BARRA
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Convex Optimization
Homework 2

Exercise 1 - LP Duality

1. (P) $\min_x c^T x$
s.t. $Ax = b$
$x \geq 0$

The Lagrangian can be written as (for $x \in \mathbb{R}^d$, $\lambda \in \mathbb{R}_+^d$, $v \in \mathbb{R}^n$):

$$\mathcal{L}(x, \lambda, v) = c^T x - \lambda^T x + v^T (b - Ax) = (c - \lambda - A^T v)^T x + v^T b$$

Then the dual function is:

$$g(\lambda, v) = \inf_x \mathcal{L}(x, \lambda, v)$$

$$= \begin{cases} (b^T v)^T & \text{if } c - \lambda - A^T v = 0, \ \lambda \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

$$= \begin{cases} (b^T v)^T & \text{if } A^T v = c - \lambda, \ \lambda \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

Since $\lambda \geq 0$, then the condition $A^T v = c - \lambda$ can be written as $A^T v \leq c$.

Finally, the dual problem of (P) is: $\boxed{\begin{array}{l} \max_v b^T v \\ \text{s.t } A^T v \leq c \end{array}}$ (D).

2. (D) $\max_y b^T y$
s.t $A^T y \leq c$

The Lagrangian can be written as (for $y \in \mathbb{R}^n$, $\lambda \in \mathbb{R}_+^d$):

$$\mathcal{L}(y,\lambda) = b^T y + \lambda^T (c - A^T y)$$
$$= (b - A\lambda)^T y + \lambda^T c$$

Then, the dual function is:

$$g(\lambda) = \sup_y \mathcal{L}(y,\lambda)$$
$$= \begin{cases} (c^T\lambda)^T & \text{if } A\lambda = b,\ \lambda \geq 0 \\ +\infty & \text{otherwise} \end{cases}$$

Thus, the dual of the problem (D) is:

$$\boxed{\begin{array}{ll} \min\limits_{\lambda} & c^T\lambda \\ \text{s.t} & A\lambda = b \\ & \lambda \geq 0 \end{array}} \quad (P).$$

3. $$\min_{x,y} \ c^T x - b^T y \qquad \text{(Self-Dual)}$$

$$\text{s.t.} \quad Ax = b$$
$$x \geq 0$$
$$A^T y \leq c$$

For $x \in \mathbb{R}^d$, $y \in \mathbb{R}^n$, $\lambda_1, \lambda_2 \in \mathbb{R}_+^d$, $v \in \mathbb{R}^n$, the Lagrangian of this problem is:

$$\mathcal{L}(x,y,\lambda_1,\lambda_2,v) = c^T x - b^T y - \lambda_1^T x + \lambda_2^T (A^T y - c) + v^T (b - Ax)$$
$$= (c - \lambda_1 - A^T v)^T x + (A\lambda_2 - b)^T y + v^T b - \lambda_2^T c.$$

The dual function is then:

$$\inf_{x,y} \mathcal{L}(x,y,\lambda_1,\lambda_2,v) = \begin{cases} (b^T v - c^T \lambda_2)^T & \text{if } \begin{cases} A^T v = c - \lambda_1, & \lambda_1 \geq 0 \\ A\lambda_2 = b, & \lambda_2 \geq 0 \end{cases} \\ -\infty & \text{otherwise} \end{cases}$$

Since $\lambda_1 \geq 0$, the condition $A^T v = c - \lambda_1$ can be written as $A^T v \leq c$.

Thus, the dual of the problem is:

$$\max_{\lambda_2, v} \; b^T v - c^T \lambda_2 \quad \text{s.t} \quad A^T v \leq c, \quad A\lambda_2 = b, \quad \lambda_2 \geq 0$$

which is equivalent to:

$$\boxed{\min_{\lambda_2, v} \; c^T \lambda_2 - b^T v \quad \text{s.t} \quad A^T v \leq c, \quad A\lambda_2 = b, \quad \lambda_2 \geq 0}$$

[Self-Dual]

Therefore, the problem is self-dual.

4. • Let x^P be the optimal solution to (P) and y^D the optimal solution to (D).

