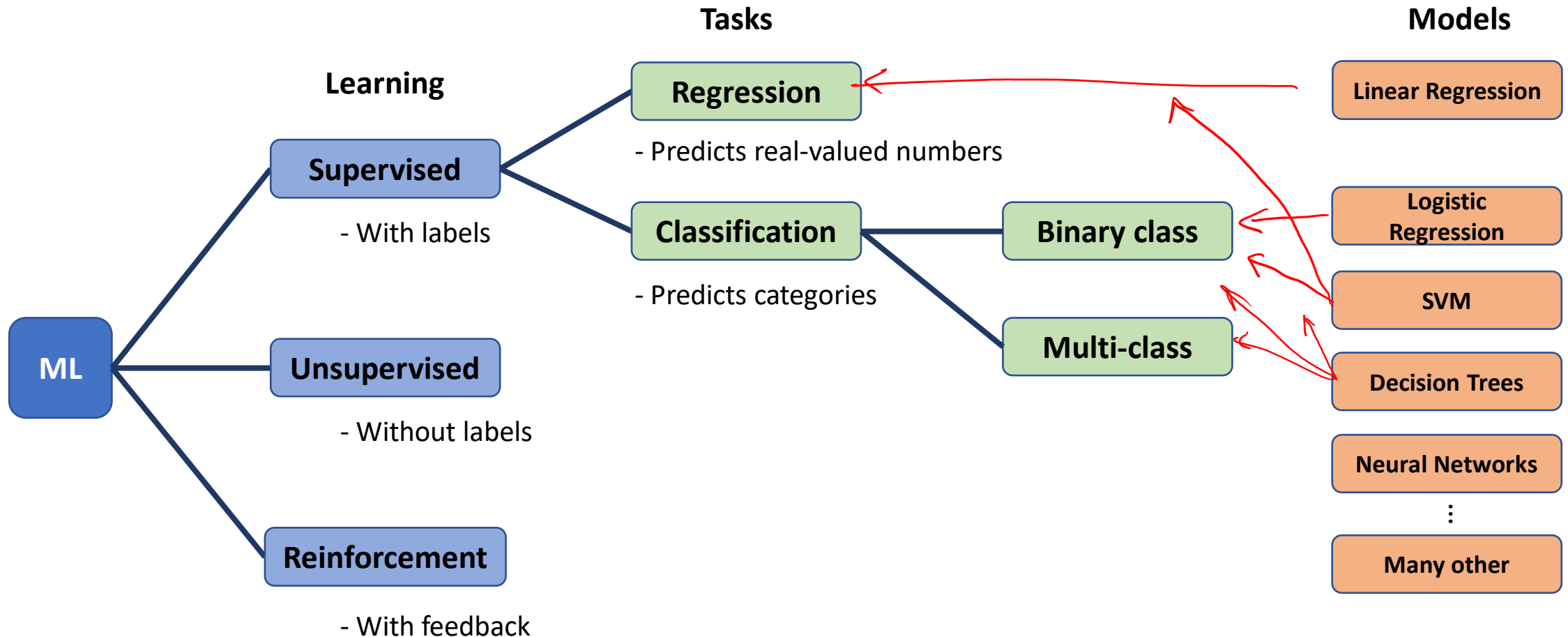




Logistic Regression

Introduction

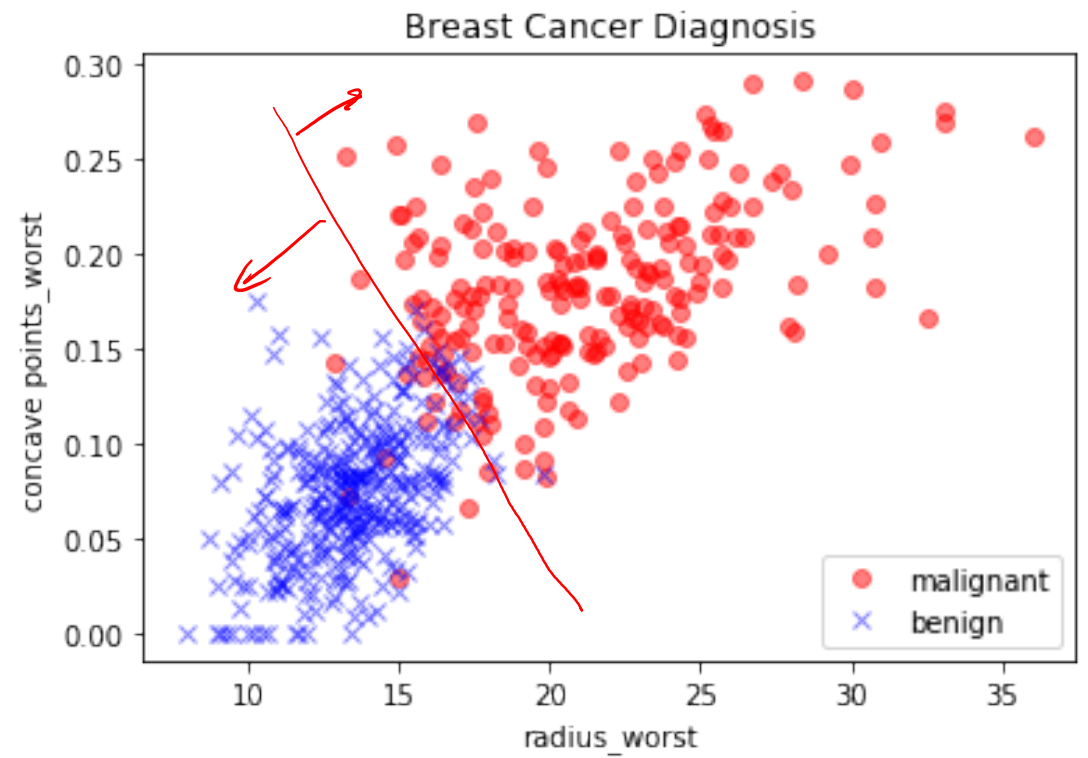
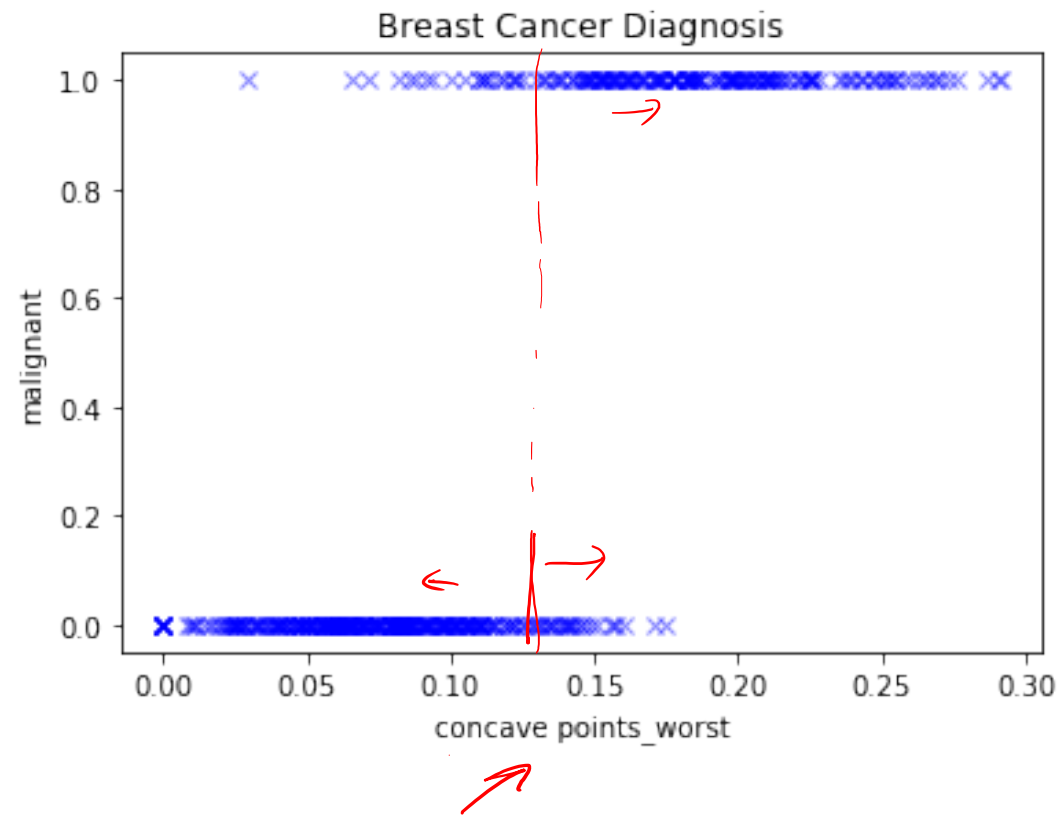
Review- types of machine learning problems



Binary Classification

Yes or No problem

- Creditcard Default
- Fraudulent Insurance Claim
- Spam Filtering
- Medical Diagnosis
- Survival Prediction
- Customer Retention
- Image Recognition
- Sentiment Analysis



