Limit Cycles in the Renormalization Group of the BCS and the sine-Gordon models

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Based on collaborations with:

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Outline

I. Brief introduction to RG

II. Historical motivation and recent developments

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V. Cyclic Kosterlitz-Thouless Flows

Introduction

The Renormalization Group is a central concept in Particle Physics, Statistical Mechanics and Condensed Matter. It explains the universal properties observed in a large variety of systems near critical points. For example the specific heat $C(T)$ of a spin system behaves near the critical temperature T_c as

$$\boxed{C(T) \sim (T - T_c)^{-\alpha}}$$

The critical exponent α depends on the Universality class ($\alpha = 0$ for 2D-Ising, $\alpha \sim 0.133$ for 3D-Ising).

Scaling behaviour is associated to fixed points of the RG transformation. Let H be the Hamiltonian with spacing $\sim 1/\Lambda$. Under coarse-graining it becomes $H' = R_b(H)$ where $b = \Lambda/\Lambda'$. Near a fixed point Hamiltonian, $H^* = R_b(H^*)$ can be expanded as

$$\boxed{H = H^* + \sum_k g_k O_k}$$

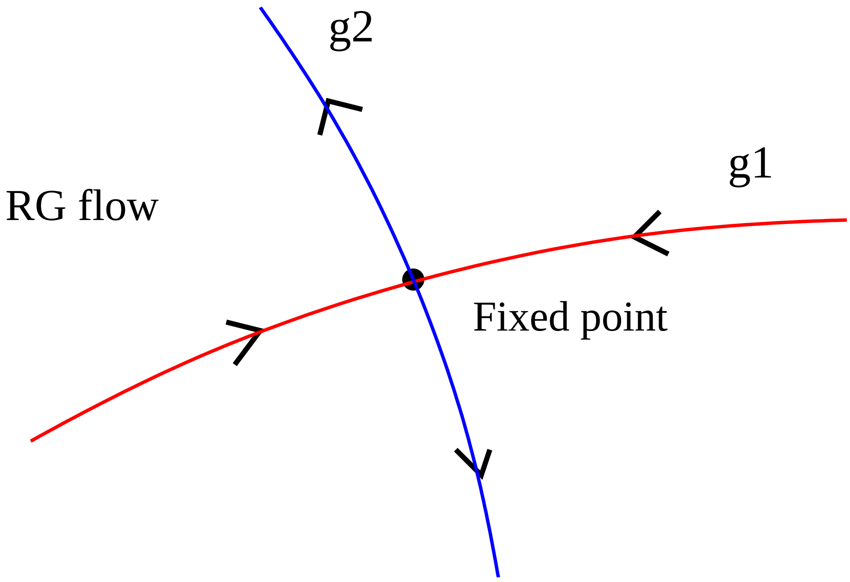
where O_k are scaling fields and g_k coupling constants that transform under the RG as

$$g'_k = b^{y_k} g_k, \quad b > 1$$

This yields a classification of the scaling fields, i.e.

$$\boxed{O_k : \begin{cases} y_k < 0 \text{ irrelevant} \\ y_k > 0 \text{ relevant} \\ y_k = 0 \text{ marginal} \end{cases}}$$

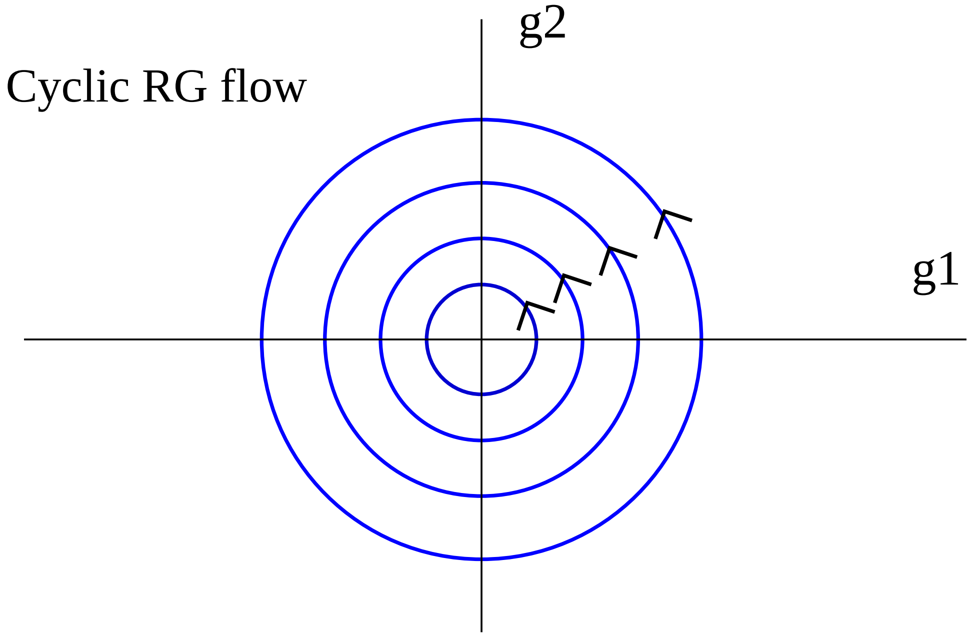
A model with one irrelevant, g_1, and one relevant, g_2, couplings has a RG flow,



y_k are real eigenvalues of the RG transformation.

For complex eigenvalues, $y = ih$, the RG evolution of the complex coupling $g = g_1 + ig_2$ is periodic in RG scale s

$$g(s) = e^{ihs} g_0, \quad b = e^s$$



with a period

$$\boxed{g(s + \frac{2\pi}{h}) = g(s)}$$

The renormalized Hamiltonian repeats itself after a finite scale transformation $b = e^{2\pi/h}$.

$$\boxed{\text{RG limit cycles} \rightarrow \text{self-similarity}}$$

Brief History

In 1971 K. G. Wilson suggested the possible existence of limit cycles in strong interactions involving two or more coupling constants:

"The $e^+ - e^-$ annihilation experiments above 1-GeV energy may distinguish a fixed point from a limit cycle or other asymptotic behaviour"

In the case of limit cycles,

"one will see perpetual oscillations in the $e^+ - e^-$ total hadronic cross section in the limit of large momentum transfer q^2".

