Effective Field Theory Approach to Ferromagnets and Antiferromagnets in Crystalline Solids

José María Román and *Joan Soto*

Departament d'Estructura i Constituents de la Matèria

and

Institut de Física d'Altes Energies

Universitat de Barcelona

Diagonal, 647

E-08028 Barcelona, Catalonia, Spain.

e-mails: roman@ecm.ub.es, soto@ecm.ub.es

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Abstract

We present a systematic construction of effective lagrangians for the low energy and momentum region of ferromagnetic and antiferromagnetic spin waves in crystalline solids. We fully exploit the spontaneous symmetry breaking pattern $SU(2) \to U(1)$, the fact that spin waves are its associated Goldstone modes, the crystallographic space group and time reversal symmetries. We show how to include explicit $SU(2)$ breaking terms due to spin-orbit and magnetic dipole interactions. The coupling to electromagnetic fields is also discussed in detail. For definiteness we work with the space group $R\bar{3}c$ and present our results to next to leading order.

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1 Introduction

Whenever we have a lagrangian (hamiltonian) describing an infinite number of degrees of freedom with a given symmetry group G the ground state of which has a smaller symmetry group H, we say that we are in a situation of spontaneous symmetry breaking (SSB, long range diagonal order). If in addition the symmetry corresponds to a continuous group we have gapless excitations. This is a consequence of the Goldstone's theorem, which was first proven for relativistic quantum field theories in [1], and then extended to nonrelativistic (condensed matter) systems [2, 3, 4]. We shall call Goldstone modes to the lowest lying (gapless) excitations. In particle physics these correspond to massless particles and are known as Goldstone bosons. For instance in strong interaction physics, pions and kaons are the approximate Goldstone bosons of the approximate flavor $G = SU(3)_L \otimes SU(3)_R$ chiral symmetry of the QCD lagrangian which breaks spontaneously down to its vector part $H = SU(3)_V$ [5]. In condensed matter (CM) Goldstone modes appear in a large variety of systems describing quite distinct physics. Among the popular ones, which we shall be concerned with in the rest of the paper, are the magnons or spin waves in ferromagnets and antiferromagnets. These correspond to a $G = SU(2)$ spin symmetry being broken down to $H = U(1)$. Two more examples are the sound waves in a superfluid which correspond to the spontaneous breaking down of the $G = U(1)$ particle number conservation to $H = Z_1$, and the phonons in a crystal which are the Goldstone modes of the $G = T_3$ continuous translational symmetry to the discrete group H of primitive translations of the crystal [6]. At somewhat more complicated level it is perhaps worth mentioning superfluid ^3He where $G = SO(3) \otimes SO(3) \otimes U(1)$ breaks down to $H = U(1) \otimes U(1)$, which presents interesting analogies with the spontaneous symmetry breaking pattern of the electroweak theory [7].

If one probes a physical system in the SSB phase by external sources (e.g. electromagnetic waves) with small energy and momentum at low temperatures, the only relevant degrees of freedom are the Goldstone modes. There is a systematic way to write down an effective lagrangian for the Goldstone modes which does not depend on the details of the microscopic dynamics but only on the symmetry breaking pattern. For relativistic theories this is known from the late 60's [8], but it has only been used extensively to next to leading order for the last ten years [9] (see [10] for a review). It has been pointed out that the same techniques can be applied to condensed matter systems [6], and lagrangians to the lowest order have been provided for the ferromagnetic and antiferromagnetic spin waves, and phonons [11]. In fact for antiferromagnetic spin waves the effective lagrangian at leading and next to leading order has already been used in [12] and [13] respectively. However, the important rôle of the chrystallographic space group has been somewhat neglected. For instance, in ref. [6] it is claimed that symmetries alone cannot distinguish the effective lagrangian of the ferromagnet

from that of the antiferromagnet, and extra dynamical inputs (like the vanishing of the total magnetisation for the antiferromagnet) are required. We shall see that once the space group is taken into account this is no longer necessary. In ref. [23] the distinction between the ferromagnetic and antiferromagnetic effective lagrangians is made by imposing time reversal over macroscopic scales. We shall also see that this requirement is too strong: whereas for some space groups having an antiferromagnetic ground state implies time reversal invariance at macroscopic scales for others it does not. In the next to leading order calculation of ref. [13] Poincaré invariance is assumed instead of a crystallographic space group. We shall see that the crystal symmetries become even more important at next to leading order, allowing for new invariant terms in addition to the Poincaré invariant ones.

The power of these techniques relies on the fact that, once the SSB pattern and space-time symmetries have been identified, one can make a *controlled* expansion for any observable in terms of the typical energy and momentum of the Goldstone modes over the typical energy and momentum of the first gapped excitation. The lagrangian is thus organised in terms of time and space derivatives. For a given precision we only have to take into account derivatives until a given order. Then the lagrangian is a function of a few unknown constants which may be obtained from experimental data and be used later on to predict further experimental results. These constants may also be obtained by a complementary calculation *ab initio* starting from a particular model. Many examples of these calculations exist in the literature (see for instance [15, 16, 17, 18]). They are usually hard, require some approximations (typically Hartree-Fock or mean-field), and, of course, the outcome depends on the particular model choosen for the microscopic dynamics. We would like to stress that the effective lagrangian for the Goldstones modes encodes the low energy and momentum physics of *any* microscopic model with the same symmetry breaking pattern and space group, and hence it provides a model independent description of the physical system to be studied. Different microscopic models just give different values for the unkwown constants.

Recently, we have shown that the effective lagrangian techniques allow for an efficient description of non-reciprocal (time-reversal violating) effects in antiferromagnets in the microwave region [14]. The contribution of the spin waves to various (non linear) electric and magnetic susceptibilities has been calculated, which requires the knowledgement of the effective lagrangian at higher orders as well as the introduction of explicit symmetry breaking terms and the coupling to the electromagnetic field. Such a calculation would be extremelly difficult to carry out *ab initio* from any realistic microscopic model.

In view of the above, and in order to stimulate further non-trivial applications of the effective lagrangian techniques, we feel it is worth to illustrate here how the effective lagrangian for the spin waves in the ferromagnetic and antiferromagnetic crystals can be built in a systematic way by solely exploiting the symmetry breaking pattern and the relevant

space-time group. We present our lagrangians at next to leading order. We will also show how the electromagnetic interactions can be included in the effective lagrangian. We will restrict ourselves to the space group $R\bar{3}c$, whose crystallographic point group is $\bar{3}m$, in order to make easy the comparation with ref. [14], but it will be clear at any stage how to proceed for any other crystallographic group. The paper is intended to be self-contained.

In order to simplify the notation we will take $\hbar = c = 1$ which leads to a relativistic notation. So $x_i = (t, \mathbf{x}_i)$, where subindex i represents a lattice position, subindices $\mu = 0, 1, 2, 3$, where the first one represents the time component.

We distribute the paper as follows. In section 2 we describe the basic fields and symmetries. In section 3 we explain how to systematically construct the effective lagrangian to a given order of space and time derivatives. In sections 4 and 5 we present the effective lagrangian to next-to-leading order for the ferromagnet and antiferromagnet respectively. In sections 6 and 7 we show how to include $SU(2)$ breaking and magnetic dipole interactions respectively. In section 8 the coupling to electromagnetic fields is included. Section 9 is devoted to a discussion of our results. In Appendix A we discuss the effect of non-trivial primitive translations in antiferromagnets. Appendix B contains technical details. In Appendix C we show some particular features for the fundamental representation, $s = 1/2$. Appendix D contains a proof of the equivalence between our formulation and the so-called $O(3)$-sigma model as well as a brief comment on the different forms that a certain topological term can be found in the literature. In Appendix E we show how constant electric and magnetic fields modify the spin wave dispersion relation.

2 Effective Fields and Crystal Symmetries

2.1 Internal symmetries

When G is a compact internal symmetry group, namely, a compact group disentangled from the space-time symmetry group, a general analysis was provided in ref. [8]. Although the construction of the effective lagrangian was carried out for relativistic theories, it readily applies to non-relativistic theories by just changing the Poincaré group by the relevant space-time group. The outcome of ref. [8] is that the effective lagrangian for the Goldstone modes can always be written in terms of a matrix field $U(x)$ taking values in the coset space G/H, determined by the pattern of symmetry breaking $G \to H$ (recall that H is the internal symmetry group of the ground state). We are searching for an effective lagrangian invariant under $SU(2)$. The field $U(x)$ transforms non-linearly under $SU(2)$ as follows:

$$U(x) \to g U(x) h^\dagger(g, U). \tag{2.1}$$

When $g \in H$, the unbroken subgroup, then $h^\dagger = g^\dagger$ and $U(x)$ transforms linearly.

In order to have an intuitive picture of the above mathematical formulation, it is helpful to take the Heisenberg model as a microscopic model,

$$H = \sum_{<i,j>} J_{ij} \mathbf{S}_i \mathbf{S}_j. \tag{2.2}$$

Our discussion however is general and holds for more complicated models, like the Hubbard model, $t-J$ model, etc., with the only requirement that they have an $SU(2)$ spin symmetry which breaks spontaneously down to $U(1)$ in the ground state. (2.2) can be written in the second quantisation language in terms of the real space creation and annihilation operators, namely, $\psi^\dagger(x)$ and $\psi(x)$,

$$H = \sum_{<i,j>} J_{ij} \left(\psi^\dagger(x_i) \mathbf{S} \psi(x_i)\right) \left(\psi^\dagger(x_j) \mathbf{S} \psi(x_j)\right). \tag{2.3}$$

This hamiltonian realises the internal $SU(2)$ symmetry as follows:

$$\psi(x_i) \longrightarrow g\psi(x_i) \quad , \quad g \in SU(2). \tag{2.4}$$

As a classical field theory, the ground state configurations $\psi_0(x_i)$ of (2.3) are those with a maximum spin in a given direction, say the third direction. For spin 1/2 we have $\psi_0^\dagger(x_i) = (1\ 0)$ for all i in the ferromagnet, whereas in the antiferromagnet half of the is would have the above configuration whereas the remaining half would have $\psi_0^\dagger(x_i) = (0\ 1)$. The symmetry of the ground state configurations is clearly $U(1) = \langle e^{i\theta S^3} \rangle$. We can think of a classical configuration close to the ground state as $\psi(x_i) \sim \tilde{U}(x_i)\psi_0(x_i)$, $\tilde{U}(x_i)$ being slowly varying through the lattice. $\tilde{U}(x_i) \in SU(2)$ admits a unique decomposition $\tilde{U}(x_i) = U(x_i)h(\tilde{U}(x_i))$ where $U(x_i) \in SU(2)/U(1)$ and $h \in U(1)$. Since the ground state configuration is $U(1)$ invariant we can always write $\psi(x_i) \sim U(x_i)\psi_0(x_i)$. Taking into account this relation, (2.4) and the $U(1)$ invariance of the ground state configuration, the non-linear transformations (2.1) are justified ($h^\dagger(g, U)$ is the suitable factor that left multiplied by gU gives an element of the coset).