Logistic Function

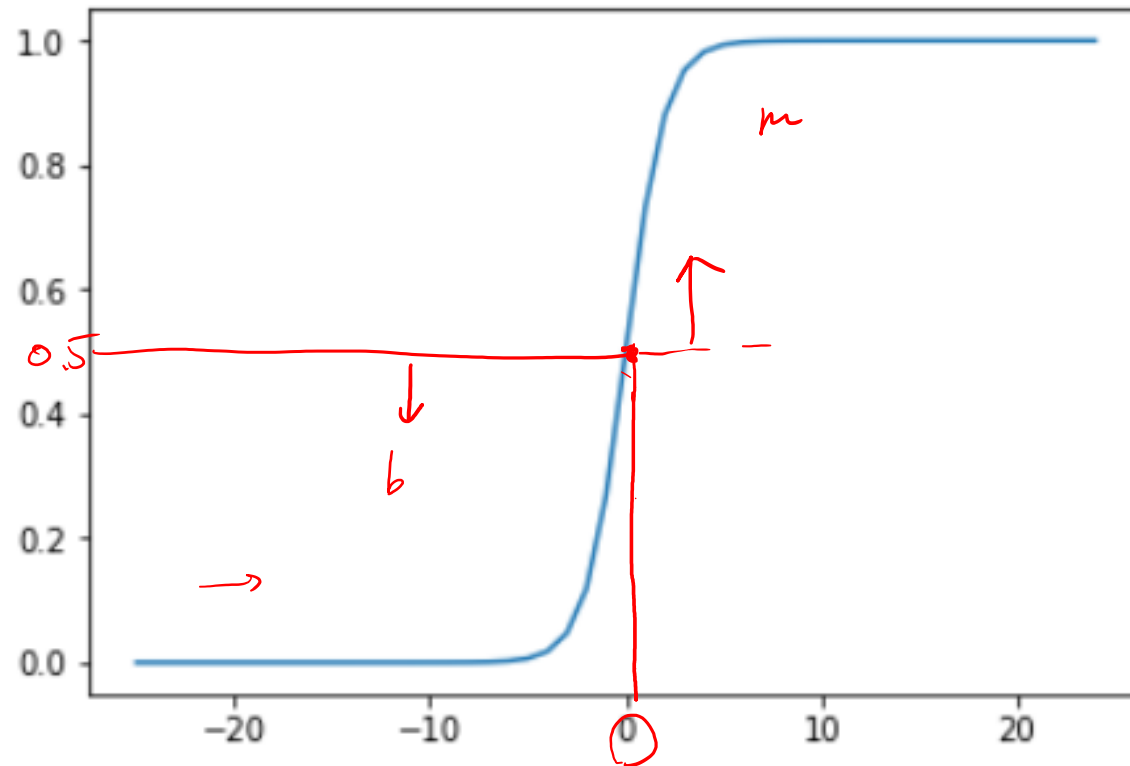
$$P^{(i)} = \sigma(z^{(i)})$$

$$\sigma(z) = \frac{1}{1 + e^{-z}} = \frac{1}{2}$$

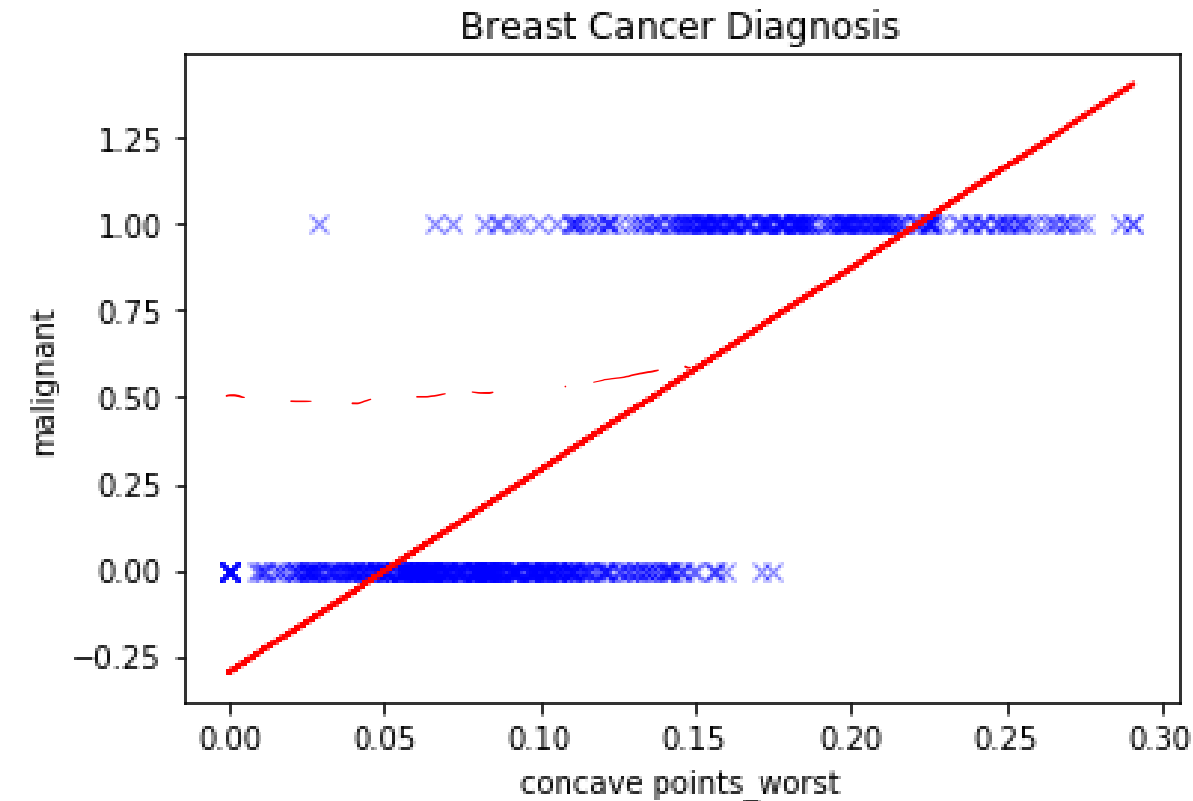
$$z^{(i)} = \underline{W} \cdot \underline{X} + \underline{b} = 0$$