This effect has not been seen, but it remains as a possibility for other Physical systems.

More recent developments:

- In 1993 Bedaque, Hammer and van Kolck studied a QM Hamiltonian in Nuclear Physics with two- and three-body delta function potentials that exhibits limit cycle behaviour.
- In 2001 Bernard and LeClair found cyclic Kosterlitz-Thouless flows in an anisotropic current-current WZW model.
- In 2002 Glazek and Wilson defined a discrete QM Hamiltonian with two couplings whose RG has limit cycles and chaotic behaviour.
- In 2002-03 LeClair, Román, Sierra proposed an extension of the BCS model of superconductivity with limit cycle behaviour and a S-matrix theory for the "cyclic sine-Gordon model".

The Glazek-Wilson model

Warm up: The Delta function potential in 2D.

$$H = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - g\,\delta(x)\delta(y)$$

where $g > 0$ is a dimensionless coupling. This QM Hamiltonian requires regularization/renormalization as a ordinary QFT. The spectrum contains a unique bound state with energy E_0,

$$E_0 = \Lambda\, e^{-1/g}$$

with Λ a mass generated dynamically (toy-QCD).

A regularized lattice version of H is given by the matrix

$$H_{n,m}(g) = b^{n+m}(\delta_{n,m} - g)$$

where $b > 1$ and $-M < n, m \leq N$. M (N) is an infrared (ultraviolet) cutoff.

The GW-model with limit cycles is a perturbation of the discrete δ-potential:

$$H_{n,m}(g_N, h_N) = b^{n+m}(\delta_{n,m} - g_N - ih_N \,\mathrm{sign}(n-m))$$

The RG analysis starts from the Schrödinger eq.

$$\sum_{m=-\infty}^{N} H_{n,m}\psi_m = E\,\psi_n$$

and eliminates the highest energy component, ψ_N, in terms of the low energy ones, $\psi_{n<N}$. The new Hamiltonian is defined by

$$g_{N-1} = g_N + \frac{g_N^2 + h_N^2}{1 - g_N}, \quad h_{N-1} = h_N \equiv h$$

The coupling h is invariant under the RG.

After p RG-iterations one gets

$$g_{N-p} = h \tan\left(\tan^{-1}\left(\frac{g_N}{h}\right) + p\beta\right), \;\; \beta = \tan^{-1} h$$

If $\pi/\beta = p$ there is a limit cycle with period p, i.e.

$$\boxed{g_{N-p} = g_N}$$

If π/β is irrational the flow of g_N is chaotic.

The main physical consequence is the existence of an infinite number of bound states,

$$\boxed{E_{n+1}(g) = e^{-\lambda} E_n(g), \quad e^{\lambda} = b^p}$$

which accumulate at the origin,

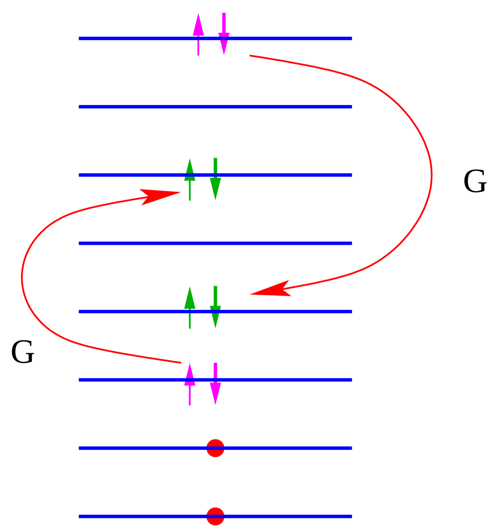
$$E_0 < E_1 < \ldots < E_\infty = 0$$

This is a generic feature of models with limit cycles and bound states (Russian doll scaling).

The standard BCS model of Superconductivity:

$$\boxed{H_{BCS} = \sum_{j=1}^{N} \varepsilon_j \; b_j^\dagger b_j \;\; - \;\; \sum_{j,j'=1}^{N} \; V_{jj'} \, b_j^\dagger b_{j'}}$$

- $b_j = c_{j,-}c_{j,+}$, $b_j^\dagger = c_{j,+}^\dagger c_{j,-}^\dagger$: Cooper pair operators
- The ε_j are equally spaced energy levels $-\omega < \varepsilon_j < \omega$ with level spacing 2δ.



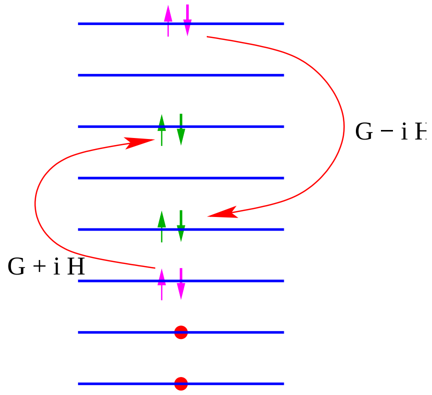
Standard choice $V_{jj'} = G > 0 \Rightarrow$ unique ground state (condensate) characterized by the energy gap

$$\Delta_0 \sim 2\omega \, e^{-1/g}, \quad g = G/\delta.$$

Russian Doll BCS model

LeClair, Román, Sierra (cond-mat/0211338)

$$V_{jj'} = \begin{cases} G + i\mathcal{H} & \text{if} \quad \varepsilon_j > \varepsilon_{j'} \\ G & \text{if} \quad \varepsilon_j = \varepsilon_{j'}\,, \\ G - i\mathcal{H} & \text{if} \quad \varepsilon_j < \varepsilon_{j'} \end{cases} \quad \begin{matrix} G = g\delta \\ \mathcal{H} = h\delta \end{matrix}.$$