If the magnetisation occurs in the third direction then

$$U(x) = exp\left\{ \frac{i\sqrt{2}}{f_\pi} \left[\pi_1(x) S^1 + \pi_2(x) S^2\right] \right\}, \tag{2.5}$$

where S^i, are generators of $SU(2)$ for any representation and $\pi_i(x)$ are the fields describing the spin waves (f_π is a dimensionful factor).

2.2 Space-time symmetries

Now that we have our basic field and know what its transformations are under the internal symmetry group, we have to find out how it transforms under the space-time symmetries. The space group and time reversal must be respected by the dynamics. The space-time symmetry will be broken by the ground state configuration. In fact the microscopic hamiltonian presents the symmetry of the paramagnetic fase $\mathcal{S} \otimes T$, i.e., space group and time reversal, which breaks down to the ground state symmetry given by the magnetic space group [19]. As in the case of the internal symmetry we impose our effective lagrangian to be invariant under the unbroken symmetry $\mathcal{S} \otimes T$. For definiteness we will be concerned with the space group $R\bar{3}c$, whose crystallographic point group is $\bar{3}m$, together with time reversal. The $\bar{3}m$ group is generated by a rotation of $2\pi/3$ around the z-axis (C_{3z}^{+}), the inversion (I) and a reflexion plane perpendicular to the y-axis (σ_y). The time reversal symmetry just reverses the sign of time. If we introduce holomorphic coordinates $z = x + iy$ and $\bar{z} = x - iy$ these transformations read

$$
\begin{aligned}
C_{3z}^{+}: &\quad \begin{cases} z & \to & e^{i2\pi/3} z \\ \bar{z} & \to & e^{-i2\pi/3} \bar{z} \\ x^3 & \to & x^3 \end{cases} \\
I: &\quad \begin{cases} z & \to & -z \\ \bar{z} & \to & -\bar{z} \\ x^3 & \to & -x^3 \end{cases} \\
\sigma_y: &\quad \begin{cases} z & \to & \bar{z} \\ \bar{z} & \to & z \\ x^3 & \to & x^3. \end{cases}
\end{aligned}
\tag{2.6}
$$

We will consider $U(x)$ in the continuum, which is always a good approximation for small momentum. Effects due to finite lattice spacing are encoded in higher space derivative terms. In the continuum approach only the crystal point group and the primitive translations (τ) of the full space group are relevant.

The ground state configuration can be arranged ferromagnetically or antiferromagnetically. This leads to different transformations of the $U(x)$ field under the crystal point group and primitive translations.

If the ground state is ferromagnetic the local magnetisation points to the same direction everywhere. This indicates that we have to assign trivial transformation properties to $U(x)$

under both the point group and the primitive translations.

$$\begin{aligned} C_{3z}^{+}: \;& U(x) \;\rightarrow\; g_3 U(x) h_3^{\dagger} \\ I: \;& U(x) \;\rightarrow\; U(x) \\ \sigma_y: \;& U(x) \;\rightarrow\; g_2 U(x) h_2^{\dagger} \\ \\ \tau: \;& U(x) \;\rightarrow\; U(x), \end{aligned} \tag{2.7}$$

where (g_i, h_i) are the non-linear $SU(2)$ transformations induced by the rotations. Notice that only the rotational part of the roto-translational elements which may exist in the space group is relevant and this is already included in (2.7). This is a general feature independent of the particular space group.

On the other hand, if we have an antiferromagnetic ground state the local magnetisation points to opposite directions depending on the point of the space where the magnetic ion is located. This must be reflected in the transformation properties of $U(x)$. Those depend in turn on how the magnetic ions are distributed in the crystal. In order to make it definite, let us consider the Cr_2O_3 crystal, which enjoys the $R\bar{3}c$ space group with $\bar{3}m$ point symmetry group. The rhombohedrical unit cell contains four Cr atoms located along the z-axis which play the role of the magnetic ions, and six oxygen atoms which play no role as far as spin is concerned [20]. However, the presence of oxygen atoms is crucial for the absence of primitive translations which map points with opposite magnetisations. Hence all primitive translations must be implemented trivially. The point group symmetries C_{3z}^{+} and σ_y map points with the same local magnetisation, while the inversion I maps points with opposite local magnetisations. This indicates that we may assign to $U(x)$ the following transformation properties under the $\bar{3}m$ group:

$$\begin{aligned} C_{3z}^{+}: \;& U(x) \;\rightarrow\; g_3 U(x) h_3^{\dagger} \\ I: \;& U(x) \;\rightarrow\; U(x) C h_I^{\dagger} \\ \sigma_y: \;& U(x) \;\rightarrow\; g_2 U(x) h_2^{\dagger} \quad , \quad C = e^{-i\pi S^2}, \\ \\ \tau: \;& U(x) \;\rightarrow\; U(x) \end{aligned} \tag{2.8}$$

h_I is a compensating $U(1)$ element that keeps the transformed field in the coset and C turns a spin up into a spin down.

Even though the Cr_2O_3 does not have primitive translations which map points with opposite local magnetisations there are antiferromagnetic materials which do have them. In that case non-trivial transformations should be assigned to $U(x)$. This is discussed in detail in the Appendix A.

Notice then, that in the antiferromagnetic case, the transformations of $U(x)$ are dictated

not only by the space symmetry group but also by the precise local magnetisation of the points related by the symmetry operation.

Let us finally discuss time-reversal symmetry. This symmetry is spontaneously broken in both ferromagnetic and antiferromagnetic ground states. Time reversal changes the sign of the spin. This transformation is implemented on a wave function by $\psi(x) \to C\psi^*(x)$ [21], which translates for $U(x)$ field in

$$T: \quad U(x) \quad \to \quad U(x)Ch_t^\dagger \quad , \quad C = e^{-i\pi S^2}, \tag{2.9}$$

h_t is again a compensating $U(1)$ element that keeps the transformed field in the coset.

It can be shown that when the space-time transformations associated to the magnetic space group, the unbroken subgroup of the space-time group, are considered the transformations of $U(x)$ are linear.

3 Construction of the Effective Lagrangian

3.1 Building blocks: Internal transformations

After having established the field transformations in the previous section, we shall proceed to construct the effective lagrangian in terms of $U(x)$ and its derivatives. Following [8] we consider $U^\dagger(x)i\partial_\mu U(x)$. This object belongs to the Lie algebra of $SU(2)$ and hence can be decomposed as

$$U^\dagger(x)i\partial_\mu U(x) = a_\mu^-(x)S_+ + a_\mu^+(x)S_- + a_\mu^3(x)S^3, \tag{3.1}$$

where we have considered the S_+ and S_- bases for convenience, which verifies

$$\begin{array}{l} S_+ = S^1 + iS^2 \\ S_- = S^1 - iS^2 \end{array} \quad , \quad \begin{array}{l} [S^3, S_+] = S_+ \\ [S^3, S_-] = -S_- \\ [S_+, S_-] = 2S^3. \end{array} \tag{3.2}$$

Under the $SU(2)$ transformations

$$U^\dagger i\partial_\mu U \quad \to \quad h(U^\dagger i\partial_\mu U)h^\dagger + \partial_\mu \theta S^3 \quad , \quad h = e^{i\theta S^3}, \tag{3.3}$$

and hence

$$\begin{array}{rcl} a_\mu^-(x) & \to & e^{i\theta(x)}a_\mu^-(x) \\ a_\mu^+(x) & \to & e^{-i\theta(x)}a_\mu^+(x) \\ a_\mu^3(x) & \to & a_\mu^3(x) + \partial_\mu\theta(x).. \end{array} \tag{3.4}$$

Namely, $a_\mu^\pm$ transforms covariantly whereas a_μ^3 transforms like a connexion under an effective $U(1)_{local}$ group, associated to the non-linear $SU(2)$ transformations. From now on any

reference to the $U(1)_{local}$ transformation must be understood as the effective transformation of the non-linear $SU(2)$ transformation over the fields in (3.4). These are taken as the basic building blocks of our construction together with their derivatives. In order to construct the effective lagrangian a covariant derivative over $a^-_\mu(x)$ can be defined as

$$D_\mu \equiv \partial_\mu - i a^3_\mu(x). \tag{3.5}$$

Although the connexion does not transform covariantly under $SU(2)$, an invariant field strength can be constructed in the standard way,

$$F_{\mu\nu}(x) \equiv \partial_\mu a^3_\nu(x) - \partial_\nu a^3_\mu(x). \tag{3.6}$$

Hence we can in principle construct all invariant terms in the effective lagrangian out of $a^-_\mu(x), a^+_\mu(x), F_{\mu\nu}(x)$ and D_μ. There are however terms which are invariant up to a total derivative which have to be included in the effective lagrangian. These terms are built out of $a^3_\mu(x)$ and are usually called topological. They read

$$\begin{array}{l} a^3_\mu \\ \epsilon^{\mu\nu\rho} a^3_\mu \partial_\nu a^3_\rho. \end{array} \tag{3.7}$$

The first term will be important later on. The second term is the well known abelian Chern-Simons form. As we will see our space-time symmetries do not allow this term. However it arises in other condensed matter systems as for instance in quantum Hall ferromagnets [22].

Given the projector P_+ over the higher spin state the fields in (3.1) can be written as

$$\begin{aligned} a^-_\mu(x) &= \frac{1}{2s} tr([U^\dagger i\partial_\mu U, S_-]P_+) \\ a^+_\mu(x) &= -\frac{1}{2s} tr([U^\dagger i\partial_\mu U, S_+]P_+) \\ a^3_\mu(x) &= \frac{1}{s} tr(U^\dagger i\partial_\mu U P_+). \end{aligned} \tag{3.8}$$

Once we have an explicit representation for $a^\pm_\mu$ and a^3_μ some relevant properties can be proved, namely, $F_{\mu\nu} \sim (a^+_\mu a^-_\nu - a^+_\nu a^-_\mu)$ and $D_\mu a^-_\nu = D_\nu a^-_\mu$ (see Appendix B). These properties ensure that the invariant terms in the effective lagrangian can be constructed in terms of $a^\pm_\mu$ and symmetrised covariant derivatives acting on them only.

The expert reader may wonder why we do not use the standard $O(3)$ sigma model formulation where the effective lagrangian is built out of $n^a(x)$, an $SU(2)$ vector such that $n^a n^a = 1$. The reason is simple: any local invariant that can be constructed in the $O(3)$ sigma formulation can be constructed in the formulation above (we prove this in appendix D). However, the opposite is not true. The essential difference comes from topological terms,

namely, terms that are invariant up to a total derivative. Those are very elusive in the $O(3)$ sigma model formulation but well under control in our formulation, as it should be clear from (3.7).

3.2 Building blocks: Space-time transformations

The most efficient procedure to construct the effective lagrangian is the following:

1. First we construct all the $SU(2)$ invariants under the transformations (3.4), order by order in the derivative expansion.