Called "logit" and is related to the decision boundary

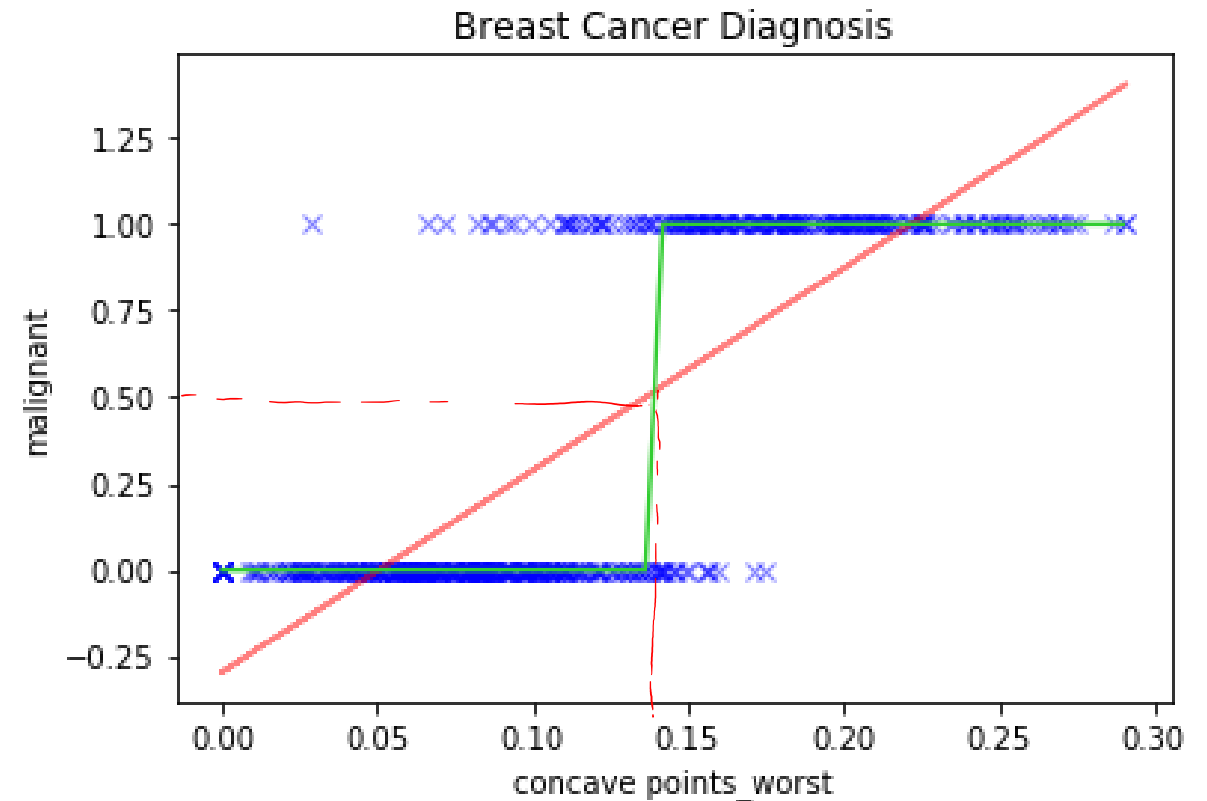
$$P^{(i)} \in \mathbb{R}[0, 1]$$



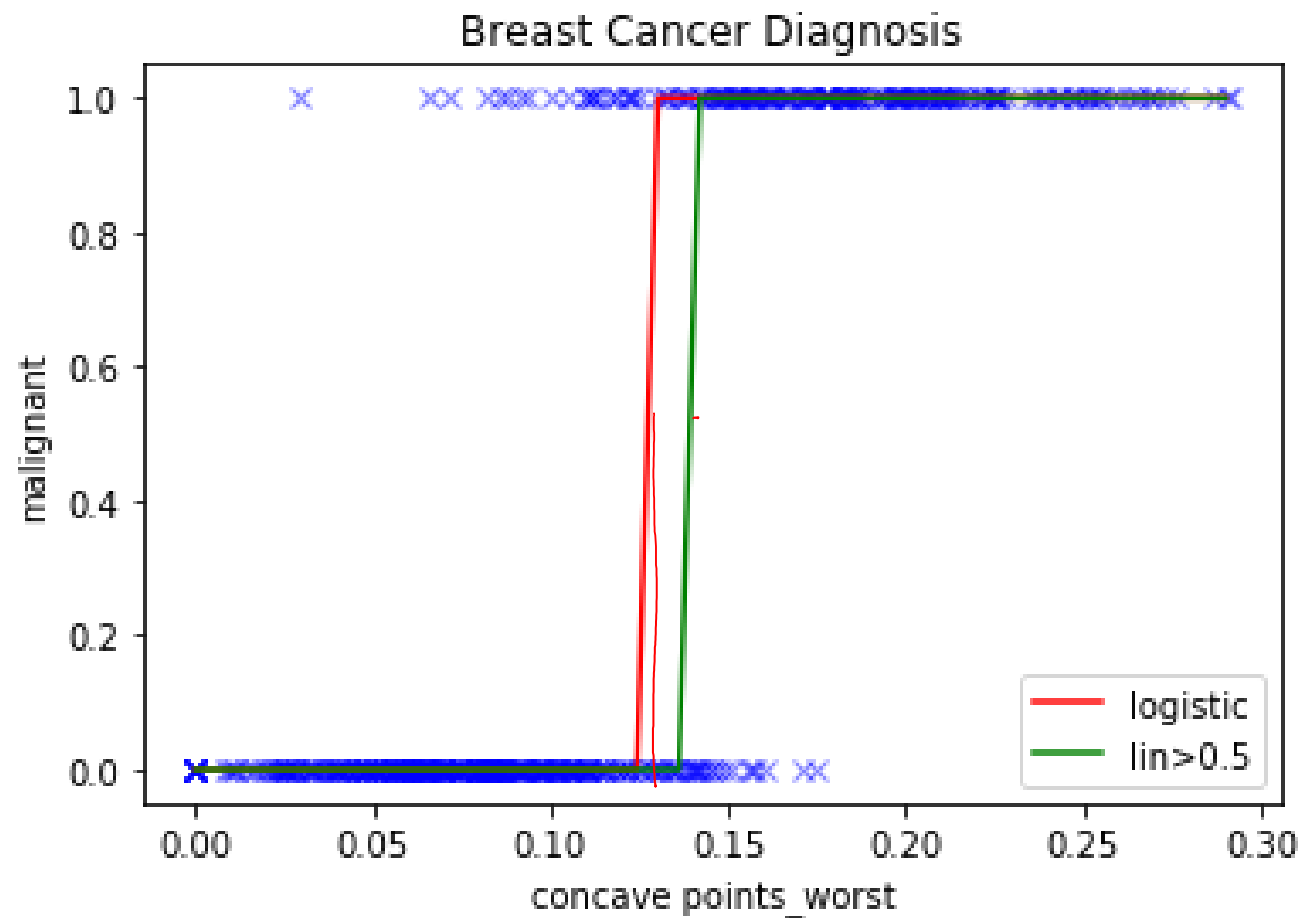
Logistic function as probability



linreg.predict(x)

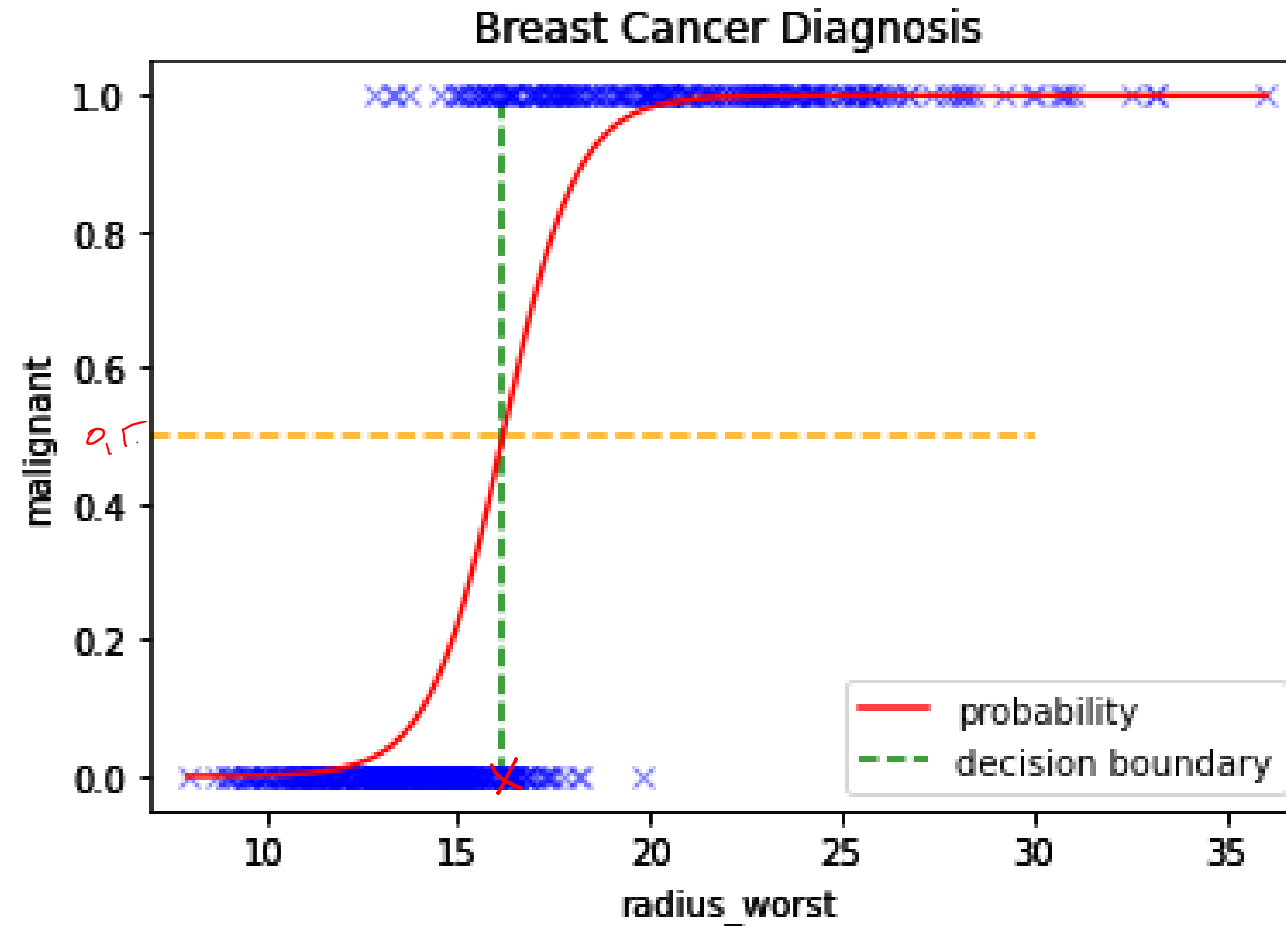


linreg.predict(x) > 0.5



Decision boundary- Univariate

radius_worst	label
13.05	0
16.39	1
10.85	0
21.86	1
21.31	1

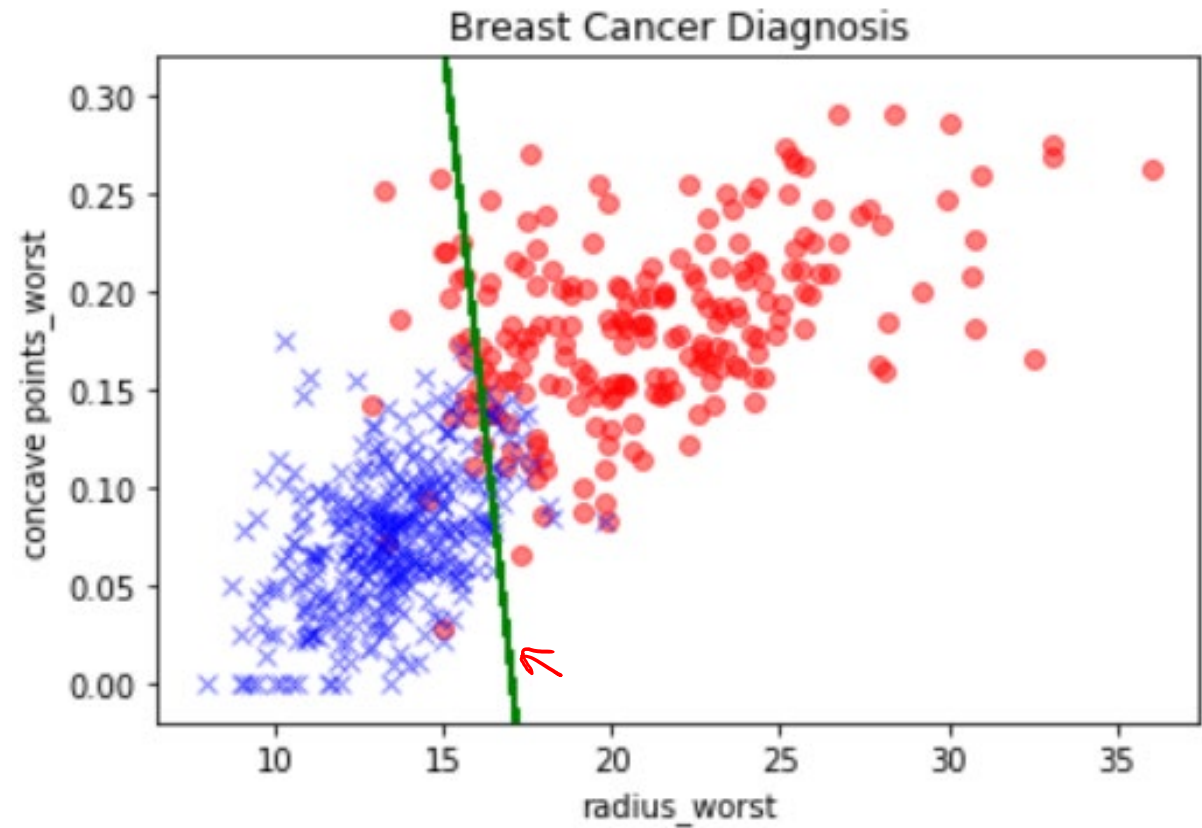


$$z = 1.05 X - 16.99 = 0$$

Decision boundary- Multivariate

Breast Cancer Diagnosis

radius_worst	concave points_worst	label
13.05	0.08263	0
16.39	0.16730	1
10.85	0.14650	0
21.86	0.15100	1
21.31	0.14900	1



$$z = -16.7 + 1.02 x_1 + 2.13 x_2 = 0$$

What if we have multiple categories?