The BCS variational ansatz for this model is

$$|\psi_{\rm BCS}\rangle = \prod_{j=1}^{N} \left(u_j + v_j\ b_j^\dagger \right) |0\rangle$$

$$u_j^2 = \frac{1}{2}\left(1 + \frac{\xi_j}{E_j}\right), \qquad v_j^2 = \frac{1}{2} e^{2i\phi_j}\left(1 - \frac{\xi_j}{E_j}\right),$$

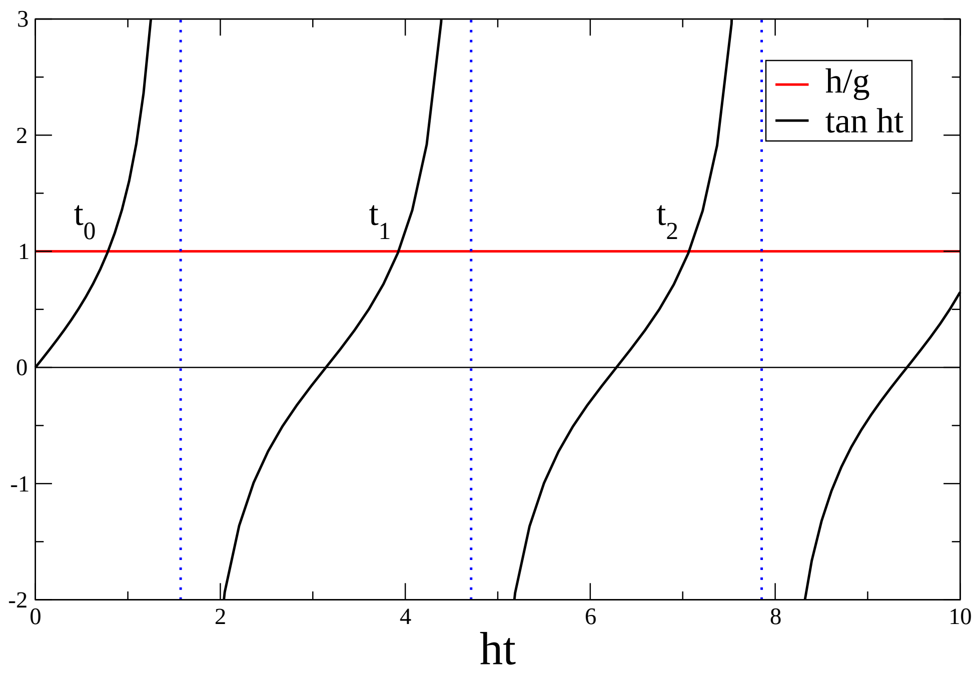
$$E_j = \sqrt{\xi_j^2 + \Delta_j^2}, \qquad \xi_j = \varepsilon_j - \mu - V_{jj}$$

Gap equation:

$$\widetilde{\Delta}_j = \sum_{j' \neq j} V_{jj'} \frac{\widetilde{\Delta}_{j'}}{E_{j'}}, \qquad \widetilde{\Delta}_j \equiv \Delta_j e^{i\phi_j}$$

In the continuum limit, $\widetilde{\Delta}(\varepsilon) = \Delta(\epsilon) e^{i\phi(\varepsilon)}$ the solution is given by

- $\Delta(\varepsilon) = \Delta, \qquad \phi(\varepsilon) = h \sinh^{-1} \frac{\varepsilon}{\Delta}$
- $\Delta = \omega / \sinh t \quad \rightarrow \quad \tan(ht) = \frac{h}{g}$



This yields an infinite number of solutions Δ_n which can be parameterized as follows:

$$\boxed{\Delta_n = \frac{\omega}{\sinh t_n}, \quad t_n = t_0 + \frac{n\pi}{h}, \quad n = 0, 1, 2, \ldots,}$$

The gaps satisfy $\Delta_0 > \Delta_1 > \cdots > \Delta_\infty = 0$.

Weak coupling $\Delta_n \ll \omega$ yields Russian Doll scaling:

$$\boxed{\Delta_n \sim 2\omega\, e^{-t_0 - n\pi/h} \rightarrow \Delta_{n+1} = e^{-\lambda}\Delta_n}$$

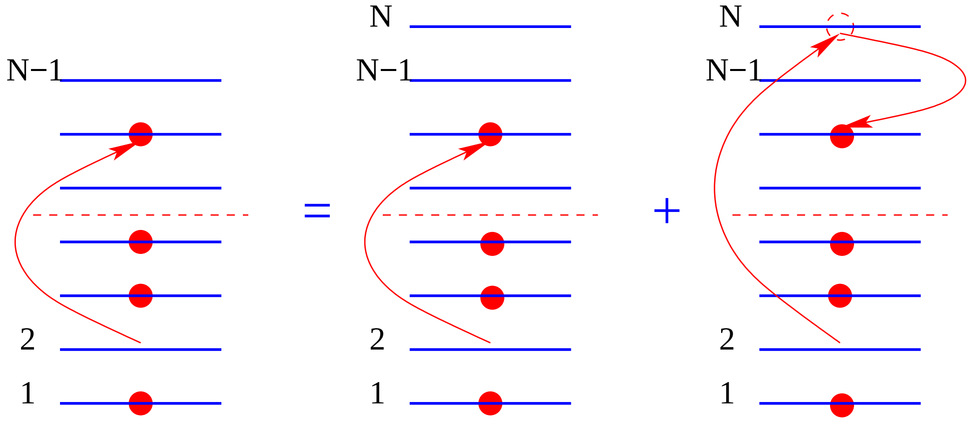
with

$$\boxed{\lambda = \frac{\pi}{h}}$$

This suggest that the RG of this model must have limit cycles.

