q_2]_{n-1}}{[q_1]_{n-2}}} + \frac{r_1}{\frac{[q_2]_{n-1}}{[q_1]_{n-2}} \cdot \frac{[q_1]_{n-2}}{[q_2]_{n-3}}} + \frac{s_1}{\frac{[q_2]_{n-1}}{[q_1]_{n-2}} \cdot \frac{[q_1]_{n-2}}{[q_2]_{n-3}} \cdot \frac{[q_2]_{n-3}}{[q_1]_{n-4}}}. \end{aligned} \quad (5)$$

In this equality, the parapermanent of order i with the upper element $q_j, j = 1, 2$ is denoted by $[q_j]_i$. Likewise, we decompose the numerator of the fraction $\frac{[q_2]_{n-1}}{[q_1]_{n-2}}$ by the elements of the first column

$$\frac{[q_2]_{n-1}}{[q_1]_{n-2}} = q_2 + \frac{p_1}{\frac{[q_1]_{n-2}}{[q_2]_{n-3}}} + \frac{r_2}{\frac{[q_1]_{n-2}}{[q_2]_{n-3}} \cdot \frac{[q_2]_{n-3}}{[q_1]_{n-4}}} + \frac{s_1}{\frac{[q_1]_{n-2}}{[q_2]_{n-3}} \cdot \frac{[q_2]_{n-3}}{[q_1]_{n-4}} \cdot \frac{[q_1]_{n-4}}{[q_2]_{n-5}}}. \quad (6)$$

Let us take the limits

$$\lim_{m \rightarrow \infty} \frac{[q_1]_m}{[q_2]_{m-1}} = x, \quad \lim_{m \rightarrow \infty} \frac{[q_2]_m}{[q_1]_{m-1}} = y.$$

Passing n to infinity in the equalities (5), (6), we get simultaneous equations

$$\begin{cases} x = q_1 + \frac{p_2}{y} + \frac{r_1}{xy} + \frac{s_2}{xy^2}, \\ y = q_2 + \frac{p_1}{x} + \frac{r_2}{xy} + \frac{s_1}{x^2y}, \end{cases}$$

from which we find that

$$y = \frac{q_2x + p_1 + \sqrt{(q_2x + p_1)^2 + 4(r_2x + s_1)}}{2x},$$

and x is the positive root of the equation of order four

$$\begin{aligned} (q_2p_2r_2 + r_2^2 - q_2^2s_2)x^4 &= (q_1q_2p_2r_2 + p_2^2r_2 + 2q_1r_2^2 + 2q_2p_1s_2 + 2r_2s_2 - p_1p_2r_2 - q_2r_1r_2 \\ &- 2r_2s_1 - q_2p_2s_1 - q_1q_2^2s_2 - q_2p_2s_2)x^3 + (q_1p_1p_2r_2 + q_1q_2r_1r_2 + 2p_2r_1r_2 + p_2^2s_1 + q_1q_2p_2s_1 \\ &+ 4q_1r_2s_1 + p_1^2s_2 + 2s_1s_2 - q_1^2r_2^2 - p_1r_1r_2 - q_2r_1s_1 - p_1p_2s_1 - s_1^2 - s_2^2 - 2q_1q_2p_1s_2 \\ &- 2q_1r_2s_2 - p_1p_2s_2 - q_2r_1s_2)x^2 + (q_1p_1p_2s_1 + q_1q_2r_1s_1 + 2p_2r_1s_1 + r_1^2r_2 + q_1p_1r_1r_2 \\ &+ 2q_1s_1^2 - 2q_1^2r_2s_1 - p_1r_1s_1 - q_1p_1^2s_2 - 2q_1s_1s_2 - p_1r_1s_2)x + s_1(q_1^2s_1 - q_1p_1r_1 - r_1^2). \end{aligned} \quad (7)$$

Thus, the following theorem is proved.

Theorem 1. If q_i, p_i, r_i, s_i are some rational positive numbers and there are limits

$$\lim_{m \rightarrow \infty} \frac{[q_1]_m}{[q_2]_{m-1}} = x, \quad \lim_{m \rightarrow \infty} \frac{[q_2]_m}{[q_1]_{m-1}} = y,$$

then x is the positive root of the equation (7) of order four.

