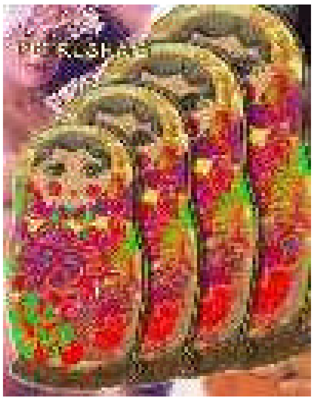
Russian Doll Renormalization Group and Superconductivity



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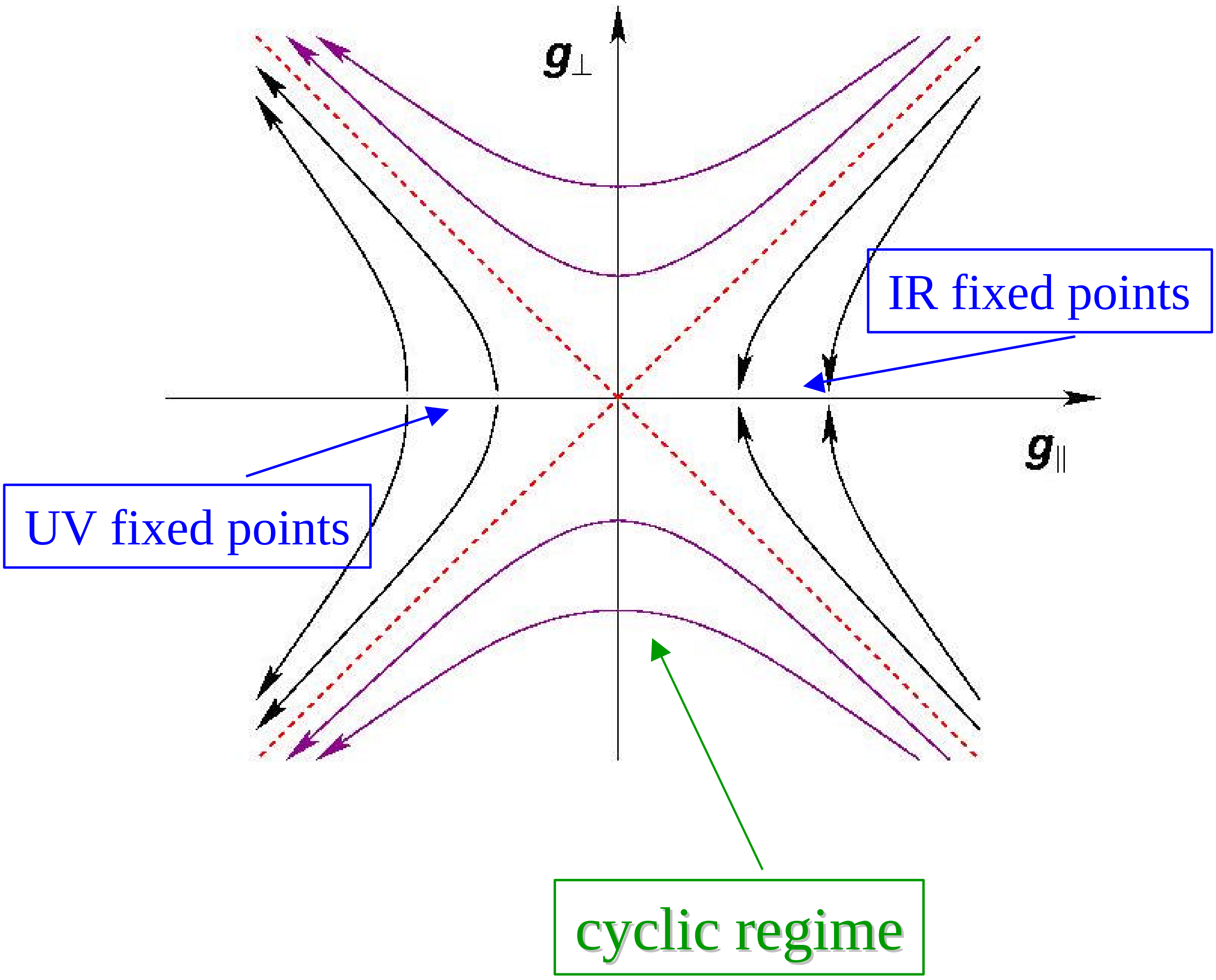
Outline:

1. Introduction: RG fixed points and cyclic limits
2. Extended BCS Hamiltonian
3. Mean-field solution
4. Beta function and cyclic RG
5. Synchronicity of the mean-field and the RG
6. Conclusions and prospects

1. Introduction: RG fixed points and cyclic limits

The Kosterlitz-Thouless diagram exhibits a rich behavior in the RG flows

2D XY spin systems



- P.F. Badeque, H.-W. Hamer and U. van Kolck, PRL 82 (1999) 463.
- S.D. Glazek and K.G. Wilson, hep-th/0203088.

2. Extended BCS Hamiltonian

The Hamiltonian for the study of a superconducting system with *N*-levels is given by:

$$H = \sum_{i=1}^{N} \sum_{s=\uparrow,\downarrow} \frac{e_i}{2} c_{i,s}^{\dagger} c_{i,s} - \sum_{i,j=1}^{N} V_{ij} c_{i,\uparrow}^{\dagger} c_{i,\downarrow}^{\dagger} c_{j,\downarrow} c_{j,\uparrow} \qquad V_{ij} = V_{ji}^{*}$$

$$\left.\begin{aligned} b_i^{\dagger} &= c_{i,\uparrow}^{\dagger} c_{i,\downarrow}^{\dagger} \\ b_i &= c_{i,\downarrow} c_{i,\uparrow} \end{aligned}\right\} \Rightarrow$$

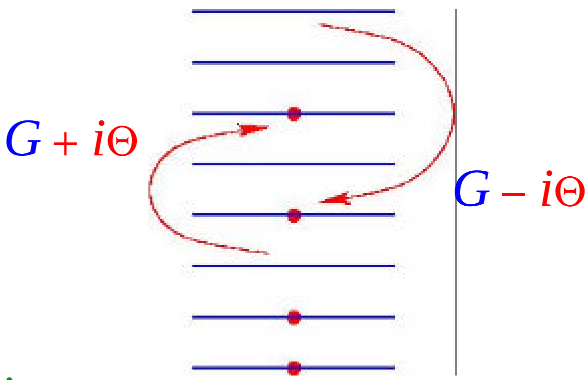
PAIR OPERATORS

Single electrons decouple

$$H\, c_{i,\uparrow}^{\dagger} \left| y \right\rangle = \frac{e_i}{2} c_{i,\uparrow}^{\dagger} \left| y \right\rangle + c_{i,\uparrow}^{\dagger} H \left| y \right\rangle$$

We consider just a hamiltonian for pairs

$$\boxed{V_{ij} = \begin{cases} G + i\Theta & if \quad e_i > e_j \\ G & if \quad e_i = e_j \\ G - i\Theta & if \quad e_i < e_j \end{cases}}$$



The extended BCS Hamiltonian:

$$H_{BCS} = \sum_{i=1}^{N} (e_i - G) b_i^{\dagger} b_i - (G + i\Theta) \sum_{i>j=1}^{N} b_i^{\dagger} b_j - (G - i\Theta) \sum_{i<j=1}^{N} b_i^{\dagger} b_j$$

3. Mean-field solution

The BCS variational ansatz:

$$\left| y_{BCS} \right\rangle = \prod_{i=1}^{N} \left(u_i + v_i b_i^{\dagger} \right) \left| 0 \right\rangle$$

yields

$$u_i^2 = \frac{1}{2}\left(1 + \frac{x_i}{\sqrt{x_i^2 + \Delta_i^2}} \right) \qquad x_i \equiv e_i - m - V_{ii}$$

$$v_i^2 = \frac{1}{2} e^{2if_i} \left(1 - \frac{x_i}{\sqrt{x_i^2 + \Delta_i^2}} \right)$$

$$\tilde{\Delta}_i = \sum_j \frac{V_{ij} \tilde{\Delta}_j}{\sqrt{e_j^2 + \Delta_j^2}} \qquad \tilde{\Delta}_i \equiv \Delta_i e^{if_i}$$

$$G = gd$$

$$\Theta = qd$$

Taking the continuum limit: $\begin{cases} \Delta_i \to \Delta(e) \\ f_i \to f(e) \end{cases}$

$$\tilde{\Delta}(e) = g \int_{-w}^{w} \frac{de'}{2} \frac{\tilde{\Delta}(e)}{\sqrt{e'^2 + \Delta^2}} + iq \left[\int_{-w}^{e} - \int_{e}^{w} \right] \frac{de'}{2} \frac{\tilde{\Delta}(e)}{\sqrt{e'^2 + \Delta^2}}$$

$$\Delta(e) = \Delta = const. \qquad \frac{df}{de} = \frac{q}{\sqrt{e^2 + \Delta^2}}$$