Renormalization Group of BCS:

$$H(G_N, \mathcal{H}_N) \to H(G_{N-1}, \mathcal{H}_{N-1})$$



$$G_{N-1}+i\mathcal{H}_{N-1} = G_N+i\mathcal{H}_N+\frac{1}{N\delta}(G_N+i\mathcal{H}_N)(G_N-i\mathcal{H}_N)$$

Hence $\mathcal{H}_N = \mathcal{H}_{N-1}$ is an RG invariant. In the large N limit one can define a variable $s = \log N_0/N$, where N_0 is the initial size of the system.

$$\frac{dg}{ds} = (g^2 + h^2), \qquad s \equiv \log \frac{N_0}{N}.$$

The solution to the above equation is

$$g(s) = h \tan \left[hs + \tan^{-1}\left(\frac{g_0}{h}\right) \right], \quad g_0 = g(N_0).$$

which exhibits the limit cycle behaviour

$$\boxed{g(s+\lambda) = g(s) \iff g(e^{-\lambda}N) = g(N), \quad \lambda \equiv \frac{\pi}{h}}$$

Sinchronicity between the mean-field and the RG:

$$\Delta_n = \frac{\omega}{\sinh t_n}, \quad \tan(h\, t_n) = \frac{h}{g}$$

one can write

$$g(s) = h \tan \left[h(s - t_n) + \frac{\pi}{2} \right]$$

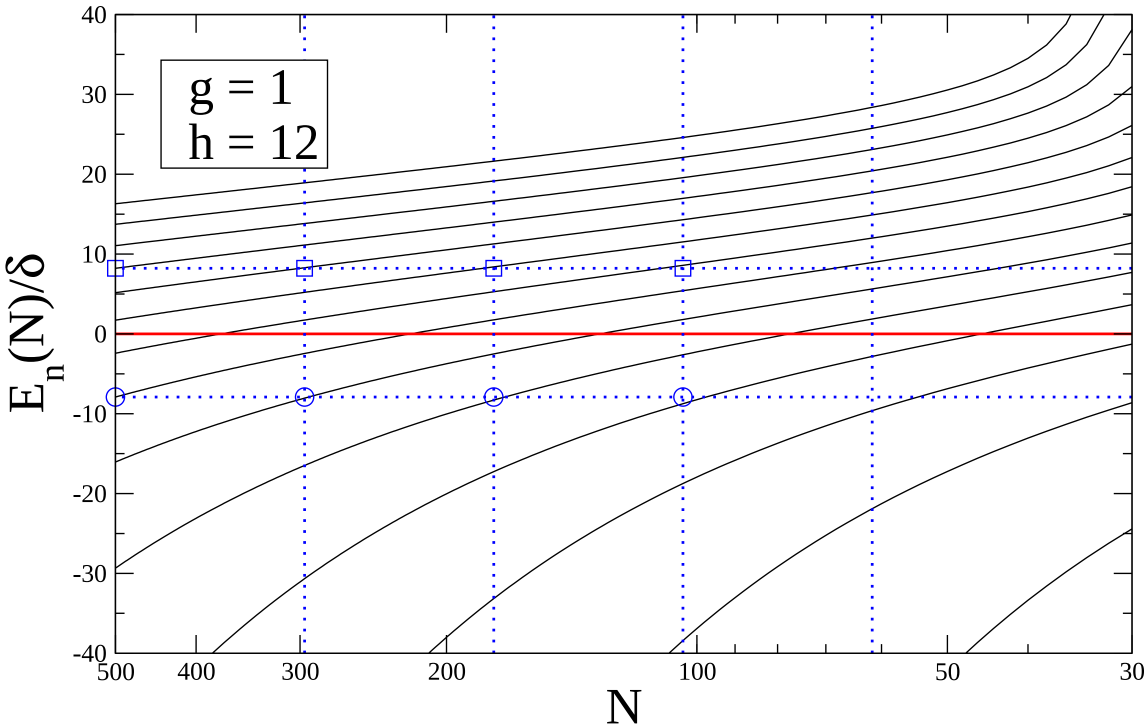
At $s = t_n$ the coupling $g(s)$ jumps from ∞ to $-\infty$. Every jump eliminates a condensate from the spectrum.

Numerical Work-One Copper Pair problem:

The BCS Hamiltonian becomes the $N \times N$ matrix,

$$H_{j,k} = \varepsilon_j \, \delta_{j,k} - (G + i\,\mathcal{H}\,\mathrm{sign}(j-k))$$

Exact eigenstates of one-pair Hamiltonian for N levels ($N_0 = 500$ down to 30).



Cyclicity of the spectrum (self-similarity):

$$E_{n+1}(N) = E_n(e^{-\lambda_1} N), \qquad E_{n+1}(N) = e^{-\lambda_1} E_n(N)$$

$$\lambda_1 = \frac{2\pi}{h}$$

Finite temperature:

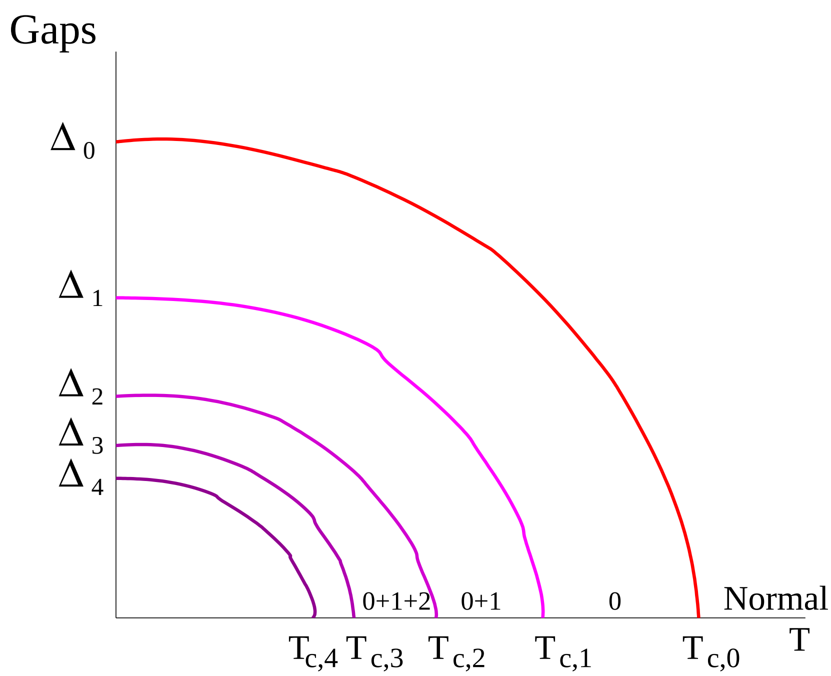
Gap equation:

$$\widetilde{\Delta}_j(T) = \sum_{j' \neq j} V_{jj'} \frac{\widetilde{\Delta}_{j'}(T)}{E_{j'}} \tanh(\beta E_{j'}/4), \qquad \beta = 1/T$$

There is a critical temperature $T_{c,n}$ for every gap $\Delta_n(0)$. For weak couplings,

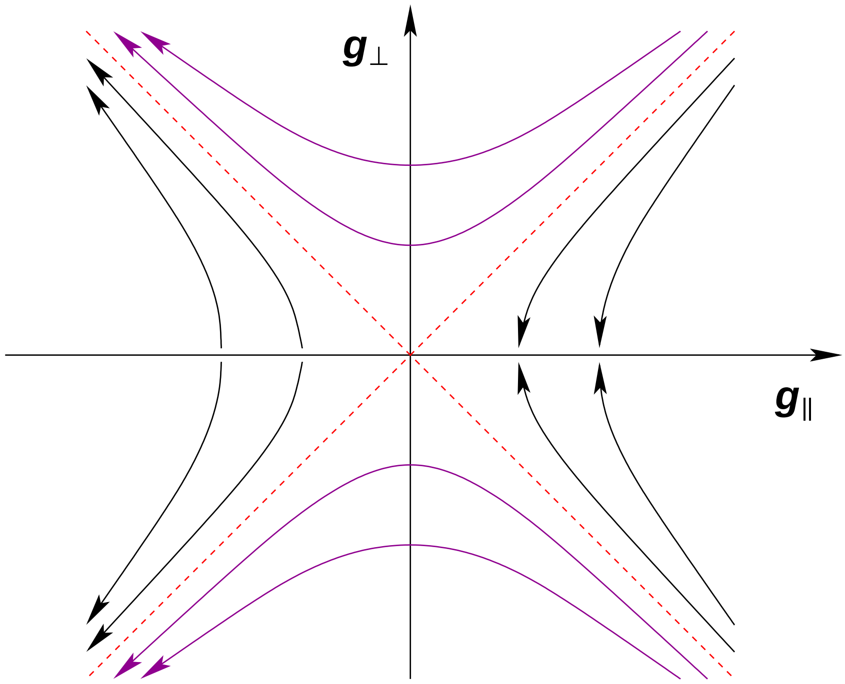
$$\Delta_n(0)/T_{c,n} = 3.52, \qquad n = 0, 1, 2, \ldots$$

"Phase diagram" of the Russian-doll SC's,



Cyclic regime of Kosterlitz-Thouless flows

The Kosterlitz-Thouless flows arise in a multitude of systems.



As a continuum field theory, it corresponds to anisotropic current-current interactions for $su(2)$. The action is

$$S = S_{WZW} + \int \frac{d^2x}{2\pi} \left(4g_\perp (J^+ \overline{J}^- + J^- \overline{J}^+) - 4g_\parallel J_3 \overline{J}_3 \right)$$

To one loop the beta functions are:

$$\frac{dg_{\|}}{dl} = -4g_{\perp}^2, \qquad \frac{dg_{\perp}}{dl} = -4g_{\perp}g_{\|}$$

There exists the RG invariant:

$$Q = g_{\|}^2 - g_{\perp}^2 \quad \equiv \quad -\frac{h^2}{16}$$

Eliminating $g_{\perp}$ and defining h as above one gets

$$\frac{dg_{\|}}{dl} = -4\left(g_{\|}^2 + \frac{h^2}{16}\right)$$

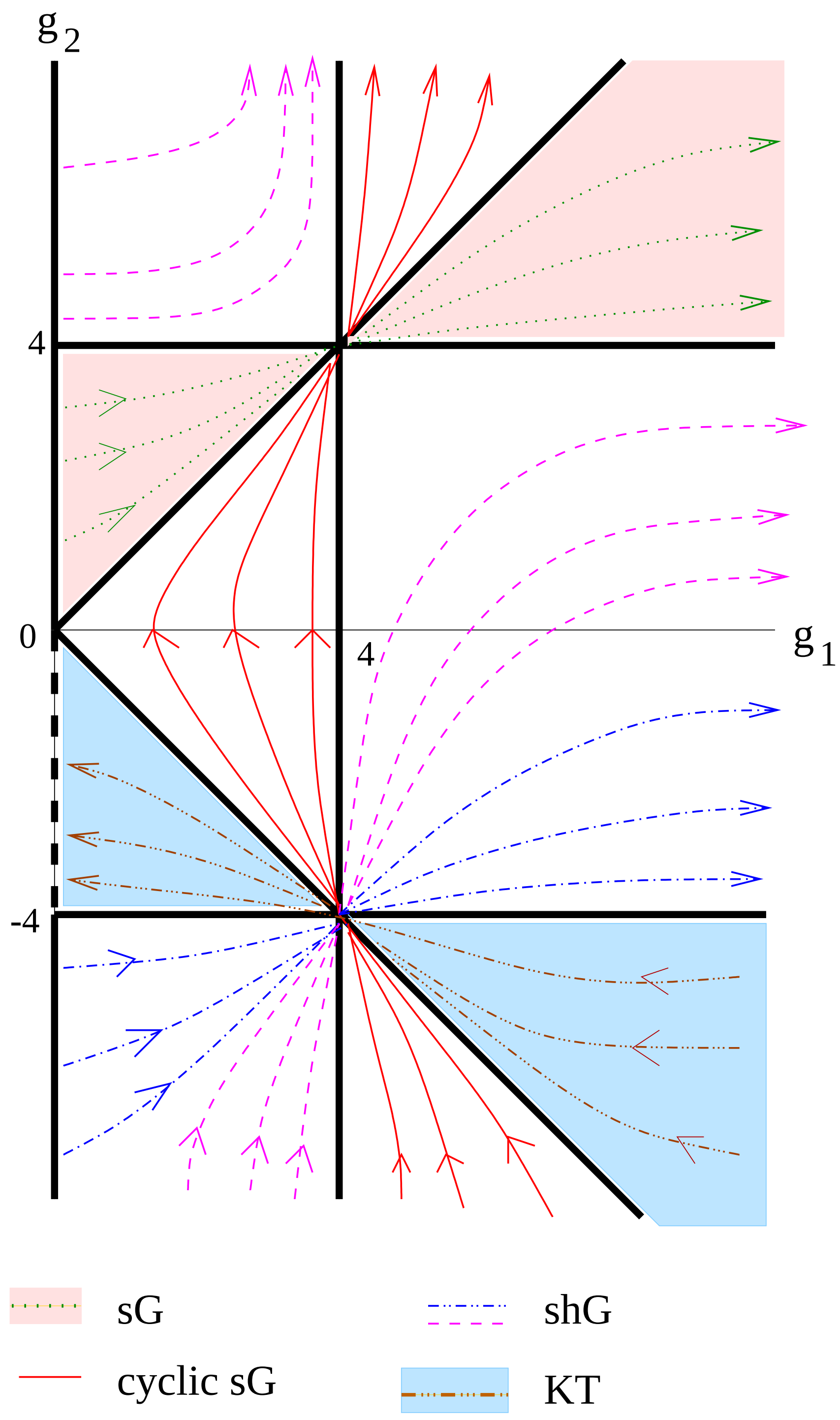
The coupling $g_{\|}$ as a function of scale $L = \exp(l)$ is:

$$g_{\|} = -\frac{h}{4}\tan(h(l - l_0))$$

Thus one observes the periodicity:

$$\boxed{g_{\|}(e^{\lambda}L) = g_{\|}(L), \qquad \lambda_{1-loop} = \frac{\pi}{h}}$$

DOES THIS BEHAVIOR PERSIST NON-PERTURBATIVELY??



Bernard-LeClair

The cyclic sine-Gordon model

LeClair, Román, Sierra (hep-th/0301042)

The cyclic regime of the KT flows can be mapped onto the sine-Gordon theory:

$$S = \int \frac{d^2x}{4\pi} \; \frac{1}{2}(\partial\phi)^2 + \Lambda \cos b\phi$$

$$b^2 = \frac{2}{1 + ih/2}$$

Quantum affine symmetry of sine-Gordon model

Bernard-LeClair

$$q = e^{2\pi i/b^2} = -e^{-\pi h/2}$$

Unitary and crossing yields a unique S-matrix satisfying,

$$S\left(\beta + \frac{2\pi}{h}\right) = S(\beta)$$

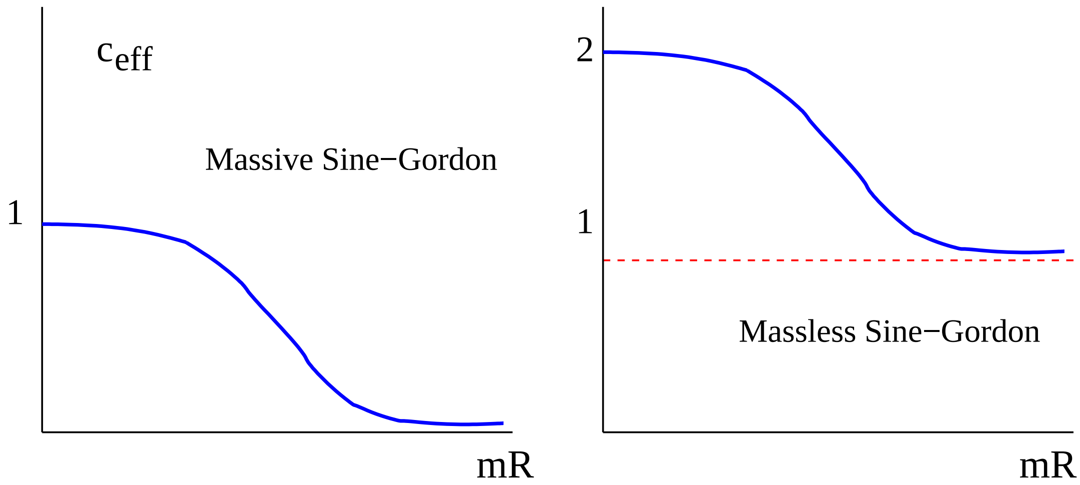
which is a signature of a cyclic RG.

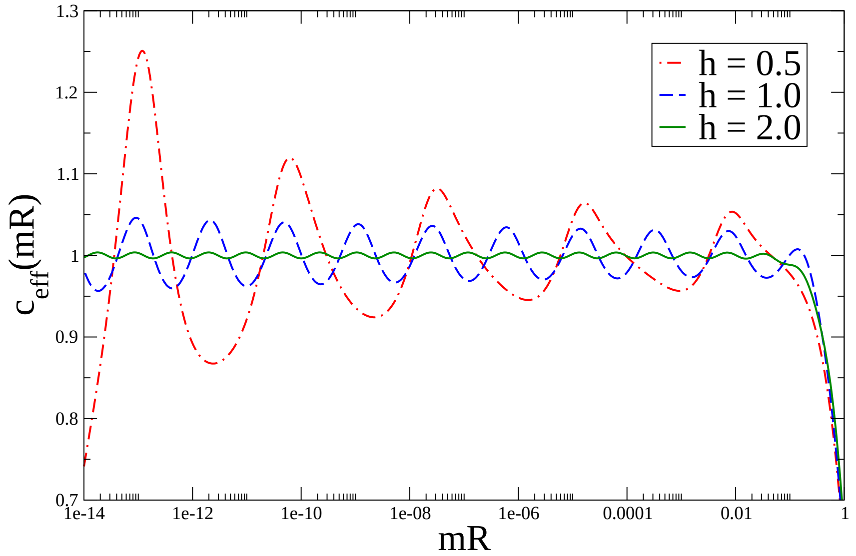
Finite size effects in free energy:

The ground state energy in a box of size R

$$E_0(R) = -\frac{\pi}{6}\frac{c_{\rm eff}(R)}{R}$$

$c_{\rm eff}$ is the effective central charge: $c = 1$ (boson), $c = 1/2$ Majorana fermion (critical Ising model), etc.

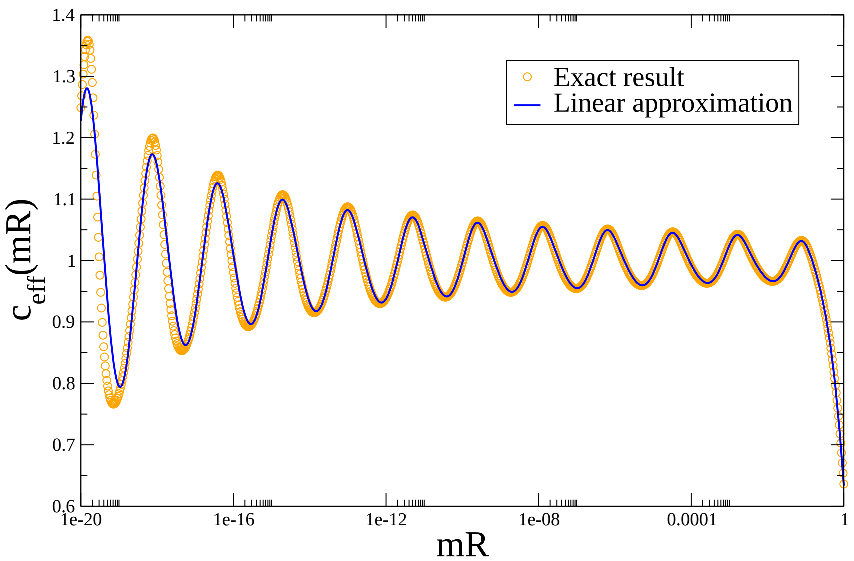




Log-periodic behaviour of c:

- Oscillates in $s = \log(2\pi/mR)$ with a period π/h
- The amplitude of the oscillations around $c = 1$ increases from the IR to the UV regions
- A linear approximation of the DdV equations

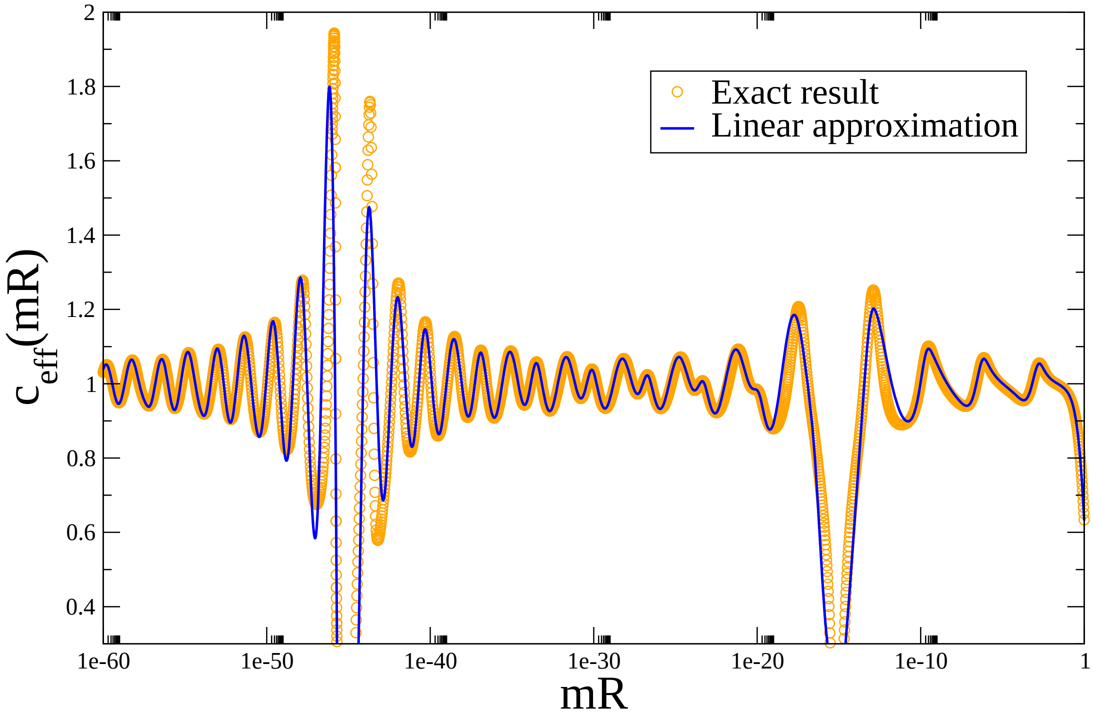
$$c_1 = \frac{24}{\pi} \sum_{n=1}^{\infty} \frac{\mathrm{Im}\left(e^{2inhs}(1-2^{inh})^2\zeta^2(-inh)\right)}{n(1+e^{n\pi h} - \frac{h}{\pi}(s-\gamma-\log 2))}$$