Example 2. If $q_1 = 3, p_1 = 3, r_1 = 3, s_1 = 3, q_2 = 2, p_2 = 2, r_2 = 2, s_2 = 2$, then the recurrence fraction is written as

$$\left[\begin{array}{c|cccccc} 3 & & & & & \\ 2 & 2 & & & & \\ 3 & 3 & 3 & & & \\ 2 & 2 & 2 & 2 & & \\ 3 & 3 & 3 & 3 & 3 & \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 & 3 & \\ 0 & 0 & 2 & 2 & 2 & 2 \\ \vdots & \dots & \dots & \dots & \dots & \ddots \end{array} \right]$$

and the rational contractions, which approximate the maximum modulo real root

$$x = \frac{1}{2} + \frac{1}{2} \left(\sqrt{-\frac{29}{4} + 2y} + \sqrt{\frac{87}{4} - 2y + \frac{27}{\sqrt{-\frac{29}{4} + 2y}}} \right) \approx 3,978743113,$$

where

$$y = \frac{1}{24} \left(145 + \sqrt[3]{-63197 + 972\sqrt{7226}} - \frac{1415}{\sqrt[3]{-63197 + 972\sqrt{7226}}} \right),$$

of the fourth order equation

$$4x^4 = 8x^3 + 23x^2 + 27x + 27,$$

are equal to

$$\begin{aligned} \delta_1 &= \frac{3}{1} = 3, \quad \delta_2 = \frac{8}{2} = 4, \quad \delta_3 = \frac{36}{9} = 4, \quad \delta_4 = \frac{96}{24} = 4, \quad \delta_5 = \frac{429}{108} \approx 3,9722, \\ \delta_6 &= \frac{1138}{286} \approx 3,9790, \quad \delta_7 = \frac{5097}{1281} \approx 3,97892, \quad \delta_8 = \frac{13520}{3398} \approx 3,97881, \\ \delta_9 &= \frac{60552}{15219} \approx 3,978711, \quad \delta_{10} = \frac{160614}{40368} \approx 3,9787455, \quad \delta_{11} = \frac{719349}{180798} \approx 3,9787442, \\ \delta_{12} &= \frac{1908070}{479566} \approx 3,97874328, \quad \delta_{13} = \frac{8545755}{2147853} \approx 3,97874296, \quad \delta_{14} = \frac{22667576}{5697170} \approx 3,97874313, \\ \delta_{15} &= \frac{101522250}{25516161} \approx 3,978743118, \quad \delta_{16} = \frac{269287302}{67681500} \approx 3,9787431129. \end{aligned}$$

2 ALGORITHM FOR CALCULATING RATIONAL CONTRACTIONS OF PERIODIC RECURRENCE FRACTIONS OF ORDER FOUR

Let us construct a new algorithm for calculating rational contractions of periodic recurrence fractions of order four.

Let k be the period of a recurrence fraction, and n — the order of the parapermanent of its rational contraction, while $n = sk$, $s = 1, 2, 3, \dots$

Then the following theorem is true.

Theorem 2. *The rational contraction*

$$\delta_n = \frac{P_n}{O_n}$$

of the periodic recurrence fraction (1) of order four, with the period $k \geq 2$, the elements of which satisfy the conditions (3), is equal to the value of the expression

$$q_0 + p_1 \cdot \frac{B_{sk-1}^{s-1}}{A_{sk}^s} + r_2 \cdot \frac{C_{sk-2}^{s-1}}{A_{sk}^s} + s_2 \cdot \frac{D_{sk-3}^{s-1}}{A_{sk}^s}, \quad (8)$$

where A_{sk}^s , B_{sk-1}^{s-1} , C_{sk-2}^{s-1} and D_{sk-3}^{s-1} are defined by the recurrence equalities

$$A_{sk}^{s=s_3} \varphi_{k-1} D_{k(s-1)-3}^{s-2} + (s_2 \varphi_{k-2} + r_2 \varphi_{k-1}) C_{k(s-1)-2}^{s-2} + (s_1 \varphi_{k-3} + r_1 \varphi_{k-2} + p_1 \varphi_{k-1}) B_{k(s-1)-1}^{s-2} + \varphi_k A_{k(s-1)}^{s-1}, \quad (9)$$