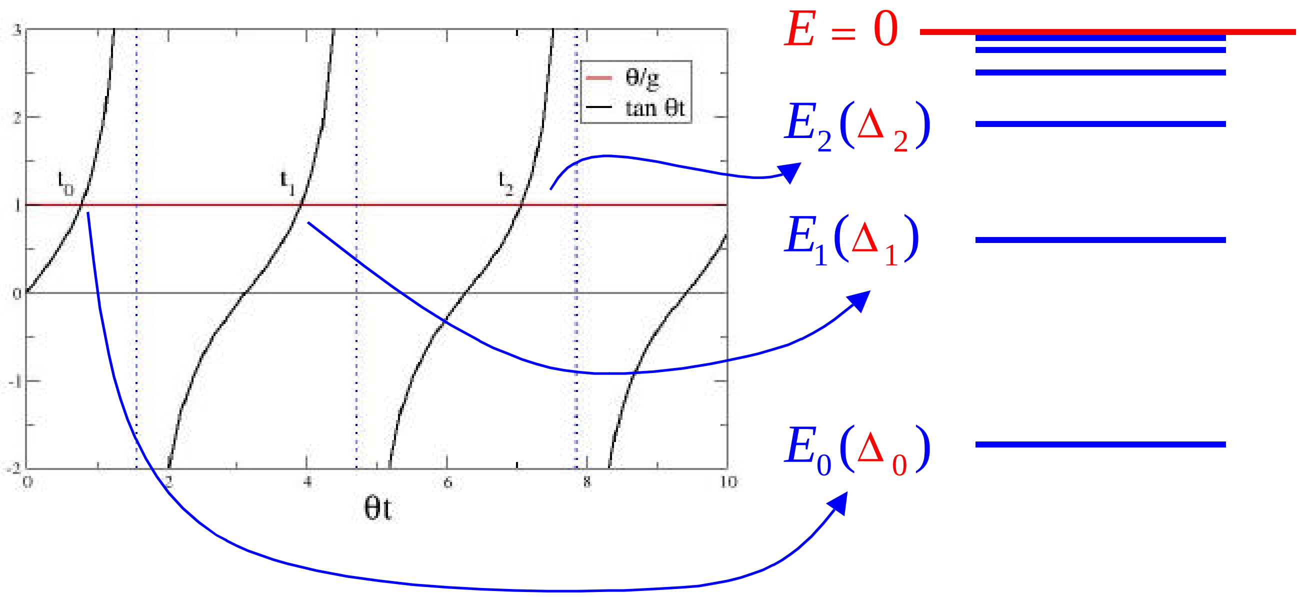
General solution to the gap equation:

$$\Delta_n = \frac{w}{\sinh t_n} \qquad t_n = t_0 + \frac{np}{q} \qquad n = 0, 1, 2, \ldots, \infty$$

$$\tan(q\, t_n) = \frac{q}{g}$$

Infinite number of solutions

Spectrum



In the low energy regime: $\Delta_n \ll w \quad \Rightarrow \quad n \gg \frac{q}{p}$