Logit

Logistic

$\begin{matrix} & A & B & C \\ \text{LR1} & A & !A & \\ \text{LR2} & & B \text{ vs } !B & \\ \text{LR3} & & C \text{ vs } !C & \end{matrix}$

→ One versus Rest

Softmax = multinomial LR

$$z^{(i)} = \mathbf{W} \cdot \mathbf{X}^{(i)} + b = \sum_j^p W_j X_j^{(i)} + b$$

$$z_{\underbrace{k}_{\text{category}}}^{(i)} = \mathbf{W}_{\underbrace{k}_{\text{category}}} \cdot \mathbf{X}^{(i)} + b = \sum_j^p \underbrace{W_{jk}}_{\text{category}} X_j^{(i)} + b$$

Probability

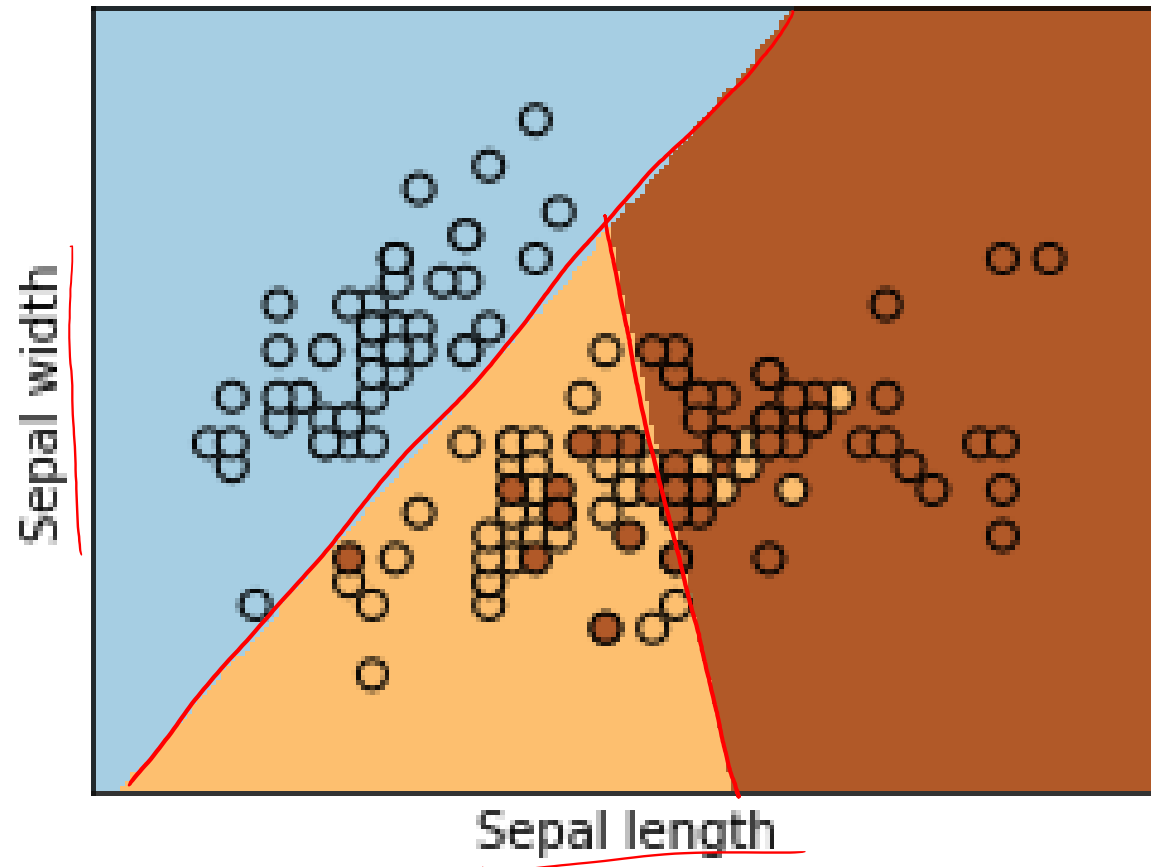
$$\sigma(z) = \frac{1}{1 + e^{-z}} = \frac{e^z}{1 + e^z}$$

$$\text{softmax}(z_k) = \frac{e^{z_k}}{\sum_{k'}^K e^{z_{k'}}}$$

#1 $P_A + P_B + P_C = 1$

A B C → multi label problem

Softmax regression



↙

https://scikit-learn.org/stable/auto_examples/linear_model/plot_iris_logistic.html



Logistic Regression

Optimization

Estimating parameters in logistic regression

Maximum Likelihood

$$P(y=1|x) = \sigma(z) = \frac{1}{1+e^{-z}}$$

$$z = \underline{w} \cdot \underline{x} + \underline{b}$$

$$\max \log [\ell(\beta_0, \beta)] = \log \left[\prod_{i:y_i=1} p(x_i) \prod_{i:y_i=0} (1 - p(x_i)) \right]$$

Diagram illustrating the relationship between input features, predicted probabilities, and the likelihood function.

Input features: y_1, y_2, y_3, y_4, y_5

Observed values: $1, 1, 0, 0, 1$

Predicted probabilities: P_1, P_2, P_3, P_4, P_5


Relationship: $x \rightarrow \boxed{\text{LR}} \rightarrow \hat{P}$

Likelihood function: $P_1 \times P_2 \times (1 - P_3) \times (1 - P_4) \times P_5$

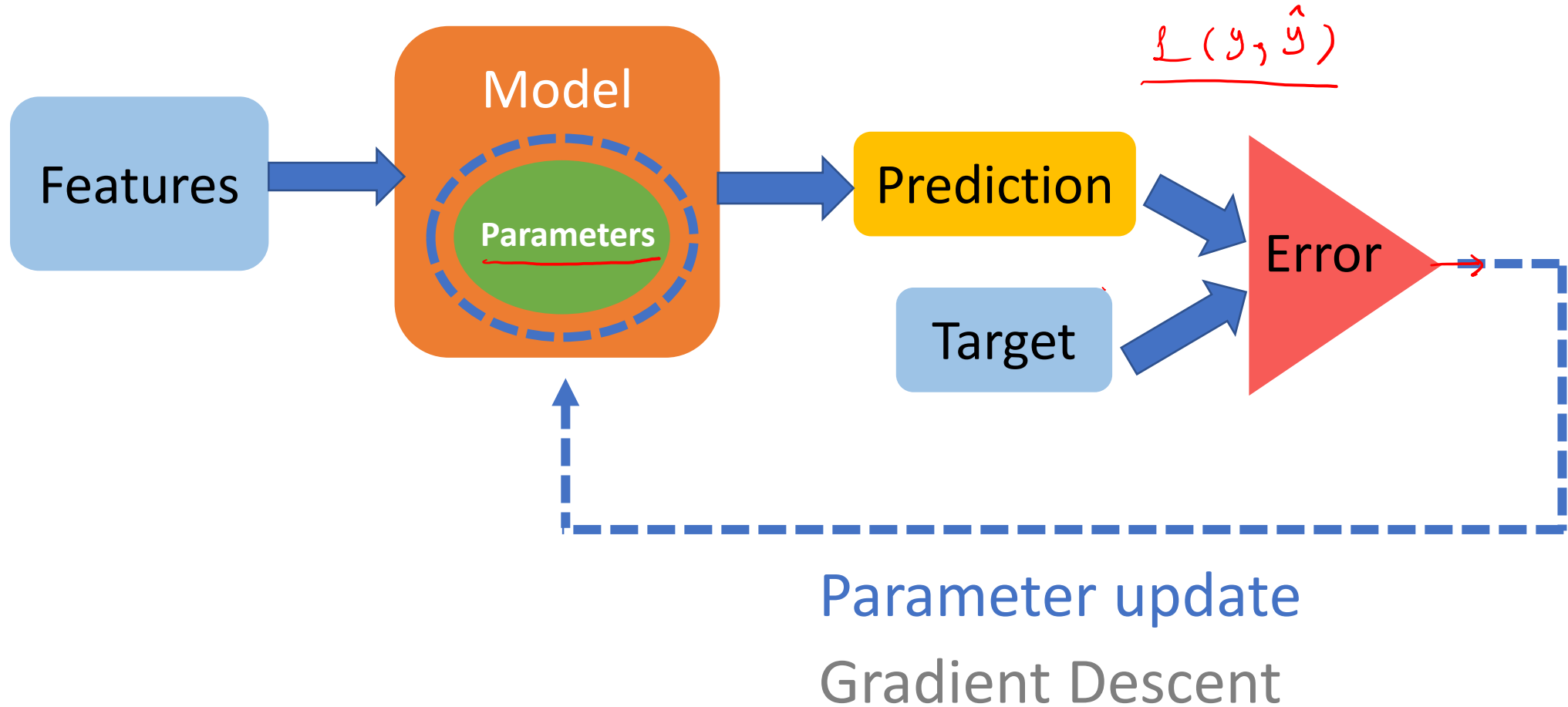
$$\mathcal{L}_{\text{BCE}} = - \sum_i \left(y_i \log P_i + (1 - y_i) \cdot \log(1 - P_i) \right)$$