$$B_{sk-1}^{s-1} = s_3 \psi_{k-2} D_{k(s-1)-3}^{s-2} + (s_2 \psi_{k-3} + r_2 \psi_{k-2}) C_{k(s-1)-2}^{s-2} + (s_1 \psi_{k-4} + r_1 \psi_{k-3} + p_1 \psi_{k-2}) B_{k(s-1)-1}^{s-2} + \psi_{k-1} A_{k(s-1)}^{s-1}, \quad (10)$$

$$C_{sk-2}^{s-1} = s_3 \tau_{k-3} D_{k(s-1)-3}^{s-2} + (s_2 \tau_{k-4} + r_2 \tau_{k-3}) C_{k(s-1)-2}^{s-2} + (s_1 \tau_{k-5} + r_1 \tau_{k-4} + p_1 \tau_{k-3}) B_{k(s-1)-1}^{s-2} + \tau_{k-2} A_{k(s-1)}^{s-1}, \quad (11)$$

$$D_{sk-3}^{s-1} = s_3 \zeta_{k-4} D_{k(s-1)-3}^{s-2} + (s_2 \zeta_{k-5} + r_2 \zeta_{k-4}) C_{k(s-1)-2}^{s-2} + (s_1 \zeta_{k-6} + r_1 \zeta_{k-5} + p_1 \zeta_{k-4}) B_{k(s-1)-1}^{s-2} + \zeta_{k-3} A_{k(s-1)}^{s-1}, \quad (12)$$

where

$$\varphi_k = \begin{bmatrix} \frac{q_1}{p_2} & & & & & & & & & & \\ \frac{q_2}{r_3} & q_2 & & & & & & & & & \\ \frac{p_3}{s_4} & \frac{p_3}{r_4} & q_3 & & & & & & & & \\ \frac{s_4}{r_4} & \frac{r_4}{p_4} & \frac{p_4}{q_4} & q_4 & & & & & & & \\ 0 & \frac{s_5}{r_5} & \frac{r_5}{p_5} & \frac{p_5}{q_5} & q_5 & & & & & & \\ \vdots & \dots & \dots & \dots & \dots & \ddots & & & & & \\ 0 & 0 & 0 & 0 & 0 & \dots & q_{k-3} & & & & \\ 0 & 0 & 0 & 0 & 0 & \dots & \frac{p_{k-2}}{r_{k-2}} & q_{k-2} & & & \\ 0 & 0 & 0 & 0 & 0 & \dots & \frac{q_{k-2}}{r_{k-1}} & \frac{p_{k-1}}{q_{k-1}} & q_{k-1} & & \\ 0 & 0 & 0 & 0 & 0 & \dots & \frac{r_{k-1}}{p_{k-1}} & \frac{r_k}{p_k} & \frac{p_k}{q_k} & q_k & \\ 0 & 0 & 0 & 0 & 0 & \dots & \frac{s_k}{r_k} & \frac{r_k}{p_k} & \frac{p_k}{q_k} & q_k & \end{bmatrix}_k, \quad (13)$$

(14)

(15)

(16)

and if $k = 2, 3, 4$, we assume that

$$\tilde{\zeta}_{<0} = \tau_{<0} = \psi_{<0} = \varphi_{<0} = 0,$$

$$\varphi_0 = \psi_0 = \tau_0 = \xi_0 = 1.$$

Proof. If $n = sk$ the numerator and the dominator of the n -th rational contraction of the periodic recurrence fraction (1) of order four, with the period of $k \geq 2$, the elements of which satisfy the conditions (3), are respectively in the form of

Let us denote the parapermanent, formed from the parapermanent (17) as a result of deleting the first column, by A_{sk}^s , the parapermanent, formed as a result of deleting the first two columns, — by B_{sk-1}^{s-1} , the parapermanent, formed as a result of deleting the first three columns, — by C_{sk-2}^{s-1} , and the parapermanent, formed as a result of deleting the first four columns, — by D_{sk-3}^{s-1} (in the four cases, the superscript denotes the number of complete periods containing these parapermanents).