the condensation energies: $E_C^{(n)} \sim -\frac{\Delta_n^2}{8d}$

$$\Delta_n \sim 2Nd \exp(-t_0 - np/q)$$

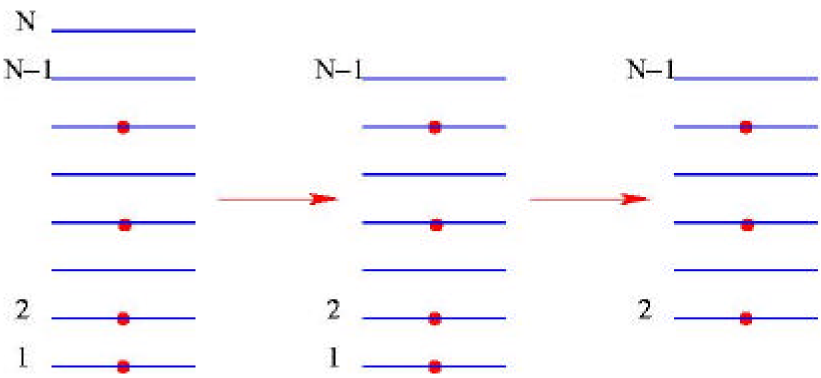
$$E_C^{(n)} \sim -\frac{1}{2} N^2 d \exp(-2t_0 - 2np/q)$$

Scaling behavior

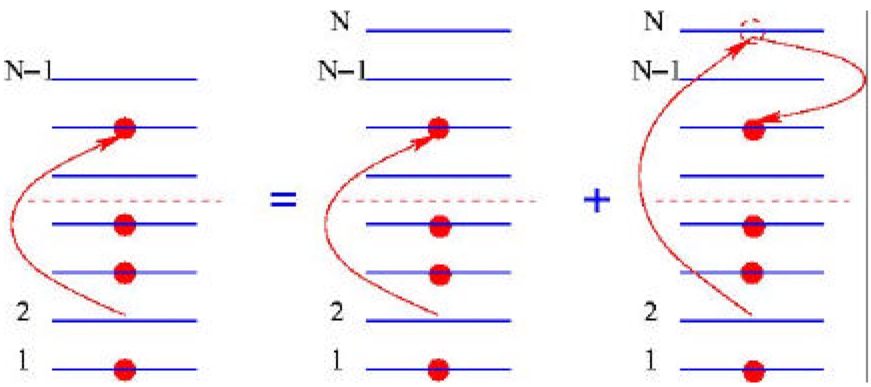
4. Beta function and cyclic RG

In a discrete system the RG eliminates energy levels

$$H_N \rightarrow H_{N-1} \rightarrow H_{N-2}$$



An effective hamiltonian can be constructed by considering the virtual processes involving the N-th level



Effective coupling:

$$V_{ij}^{(N-1)} = V_{ij}^{(N)} + \frac{1}{2} V_{iN}^{(N)} V_{Nj}^{(N)} \left(\frac{1}{x_N - x_i} + \frac{1}{x_N - x_j} \right)$$

$$V_{ij} = V_{ji}^{*} \qquad\qquad x_i = e_i - m - V_{ii}$$

In our problem:

$$\begin{cases} V_{ij} = G \pm i\Theta \\ e_N = e_1 \approx w = Nd \gg e_i \end{cases} \qquad \begin{cases} G = gd \\ \Theta = qd \end{cases}$$

$$\boxed{g_{N-1} = g_N + \frac{1}{N}\left(g_N^2 + q_N^2\right) \qquad q_{N-1} = q_N}$$

RG invariant

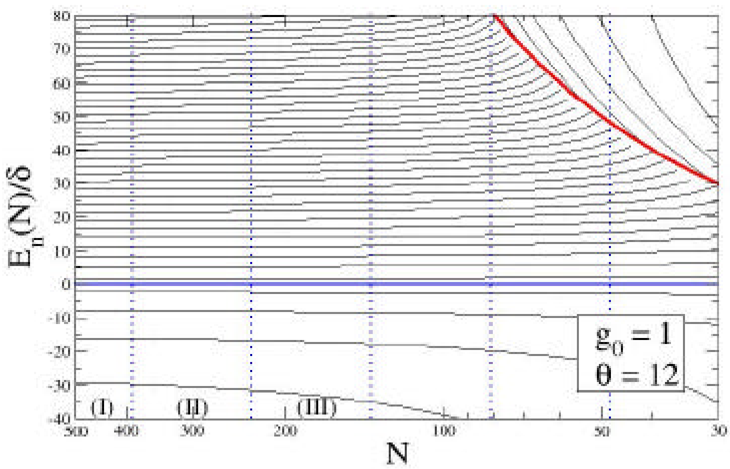
In the large-N limit we define a RG scale: $\boxed{s \equiv \log \frac{N_0}{N}}$