2. After that we search for invariants under the space-time transformations (2.7)-(2.9) among those terms.

This procedure allows us to ignore the $SU(2)$ spin transformations induced by the space-time transformations in (2.7)-(2.9). Hence, the effective space transformations for the ferromagnetic systems read

$$\xi : \{C_{3z}^{+}, I, \sigma_y\} : \quad \begin{cases} a_\mu^- & \to & a_{\xi\mu}^- \\ a_\mu^3 & \to & a_{\xi\mu}^3, \end{cases} \tag{3.9}$$

where the symbol $\xi\mu$ stands for the transformed index μ under the space transformation ξ together with the appropriate coefficient in each case. Recall that the subindex μ corresponds to a derivative, which transforms with the inverse representation of the space points given in (2.6). Whereas for the antiferromagnetic systems they become

$$\xi : \{C_{3z}^{+}, \sigma_y\} : \quad \begin{cases} a_\mu^- & \to & a_{\xi\mu}^- \\ a_\mu^3 & \to & a_{\xi\mu}^3 \end{cases} \tag{3.10a}$$

$$\xi : \{I\} : \quad \begin{cases} a_\mu^- & \to & -a_{\xi\mu}^+ \\ a_\mu^3 & \to & -a_{\xi\mu}^3. \end{cases} \tag{3.10b}$$

The effective time reversal transformation for both ferromagnet and antiferromagnet reads

$$T : \quad \begin{cases} a_\mu^- & \to & -a_{t\mu}^+ \\ a_\mu^3 & \to & -a_{t\mu}^3, \end{cases} \tag{3.11}$$

again $t\mu$ represents the transformation of the index μ under time reversal symmetry T. Let us remark at this point that because of the different transformation properties under the space symmetries the effective lagrangians for the ferromagnetic and antiferromagnetic spin waves will be different. This is in fact not surprising since it is well known that the low momentum dispersion relation of the spin waves is quadratic for the ferromagnet but linear for the antiferromagnet. We shall obtain this result from symmetry considerations only.

The difference arises from the different transformations given by (3.9) for the ferromagnet and by (3.10) for the antiferromagnet. In fact only the terms with an odd number of time derivatives lead eventually to the above mentioned differences. In order to prove this let us consider the set of generators $\{C_{3z}^{+}, \sigma_y, I, T\}$, as displayed in (3.9) and (3.11), for the ferromagnet and $\{C_{3z}^{+}, \sigma_y, TI, T\}$ for the antiferromagnet, where we choose TI instead of I in (3.10b) as a generator. Notice that $\{C_{3z}^{+}, \sigma_y, T\}$ act identically in the ferromagnetic and antiferromagnetic case, and hence the only differences may arise due to action of I or TI. Consider next TI on the space derivatives $p = z, \bar{z}, 3$ for the antiferromagnetic case,

$$T\xi : \{TI\} : \quad \left\{ \begin{array}{lcl} a_p^- & \rightarrow & a_{\xi p}^- \\ a_p^3 & \rightarrow & a_{\xi p}^3, \end{array} \right. \tag{3.12}$$

and notice that these transformations are identical to those of I on space derivatives for the ferromagnetic case (3.9). Finally the action of TI on time derivatives for the antiferromagnet reads

$$T\xi : \{TI\} : \quad \left\{ \begin{array}{lcl} a_0^- & \rightarrow & -a_0^- \\ a_0^3 & \rightarrow & -a_0^3. \end{array} \right. \tag{3.13}$$

which only differs by a sign from the action of I on time derivatives for ferromagnets (3.11). Therefore the invariants with an even number of time derivatives are the same for both ferromagnetic and antiferromagnetic spin waves. Notice that the proof which we carried out is independent of the particular point group we choose; the only thing we have to do is to substitute the space transformation ξ which maps points with opposite magnetisation (in our case the inversion I) by itself followed by the time reversal transformation $T\xi$.

It is worth mentioning at this point that if primitive translations which map points with opposite magnetisation existed, terms with an odd number of time derivatives would not be allowed (see Appendix A). Only in this case the ground state of the antiferromagnet appears to be time reversal invariant at macroscopic scales, which enforces the above restriction on the effective lagrangian [23]. However, as it should be clear from the above, this is not the most general situation, although it is the most usual one.

3.3 Derivative expanssion: Power counting

The organisation of the effective lagrangian in terms of derivatives is easier in relativistic theories than in non-relativistic ones. Energy and momentum are universally related in the former, which allows to control the derivative expansion by means of a single dimensionful parameter. This parameter may be thought of as the energy of the first massive excitation. For non-relativistic theories, energy and momentum need not fit in any precisely given form, so we may expect the derivative expansion to be controled by at least two independent

parameters: one with dimensions of energy for the time derivatives and one with dimensions of momentum for the space derivatives. We may identify the former as the energy of the first gapped excitation J and the latter as the typical inverts lattice spacing $1/a$. A natural way to relate the time derivative expansion to the space derivative expansion arises once the lowest order terms are written down. It consist of counting the lowest order in time derivatives as being equally important as the lowest order in space derivatives, no matter how many derivatives are in either. This procedure enforces that time and space derivatives are related in the same fashion as energy and momentum in the dispersion relation. This is the right way to proceed as far as there is no external source probing the system (for instance if we wish to calculate the magnetic susceptibility) or the external source has a typical energy and momentum compatible with the dispersion relation. Otherwise the counting should be rearranged according to the the typical energy and momentum of the external source (this will be the case when probing the system by electromagnetic radiation). The latter situation never occurs in relativistic theories because, as mention before, energy and momentum are universally related.

4 Effective Lagrangian for the Ferromagnet

As we mention in the introduction the effective lagrangian for the Goldstone modes must be constructed order by order in space and time derivatives.

The lowest order terms in space derivatives being invariant both under $SU(2)$ and the space-time symmetries read

$$\begin{array}{l} a_z^+ a_{\bar{z}}^- + a_z^- a_{\bar{z}}^+ \\ a_3^+ a_3^- . \end{array} \tag{4.1}$$

We call these terms $O(p^2)$. It is very easy to convince oneself that there are no $O(p^1)$ terms (i.e., with a single space derivative) which are invariant. However, (3.7) provides a term which is invariant up to a total time derivative. It reads

$$a_0^3. \tag{4.2}$$

This term appears in the literature [11, 24, 25] in a variety of forms, none of which being local as above, which we will show to be equivalent to (4.2) in the Appendix C. There, we also discuss its relation to certain topological objects.

Equations (4.1) and (4.2) indicate how time derivatives must be counted in relation to space derivatives. A time derivative must be counted as $O(p^2)$. Let us at this point elaborate a bit on the lowest order lagrangian in order to see how it produces the usual

dispersion relation for ferromagnetic spin waves together with their interaction. We write

$$\mathcal{L}(x) = f_\pi^2 \left[\frac{1}{2} a_0^3 - \frac{1}{m}(a_z^+ a_{\bar{z}}^- + a_z^- a_{\bar{z}}^+) - \frac{1}{2\gamma m} a_3^+ a_3^- \right], \tag{4.3}$$

where f_π, m and γ are free parameters. The connexion $U^\dagger i\partial_\mu U$ is expanded in terms of the Goldstone modes field,

$$U^\dagger i\partial_\mu U = -\frac{1}{f_\pi^2} \left[(f_\pi \partial_\mu \pi^- + \cdots) S_+ + (f_\pi \partial_\mu \pi^+ + \cdots) S_- + (i(\pi^+ \partial_\mu \pi^- - \pi^- \partial_\mu \pi^+) + \cdots) S^3 \right], \tag{4.4}$$

where $\pi^\pm = (\pi^1 \pm i\pi^2)/\sqrt{2}$, giving rise to

$$\mathcal{L}(x) = \pi^- i\partial_0 \pi^+ - \frac{1}{2m} \partial_i \pi^- \partial_i \pi^+ - \frac{1}{2\gamma m} \partial_3 \pi^- \partial_3 \pi^+, \tag{4.5}$$

up to quadratic order in the spin wave fields. From (4.5) we see clearly that a time derivative must be counted as $1/2m$ two space derivatives. Observe that because (4.5) is first order in time derivatives, the remaining $U(1)$ spin symmetry implies that the number of ferromagnetic spin waves is conserved. It is also remarkable that the interaction of any number of spin waves at this order, which is obtained by keeping more terms of the expansion (4.4) in (4.5), is given by just three constants, namely f_π, m and γ.

The order of magnitude of the constants above follows from the fact that the effective lagrangian is an expansion for low energy and momentum controlled by the parameters J and $1/a$, namely the energy of the first gapped excitation and the typical lattice spacing respectively. J suppresses the time derivatives and $1/a$ the space ones. Since the lagrangian density has dimensions of $(energy).(momentum)^3$ we can estimate the size of each term by writing $\mathcal{L}(x) \sim J/a^3 \times (dimensionless\ quantities)$ and taking into account that the dimensionless quantities are built out of time derivatives over J and space derivatives over $1/a$. We obtain $f_\pi^2 \sim 1/a^3$ and $1/m \sim a^2 J$, in accordance with standard microscopic calculations in the Heisenberg model.

The following non-trivial order is $O(p^4)$, which gives rise to the terms below:

$$\begin{aligned} &a_0^+ a_0^- \\ &i(D_0 a_3^+ a_3^- - D_0 a_3^- a_3^+) \\ &i[(D_0 a_z^+ a_{\bar{z}}^- - D_0 a_z^- a_{\bar{z}}^+) - (D_0 a_{\bar{z}}^- a_z^+ - D_0 a_{\bar{z}}^+ a_z^-)] \end{aligned} \tag{4.6}$$

$$
\begin{array}{l}
a_3^+ a_3^- a_3^+ a_3^- \\
a_3^+ a_3^- (a_z^+ a_{\bar{z}}^- + a_z^- a_{\bar{z}}^+) \\
a_3^+ a_z^- a_3^+ a_{\bar{z}}^- + a_3^- a_{\bar{z}}^+ a_3^- a_z^+ \\
a_z^+ a_z^- (a_3^+ a_z^- + a_3^- a_z^+) + a_{\bar{z}}^- a_{\bar{z}}^+ (a_3^- a_{\bar{z}}^+ + a_3^+ a_{\bar{z}}^-) \\
a_z^+ a_{\bar{z}}^- a_z^+ a_{\bar{z}}^- + a_z^- a_{\bar{z}}^+ a_z^- a_{\bar{z}}^+ \\
a_z^+ a_{\bar{z}}^- a_{\bar{z}}^+ a_z^- \\
D_3 a_3^+ D_3 a_3^- \\
D_3 a_z^+ D_3 a_{\bar{z}}^- + D_3 a_z^- D_3 a_{\bar{z}}^+ \\
(D_3 a_z^+ D_z a_z^- + D_3 a_z^- D_z a_z^+) + (D_3 a_{\bar{z}}^- D_{\bar{z}} a_{\bar{z}}^+ + D_3 a_{\bar{z}}^+ D_{\bar{z}} a_{\bar{z}}^-) \\
D_z a_{\bar{z}}^+ D_z a_{\bar{z}}^- + D_{\bar{z}} a_z^- D_{\bar{z}} a_z^+ .
\end{array}
\tag{4.7}
$$

Notice that imposing rotational invariance would reduce (4.6) and (4.7) to two and three terms respectively.