Let us decompose the parapermanent (17) by the elements of the first column and get the equality

$$P_{sk} = q_0 A_{sk}^s + p_1 B_{sk-1}^{s-1} + r_2 C_{sk-2}^{s-1} + s_3 D_{sk-3}^{s-1}. \quad (18)$$

Let us decompose the parapermanent A_{sk}^s by the elements of the inscribed rectangular table $T(k+1)$, then we get the recurrence (9). In the same way, let us decompose the parapermanents B_{sk-1}^{s-1} , C_{sk-2}^{s-1} , and D_{sk-3}^{s-1} by the elements of the tables $T(k)$, $T(k-1)$, and $T(k-2)$. At that we get the recurrences (10), (11), (12).

As $Q_{sk} = A_{sk}^s$, considering (18), we conclude that the rational contraction $\delta_n = \frac{P_n}{Q_n}$ of the periodic recurrence fraction is equal to

$$\frac{P_{sk}}{Q_{sk}} = \frac{q_0 A_{sk}^s + p_1 B_{sk-1}^{s-1} + r_2 C_{sk-2}^{s-1} + s_3 D_{sk-3}^{s-1}}{A_{sk}^s} = q_0 + p_1 \frac{B_{sk-1}^{s-1}}{A_{sk}^s} + r_2 \frac{C_{sk-2}^{s-1}}{A_{sk}^s} + s_3 \frac{D_{sk-3}^{s-1}}{A_{sk}^s}.$$

□

Example 3. Let us have a periodic recurrence fraction of order four with the period, where $q_1 = 1, p_2 = 1, r_3 = 1, s_4 = 1, q_2 = 1, p_3 = 1, r_4 = 1, s_5 = 1, q_3 = 2, p_4 = 2, r_5 = 2, s_1 = 2, q_4 = 1, p_5 = 1, r_1 = 1, s_2 = 1, q_5 = 2, p_1 = 2, r_2 = 2, s_3 = 2$.

This periodic recurrence fraction approximates to the maximum modulo real root

$$x = \frac{1}{4} + \frac{1}{2} \left(\sqrt{-\frac{11}{8} + 2y} + \sqrt{\frac{33}{8} - 2y + \frac{\frac{109}{36}}{\sqrt{-\frac{11}{8} + 2y}}} \right) \approx 1,969558741906025,$$

where

$$y = \frac{1}{2} \left(\frac{55}{24} + \frac{1}{9} \sqrt[3]{-2007 + 144\sqrt{622}} - \frac{23}{\sqrt[3]{-2007 + 144\sqrt{622}}} \right),$$

the equation of order four

$$9x^4 - 9x^3 - 9x^2 - 8x - 16 = 0. \quad (19)$$

Let us find the rational contractions (20) of the relevant recurrence fraction first with the help of the recurrences (2), and then by the algorithm of the theorem 2.

$$\frac{P_n}{Q_n} = \left[\begin{array}{c|cccccccccccc} 1 & & & & & & & & & & & \\ 1 & 1 & & & & & & & & & & \\ 1 & \frac{1}{2} & 2 & & & & & & & & & \\ 1 & \frac{1}{2} & 2 & 1 & & & & & & & & \\ 0 & \frac{1}{2} & 2 & \frac{1}{2} & 2 & & & & & & & \\ 0 & 0 & 2 & \frac{1}{2} & 2 & 1 & & & & & & \\ 0 & 0 & 0 & \frac{1}{2} & 2 & 1 & 1 & & & & & \\ 0 & 0 & 0 & 0 & 2 & 1 & \frac{1}{2} & 2 & & & & \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} & 2 & 1 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 2 & \frac{1}{2} & 2 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & \frac{1}{2} & 2 & 1 & \\ \vdots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \ddots & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 2 \end{array} \right] \quad (20)$$