Beta function: $\frac{dg}{ds} = g^2 + q^2$ $\qquad q = const$

RG solution:

$$g(s) = q \tan\left[q\, s + \tan^{-1}\frac{g_0}{q}\right]$$

$$g_0 = g(N_0)$$



Cyclic behavior of the RG:

$$g(s + l) = g(s) \quad \Leftrightarrow \quad g(e^{-l} N) = g(N) \qquad l \equiv \frac{p}{q}$$

The cyclicity is shown in the spectrum:

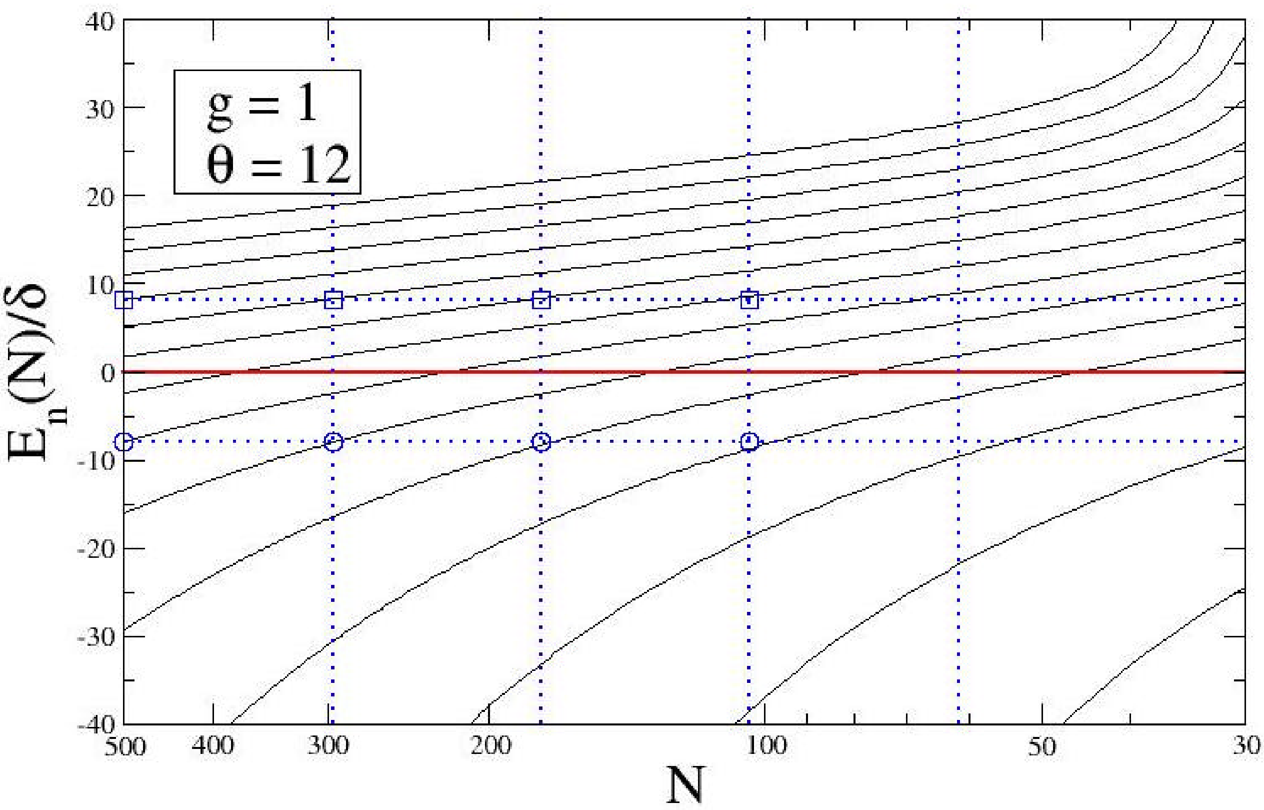
- After a cycle λ we recover the same coupling
- Two systems with sizes N and $N' = Ne^{-l}$ and the same couplings have the same spectrum

$$\{E(g,q,e^{-l}N)\} = \{E(g,q,N)\}$$

g, q fixed

Agrees with mean-field solution

One Cooper pair for fixed couplings and different sizes



5. Synchronicity of the mean-field and the RG

Running coupling constant: $g(s) = q \tan\left[q\, s + \tan^{-1}\frac{g_0}{q}\right]$

What does it happen at points where $g(s) \to \infty$?