5 Effective Lagrangian for the Antiferromagnet

In this section we follow exactly the same logical steps as in the ferromagnetic case, but taking into account that the transformation properties of $U(x)$ under the point group are different.

The lowest order terms in space derivatives are exactly the same as in (4.1). Namely,

$$
\begin{array}{l}
a_z^+ a_{\bar{z}}^- + a_z^- a_{\bar{z}}^+ \\
a_3^+ a_3^- .
\end{array}
\tag{5.1}
$$

The lowest order term in time derivatives is not (4.2) anymore. Indeed, the transformation properties under I now forbid this term. Therefore as it was pointed out in section 3 the difference between ferromagnetic and antiferromagnetic spin waves arises from terms containing an odd number of time derivatives. Then the lowest order term in time derivatives is in this case

$$
a_0^+ a_0^- . \tag{5.2}
$$

Hence the effective lagrangian to lowest order reads

$$
\mathcal{L}(x) = f_\pi^2 \left[a_0^+ a_0^- - 2v^2 (a_z^+ a_{\bar{z}}^- + a_z^- a_{\bar{z}}^+) - (\gamma v)^2 a_3^+ a_3^- \right], \tag{5.3}
$$

where f_π is the spin stiffness, v the spin wave velocity in $x-y$ plain and γv the spin wave velocity in the z direction. In terms of the spin wave fields (4.4), the lagrangian above reads

$$
\mathcal{L}(x) = \partial_0 \pi^+ \partial_0 \pi^- - v^2 \partial_i \pi^+ \partial_i \pi^- - (\gamma v)^2 \partial_3 \pi^+ \partial_3 \pi^- , \tag{5.4}
$$

where it is apparent that now we have a linear dispersion relation. It also becomes apparent that the time derivatives must be counted as v times a space derivative. Observe that because (5.4) is second order in derivatives it describes two degrees of freedom. The remaining $U(1)$ spin symmetry tells us that antiferromagnetic spin waves can only be produced (or annihilated) in pairs. This is completely analogous to a relativistic theory: one of the spin waves plays the role of a (massless) particle and the other of an antiparticle. The total number of particles plus antiparticles is conserved due to the $U(1)$ symmetry. It is again remarkable that the interaction between spin waves at this order, which we would obtain by keeping further terms in the expansion (4.4), is given in terms of three parameters only, namely f_π, v and γ [13].

The order of magnitude of the constants above is estimated as in the ferromagnetic case. Now we obtain $f_\pi^2 \sim 1/Ja^3$ and $v \sim Ja$, which is again in accordance with microscopic calculations in the Heisenberg model.

It is worth mentioning that the following extra term appears at the lowest order

$$F_{03} \sim (a_0^+ a_3^- - a_0^- a_3^+). \tag{5.5}$$

This term is a total derivative and will be dropped. This is fine as far as we stay within a perturbative approach. However, since it is a total derivative of an object which is not $SU(2)$ invariant, it may become relevant if a non-perturbative analysis is attempted.

The next to leading non-trivial order is $O(p^4)$, which reads

$$\begin{aligned}
&a_0^+ a_0^- a_0^+ a_0^- \\
&i a_0^+ a_0^- (a_0^+ a_3^- - a_0^- a_3^+) \\
&a_0^+ a_0^- a_3^+ a_3^- \\
&a_0^+ a_3^- a_0^+ a_3^- + a_0^- a_3^+ a_0^- a_3^+ \\
&i(a_0^+ a_3^- - a_0^- a_3^+) a_3^+ a_3^- \\
&a_0^+ a_0^- (a_z^+ a_{\bar z}^- + a_z^- a_{\bar z}^+) \\
&a_0^+ a_z^- a_0^+ a_{\bar z}^- + a_0^- a_{\bar z}^+ a_0^- a_z^+ \\
&i(a_0^+ a_3^- - a_0^- a_3^+)(a_z^+ a_{\bar z}^- + a_z^- a_{\bar z}^+) \\
&i(a_0^+ a_z^- a_3^+ a_{\bar z}^- - a_0^- a_{\bar z}^+ a_3^- a_z^+) \\
&i[a_z^+ a_z^- (a_0^+ a_z^- - a_0^- a_z^+) - a_{\bar z}^- a_{\bar z}^+ (a_0^- a_{\bar z}^+ - a_0^+ a_{\bar z}^-)] \\
&D_0 a_0^+ D_0 a_0^- \\
&D_0 a_3^+ D_0 a_3^- \\
&D_0 a_z^+ D_0 a_{\bar z}^- + D_0 a_z^- D_0 a_{\bar z}^+ \\
&i[D_0 a_3^+ (D_z a_{\bar z}^- + D_{\bar z} a_z^-) - D_0 a_3^- (D_{\bar z} a_z^+ + D_z a_{\bar z}^+)]
\end{aligned} \tag{5.6}$$

together with the space derivatives terms given in (4.7). Notice that the above terms with an odd number of time derivatives would not appear if a primitive translation mapping

points with opposite magnetisations existed in the Cr_2O_3 (see Appendix A). Notice also that imposing rotational invariance would reduce (5.6) to five terms.

6 Spin-orbit Corrections

The spin-orbit coupling of the electrons in the magnetic ions is usually the main source of explicit $SU(2)$ breaking. This explicit breaking can be taken into account in Heisenberg-type models by the inclusion of two new terms in the hamiltonian [26]

$$H = \sum_{<i,j>} J_{ij}\mathbf{S}_i\mathbf{S}_j + \sum_{<i,j>} \mathbf{D}_{ij}(\mathbf{S}_i \times \mathbf{S}_j) + \sum_{<i,j>} M^{ab}_{ij} S^a_i S^b_j, \tag{6.1}$$

where $D^a \sim (\Delta g/g)J$ and $M^{ab} \sim (\Delta g/g)^2 J$ are the antisymmetric and symmetric anysotropies respectively (M^{ab} is symmetric with respect the spin indices). These are related to the (super)exchange coupling through the change in the effective gyromagnetic factor due to the spin-orbit interaction.. Typically $\Delta g \sim 10^{-2} g$ [27].

In order to incorporate the effects of (6.1) in the effective theory we promote $\mathbf{D}_{ij}$ and M^{ab}_{ij} to source fields $D^a(\mathbf{x}_i, \mathbf{x}_j)$ and $M^{ab}(\mathbf{x}_i, \mathbf{x}_j)$ and assign transformation properties to them such that (6.1) becomes $SU(2)$ invariant,

$$\begin{array}{rcl} D^a(\mathbf{x}_i, \mathbf{x}_j) & \rightarrow & R^a{}_b D^b(\mathbf{x}_i, \mathbf{x}_j) \\ M^{ab}(\mathbf{x}_i, \mathbf{x}_j) & \rightarrow & R^a{}_c R^b{}_d M^{cd}(\mathbf{x}_i, \mathbf{x}_j). \end{array} \tag{6.2}$$

Now, if we could derive our effective lagrangian for the spin waves from the microscopic model it would be a functional of $U(x)$, D^a and M^{ab} invariant under $SU(2)$ transformations. Once we particularise the sources to reproduce the anisotropic spin-orbit tensors we automatically obtain the effects of the latter in the effective theory.

Since the spin-orbit corrections lead to short range interactions in (6.1), the local limit of the sources will be taken. In this limit the leading contribution compatible with the crystal symmetries of the antisymmetric anisotropy is represented by a tensor with one $SU(2)$ and two symmetric space indices, D^a_{pq}, where $p, q = z, \bar{z}, 3$ (these indices transform with the inverse representation of the space points (2.6)). The symmetric tensor is represented by a second order tensor of $SU(2)$ with no space indices, M^{ab}.

Let us next consider the following objects which transform covariantly under $SU(2)$,

$$\begin{array}{rcl} D_{pq} \equiv D^a_{pq} S^a & \rightarrow & g D_{pq} g^\dagger \\ M \equiv M^{ab}(S^a \otimes S^b + S^b \otimes S^a) & \rightarrow & (g \otimes g) M (g^\dagger \otimes g^\dagger). \end{array} \tag{6.3}$$

When these sources are set to their most general form compatible with the crystal symmetries only a few non-vanishing terms remain. Namely,

$$
\begin{aligned}
D_{zz} &= D^-_{zz} S_+ \qquad & D^-_{zz} &= -D^+_{\bar{z}\bar{z}} \\
D_{\bar{z}\bar{z}} &= D^+_{\bar{z}\bar{z}} S_- & & \\
D_{3z} &= D^+_{3z} S_- & D^+_{3z} &= -D^-_{3\bar{z}} \\
D_{3\bar{z}} &= D^-_{3\bar{z}} S_+ & &
\end{aligned}
\tag{6.4}
$$

$$
M = M^{-+}(S_+ \otimes S_- + S_- \otimes S_+) + M^{33}(S^3 \otimes S^3),
$$

where $D^+_{\bar{z}\bar{z}}$, D^+_{3z}, M^{-+} and M^{33} are free parameters.

6.1 Spin-orbit sources: Internal transformations

Since our basic building blocks transform in a simple way under the $U(1)_{local}$ in the $SU(2)$ non-linear realisation, it is convenient to introduce new sources with simple transformation properties under $U(1)_{local}$ in the following way:

$$
U^\dagger(x) D_{pq} U(x) = d^-_{pq}(x) S_+ + d^+_{pq}(x) S_- + d^3_{pq}(x) S^3. \tag{6.5}
$$

These new sources transform as

$$
\begin{aligned}
d^-_{pq}(x) &\rightarrow e^{i\theta(x)} d^-_{pq}(x) \\
d^+_{pq}(x) &\rightarrow e^{-i\theta(x)} d^+_{pq}(x) \\
d^3_{pq}(x) &\rightarrow d^3_{pq}(x).
\end{aligned}
\tag{6.6}
$$

The same can be done for the symmetric source,

$$
\begin{aligned}
\Big(U^\dagger(x) \otimes U^\dagger(x)\Big) M \Big(U(x) \otimes U(x)\Big) = {} & m^{--}(x)(S_+ \otimes S_+) \\
& + m^{++}(x)(S_- \otimes S_-) \\
& + m^{33}(x)(S^3 \otimes S^3) \\
& + m^{-+}(x)(S_+ \otimes S_- + S_- \otimes S_+) \\
& + m^{-3}(x)(S_+ \otimes S^3 + S^3 \otimes S_+) \\
& + m^{+3}(x)(S_- \otimes S^3 + S^3 \otimes S_-).
\end{aligned}
\tag{6.7}
$$

The transformation properties for these components are

$$
\begin{aligned}
m^{--}(x) &\rightarrow e^{2i\theta(x)} m^{--}(x) \\
m^{++}(x) &\rightarrow e^{-2i\theta(x)} m^{++}(x) \\
m^{-3}(x) &\rightarrow e^{i\theta(x)} m^{-3}(x) \\
m^{+3}(x) &\rightarrow e^{-i\theta(x)} m^{+3}(x) \\
m^{-+}(x) &\rightarrow m^{-+}(x) \\
m^{33}(x) &\rightarrow m^{33}(x).
\end{aligned}
\tag{6.8}
$$

An explicit representation for the sources $d^a_{pq}(x)$ introduced in (6.5) is given by

$$\begin{aligned} d^-_{pq}(x) &= \frac{1}{2s} tr([U^\dagger D_{pq} U, S_-]P_+) \\ d^+_{pq}(x) &= -\frac{1}{2s} tr([U^\dagger D_{pq} U, S_+]P_+) \\ d^3_{pq}(x) &= \frac{1}{s} tr(U^\dagger D_{pq} U P_+), \end{aligned} \tag{6.9}$$

and a similar but lengthier expression can be given for $m^{ab}(x)$ in (6.7). From the explicit representation (6.9) for the sources d^a_{pq} together with (6.4) it is obvious that $d^a_{zz} d^b_{3\bar{z}} = d^b_{zz} d^a_{3\bar{z}}$, $d^a_{\bar{z}\bar{z}} d^b_{3z} = d^b_{\bar{z}\bar{z}} d^a_{3z}$ and $d^a_{zz} d^b_{3z} = d^b_{\bar{z}\bar{z}} d^a_{3\bar{z}}$. Moreover one can easily prove that derivatives over these sources can be written as the source itself multiplied by $a^\pm_\mu(x)$ or $a^3_\mu(x)$. Therefore only $d^a_{pq}(x)$ and $m^{ab}(x)$ and not their derivatives have to be used in addition to our basic building blocks (3.1) to construct the effective lagrangian.