Estimating parameters in logistic regression

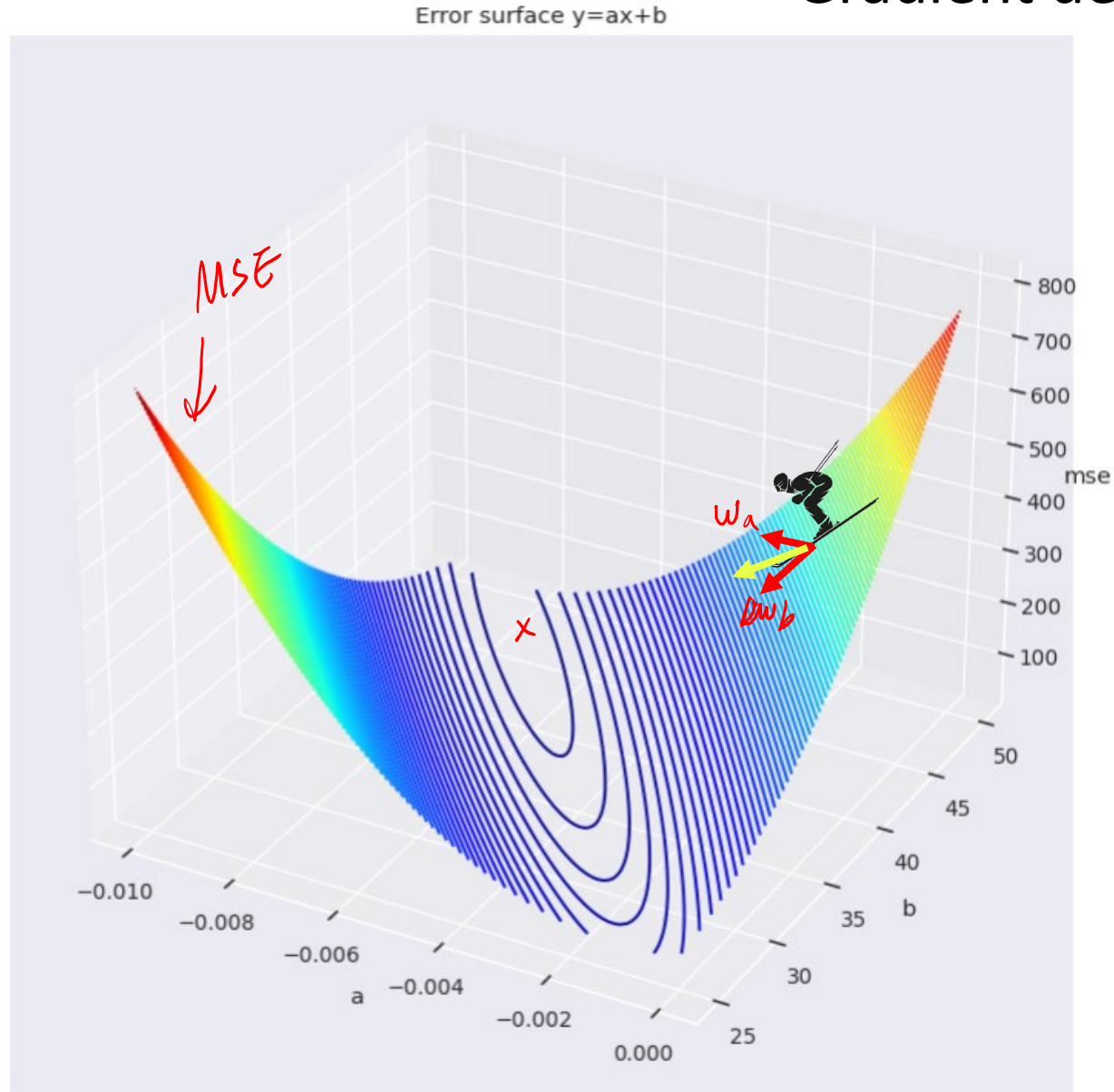
Cross Entropy

$$\begin{aligned}\mathcal{H}(P, Q) &= - \sum_{i,j} P_{i,j} \log(Q_{i,j}) \quad + \quad \cdot \\ &= - \frac{1}{m} \sum_i^m y_i \log \hat{p}_i + (1 - y_i) \log (1 - \hat{p}_i)\end{aligned}$$


Searching Parameters

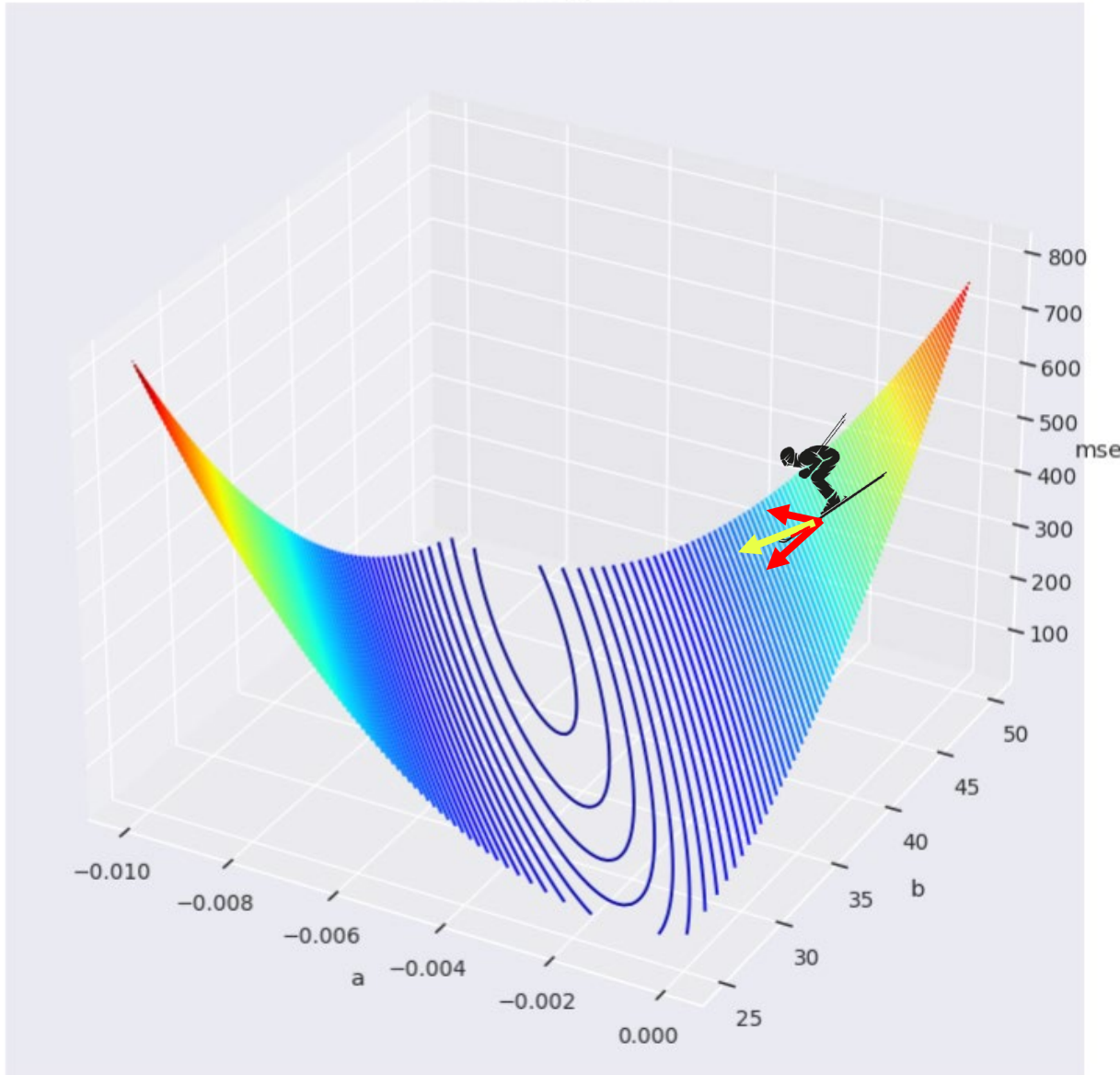


Gradient descent



Gradient descent (MSE loss)

Error surface $y=ax+b$



Loss function

$$L = \frac{1}{2n} \sum_i^n (y_i - f(x_i))^2 = \frac{1}{2n} \sum_i^n (y_i - (ax_i + b))^2$$

$\frac{\partial}{\partial a} f^2 = 2f \cdot \frac{\partial f}{\partial a}$

Gradients

$$\left\{ \begin{aligned} \nabla_a L &= \frac{\partial L}{\partial a} = -\frac{1}{n} \sum_i^n (y_i - (ax_i + b))x_i \\ \nabla_b L &= \frac{\partial L}{\partial b} = -\frac{1}{n} \sum_i^n (y_i - (ax_i + b)) \end{aligned} \right.$$

$\frac{\partial}{\partial x} g(f(x))$

Parameter(weight) update rule

$$\omega = \omega - \boxed{\alpha} \nabla_{\omega} L$$

learning rate
 $\omega = -\alpha$
 a, b

Gradient descent (BCE loss)

$$\log'(x) = \frac{1}{x}$$

$$\mathcal{L}_{BCE} = -\frac{1}{m} \sum_i y_i \log \hat{p}_i + (1 - y_i) \log (1 - \hat{p}_i)$$

$$\sigma' = \sigma(1 - \sigma)$$

$z = wx + b$

$$\sigma(z) = \frac{1}{1 + e^{-z}}$$

$$\boxed{\frac{\partial \mathcal{L}(\sigma)}{\partial w}} = \boxed{\frac{\partial \mathcal{L}}{\partial \sigma}} \left(\frac{\partial \sigma}{\partial w} \right) \rightarrow \boxed{\sigma(1 - \sigma) \cdot x}$$

\downarrow $-\left[\frac{y}{\sigma} + \frac{(1-y)}{(1-\sigma)}(-1) \right] \cdot x$

$$\left\{ \frac{\partial \mathcal{L}}{\partial w} = \left[-\frac{y}{\sigma} + \frac{(1-y)}{(1-\sigma)} \right] \cdot \sigma(1 - \sigma) \cdot \underline{x} \right.$$

$$\left. \left\{ \frac{\partial \mathcal{L}}{\partial b} = \left[\begin{array}{c} \text{''} \end{array} \right] \right. \right.$$

Newton's Method

Gradient
descent

$$w \leftarrow w - \alpha \nabla_w \ell$$

$$O(n), O(n)$$

Newton's
Method

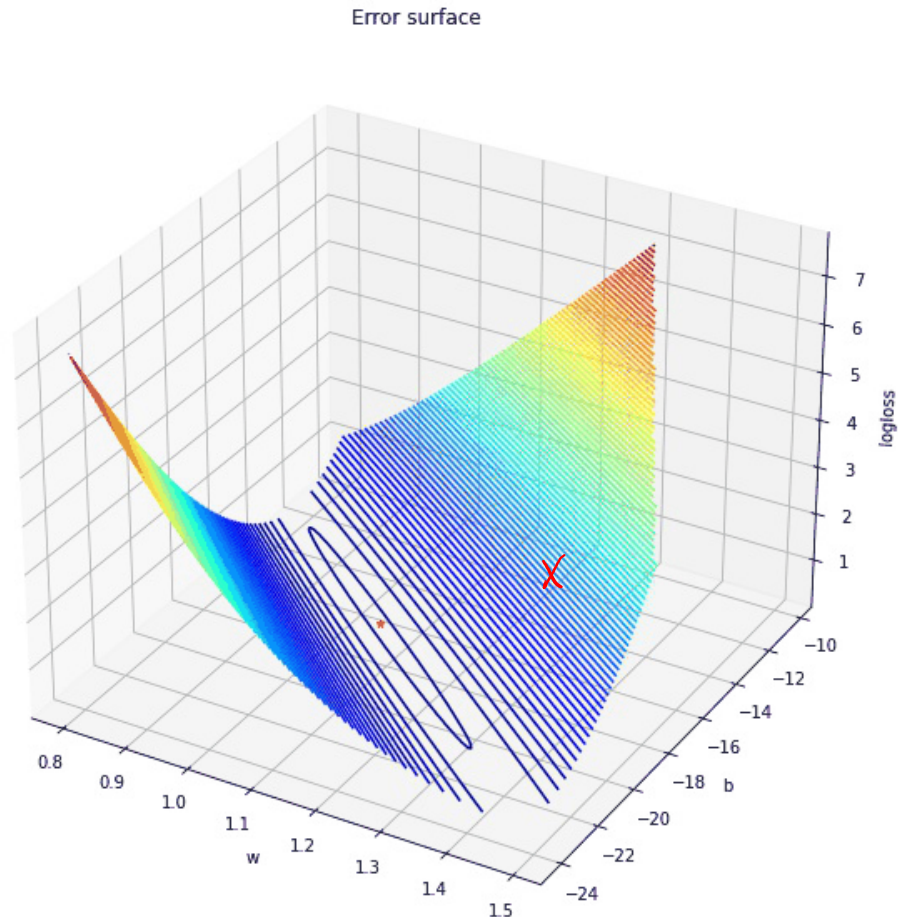
$$w \leftarrow w - \alpha \frac{\nabla_w \ell}{\nabla_w^2 \ell}$$