By means of the recurrences (2) we shall have:

$$\delta_1 = \frac{2}{1} = 2, \quad \delta_7 = \frac{396}{201} \approx 1.970149, \quad \delta_{13} = \frac{49692}{25230} \approx 1.96956005,$$

$$\begin{aligned}
\delta_2 &= \frac{6}{3} = 2, \quad \delta_8 = \frac{768}{390} \approx 1.9692308, \quad \delta_{14} = \frac{146807}{74538} \approx 1.96955915, \\
\delta_3 &= \frac{12}{6} = 2, \quad \delta_9 = \frac{2269}{1152} \approx 1.969618, \quad \delta_{15} = \frac{289145}{146807} \approx 1.96955867, \\
\delta_4 &= \frac{35}{18} \approx 1.9444, \quad \delta_{10} = \frac{4469}{2269} \approx 1.9695901, \quad \delta_{16} = \frac{560950}{284810} \approx 1.969558653, \\
\delta_5 &= \frac{69}{35} \approx 1.97142, \quad \delta_{11} = \frac{8670}{4402} \approx 1.96955929, \quad \delta_{17} = \frac{1657236}{841425} \approx 1.969558784, \\
\delta_6 &= \frac{134}{68} \approx 1.97059, \quad \delta_{12} = \frac{25614}{13005} \approx 1.96955017, \quad \delta_{18} = \frac{3215088}{1632390} \approx 1.969558745, \\
\delta_{19} &= \frac{9498457}{4822632} \approx 1.969558739, \quad \delta_{20} = \frac{18707769}{9498457} \approx 1.9695587399, \\
\delta_{21} &= \frac{36293638}{18427294} \approx 1.9695587426, \quad \delta_{22} = \frac{107223678}{54440457} \approx 1.969558741948, \\
\delta_{23} &= \frac{208017180}{105616134} \approx 1.96955874185, \quad \delta_{24} = \frac{614553083}{312025770} \approx 1.96955874189, \\
\delta_{25} &= \frac{1210398397}{614553083} \approx 1.969558741926, \quad \delta_{26} = \frac{2348209518}{1192251578} \approx 1.9695587419051, \\
\delta_{27} &= \frac{6937404876}{3522314277} \approx 1.9695587419044, \quad \delta_{28} = \frac{13458775392}{6833396286} \approx 1.969558741906103, \\
\delta_{29} &= \frac{39761773093}{20188163088} \approx 1.96955874190628, \quad \delta_{30} = \frac{78313147789}{39761773093} \approx 1.96955874190598.
\end{aligned}$$

Let us do similar calculations with the help of the algorithm of the theorem 2.

We shall calculate $\xi_{-1}, \xi_0, \xi_1, \xi_2, \tau_0, \tau_1, \tau_2, \tau_3, \psi_1, \psi_2, \psi_3, \psi_4, \varphi_2, \varphi_3, \varphi_4, \varphi_5$ from the equalities (13), (14), (15), (16):

$$\begin{aligned}
\varphi_5 &= \begin{bmatrix} 1 \\ \frac{1}{2} & 2 \\ \frac{1}{2} & 2 & 1 \\ \frac{1}{2} & 2 & \frac{1}{2} & 2 \\ 0 & 2 & \frac{1}{2} & 2 & 1 \end{bmatrix} = 35, \quad \varphi_4 = \begin{bmatrix} 1 \\ \frac{1}{2} & 2 \\ \frac{1}{2} & 2 & 1 \\ \frac{1}{2} & 2 & \frac{1}{2} & 2 \end{bmatrix} = 18, \quad \varphi_3 = \begin{bmatrix} 1 \\ \frac{1}{2} & 2 \\ \frac{1}{2} & 2 & 1 \end{bmatrix} = 6, \\
\varphi_2 &= \begin{bmatrix} 1 \\ \frac{1}{2} & 2 \end{bmatrix} = 3, \quad \psi_4 = \begin{bmatrix} 2 \\ 2 & 1 \\ 2 & \frac{1}{2} & 2 \\ 2 & \frac{1}{2} & 2 & 1 \end{bmatrix} = 24, \quad \psi_3 = \begin{bmatrix} 2 \\ 2 & 1 \\ 2 & \frac{1}{2} & 2 \end{bmatrix} = 12, \\
\psi_2 &= \begin{bmatrix} 2 \\ 2 & 1 \end{bmatrix} = 4, \quad \psi_1 = 2, \quad \tau_3 = \begin{bmatrix} 1 \\ \frac{1}{2} & 2 \\ \frac{1}{2} & 2 & 1 \end{bmatrix} = 6, \quad \tau_2 = \begin{bmatrix} 1 \\ \frac{1}{2} & 2 \end{bmatrix} = 3, \quad \tau_1 = 1, \quad \tau_0 = 1, \\
\xi_2 &= \begin{bmatrix} 2 \\ 2 & 1 \end{bmatrix} = 4, \quad \xi_1 = 2, \quad \xi_0 = 1, \quad \xi_{-1} = 0.
\end{aligned}$$