6.2 Spin-orbit sources: Space-time transformatinos

Let us next see how these sources transform under the space-time symmetries. Recall first that a space transformation induces a $SU(2)$ spin transformation also in the sources. However, as it was pointed out in section 3, the $SU(2)$ transformations induced by space-time transformations can be ignored because the lagrangian is first constructed to be $SU(2)$ invariant.

For the ferromagnet the effective $\bar{3}m$ transformations are given by

$$\xi : \{C^+_{3z}, I, \sigma_y\} : \quad \left\{ \begin{array}{lcl} d^-_{pq} & \to & d^-_{\xi p \xi q} \\ d^3_{pq} & \to & d^3_{\xi p \xi q} \\ m^{ab} & \to & m^{ab}, \end{array} \right. \tag{6.10}$$

and for the antiferromagnet

$$\xi : \{C^+_{3z}, \sigma_y\} : \quad \left\{ \begin{array}{lcl} d^-_{pq} & \to & d^-_{\xi p \xi q} \\ d^3_{pq} & \to & d^3_{\xi p \xi q} \\ m^{ab} & \to & m^{ab} \end{array} \right. \tag{6.11a}$$

$$\xi : \{I\} : \quad \left\{ \begin{array}{lcl} d^-_{pq} & \to & -d^+_{\xi p \xi q} \\ d^3_{pq} & \to & -d^3_{\xi p \xi q} \\ m^{--} & \to & m^{++} \\ m^{-+} & \to & m^{-+} \\ m^{-3} & \to & m^{+3} \\ m^{33} & \to & m^{33}. \end{array} \right. \tag{6.11b}$$

Time reversal, like in (3.11), gives the same transformations both for the ferromagnet and the antiferromagnet,

$$T: \quad \begin{cases} d^-_{pq} & \to & -d^+_{pq} \\ d^3_{pq} & \to & -d^3_{pq} \\ m^{--} & \to & m^{++} \\ m^{-+} & \to & m^{-+} \\ m^{-3} & \to & m^{+3} \\ m^{33} & \to & m^{33}. \end{cases} \tag{6.12}$$

Again, as in section 3, it is easy to prove that the transformations (6.10) and (6.12) give the same invariants as (6.11) and (6.12). Consider instead of (6.11b) the transformation given by $T\xi$,

$$T\xi : \{TI\} : \quad \begin{cases} d^-_{pq} & \to & d^-_{\xi p \xi q} \\ d^3_{pq} & \to & d^3_{\xi p \xi q} \\ m^{ab} & \to & m^{ab}. \end{cases} \tag{6.13}$$

Therefore all the invariants constructed out of ds, ms and as with an arbitrary number of space derivatives and an even number of time derivatives are the same for ferromagnetic and antiferromagnetic spin waves.

6.3 Invariant terms

The previous transformation properties lead to the following invariants at leading order in terms of ds and ms,

$$\begin{array}{l} d^+_{zz} d^-_{\bar{z}\bar{z}} + d^-_{zz} d^+_{\bar{z}\bar{z}} \\ (d^+_{3z} d^-_{zz} + d^-_{3z} d^+_{zz}) + (d^-_{3\bar{z}} d^+_{\bar{z}\bar{z}} + d^+_{3\bar{z}} d^-_{\bar{z}\bar{z}}) \\ d^+_{3z} d^-_{3\bar{z}} + d^-_{3z} d^+_{3\bar{z}} \\ d^3_{zz} d^3_{\bar{z}\bar{z}} \\ d^3_{3z} d^3_{zz} + d^3_{3\bar{z}} d^3_{\bar{z}\bar{z}} \\ d^3_{3z} d^3_{3\bar{z}} \\ \\ m^{-+} \\ m^{33}. \end{array} \tag{6.14}$$

If we expand the ds and ms sources in terms of the spin wave fields we obtain

$$\begin{aligned} U^\dagger D_{pq} U &= \left[D^-_{pq} + \frac{1}{f_\pi^2} (D^+_{pq} \pi^- \pi^+ - D^-_{pq} \pi^+ \pi^-) + \cdots \right] S_+ \\ &+ \left[D^+_{pq} + \frac{1}{f_\pi^2} (D^-_{pq} \pi^+ \pi^+ - D^+_{pq} \pi^+ \pi^-) + \cdots \right] S_- \end{aligned} \tag{6.15}$$

$$+ \quad \left[-\frac{2i}{f_\pi}(D^+_{pq}\pi^- - D^-_{pq}\pi^+) + \cdots \right] S^3,$$

$$\begin{aligned}
(U^\dagger \otimes U^\dagger) M (U \otimes U) = {} & [(2M^{-+} - M^{33})\pi^-\pi^- + \cdots](S_+ \otimes S_+) \\
& + [(2M^{-+} - M^{33})\pi^+\pi^+ + \cdots](S_- \otimes S_-) \\
& + [M^{-+} - 2(2M^{-+} - M^{33})\pi^+\pi^- + \cdots](S_+ \otimes S_- + S_- \otimes S_+) \\
& + [-i(2M^{-+} - M^{33})\pi^- + \cdots](S_+ \otimes S^3 + S^3 \otimes S_+) \\
& + [i(2M^{-+} - M^{33})\pi^+ + \cdots](S_- \otimes S^3 + S^3 \otimes S_-) \\
& + [2(2M^{-+} - M^{33})\pi^+\pi^- + M^{33} + \cdots](S^3 \otimes S^3).
\end{aligned} \tag{6.16}$$

From these two expansions we can easily see that the terms (6.14) at quadratic order in the spin waves fields produce a gap in the dispersion relation. Notice however that the energy gap for the spin waves is $\sim d^2$ for the ferromagnet, since its spin waves verify a Schrödinger like equation, whereas it is $\sim d$ for the antiferromagnet, since its spin waves verify a Klein-Gordon type equation.

Notice also that the terms in (6.14) contribute to the ground state energy and force the (staggered) magnetisation to be along the third direction. This is why we took the direction of the spontaneously symmetry breaking along the third axes in (2.5). Otherwise terms with a single spin wave field would appear in the expansion of (6.14) indicating us that we chose a wrong direction for the (staggered) magnetisation.

The above considerations allow us to decide how to take the relative order of magnitude for the derivatives and the anisotropy tensors. If we consider, for instance, the typical energy of the spin waves of the order of their energy gap, the counting for the ferromagnet becomes $a_0 \sim a_i^2 \sim d^2 \sim m$, which leads to the following $O(p^3)$ terms:

$$\begin{aligned}
& i[d^3_{\bar{z}\bar{z}}(a^+_{\bar{z}}a^-_3 - a^-_{\bar{z}}a^+_3) - d^3_{zz}(a^-_z a^+_3 - a^+_z a^-_3)] \\
& i[d^3_{3z}(a^+_{\bar{z}}a^-_3 - a^-_{\bar{z}}a^+_3) - d^3_{3\bar{z}}(a^-_z a^+_3 - a^+_z a^-_3)],
\end{aligned} \tag{6.17}$$

whereas for the antiferromagnet an analogous counting implies $a_0^2 \sim a_i^2 \sim d^2 \sim m$, which in addition to the previous (6.17) terms leads to new $O(p^3)$ terms:

$$\begin{aligned}
& d^3_{\bar{z}\bar{z}}(a^+_{\bar{z}}a^-_0 + a^-_{\bar{z}}a^+_0) + d^3_{zz}(a^-_z a^+_0 + a^+_z a^-_0) \\
& d^3_{3z}(a^+_{\bar{z}}a^-_0 + a^-_{\bar{z}}a^+_0) + d^3_{3\bar{z}}(a^-_z a^+_0 + a^+_z a^-_0).
\end{aligned} \tag{6.18}$$

Notice again that if primitive translations which map points with opposite magnetisations existed these terms would be forbidden.

7 Magnetic Dipole Corrections

A second source of explicit $SU(2)$ breaking are the magnetic dipole interactions. They have the form [27]

$$H = \mu^2 \sum_{i \neq j} \left(\frac{\mathbf{S}_i \mathbf{S}_j}{|\mathbf{x}_i - \mathbf{x}_j|^3} - \frac{(\mathbf{S}_i \hat{\mathbf{x}}_i)(\mathbf{S}_j \hat{\mathbf{x}}_j)}{|\mathbf{x}_i - \mathbf{x}_j|^3} \right). \tag{7.1}$$

The first term is $SU(2)$ invariant, and has the form of the (super)exchange parameter in the Heisenberg model, while the second term has the form of the symmetric anisotropy in (6.1),

$$\frac{1}{|\mathbf{x}_i - \mathbf{x}_j|^3} \sim J_{ij} \qquad , \qquad \frac{\hat{x}_i^a \hat{x}_j^b}{|\mathbf{x}_i - \mathbf{x}_j|^3} \sim M_{ij}^{ab}, \tag{7.2}$$

Let us point out however that whereas the superexchange and anisotropic terms lead to short range interactions the terms above lead to long range interactions. In spite of this, the local limit will be taken since the strength of the interactions decays like the third power of the distance between the magnetic dipoles and furthermore it is usually very small compared to the spin-orbit terms. In fact this long range interaction starts playing a crucial role at very long distance phenomena, like in the formation of domain walls in ferromagnets [27], which lie beyond the scope of this work. If we restrict ourselves to the local limit, then both terms have been already taken into account in the $SU(2)$ invariant structure (J_{ij}) and in the explicit symmetry breaking source (M^{ab}).

8 The Coupling to Electromagnetic Fields

If we probe our ferromagnet or antiferromagnet by electromagnetic waves the energy of which is much smaller than the energy of the first gaped excitation, the only relevant magnetic degrees of freedom are the spin waves. It is then relevant how to include the electromagnetic fields in the effective lagrangians built in the previous sections.

