GENERALIZATION OF CASSINNI FORMULAS FOR BALANCING AND LUCAS-BALANCING NUMBERS

Prasanta Kumar Ray
International Institute of Information Technology Bhubaneswar, Odisha, India
prasanta@iiit-bh.ac.in

Kaberi Parida
National Institute of Technology Rourkela, Odisha, India
parida.kaberi@gmail.com

Abstract

The mathematical identity that connects three adjacent balancing numbers is well known under the name Cassini formula, and is used to establish many important identities involving balancing numbers and their related sequences. This article is an attempt to draw attention to some of the unusual properties of generalized balancing numbers, in particular, to the generalized Cassini formula.

1. Introduction

Recently, Behera et.al [1] introduced the sequence of balancing numbers as follows: A positive integer n is called a balancing number with balancer r, if it is the solution of Diophantine equation

$$1 + 2 + 3 + \ldots + (n-1) = (n+1) + (n+2) + \ldots (n+r).$$

The balancing numbers though obtained from a simple Diophantine equation, are very useful for the computation of square triangular numbers. An important result about balancing numbers is that, n is balancing number if and only if n^2 is a triangular number i.e. $8n^2+1$ is a perfect square. The square root of $8n^2+1$ also generates a sequence of numbers which are called as Lucas-balancing numbers. Balancing and Lucas-balancing numbers can be generated by the recurrence formulas

$$B_{n+1} = 6B_n - B_{n-1}, \quad n \geq 2, \tag{1}$$

and

$$C_{n+1} = 6C_n - C_{n-1}, \quad n \geq 2, \tag{2}$$

with their respective initial terms $B_1 = 1,\ B_2 = 6$ and $C_1 = 3,\ C_2 = 17$ [1, 10]. The closed forms which are also called as Binet's formulas for balancing and Lucas-balancing numbers are respectively given by

$$B_n = \frac{\lambda^n - \lambda^{-n}}{2\sqrt{8}}, \qquad C_n = \frac{\lambda^n + \lambda^{-n}}{2}, \tag{3}$$

where $\lambda = 3 + \sqrt{8}$ and $\lambda^{-1} = 3 - \sqrt{8}$ are the roots of the auxiliary equation $\lambda^2 - 6\lambda + 1 = 0$.

G.K. Panda [8] established many important identities concerning balancing numbers and their related sequences. In [4], K. Liptai et al. added another interesting result to the theory of balancing numbers by generalizing these numbers. A. Berczes et al. [2] and P. Olajos [5] surveyed many interesting properties of generalized balancing numbers. Recently, G.K. Panda et al. [9] introduced gap balancing numbers and established many properties of these numbers. Some curious congruence properties of balancing numbers are also studied in [12]. In [10, 11], Ray established new product formulae to generate both balancing and Lucas-balancing numbers. Recently, R. Keskin and O. Karaath [3] obtained some new properties for balancing numbers and square triangular numbers.

Among all the important identities for Fibonacci numbers, one of the most famous identity is Cassini formula. The Cassini formulas for Fibonacci numbers and their related sequences are available in literature. They play an important role for finding new identities for these numbers. Cassinni formulas for Fibonacci numbers, Lucas numbers, Pell numbers, balancing numbers and Lucas-balancing numbers are respectively given by

$$\begin{aligned}
F_{n+1}F_{n-1} - F_n^2 &= (-1)^n, \\
L_{n+1}L_{n-1} - L_n^2 &= 5(-1)^{n-1}, \\
P_{n+1}P_{n-1} - P_n^2 &= (-1)^n, \\
B_{n+1}F_{n-1} - B_n^2 &= -1, \\
C_{n+1}C_{n-1} - C_n^2 &= 8.
\end{aligned}$$

This article is an attempt to draw attention to some of the unusual properties of generalized balancing numbers, in particular, to the generalized Cassini formula.