Consequently, the recurrences (9), (10), (11), (12) will be written as:

$$\begin{aligned}
A_{5s}^s &= 18D_{5s-8}^{s-2} + 30C_{5s-7}^{s-2} + 33B_{5s-6}^{s-2} + 35A_{5s-5}^{s-1}, \\
B_{5s-1}^{s-1} &= 12D_{5s-8}^{s-2} + 20C_{5s-7}^{s-2} + 22B_{5s-6}^{s-2} + 24A_{5s-5}^{s-1}, \\
C_{5s-2}^{s-1} &= 3D_{5s-8}^{s-2} + 5C_{5s-7}^{s-2} + 6B_{5s-6}^{s-2} + 6A_{5s-5}^{s-1}, \\
D_{5s-3}^{s-1} &= 2D_{5s-8}^{s-2} + 4C_{5s-7}^{s-2} + 4B_{5s-6}^{s-2} + 4A_{5s-5}^{s-1}.
\end{aligned}$$

The s -th approximation to the value of the given recurrence fraction, by the algorithm of the theorem 2 is of the form

$$\gamma_s = 1 + \frac{B_{5s-1}^{s-1}}{A_{5s}^s} + \frac{C_{5s-2}^{s-1}}{A_{5s}^s} + \frac{D_{5s-3}^{s-1}}{A_{5s}^s}.$$

Since, $D_2^0 = \zeta_2 = 4$, $C_3^0 = \tau_3 = 6$, $B_4^0 = \psi_4 = 24$, $A_5^1 = \varphi_5 = 35$, then

$$\gamma_1 = \frac{69}{35} = 1.97143, \quad \gamma_2 = \frac{4469}{2269} \approx 1.9695901, \quad \gamma_3 = \frac{289145}{146807} \approx 1.969558672,$$

$$\gamma_4 = \frac{18707769}{9498457} \approx 1.9695587399, \quad \gamma_5 = \frac{1210398397}{614553083} \approx 1.969558741926,$$

$$\gamma_6 = \frac{78313147789}{39761773093} \approx 1.96955874190598.$$

Thus, from this example it is clear that the s -th approximation γ_s , found by means of the algorithm of Theorem 2 coincides with the $(5s)$ -th approximation δ_{5s} , found by the algorithm (2).

3 CONCLUSIONS

Therefore, recurrence fractions of order four are natural generalization of chain fractions. Periodic recurrence fractions of order four are connected with corresponding algebraic equations of order four and show irrationalities of order four, while Theorem 2 provides an effective algorithm for constructing rational approximations to these irrationalities.

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Вивчаються рекурентні дробі четвертого порядку, встановлюються їх зв'язки з алгебраїчними рівняннями четвертого порядку і будуються алгоритми обчислення раціональних наближень.

Ключові слова і фрази: періодичний дріб, трикутна матриця, параперманент, парадетермінант, раціональне наближення.

Заторский Р.А., Семенчук А.В. Алгоритм вычисления рациональных приближений периодической рекуррентной дроби 4-го порядка // Карпатские матем. публ. — 2014. — Т.6, №1. — С. 32–43.

Изучаются рекуррентные дроби четвертого порядка, устанавливаются их связи с алгебраическими уравнениями четвертого порядка и строятся алгоритмы вычисления их рациональных приближений.

Ключевые слова и фразы: периодическая рекуррентная дробь, треугольная матрица, параперманент, парадетерминант, рациональное приближение.