$\nabla_w^2 \ell \rightarrow$ Hessian

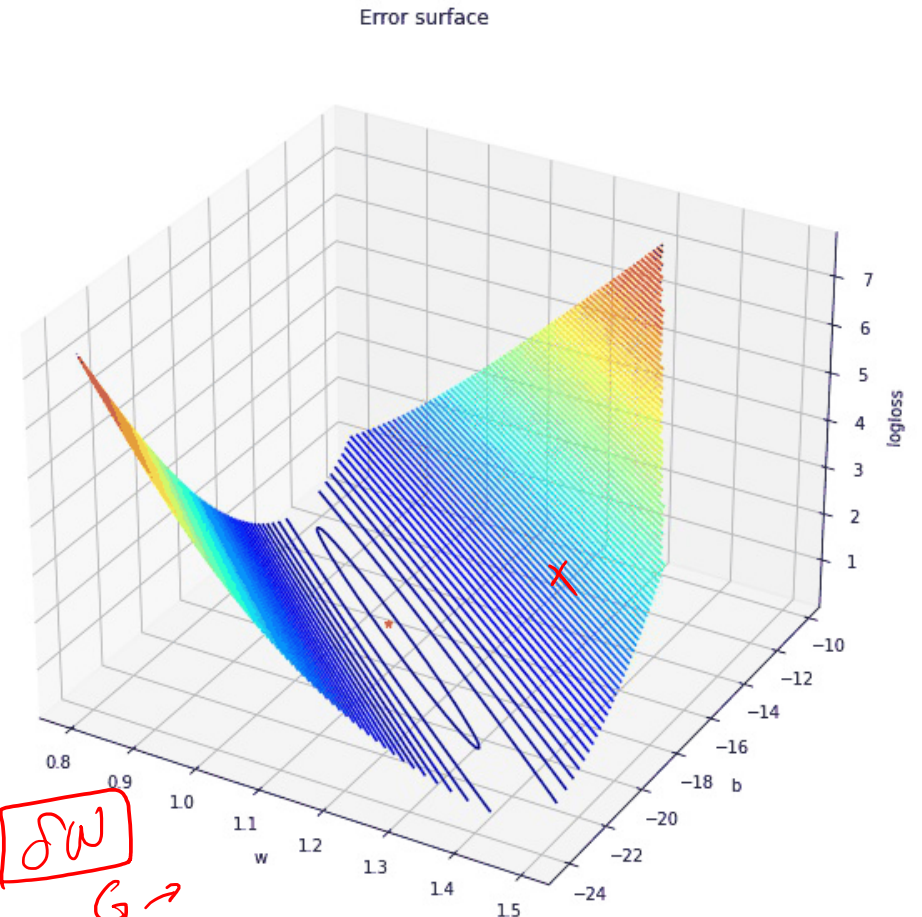
$$w \leftarrow w - \alpha \begin{bmatrix} G \\ H \end{bmatrix}$$

$$O(n^3), O(n^2)$$

Gradient descent vs. Newton's method



$-\alpha G$



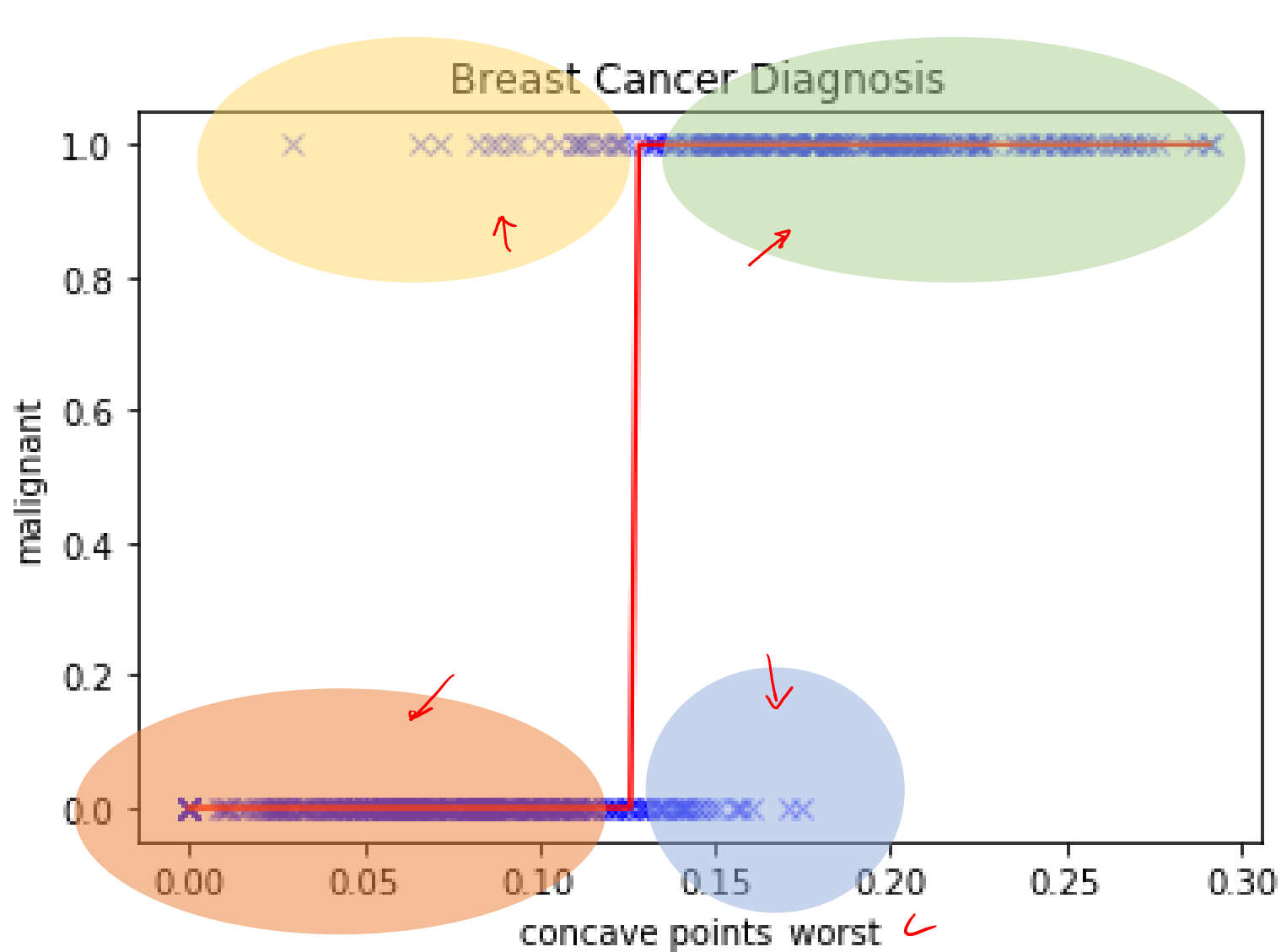
$\boxed{\partial w}$
 $-\alpha \frac{G}{H}$



Logistic Regression

Performance Metrics

Interpreting Logistic Regression Result



$Y_t = 1, Y_p = 1$
True Positive

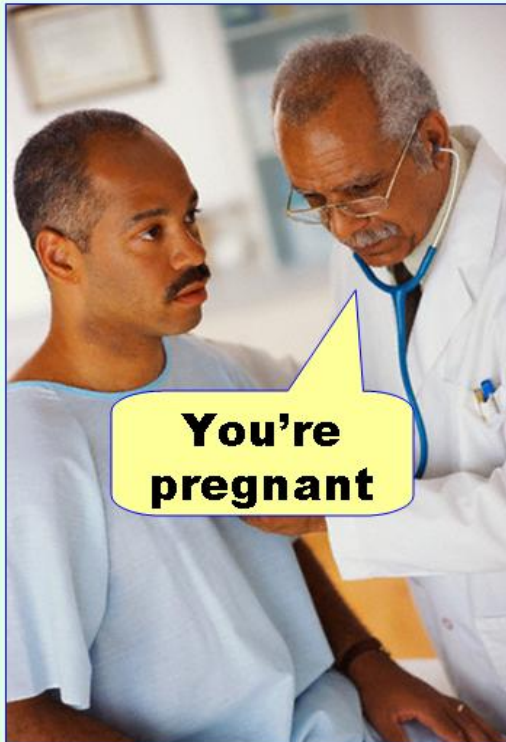
$Y_t = 0, Y_p = 0$
True Negative

$Y_t = 0, Y_p = 1$
False Positive

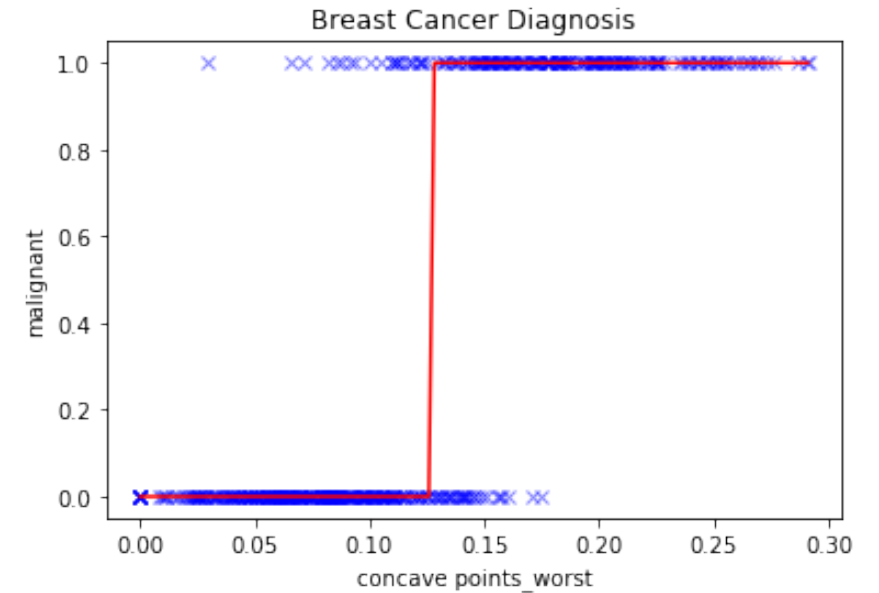
$Y_t = 1, Y_p = 0$
False Negative

Type I error and Type II error

Type I error
(false positive)