2. The Balancing λ-numbers

For any real number $\lambda > 0$, consider the recurrence relation

$$B_\lambda(n+2) = 6\lambda B_\lambda(n+1) - B_\lambda(n), \tag{4}$$

with initial terms $B_\lambda(0) = 0,\ B_\lambda(1) = 1$. The recurrence relation (4) generates an infinite number of numerical sequences for any λ. For $\lambda = 1$, (4) reduces to the recurrence relation for balancing numbers. For $\lambda = 2$, (4) reduces to the following recurrence relation:

$$B_2(n+2) = 12B_2(n+1) - B_2(n); \quad B_2(0) = 0,\ B_2(1) = 1,$$

which generates the sequence $12, 143, 1704, 20305, 241956 \ldots$ and so on. Table-1 shows the four expanded sequences of the balancing λ-numbers corresponding to the values $\lambda = 1, 2, 3, 4$

Table-1

n	0	1	2	3	4	5	6
$B_1(n)$	0	1	6	35	204	1189	6930
$B_1(-n)$	-0	-1	-6	-35	-204	-1189	-6930
$B_2(n)$	0	1	12	143	1704	20305	241956
$B_2(-n)$	-0	-1	-12	-143	-1704	-20305	-241956
$B_3(n)$	0	1	18	323	5796	104005	1866294
$B_3(n)$	-0	-1	-18	-323	-5796	-104005	-1866294
$B_4(n)$	0	1	24	575	13776	3300489	7907400
$B_4(-n)$	-0	-1	-24	-575	-13776	-3300489	-7907400

2.1. Generating function for balancing λ-numbers

In this section, we will first find the generating function for the balancing λ-numbers and then using this function, we will establish the Binet's formulas for these numbers. Let

$$G(x) = \sum_{n=0}^{\infty} B_\lambda(n)x^n = B_\lambda(0) + B_\lambda(1)x + B_\lambda(2)x^2 + \ldots + B_\lambda(n)x^n + \ldots$$

be the generating function for balancing λ-numbers. By simple algebraic manipulation, we obtain

$$G(x) - 6\lambda x G(x) + x^2 G(x) = x,$$

which follows that

$$G(x) = \frac{x}{1 - 6\lambda x + x^2}.$$

Using the generating function for balancing λ-numbers, we can find the Binet's formula as follows: Rewrite the factor $1 - 6\lambda x + x^2$ as

$$1 - 6\lambda x + x^2 = (1 - \alpha x)(1 - \beta x),$$

where $\alpha = 3\lambda + \sqrt{9\lambda^2 - 1}$, $\quad \beta = 3\lambda - \sqrt{9\lambda^2 - 1}$. Therefore, $G(x)$ can be written as

$$G(x) = \frac{x}{1 - 6\lambda x + x^2} = \frac{A}{(1 - \alpha x)} + \frac{B}{(1 - \beta x)},$$

which implies that

$$x = A(1 - \beta x) + B(1 - \alpha x) = (A + B) - x(A\beta + B\alpha).$$

Therefore,

$$A + B = 0 \ \text{ and } \ A\beta + B\alpha = -1.$$

Solving these equations, we get

$$A = \frac{1}{2\sqrt{9\lambda^2 - 1}}, \quad B = -\frac{1}{2\sqrt{9\lambda^2 - 1}}.$$

Thus, we have

$$G(x) = \frac{x}{1 - 6\lambda x + x^2} = \frac{1}{\alpha - \beta}\left[\frac{1}{1 - \alpha x} - \frac{1}{1 - \beta x}\right]$$
$$= \frac{1}{\alpha - \beta}\sum\left(\alpha^n - \beta^n\right)x^n,$$

follows that $B_{n_\lambda} = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, which is the Binet's formula for balancing λ-numbers. It can be observed that

$$\alpha + \beta = 6\lambda, \ \alpha - \beta = 2\sqrt{9\lambda^2 - 1} \ \text{ and } \ \alpha\beta = 1.$$

2.2. Cassini formula for the balancing λ-numbers

The Cassini formula can be generalized for the case of the balancingλ-numbers.