When the electromagnetic fields enter the game the $U(1)_{em}$ local gauge invariance is the additional symmetry that we have to take into account. Since the fields $U(x)$ have trivial transformation properties under the $U(1)_{em}$ (spin waves have no electric charge) we may naïvely expect non-minimal couplings (couplings to the field strength tensor) only. However, the Pauli term which arises in any microscopic model when a magnetic field is present breaks explicitly the $SU(2)$ spin symmetry in a very particular way. We shall first address how to introduce in the effective lagrangian the effects of the microscopic Pauli term.

8.1 Pauli coupling

For definiteness we may have in mind the Heisenberg model, but the argument we present

goes through for more complicated models like the t-J model, Hubbard model, etc.. For any microscopic model we can think of the hamiltonian in the presence of a magnetic field will be augmented, at least, by the Pauli term, which explicitly breaks the $SU(2)$ spin symmetry,

$$H = \sum_{<i,j>} J_{ij}\mathbf{S}_i\mathbf{S}_j - \mu\sum_i \mathbf{S}_i\mathbf{B}. \tag{8.1}$$

Let us assume that the remaining part of the hamiltonian is invariant under the original $SU(2)$ symmetry, as in (8.1), even when the electromagnetic fields are switched on. Then the only term which breaks this symmetry is the Pauli term. Let us then write the lagrangian in the second quantisation formalism,

$$L = \sum_i \psi^\dagger(x_i) i\partial_0 \psi(x_i) + \mu\sum_i \left(\psi^\dagger(x_i)\mathbf{S}\psi(x_i)\right)\mathbf{B}(x_i) + \cdots. \tag{8.2}$$

This lagrangian can be rewritten in a form that the Pauli term is associated to the time derivative,

$$L = \sum_i \psi^\dagger(x_i) i\left(\partial_0 - i\mu\mathbf{S}\mathbf{B}(x_i)\right)\psi(x_i) + \cdots. \tag{8.3}$$

Written in this way the term inside the parenthesis has the form of a covariant derivative for time dependent $SU(2)$ transformations. Then if we promote the Pauli term to a new source, $A_0(x) \sim \mu\mathbf{S}\mathbf{B}(x)$ such that it transforms like a connexion under time dependent $SU(2)$ transformations,

$$A_0(x) \to g(t)A_0(x)g^\dagger(t) + ig(t)\partial_0 g^\dagger(t), \tag{8.4}$$

the microscopic lagrangian becomes invariant under time dependent $SU(2)$ spin transformations. Now we can construct the effective theory with this source, and finally set it to its actual value. This is completely analogous to the procedure carried out in section 6 for the spin-orbit breaking terms. The effective theory derived from the microscopic level would be a functional of the $U(x)$ and the connexion $A_0(x)$ invariant under time dependent $SU(2)$ transformations. This is easily achieved by replacing the time derivatives acting on U by covariant time derivatives. Namely,

$$\partial_0 \to D_0 \equiv \partial_0 - iA_0(x). \tag{8.5}$$

When we particularise the $A_0(x)$ to reproduce the Pauli term we obtain its effects in the effective theory. From now on we will have to write $a_0^\pm$ and a_0^3 as

$$\begin{aligned} U^\dagger iD_0 U = -\frac{1}{f_\pi^2}\Bigg\{ &\left[f_\pi\partial_0\pi^- - \mu\left(\frac{1}{2}(f_\pi^2 - \pi^+\pi^-)B^{\bar{z}} + \frac{1}{2}\pi^-\pi^- B^z + if_\pi\pi^- B^3\right) + \cdots\right] S_+ \\ &+ \left[f_\pi\partial_0\pi^+ - \mu\left(\frac{1}{2}\pi^+\pi^+ B^{\bar{z}} + \frac{1}{2}(f_\pi^2 - \pi^+\pi^-)B^z - if_\pi\pi^+ B^3\right) + \cdots\right] S_- \\ &+ \left[i(\pi^+\partial_0\pi^- - \pi^-\partial_0\pi^+) - \mu\left(if_\pi\pi^+ B^{\bar{z}} - if_\pi\pi^- B^z + (f_\pi^2 - 2\pi^+\pi^-)B^3\right) + \cdots\right] S^3\Bigg\}, \end{aligned} \tag{8.6}$$

whereas $a_i^{\pm}$ and a_i^3 are still given by (4.4).

It is important to realise that we do not have to introduce any additional unknown constant, but the effective gyromagnetic factor of the spin degrees of freedom in the microscopic theory in the above terms. Notice, however, that space derivatives on $A_0(x)$ transform covariantly under time dependent $SU(2)$ transformations. Hence they may be used to construct further invariants in analogy to the ds in (6.5) and (6.9). We will not present these invariants explicitly since, as we will see later on, they are very suppressed in realistic situations.

8.2 Non-minimal couplings

Let us next discuss the non-minimal couplings. This is the only way the electromagnetic field can couple to non charged excitations like the spin waves, namely, by means of the field strength tensor, i.e., electric field, **E**, and magnetic field, **B**. The transformation properties associated to the electromagnetic fields are the following:

1. they both are scalars under $SU(2)$ spin transformations,
2. the electric field transforms like a vector under $\bar{3}m$,
3. the magnetic field transforms like a pseudovector under $\bar{3}m$, and
4. they both are scalars under under primitive translations,
5. under the time reversal they transform as

$$T: \quad \begin{cases} E^a & \rightarrow & E^a \\ B^a & \rightarrow & -B^a. \end{cases} \tag{8.7}$$

8.3 Power counting

Once we have the transformation properties for the electromagnetic fields let us consider their relative size in order to write the derivative expansion in the effective theory.

The expression (8.5) indicates us that the Pauli term, $\mu\mathbf{B}$, has to be suppressed by the first gapped excitation, J. Consider next the electromagnetic fields in terms which are $SU(2)$ invariant. They may arise from a microscopic model in two ways: (i) originated by minimal couplings of the electromagnetic potentials or (ii) by explicit non-minimal couplings in the microscopic model, which may arise when integrating out even higher scales of energy and momentum. The second kind of terms are suppressed by a higher energy and momentum scale and will be ignored as far as estimates are concerned. They would slightly modify the values of the parameters in the effective lagrangian, which are anyway unknown. The relative suppression of the terms originated by minimal coupling are simple to estimate: the

scalar potential goes always accompanying a time derivative and hence it will be suppressed by J, whereas the vector potential is associated to a link variable (in a lattice model) and hence suppressed by $1/a$ (in the continuum limit the space derivative is associated to the link and therefore the vector potential is associated to the space derivative). Consequently the electric fields are suppressed by J/ea whereas the magnetic fields by $1/ea^2$, where e is the electron charge.

When the electromagnetic field enters the system it fixes the relevant scales of energy and momentum. For the effective theory to make sense these must be much smaller than J and $1/a$ respectively. The amplitude of the electromagnetic field is also constrained so that we can organise our effective lagrangian in increasing powers of E and B as well as derivatives. This requires $\mu B/J$, eaE/J and ea^2B be much smaller than 1.

Nothing else can be said in general about the organisation of the effective lagrangian. Let us at this point introduce $v = Ja$ which combines the energy and momentum scales in a single parameter. This parameter allows us to write our theory in terms of a single scale, namely, J. For the antiferromagnetic spin waves v has a precise meaning: it is the velocity of propagation of the spin waves, whereas for the ferromagnetic spin waves it does not have any particular meaning. Given typical values for $J \sim 10meV$ and $a \sim 10\rho A$ [27], then $v \sim 10^{-4}$ (Recall that we have taken $c = 1$, therefore v has to be thought as v/c).

In order to further illuminate the construction of the effective lagrangian, let us consider the interaction with a classical monochromatic electromagnetic wave of energy ω. In this case the amplitude of the electric and magnetic fields is the same. Let us choose $\omega \sim 10^{-1}J$, which tells us that we must count ∂_0 as $10^{-1}J$. The momentum of the electromagnetic field is in this case ω, which tells us that for typical lattice spacings we must count $v\partial_i \sim 10^{-4}\partial_0$. Notice that the counting is very different in the absence of electromagnetic field. We still have to fix the amplitude of the electromagnetic field. For simplicity let us choose $eaE \sim 10^{-1}J$. Then the following relative suppressions hold:

$$\begin{array}{rcl} \partial_0, eaE & \sim & 10^{-1}J \\ d & \sim & 10^{-2}J \\ \mu B, m & \sim & 10^{-4}J \\ v\partial_i, eavB & \sim & 10^{-5}J. \end{array} \tag{8.8}$$

where a typical value for the gyromagnetic factor $\mu \sim \mu_B = 9.27\,10^{-24}J/T$ has been taken. With the counting above the following terms are obtained for the ferromagnetic systems:

First order $O(p^1)$:

$$a_0^3. \tag{8.9}$$

Second order $O(p^2)$:

$$\begin{aligned} &a_0^+ a_0^- \\ &E^z E^{\bar{z}} \\ &E^3 E^3. \end{aligned} \tag{8.10}$$

Fourth order $O(p^4)$:

$$\begin{aligned} &a_0^+ a_0^- a_0^+ a_0^- \\ \\ &E^z E^{\bar{z}} E^z E^{\bar{z}} \\ &E^z E^{\bar{z}} E^3 E^3 \\ &E^3 E^3 E^3 E^3 \\ \\ &\partial_0 E^z \partial_0 E^{\bar{z}} \\ &\partial_0 E^3 \partial_0 E^3 \\ \\ &a_0^+ a_0^- E^z E^{\bar{z}} \\ &a_0^+ a_0^- E^3 E^3 \\ \\ &d^+_{zz} d^-_{\bar{z}\bar{z}} + d^-_{zz} d^+_{\bar{z}\bar{z}} \\ &(d^+_{3z} d^-_{zz} + d^-_{3z} d^+_{zz}) + (d^-_{3\bar{z}} d^+_{\bar{z}\bar{z}} + d^+_{3\bar{z}} d^-_{\bar{z}\bar{z}}) \\ &d^+_{3z} d^-_{3\bar{z}} + d^-_{3z} d^+_{3\bar{z}} \\ &d^3_{zz} d^3_{\bar{z}\bar{z}} \\ &d^3_{3z} d^3_{zz} + d^3_{3\bar{z}} d^3_{\bar{z}\bar{z}} \\ &d^3_{3z} d^3_{3\bar{z}} \\ \\ &m^{-+} \\ &m^{33}. \end{aligned} \tag{8.11}$$

Notice that it is not up to fourth order that we obtain a coupling of the electric field to the spin waves. The Pauli coupling is encoded in the a_0 blocks as given in (8.6). At $O(p^4)$ there is only a contribution arising from (8.9) when we substitute the time derivative by the covariant derivative (8.5).