Type II error
(false negative)



Binary Classification Performance Metrics

Confusion Matrix

TP, TN, FP, FN

		Y _p	
		0	1
Y _t	0	70 <small>TN</small>	1 <small>FP</small>
	1	3 <small>FN</small>	40 <small>TP</small>

```
from sklearn.metrics import confusion_matrix
confusion_matrix(y_true, y_pred)
```

$$ACC = \frac{TP+TN}{TP+TN+FP+FN}$$

Accuracy

$$TPR = \frac{TP}{TP+FN} \rightarrow \frac{TP}{P(data)}$$

Recall, Sensitivity

$$TNR = \frac{TN}{TN+FP} \rightarrow \frac{TN}{N(data)}$$

Specificity, Selectivity

$$PPV = \frac{TP}{TP+FP} \rightarrow \frac{TP}{P(prediction)}$$

Positive predictive value,
Precision

$$FPR = \frac{FP}{FP+TN} \rightarrow \frac{FP}{N(data)} = 1 - TNR$$

False-positive rate,
Fall-out

$$FNR = \frac{FN}{FN+TP} = \frac{FN}{P(data)} = 1 - TPR$$

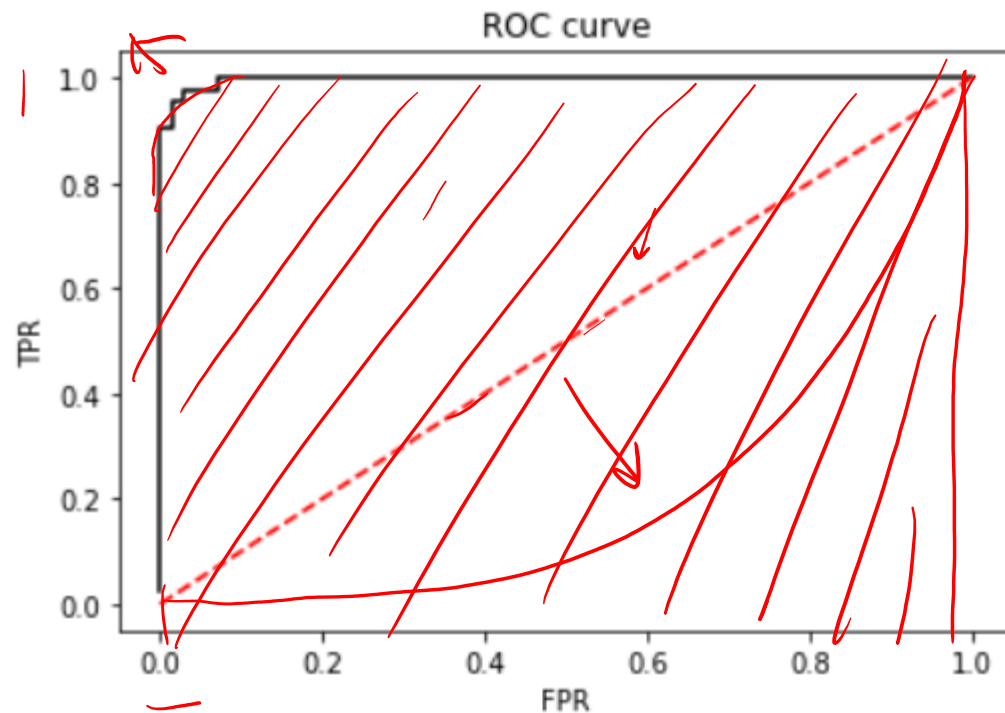
False-negative rate,
Miss rate

$$F1 = \frac{2TP}{2TP+FP+FN} = \frac{2P \cdot r}{P+r}$$

Harmonic mean of
Precision and recall

Performance Metrics- ROC, AUC

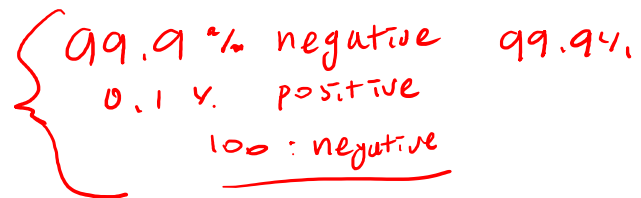





Receiver-Operating Characteristics Curve



AUC
0-1

AUC=0.99

Which Performance Metric should I choose?

- Accuracy 
 - Sensitivity, Recall, TPR 
 - Specificity, Sensitivity, TNR 
 - Precision, PPV
 - False Positive Rate (fall-out) 
 - False Negative Rate (miss rate) 
 - F1 score
 - AUC / ROC
 - Confusion matrix
- 

Why use Cross-Entropy, not Accuracy?

Cross Entropy

$$\mathcal{L}_{CE}(y, p) = -\frac{1}{N} \sum_i^N \sum_j^m y_{i,j} \log \hat{p}_{i,j}$$

Accuracy

$$\frac{\text{TP} + \text{TN}}{\text{ALL}}$$

A

Predicted probability			Target			Correct?
<u>0.4</u>	0.3	0.3	<u>1</u>	0	0	1 <u>-</u>
0.3	0.4	0.3	0	1	0	1 <u>-</u>
0.8	0.1	0.1	0	0	<u>1</u>	0 <u>-</u>

2/3

B

Predicted probability			Target			Correct?
<u>0.8</u>	0.1	0.1	<u>1</u>	0	0	1
0.1	<u>0.8</u>	0.1	0	<u>1</u>	0	1
<u>0.4</u>	<u>0.3</u>	<u>0.3</u>	0	0	1	0

2/3



Logistic Regression

library usage

Using sklearn's LogisticRegression

[sklearn.linear_model](#).LogisticRegression

```
class sklearn.linear_model.LogisticRegression(penalty='l2', dual=False, tol=0.0001, C=1.0, fit_intercept=True,  
intercept_scaling=1, class_weight=None, random_state=None, solver='lbfgs', max_iter=100, multi_class='auto',  
verbose=0, warm_start=False, n_jobs=None, l1_ratio=None)
```

(Handwritten red annotations: a checkmark above 'fit_intercept=True', a red underline under 'class_weight=None', and a red '- 1' below 'l1_ratio=None')

solver : str, {'newton-cg', 'lbfgs', 'liblinear', 'sag', 'saga'}

https://scikit-learn.org/stable/modules/generated/sklearn.linear_model.LogisticRegression.html

<https://github.com/scikit-learn/> (directory: scikit-learn/sklearn/linear_model/logistic.py)

Using sklearn's LogisticRegression

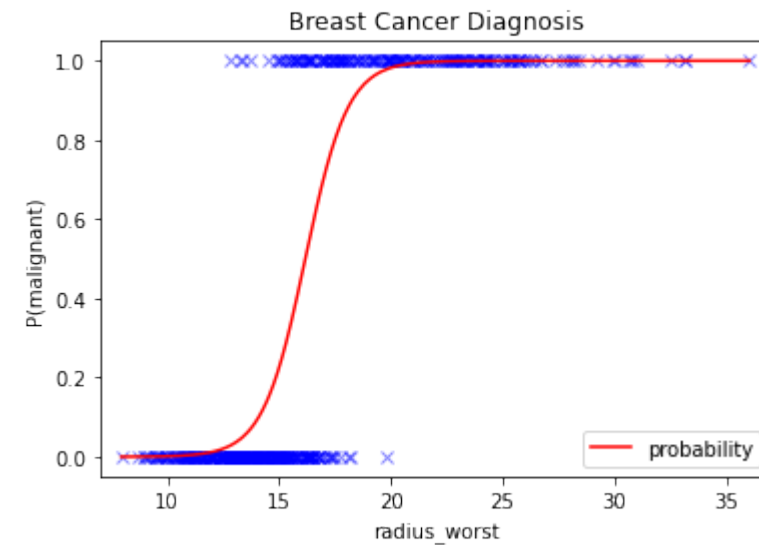
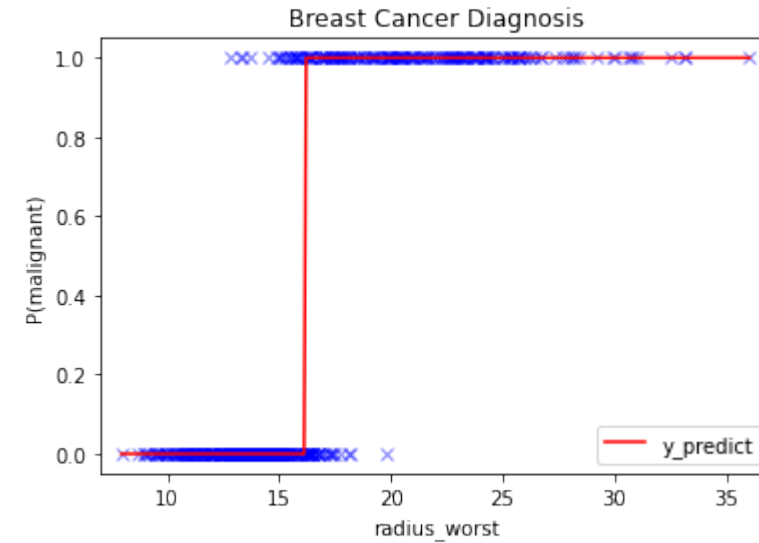
```
from sklearn.linear_model import LogisticRegression  
model = LogisticRegression().fit(X, y)
```

model.coef_

model.intercept_

$y_p \leftarrow$ model.predict(X_test)