Theorem 2.1. *If $B_\lambda(n)$ be the n^{th} balancing λ-numbers, then the generalized Cassinni formula is*

$$B_\lambda^2(n) - B_\lambda(n)B_\lambda(n-1) = 1. \tag{5}$$

Proof. The identity (5) can be proved by mathematical induction. Recurrence relation (1) takes the following values

$$B_\lambda(0) = 0, \ B_\lambda(1) = 1, \ B_\lambda(2) = 6\lambda,$$

which implies that

$$B_\lambda^2(1) - B_\lambda(2)B_\lambda(0) = 1,$$

and therefore the base of the induction is proved. We assume that the identity (5) is valid for any positive integer n and prove the validity for the case $(n+1)$, i. e the identity

$$B_\lambda^2(n+1) - B_\lambda(n+2)B_\lambda(n) = 1 \tag{6}$$

is valid too. In order to prove the identity (6), we represent the left part of (6) as follows

$$\begin{aligned} B_\lambda^2(n+1) - B_\lambda(n)B_\lambda(n+2) &= B_\lambda^2(n+1) - B_\lambda(n)\left(6B_\lambda(n+1) - B_\lambda(n)\right) \\ &= B_\lambda(n+1)\left(B_\lambda(n+1) - 6B_\lambda(n)\right) + B_\lambda^2(n) \\ &= B_\lambda^2(n) - B_\lambda(n+1)B_\lambda(n-1) = 1, \end{aligned}$$

which completes the proof. □

2.3. Numerical examples

Consider the examples of the validity of the identity (5) for the various sequences shown in Table-1. Consider the balancing 2-number $B_2(n)$ for the case of $n = 4$ to get

$$B_2(4) = 1704,\ B_2(5) = 20305,\ B_2(6) = 241956.$$

By performing calculations over them according to (5), we obtain the following result

$$B_2^2(5) - B_2(4)B_2(6) = 20305^2 - (1704 \times 241956) = 1.$$

Further, consider the $B_3(n)$ sequence from Table-1 for the case $n = 3$. For this case, we should choose the following balancing 3-numbers $B_3(n)$,

$$B_3(3) = 323,\ B_3(4) = 5796,\ B_3(5) = 104005.$$

By performing calculations over them according to (5), we obtain the following result:

$$B_3^2(4) - B_3(3)B_3(5) = 1.$$

Finally, Consider the $B_4(-n)$ sequence from Table-1 for the case $n = 4$. For this case, we should choose the following balancing 4-numbers $B_4(-n)$.

$$B_4(-4) = -13776,\ B_4(-3) = -575,\ B_4(-2) = -24.$$

By performing calculations over them according to equation (5), we obtain the following result

$$B_4^2(-3) - B_4(-2)_4(-4) = -575^2 - (-24 \times -13776) = 1.$$

3. Lucas-balancing-λ numbers

For any real number $\lambda > 0$, consider the recurrence relation

$$C_\lambda(n+2) = 6\lambda C_\lambda(n+1) - C_\lambda(n); C_\lambda(0) = 1, C_\lambda(1) = 3. \tag{7}$$

The recurrence relation (7) generates an infinite number of new numerical sequences for any real number λ. For $\lambda = 1$, (7) reduces the following relation

$$C_1(n+2) = 6C_1(n+1) - C_1(n), C_1(0) = 1, C_1(1) = 3,$$

which generates the Lucas balancing numbers. For $\lambda = 2$, $\lambda = 3$, and $\lambda = 4$, we have the following sequences:

$$\begin{gathered} 1,\ 3,\ 35,\ 417,\ 4969,\ 59211, \ldots, \\ 1,\ 3,\ 53,\ 951,\ 17065, \ldots, \\ 1,\ 3,\ 71,\ 1701,\ 40753, \ldots. \end{gathered}$$

Table-2 shows that the four expanded sequences of the Lucas-balancing λ-numbers, corresponding to the values $\lambda = 1, 2, 3, 4$.