For the antiferromagnet there is no invariant at $O(p^1)$. Second order ($O(p^2)$) invariants are the same as those for the ferromagnet in (8.10). At fourth order ($O(p^4)$) in addition to the terms given in (8.11) we have two new terms,

$$\begin{aligned} &i[(d^+_{\bar{z}\bar{z}} a_0^- - d^-_{\bar{z}\bar{z}} a_0^+) E^z - (d^-_{zz} a_0^+ - d^+_{zz} a_0^-) E^{\bar{z}}] \\ &i[(d^+_{3z} a_0^- - d^-_{3z} a_0^+) E^z - (d^-_{3\bar{z}} a_0^+ - d^+_{3\bar{z}} a_0^-) E^{\bar{z}}]. \end{aligned} \tag{8.12}$$

Notice that these new terms which appear in the antiferromagnetic case at $O(p^4)$ contain time derivatives and the spin-orbit tensor which breaks explicitly the $SU(2)$ spin symmetry, which give rise to an electromagnetic coupling mediated by the spin-orbit interaction. These terms would be forbidden if a primitive translation mapping points with opposite magnetisations existed.

9 Summary and Discussion

We have provided a systematic way to write down effective lagrangians for spin waves in ferromagnetic and antiferromagmetic crystalline solids. We have done so at next to leading order. This is achieved by fully exploiting the internal symmetry breaking pattern $SU(2) \to U(1)$ as well as the space symmetries and time reversal. We have shown how to introduce explicit symmetry breaking terms, as those induced by atomic spin-orbit couplings and magnetic dipole interactions. We have also shown how to introduce couplings to electromagnetic fields.

The alert reader may wonder about the role that the magnetic group plays in our construction. In fact, it does not play any role else that indicating us the symmetry group of the ground state configuration. Notice that when we have spontaneous symmetry breaking the magnetic group should not be important since in a first approximation the local magnetisation direction is arbitrary, and each direction corresponds to a different magnetic group. It is only after introducing explicit $SU(2)$ breaking corrections, like those induced by the spin-orbit interactions, that the local magnetisations take the direction of a crystal symmetry axis, giving rise to a non-trivial magnetic group. If we introduce a constant magnetic field in an arbitrary direction whose contribution is larger than that of the spin-orbit interactions the local magnetisations will point to the direction of the magnetic field. Then the magnetic group will be trivial again. Within our formalism all the possible situations are taken into account once we have assign the proper space-time transformations to the field $U(x)$ in (2.6)-(2.8).

A word of caution is needed when using the effective lagrangians to higher orders: quantum (loop) corrections to lower order terms give in general contributions of some higher order. Then in order to make a consistent calculation to a given higher order, in general a loop calculation at lower order is necessary. For the antiferromagnet the kind of loop calculations to be carried out is completely analogous to calculations in relativistic theories, for which there is abundant experience [9, 13]. Typically a one loop calculation at $O(p^2)$ gives $O(p^4)$, a two loop one $O(p^6)$, etc. For the ferromagnet, there are no explicit loop calculations in the literature to our knowledge. On general grounds, we expect a similar pattern though the exact way in which lower order loop contributions combine into higher orders might be

different[1].

We would like to point out a few issues in our work that we find particularly relevant.

1. Our formulation easily keeps track of elusive topological terms.

2. We find the remarkable differences between the ferromagnetic and antiferromagnetic spin wave effective lagrangian by the only use of symmetry properties.

3. Within the antiferromagnetic case, we point out that the existence of primitive translations which map points with opposite magnetisations has non-trivial consequences in the effective lagrangian.

4. We easily see that the Pauli term does not introduce any new parameter in the effective lagrangian.

We would like to stress again that an important advantadge of the effective lagrangian is that it is model independent. Any microscopic model with the same space-time group undergoing the same symmetry breaking pattern gives rise to the same effective lagrangian. Only the particular values of the constants may differ. However, in view of the relative large number of such unkwown constants that arise at next to leading order in our lagrangians, one may wonder if they can actually be of any use at all. We are confident that they will. In particular, we propose to use them as a guideline on possible interesting phenomena due to spin waves. If one suspects that a given material may have exciting magnetic properties in the spin wave region, one need not carry out a microscopic calculation to check so. By writing down the effective lagrangian one can most easily check if the expected phenomenon may occur or not. To do this one has to input realistic numbers for the J and a parameters according to the given material, changing the counting (8.8) if necessary, and proceed as above. In case it is feasible, a microscopic calculation may be supplemented to fix the unknown parameters. Recently, we have presented a non-trivial application of these techniques in [14].

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[1] A recent non-trivial loop calculation can be found in [28]

Appendix A: Primitive Translations Effects

In this Appendix we discuss the important consequences that primitive translations which map points with opposite magnetisations have in the effective lagrangian for antiferromagnets. Let us call τ to one such translations. According to the procedure in section 2, τ may be implemented by

$$\tau : \ U(x) \ \rightarrow \ U(x)Ch_{\tau}^{\dagger}, \tag{A.1}$$

where h_τ is a compensating $U(1)$ element that keeps the transformed field in the coset. This implies

$$\tau : \ \left\{ \begin{array}{lcl} a_{\mu}^{-} & \rightarrow & -a_{\mu}^{+} \\ a_{\mu}^{3} & \rightarrow & -a_{\mu}^{3}. \end{array} \right. \tag{A.2}$$

This transformation differs from T in (3.11) only in terms with time derivatives. If we consider τT instead of τ as an independent generator, we have

$$\tau T : \ \left\{ \begin{array}{lcl} a_{\mu}^{-} & \rightarrow & a_{t\mu}^{-} \\ a_{\mu}^{3} & \rightarrow & a_{t\mu}^{3}. \end{array} \right. \tag{A.3}$$

Consequently, terms with an odd number of time derivatives are forbidden. Moreover, it is worth mentioning that the remaining terms for the antiferromagnetic effective lagrangian are the same as those for the ferromagnetic one. This statement holds from considering $\tau\xi$,

$$\tau\xi : \ \left\{ \begin{array}{lcl} a_{\mu}^{-} & \rightarrow & a_{\xi\mu}^{-} \\ a_{\mu}^{3} & \rightarrow & a_{\xi\mu}^{3}, \end{array} \right. \tag{A.4}$$

where ξ stands for elements of the point group which map points with opposite magnetisations, like that in (3.10b), together with T and τT as the symmetry generators.

The transformations $\tau\xi$ and τT, given by (A.4) and (A.3) respectively, where proposed in [23] as the macroscopic transformation for the antiferromagnetic spin waves. Notice that these transformations are slightly less restrictive than the ones we have, namely, $\tau\xi$, T and τT.

The introduction of the spin-orbit interaction does not invalidate the previous statements. Indeed, as mention before the differences between T and τ come from the time indices. However, spin-orbit sources have no time indices and, therefore, they transform trivially under τT.

Nevertheless, the statements above must be generalised in the presence of electromagnetic fields. The electromagnetic fields transform trivially under τ, and therefore, under τT the magnetic field, $\mathbf{B}$, changes sign. This implies that the invariants with an odd number of time derivatives plus magnetic fields are not allowed in the effective lagrangian for the antiferromagnetic spin waves. The remaining terms in the effective lagrangian are the same as the ones for the ferromagnetic spin waves.

Appendix B: Mathematical Properties

In this appendix we discuss a few important technicalities which have been used through the paper.

The projectors $P_\pm$ are introduced in sect. 2 in order to single out the magnetisation direction. They are given by

$$P_+ = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \quad , \qquad P_- = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \tag{B.1}$$

and verify

$$tr(P_\pm S^a) = \pm s\delta^{a3}. \tag{B.2}$$

When two of these projectors are in a trace, we can split it in two pieces

$$tr\Big(P_\pm(\cdots)P_\pm(\cdots)\Big) = tr\Big(P_\pm(\cdots)\Big)tr\Big(P_\pm(\cdots)\Big). \tag{B.3}$$

This is immediate because the only non-zero element in the matrix $P_+(\cdots)P_+$ is the $(\cdots)_{11}$, or $(\cdots)_{(2s+1)\,(2s+1)}$ for the P_- case.

Another important property is that in any product of generators between two P_+ where at most two of them are different from S^3 the following substitution can be performed:

$$\begin{aligned} S^\alpha &\rightarrow \frac{\sqrt{2s}}{2}\,(\sigma^\alpha \oplus 0) \qquad\qquad (\alpha = 1,2) \\ S^3 &\rightarrow (s-1)\,(\mathbf{1} \oplus 0) + P_+, \end{aligned} \tag{B.4}$$

since in this case only the upper-left 2×2 matrix contributes to the trace. An analogous property holds for P_-.

In addition in section 2 the matrix $C = e^{-i\pi S^2}$ is introduced to implement the space and time inversions. Below we give some relevant properties of this matrix.

The time inversion implies [21]

$$C^\dagger S^a C = -(S^a)^T. \tag{B.5}$$

The action of C over the projectors is given by

$$C^\dagger P_+ C = P_-, \tag{B.6}$$

which follows from the fact that $P_\pm$ belong to the subspace spanned by arbitrary powers of S^3. We also have that

$$tr(P_+ S^{a_1} \cdots S^{a_n} P_+) \;\; = \;\; (-)^n tr(P_- S^{a_n} \cdots S^{a_1} P_-), \tag{B.7}$$

which follows immediately by inserting $C^\dagger C$ between the elements on the l.h.s. and using the properties above.

By using this property together with the clausure relations for the Pauli matrices,

$$\frac{1}{2}(\sigma_+)_{\alpha\beta}(\sigma_-)_{\gamma\delta} + \frac{1}{2}(\sigma_-)_{\alpha\beta}(\sigma_+)_{\gamma\delta} = -\delta_{\alpha\beta}\delta_{\gamma\delta} - \sigma^3_{\alpha\beta}\sigma^3_{\gamma\delta} + 2\delta_{\alpha\delta}\delta_{\gamma\beta}, \tag{B.8}$$

the relations $F_{\mu\nu} \sim a^+_\mu a^-_\nu - a^+_\nu a^-_\mu$ and $D_\mu a^-_\nu = D_\nu a^-_\mu$ mentioned in section 2 can be proved from the explicit representation (3.8).

Appendix C: Fundamental Representation

In this appendix we point out the simplifications that occur for $s = 1/2$ which follow from the properties of the Appendix B.