$p \leftarrow$ model.predict_proba(X_test)



Using sklearn's LogisticRegression

```
from sklearn.model_selection import train_test_split
(X_train, X_test, y_train, y_test) = train_test_split(X, y, test_size=0.2, random_state=42)
from sklearn.linear_model import LogisticRegression as LR

clf = LR(class_weight="balanced", solver='liblinear').fit(X_train, y_train.ravel())
clf.score(X_test, y_test)

0.9649122807017544
```

```
from sklearn.metrics import confusion_matrix, accuracy_score, f1_score, precision_score, recall_score

yp = clf.predict(X_test)
print('acc', accuracy_score(y_test, yp))
print('recall', recall_score(y_test, yp))
print('precision', precision_score(y_test, yp))
print('F1', f1_score(y_test, yp))
```

```
acc 0.9649122807017544
recall 0.9302325581395349
precision 0.975609756097561
F1 0.9523809523809524
```

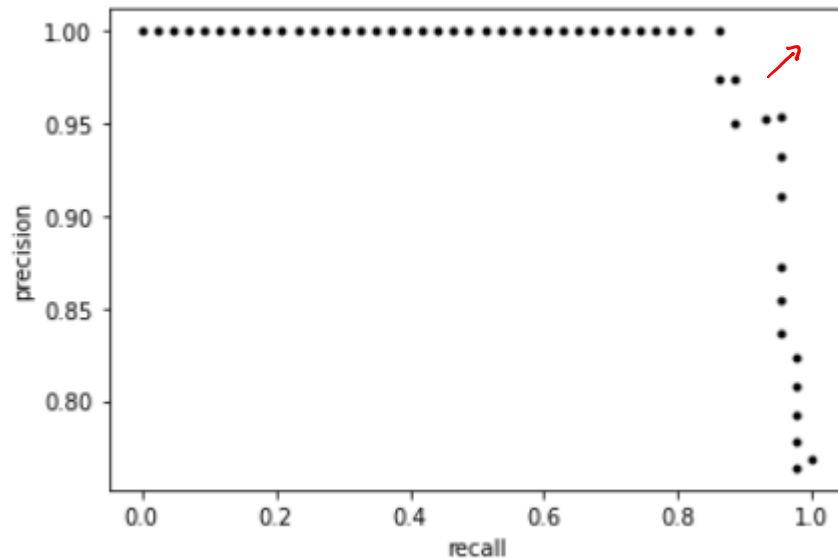
```
pd.DataFrame(confusion_matrix(y_test, yp, labels=[0, 1]))
```

y_t ↓

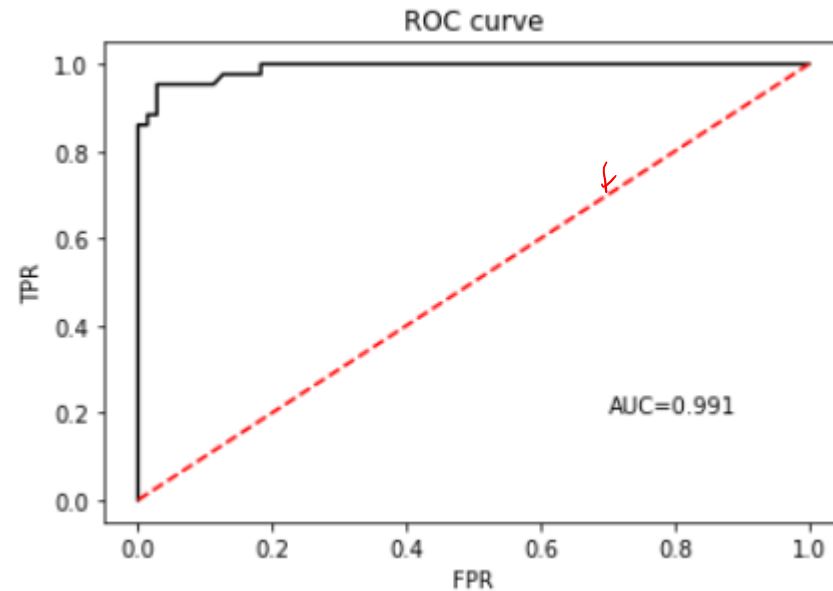
	0	1	y_p
0	70	1	
1	3	40	

Using sklearn's LogisticRegression

```
from sklearn.metrics import precision_recall_curve
ypp = clf.predict_proba(x_test)
pre, rec, th = precision_recall_curve(y_test, ypp[:,1])
plt.plot(rec, pre, 'k-')
plt.ylabel('precision')
plt.xlabel('recall')
```



```
from sklearn.metrics import roc_curve, roc_auc_score
fpr, tpr, th = roc_curve(y_test, ypp[:,1])
auc = roc_auc_score(y_test, ypp[:,1])
plt.plot(fpr, tpr, 'k-')
plt.plot(np.arange(0, 1.1, 0.1), np.arange(0, 1.1, 0.1), 'r--')
plt.title('ROC curve')
plt.xlabel('FPR')
plt.ylabel('TPR')
plt.text(0.7, 0.2, 'AUC=+' + "{:.3f}".format(auc))
```



What about the statistics?

Statsmodels library

```
import statsmodels.api as sm  
logit_model=sm.Logit(y_train,x_train)  
result=logit_model.fit()  
print(result.summary())
```

Optimization terminated successfully.
Current function value: 0.681033
Iterations 4

Logit Regression Results

Dep. Variable:	y	No. Observations:	455
Model:	Logit	Df Residuals:	454
Method:	MLE	Df Model:	0
Date:	Wed, 18 Sep 2019	Pseudo R-squ.:	-0.03232
Time:	19:23:16	Log-Likelihood:	-309.87
converged:	True	LL-Null:	-300.17
		LLR p-value:	nan

	coef	std err	z	P> z	[0.025	0.975]
x1	2.3970	0.731	3.279	0.001	0.964	3.830

What about the statistics?

Bootstrap (Resample)



What about the statistics?

sklearn.ensemble.BaggingClassifier

```
class sklearn.ensemble.BaggingClassifier(base_estimator=None, n_estimators=10, *, max_samples=1.0, max_features=1.0,  
bootstrap=True, bootstrap_features=False, oob_score=False, warm_start=False, n_jobs=None, random_state=None, verbose=0)
```

```
clf = BaggingClassifier(base_estimator=LogisticRegression(class_weight="balanced"),n_estimators=1000).fit(X,y)
```

```
1 clf.estimators_
```

```
LogisticRegression(class_weight='balanced', random_state=1952926171),  
LogisticRegression(class_weight='balanced', random_state=1761383086),  
LogisticRegression(class_weight='balanced', random_state=1449071958),  
LogisticRegression(class_weight='balanced', random_state=1910541088),  
LogisticRegression(class_weight='balanced', random_state=1341730541),
```

```
1 clf.estimators_[0].coef_
```

```
array([[1.10855636, 1.6176168 ]])
```

```
1 clf.estimators_[0].intercept_
```

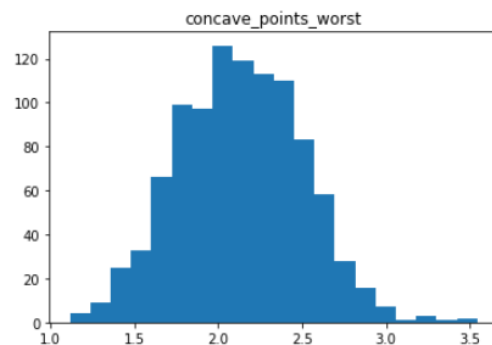
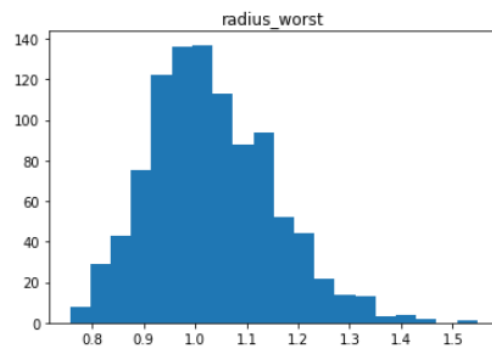
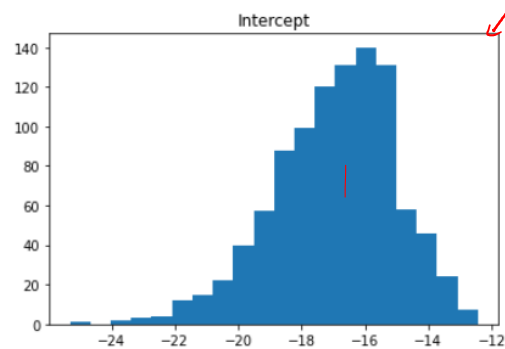
```
array([-18.31031081])
```

```
1 est=clf.estimators_[0]  
2 list(est.intercept_)+list(est.coef_[0])
```

```
[-18.310310809003678, 1.1085563574633186, 1.6176167999143638]
```

<https://scikit-learn.org/stable/modules/generated/sklearn.ensemble.BaggingClassifier.html>

What about the statistics?



scipy.stats.ttest_1samp

`scipy.stats.ttest_1samp(a, popmean, axis=0, nan_policy='propagate', alternative='two-sided')`

```
1 t = ttest_1samp(coefs[:,i], 0)
2 print(t)
3 t.statistic
```

`Ttest_1sampResult(statistic=181.38175127406623, pvalue=0.0)`

181.38175127406623

coef	t-statistic	p-value
Intercept	-278.342110	0.0
radius_worst	270.601870	0.0
concave_points_worst	181.381751	0.0

$$\begin{cases} H_0 \rightarrow c = 0 \\ H_1 \Rightarrow c \neq 0 \end{cases}$$

5%.

$$p < 0.025$$



https://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.ttest_1samp.html