Table-2

n	0	1	2	3	4	5	6
$C_1(n)$	1	3	17	99	577	3363	19601
$C_1(-n)$	1	3	17	99	577	3363	19601
$C_2(n)$	1	3	35	417	4969	59211	705563
$C_2(-n)$	1	3	35	417	4969	59211	705563
$C_3(n)$	1	3	53	951	17065	306219	5494877
$C_3(n)$	1	3	53	951	17065	306219	5494877
$C_4(n)$	1	3	71	1701	40753	976371	23392151
$C_4(-n)$	1	3	71	1701	40753	976371	23392151

3.1. Generating function for Lucas-balancing λ-numbers

In this section, we first find the generating function for Lucas-balancing numbers and then with the help of this function, we find the Binet's formula for these numbers. Let

$$g(x) = \sum_{n=0}^{\infty} C_\lambda(n) x^n$$

be the generating function for Lucas-balancing λ-numbers. By simple algebraic manipulation, we get

$$g(x) - 6\lambda x g(x) + x^2 g(x) = 1 + x(3 - 6\lambda),$$

which follows that

$$g(x) = \frac{1 + x(3 - 6\lambda)}{1 - 6\lambda x + x^2}.$$

Using this generating function and the previous described method, we can find the Binet's formula for Lucas- balancing λ-numbers as

$$C_\lambda^n = \frac{(3-\beta)\alpha^n - (3-\alpha)\beta^n}{\alpha - \beta}.$$

where $\alpha = 3\lambda + \sqrt{9\lambda^2 - 1}$ and $\beta = 3\lambda - \sqrt{9\lambda^2 - 1}$.
We notice that,

$$\alpha + \beta = 6\lambda, \ \alpha - \beta = 2\sqrt{9\lambda^2 - 1} \ \text{ and } \ \alpha\beta = 1.$$

3.2. Cassini formula for Lucas-balancing λ-numbers

The Cassini formula can also be generalized for the case of the Lucas-balancing λ-numbers.

Theorem 3.1. *If $C_\lambda(n)$ be the n^{th} Lucas-balancing λ-numbers, then the generalized Cassinni formula for Lucas-balancing λ-numbers is*

$$C_\lambda(n+1)C_\lambda(n-1) - C_\lambda^2(n) = 18\lambda - 10. \tag{8}$$

Proof. Once again induction comes into picture. According to the recurrence relation (7) takes the following values

$$C_\lambda(0) = 1,\ C_\lambda(1) = 3,\ C_\lambda(2) = 18\lambda - 1,$$

follows that the identity (8) for the case $n = 1$ is equal to

$$C_\lambda(2)C_\lambda(0) - C_\lambda^2(1) = (18\lambda - 1) - 9 = 18\lambda - 10.$$

Thus, the result is true for $n = 1$. For inductive assumption, suppose that the identity (8) is valid for any given positive integer n. We notice that

$$\begin{aligned} C_\lambda(n)C_\lambda(n+2) - C_\lambda^2(n+1) &= C_\lambda(n)\left(6C_\lambda(n+1) - C_\lambda(n)\right) - C_\lambda^2(n+1) \\ &= C_\lambda(n+1)\left(6C_\lambda(n) - C_\lambda(n+1)\right) - C_\lambda^2(n) \\ &= C_\lambda(n+1)C_\lambda(n-1) - C_\lambda^2(n) = 18\lambda - 10, \end{aligned}$$

from which the proof follows. □

3.3. Numerical examples

Consider the examples of the validity of the identity (8) for the various sequences shown in Table-1. Let us consider the $C_2(n)$ sequence for the case $n = 3$. For this case, we should choose the following Lucas-balancing 2-numbers $C_2(n)$ as follows

$$C_2(4) = 4969,\ C_2(5) = 59211,\ C_2(3) = 417.$$

By performing calculations over them according to (8), we obtain the following result

$$C_2(5)C_2(3) - C_2^2(4) = (59211 \times 417) - 4969^2 = 26.$$

And right hand side is $18\lambda - 10 = 18(2) - 10 = 26$.

Consider $C_3(n)$ sequence for the case $n = 4$. For this case, we should choose the following Lucas-balancing 3-numbers $C_3(n)$,

$$C_3(6) = 5494877,\ C_3(4) = 17065,\ C_3(5) = 306219.$$

In a similar way, we obtain the following result

$$C_3(6)C_3(4) - C_3^2(5) = (306219 \times 17065) - 5494877^2 = 44,$$

and right hand side of it is given by $18\lambda - 10 = 18(3) - 10 = 44$.

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