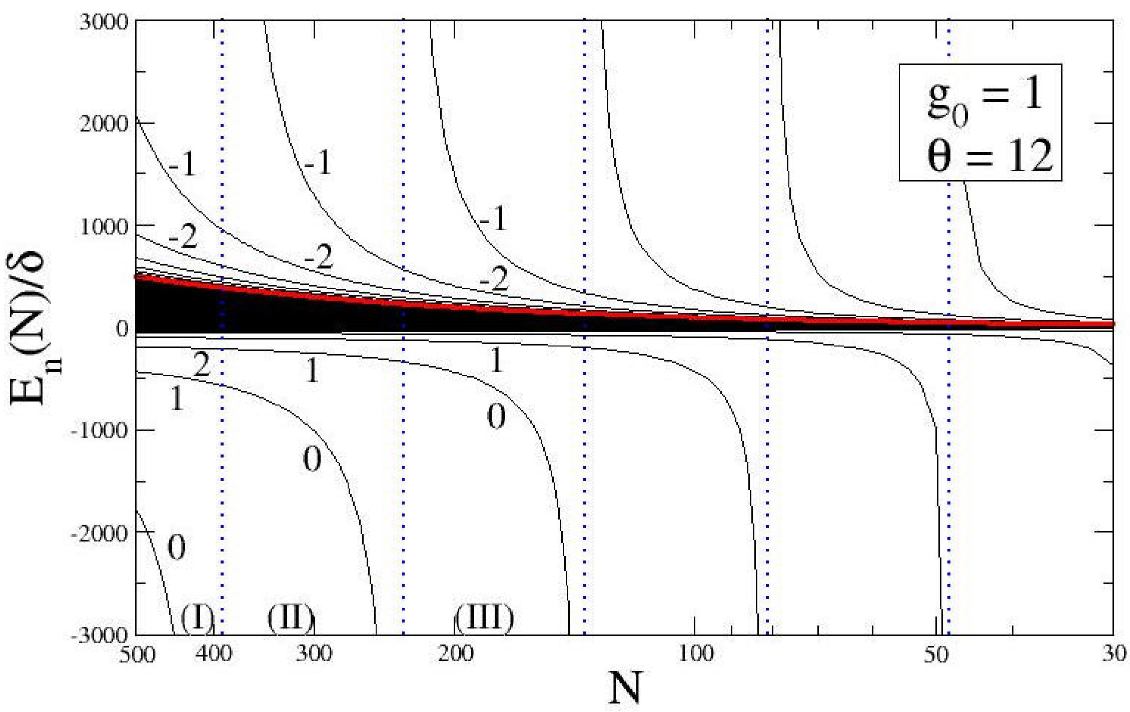
Using the MF solution: $\tan(q\, t_n) = \frac{q}{g}$ $\qquad \Delta_n = \frac{w}{\sinh t_n}$

$$\boxed{g(s) = q \tan\left[q\,(s - t_n) + \frac{p}{2}\right]}$$

$g(s) \to \infty$ for scales equal to mean-field solutions;
a condensate disappears from the spectrum

Russian-doll Renormalization Group

$$\left.\begin{array}{l}\Delta_0(g = +\infty) = +\infty \\ \Delta_{n+1}(g = +\infty) = \Delta_n(g = -\infty)\end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} E_C^{(0)}(g = +\infty) = -\infty \\ E_C^{(n+1)}(g = +\infty) = E_C^{(n)}(g = -\infty)\end{array}\right.$$



6. Conclusions and prospects

1. We have presented a complex extension $(g \pm iq)$ of the standard BCS Hamiltonian.
2. The Mean-Field solution yields an infinite number of condensates, related by a scaling transformation.
3. The coupling constant shows a cyclic behavior under RG transformations, $l \equiv p/q$.
4. This Russian-doll behavior of the RG is intimately related to the existence of the condensates.

PROSPECTS:

• The parameter q could provide for a mechanism to increase the value of the gap.

• The existence of multiple gaps would be revealed by a set of critical temperatures following: $\Delta_n(0)/T_{c,n} \simeq 3.52$

• Models as XXZ spin chain or sine-Gordon are susceptible of presenting limit cycles in their RG flow, which could be understood from a string formulation.