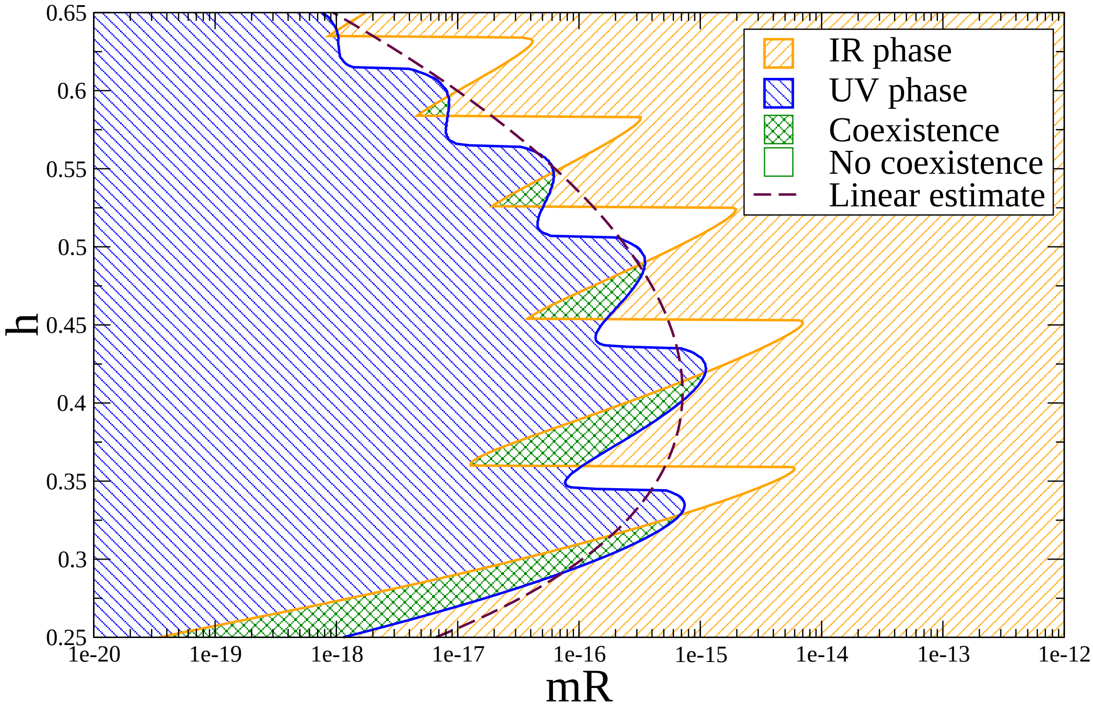
The linear approximation blows up at distances

$$mR_{c,n} = \pi e^{-\gamma}\, exp\left(-\frac{\pi}{h}(1+e^{n\pi h})\right),\ n = 1, 2, \ldots$$

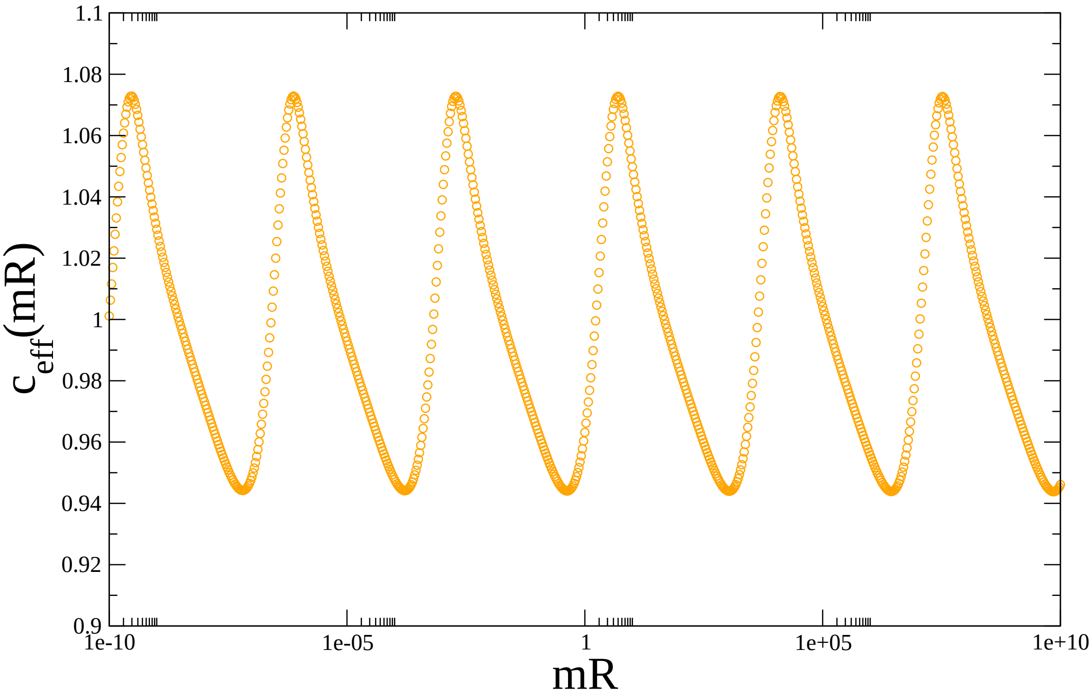
Cascade of critical distances,



Coexistence regimes,



Massless sine-Gordon log-periodic oscillatory behavior:



Questions and suggestions:

- Is there a microscopic origin of the h-interaction in BCS? In real space this interaction breaks parity, hence materials lacking an inversion center are potential candidates.

- For usual S-matrix models the UV is described by a CFT with a fix value of $c_{\mathrm{eff},\mathrm{UV}} = c_{CFT}$. What is the "field theory" that describes the UV of the cyclic sine-Gordon model? Continous scale invariance must be replaced by discrete scale invariance, which is related to complex conformal weights and exponents. This is reminiscent to self-similar fractals which are characterized by Log-periodic behaviour.

- What is the fate of the c-theorem in theories with limit cycles or chaotic RG?

- Find new models with cyclic RG's. They may lead to new Physics.