Since x^P and y^D are feasible for (P) and (D), then we have:

$$A x^P = b \quad \text{and} \quad A^T y^D \leq c$$
$$x^P \geq 0$$

Thus (x^P, y^D) is feasible for the self-dual problem

Then: $$c^T x^P - b^T y^D = \min_x c^T x - \max_y b^T y$$
$$= \min_x c^T x + \min_y -b^T y$$
$$= \min_{x,y} c^T x - b^T y.$$

Therefore $[x^P, y^D]$ is the optimal solution to the self-dual problem, thus equal to $[x^*, y^*]$. So we have seen that solving (P) and (D) leads to the optimal solution $[x^*, y^*]$.

- Under strong duality, the value p^* of (P) and the value d^* of its dual (D) are the same for the optimal solution $[x^*, y^*]$:

$$c^T x^* = p^* = d^* = b^T y^*$$

Thus, $c^T x^* - b^T y^* = 0$.

<u>The optimal value of the (Self-Dual) problem is 0.</u>

<u>Exercise 2 - Regularized Least-Square</u>

1. Let $f(x) = \|x\|_1$.

The conjugate f^* of f is defined by:

$$f^*(y) = \sup_x \left(y^T x - f(x) \right)$$

$$= \sup_x \sum_{i=1}^{n} \left(y_i x_i - |x_i| \right)$$

$$= \sup_x \sum_{i=1}^{n} |x_i| \left(y_i \operatorname{sgn}(x_i) - 1 \right)$$

- if there exist $i \in \{1, \dots, n\}$ such that $y_i > 1$, then by taking $x_i \to +\infty$, $|x_i| (\underbrace{y_i - 1}_{>0}) \to +\infty$ and the sum tends to $+\infty$.

- if there exist $i \in \{1, \dots, n\}$ such that $y_i < -1$, then by taking $x_i \to -\infty$, $|x_i| (\underbrace{-y_i - 1}_{>0}) \to +\infty$ and the sum tends to $+\infty$.

- if $\|y\|_\infty \leq 1$, then $(y_i \operatorname{sgn}(x_i) - 1) \leq 0$ for all $x_i \in \mathbb{R}$. Therefore the sup is obtained for $x_i = 0$ for $i \in \{1, \dots, n\}$.

Thus: $$\boxed{f^*(y) = \begin{cases} 0 & \text{if } \|y\|_\infty \leq 1 \\ +\infty & \text{otherwise} \end{cases}}$$

2. $\min_x \|Ax - b\|_2^2 + \|x\|_1$ (RLS)

$\Leftrightarrow \min_{x,y} \|y\|_2^2 + \|x\|_1$

s.t $y = Ax - b$.

The Lagrangian of this equivalent problem is, for $x \in \mathbb{R}^d$, $y, v \in \mathbb{R}^n$:

$$\mathcal{L}(x, y, v) = \|y\|_2^2 + \|x\|_1 + v^T(y - Ax + b)$$
$$= v^T y + \|y\|_2^2 + \|x\|_1 - v^T A x + v^T b.$$

Then: $\min_{x,y} \mathcal{L}(x, y, v) = v^T b + \min_y \left(v^T y + \|y\|_2^2\right) + \min_x \left(\|x\|_1 - v^T A x\right)$

Let $h(y) = v^T y + \|y\|_2^2$ $\quad y^T y$

$$\nabla h(y) = v + 2y = 0 \Leftrightarrow y = -\frac{1}{2} v$$

So: $\min_{x,y} \mathcal{L}(x, y, v) = v^T b - \frac{1}{2} \underbrace{v^T v}_{= \|v\|_2^2} + \|\frac{1}{2} v\|_2^2 + \min_x \left(\|x\|_1 - v^T A x\right)$

$$= v^T b - \frac{1}{4} \|v\|_2^2 - \underbrace{\max_x \left(v^T A x - \|x\|_1\right)}$$

$$= v^T b - \frac{1}{4} \|v\|_2^2 - f^*(v^T A)$$

According to question 1: $f^*(v^T A) = 0$ if $\|v^T A\|_\infty \leq 1$

Hence the dual problem is:

$$\boxed{\begin{aligned} &\max_v \; v^T b - \frac{1}{4} \|v\|_2^2 \\ &\text{s.t. } \|v^T A\|_\infty \leq 1 \end{aligned}}$$

Exercise 3 – Data Separation

1. $$\min_{w} \frac{1}{n} \sum_{i=1}^{n} \underbrace{\ell(w, x_i, y_i)}_{} + \frac{\tau}{2} \|w\|_2^2 \qquad [Sep\ 1]$$

$$\Leftrightarrow \min_{w} \frac{1}{n\tau} \sum_{i=1}^{n} \max\{0, 1 - y_i (w^T x_i)\} + \frac{1}{2} \|w\|_2^2 \qquad \times \frac{1}{\tau}$$

$$\Leftrightarrow \min_{w, z} \frac{1}{n\tau} \sum_{i=1}^{n} z_i + \frac{1}{2} \|w\|_2^2$$

$$\text{s.t. } z_i = \max\{0, 1 - y_i (w^T x_i)\} \quad i = 1, \dots, n$$

$$\Leftrightarrow \min_{w, z} \frac{1}{n\tau} \mathbf{1}^T z + \frac{1}{2} \|w\|_2^2 \qquad [Sep\ 2]$$

$$\text{s.t. } z_i \geq 1 - y_i (w^T x_i) \quad i = 1, \dots, n$$
$$z_i \geq 0 \quad i = 1, \dots, n$$

Therefore problem [Sep 2] solves problem [Sep 1].

2. For $z \in \mathbb{R}^n$, $w \in \mathbb{R}^d$, $\lambda \in \mathbb{R}^n_+$, $\pi \in \mathbb{R}^n_+$, the Lagrangian of [Sep 2] can be written as:

$$\mathcal{L}(z, w, \lambda, \pi) = \frac{1}{n\tau} \mathbf{1}^T z + \frac{1}{2} \|w\|_2^2 - \pi^T z + \sum_{i=1}^{n} \lambda_i \left(1 - y_i (w^T x_i) - z_i\right)$$

$$\inf_{z, w} \mathcal{L}(z, w, \lambda, \pi) = \inf_{z} \left(\frac{1}{n\tau} \mathbf{1} - \pi - \lambda\right)^T z + \inf_{w} \left(\frac{1}{2} \|w\|_2^2 - \sum_{i=1}^{n} \lambda_i y_i (w^T x_i)\right) + \sum_{i=1}^{n} \lambda_i$$

$$\inf_{z} \left(\frac{1}{n\tau} \mathbf{1} - \pi - \lambda\right)^T z = \begin{cases} 0 & \text{if } \frac{1}{n\tau} \mathbf{1} \leq \lambda \\ -\infty & \text{otherwise} \end{cases} \quad \text{since } \pi \geq 0$$

Let's derive $\mathcal{L}$ is w:

$$\nabla_w \mathcal{L}(z, w, \lambda, \pi) = w - \sum_{i=1}^{n} \lambda_i y_i x_i = 0 \Leftrightarrow w = \sum_{i=1}^{n} \lambda_i y_i x_i$$

Thus $$\inf_{w} \left(\frac{1}{2} \|w\|_2^2 - \sum_{i=1}^{n} \lambda_i y_i (w^T x_i)\right) = \frac{1}{2} \left\|\sum_{i=1}^{n} \lambda_i y_i x_i\right\|_2^2$$

$$- \sum_{ij} \lambda_i \lambda_j y_i y_j x_j^T x_i$$

After removing the variable w, we obtain the following dual problem:

$$\boxed{\begin{aligned} \max_{\lambda} \quad & \sum_{i=1}^{n} \lambda_i + \frac{1}{2} \left\| \sum_{i=1}^{n} \lambda_i y_i x_i \right\|_2^2 - \sum_{i,j} \lambda_i \lambda_j y_i y_j x_j^T x_i \\ \text{s.t.} \quad & \frac{1}{n\tau} \mathbf{1} - \lambda \geq 0 \\ & \lambda \geq 0 \end{aligned}}$$