All the invariants can be constructed in terms of

$$T(x) = U(x)\sigma^3 U^\dagger(x), \tag{C.1}$$

which transforms covariantly under $SU(2)$, their derivatives and topological terms, which are built out of a^3_μ alone. Notice that

$$a^3_\mu = tr(U^\dagger i\partial_\mu U\sigma^3). \tag{C.2}$$

Recall that T verifies

$$T^2 = 1 \qquad , \qquad trT = 0. \tag{C.3}$$

In the fundamental representation $P_+ = (\mathbf{1} + \sigma^3)/2$ and the building blocks are:

$$a^-_\mu(x) = \frac{1}{4}tr\left([U^\dagger i\partial_\mu U, \sigma_-]\sigma^3\right) = -\frac{i}{4}tr(U^\dagger\partial_\mu TU\sigma_-). \tag{C.4}$$

Let us consider the covariant derivative $D_\nu = \partial_\nu - ia^3_\nu$ on terms of the form (C.4),

$$tr(U^\dagger AU\sigma_-) \qquad , \qquad A \to gAg^\dagger \in \mathcal{L}(SU(2)), \tag{C.5}$$

where A transforms covariantly under $SU(2)$ and is in the Lie algebra. The result of the covariant derivative is

$$D_\nu tr(U^\dagger AU\sigma_-) = tr(U^\dagger\partial_\nu AU\sigma_-) - \frac{1}{2}tr(AT)tr(U^\dagger\partial_\nu TU\sigma_-). \tag{C.6}$$

Since all the terms can be derived from (C.4) the most general form of A is

$$A = \partial_{\mu_1}\cdots\partial_{\mu_n}T, \tag{C.7}$$

which leads us to write for the Ts

$$D_\nu tr(U^\dagger \partial_{\mu_1}\cdots\partial_{\mu_n} T U\sigma_-) = tr(U^\dagger\partial_\nu\partial_{\mu_1}\cdots\partial_{\mu_n} T U\sigma_-) - \frac{1}{2}tr(\partial_{\mu_1}\cdots\partial_{\mu_n}TT)tr(U^\dagger\partial_\nu T U\sigma_-). \quad \text{(C.8)}$$

In order to get $SU(2)$ invariants we will have products of the form

$$tr(U^\dagger AU\sigma_+)tr(U^\dagger BU\sigma_-) = \frac{1}{2}\left[tr(U^\dagger AU\sigma_+)tr(U^\dagger BU\sigma_-) + tr(U^\dagger AU\sigma_-)tr(U^\dagger BU\sigma_+)\right] + \frac{1}{2}\left[tr(U^\dagger AU\sigma_+)tr(U^\dagger BU\sigma_-) - tr(U^\dagger AU\sigma_-)tr(U^\dagger BU\sigma_+)\right], \quad \text{(C.9)}$$

which have been written as the sum of its symmetric and antisymmetric parts

Now the Pauli matrices $\sigma_\pm$ can be eliminated with the aid of the clausure relation (B.8). For the symmetric part it is immediate, leading to

$$-tr(AT)tr(BT) + 2tr(AB), \quad \text{(C.10)}$$

while for the antisymmetric part a little trick has to be used in order to be able to use the clausure relation: we perform in the first factor of each term the substitution $[\sigma^3,\sigma_\pm] = \pm 2\sigma_\pm$, and we obtain

$$\left[tr(U^\dagger AU[\sigma^3,\sigma_+])tr(U^\dagger BU\sigma_-) + tr(U^\dagger AU[\sigma^3,\sigma_-])tr(U^\dagger BU\sigma_+)\right] = -4tr([A,B]T). \quad \text{(C.11)}$$

Hence we conclude that for the $s = 1/2$ case all the invariants can be constructed from products of the following traces:

$$\begin{array}{l} tr\left(\partial_{\mu_1}\cdots\partial_{\mu_n}T\partial_{\nu_1}\cdots\partial_{\nu_m}T\right) \\ tr\left([\partial_{\mu_1}\cdots\partial_{\mu_n}T,\partial_{\nu_1}\cdots\partial_{\nu_m}T]T\right) \end{array} \qquad (n,m = 0,1,2,\ldots), \quad \text{(C.12)}$$

which correspond to take into account the symmetric and antisymmetric parts of the product $\partial_{\mu_1}\cdots\partial_{\mu_n}T\partial_{\nu_1}\cdots\partial_{\nu_m}T$ respectively.

Appendix D: Equivalence with the $O(3)$-sigma model

Let us next make contact with the so called $O(3)$ σ-model formulation, which uses a unitary vector $n^a(x)$ as the basic building block of the effective lagrangian. Recall first that a formula like (C.4) exist for any representation,

$$a_\mu^-(x) = \frac{1}{2s}tr\left([U^\dagger i\partial_\mu U, S_-]P_+\right) = -\frac{i}{2s}tr(U^\dagger\partial_\mu T_+ U S_-) \quad , \quad T_+ = UP_+U^\dagger. \quad \text{(D.1)}$$

We can relate our notation with the standard one of the unitary vector $n^a(x)$ by noting that

$$T(x) = U(x)S^3U^\dagger(x) = n^a(x)S^a. \tag{D.2}$$

It is easy to check that $\mathbf{n}^2 = 1$ and clearly n^a transforms like a vector. Recall next that P_+ can be written as

$$P_+ = \sum_{n=0}^{2s} a_n (S^3)^n \tag{D.3}$$

since $\{(S^3)^n\}$ is a bases of the subspace of diagonal matrices. Hence

$$T_+ = \sum_{n=0}^{2s} a_n (US^3U^\dagger)^n = \sum_{n=0}^{2s} a_n T^n. \tag{D.4}$$

For an arbitrary representation the covariant derivative (3.5) on a term like (D.1) yields a formula similar to that in (C.6),

$$D_\nu tr(U^\dagger AUS_-) = tr(U^\dagger \partial_\nu AUS_-) - \frac{1}{s} tr(AT) tr(U^\dagger \partial_\nu T_+ US_-), \tag{D.5}$$

where A is made of an arbitrary number of derivatives on T_+. The possible invariants will be made out of products of the form

$$\begin{aligned} tr(U^\dagger AUS_+)tr(U^\dagger BUS_-) = \\ \frac{1}{2}\Big[tr(U^\dagger AUS_+)tr(U^\dagger BUS_-) + tr(U^\dagger AUS_-)tr(U^\dagger BUS_+)\Big] \\ + \frac{1}{2}\Big[tr(U^\dagger AUS_+)tr(U^\dagger BUS_-) - tr(U^\dagger AUS_-)tr(U^\dagger BUS_+)\Big]. \end{aligned} \tag{D.6}$$

Taking into account that $U \in SU(2)$ the property below follows:

$$(US^aU^\dagger)(US^aU^\dagger) = (S^a)(S^a), \tag{D.7}$$

which can be rewritten as

$$\begin{aligned} \frac{1}{2}\Big[(US_+U^\dagger)(US_-U^\dagger) + (US_-U^\dagger)(US_+U^\dagger)\Big] = -(T)(T) \\ + \frac{1}{2}\Big[(S_+)(S_-) + (S_-)(S_+)\Big] + (S^3)(S^3). \end{aligned} \tag{D.8}$$

Using (D.8) the symmetric part of (D.6) is

$$-tr(AT)tr(BT) + \frac{1}{2}\Big[tr(AS_+)tr(BS_-) + tr(AS_-)tr(BS_+)\Big] + tr(AS^3)tr(BS^3), \tag{D.9}$$

whereas for the antisymmetric part the same trick as in Appendix C has to be used, i.e., change $S_\pm = \pm[S^3, S_\pm]$ in the first factor of each term. We obtain

$$\begin{aligned} \frac{1}{2}\Big[tr(U^\dagger AU[S^3, S_+])tr(U^\dagger BUS_-) + tr(U^\dagger AU[S^3, S_-])tr(U^\dagger BUS_+)\Big] = \\ \frac{1}{2}\Big[tr([A,T]S_+)tr(BS_-) + tr([A,T]S_-)tr(BS_+)\Big] + tr([A,T]S^3)tr(BS^3). \end{aligned} \tag{D.10}$$

Finally it has been shown that all the invariants can be written in terms of T and therefore in terms of $n^a(x)$. The reverse is also true. Suppose we have written our effective lagrangian in terms of derivatives of n^a. From (D.2)

$$n^a = \frac{3}{s(s+1)(2s+1)} tr(S^3 U^\dagger S^a U). \tag{D.11}$$

Consider a vector of the form

$$v^a = tr(A U^\dagger S^a U) \qquad , \qquad A \in \mathcal{L}(SU(2)), \tag{D.12}$$

$$\partial_\mu v^a = tr(\partial_\mu A U^\dagger S^a U) + tr([U^\dagger \partial_\mu U, A] U^\dagger S^a U). \tag{D.13}$$

Since any vector will be obtained by applying several derivatives on n^a, A in (D.12) will only contain $U^\dagger \partial_\mu U$ and its derivatives, which can be written in terms of our basic fields $a_\mu^\pm, a_\mu^3$. The remaining dependence on U is through $U^\dagger S^a U$ only which will cancel out upon contraction with other vectors, according to (D.7), in order to build a scalar lagrangian.

So far we have been talking about invariant terms in the effective lagrangian. It is not true however that terms which are invariant up to a total derivative, like a_μ^3, can be written in terms of n^a in a local form. However, we can write them locally in T_+ or n^a if we introduce an extra dimension in the following way. We interpolate smoothly the Goldstone fields $\pi^\alpha(x) \to \pi^\alpha(x,\lambda)$, $\lambda \in [0,1]$ in such a way that $\pi^\alpha(x,1) = \pi^\alpha(x)$ and $\pi^\alpha(x,0) = 0$. Let us concentrate on a_0^3 which is the only one which arises in our effective lagrangian. It is very easy to check that

$$a_0^3 = \frac{1}{s} \int_0^1 d\lambda \epsilon^{\alpha\beta} tr(T_+ \partial_\alpha T_+ \partial_\beta T_+) \sim \int_0^1 d\lambda \epsilon^{\alpha\beta} \epsilon_{abc} n^a \partial_\alpha n^b \partial_\beta n^c, \tag{D.14}$$

$(\alpha, \beta = 0, \lambda)$. The second equality follows upon using (D.2) and performing the trace. The final result must be a scalar function of n^a, $\partial_\alpha n^b$ and $\partial_\beta n^c$ antisymmetric under the exchange of α and β, being (D.14) the only possibility. To our knowledge this term was first written in the last form in [24] (see also [6]), whereas we have not been able to locate the two previous forms in the literature. We stress again that only the form that we use in our effective lagrangian is local. It is usual however to find (D.14) with a rather different aspect, namely,

$$a_0^3 \sim A^a(n) \partial_0 n^a \qquad , \qquad n^a = \epsilon_{abc} \frac{\partial A^b}{\partial n^c}, \tag{D.15}$$

where the second equation gives A^a as an implicit function of n^b [25, 29]. It is easy to check that the last expression of (D.14) and (D.15) give rise to the same equations of motion.

Appendix E: Coupling to Constant Electric and Magnetic Fields

In this Appendix we present a further example on how the effective lagrangian for spin waves coupled to electromagnetism can be used to obtain qualitative information on the system.

Let us suppose that the system is exposed to constant electric and magnetic fields and address the question on how the low momentum dispersion relations of the spin waves change in the ferromagnetic and antiferromagnetic case. For simplicity we assume that the magnetic field is on the third direction and the electric field in the $z-\bar{z}$ plane, and fix their relative size, which is now arbitrary, as $eaE/J \sim ea^2B$. Recall that the Pauli term must be counted as $\mu B/J \sim 10ea^2B$. We also neglect terms induced by spin-orbit or magnetic dipole interactions which explicitly break the $SU(2)$ symmetry since it is straightforward to take them into account if desired. We shall focus in the leading effects due to higher order terms, since those due to the lowest order terms are well known and will be easily reproduced.

For the ferromagnetic case the leading corrections due to higher order terms arise from

$$\begin{array}{l} (a_z^+ a_{\bar{z}}^- + a_z^- a_{\bar{z}}^+) E^z E^{\bar{z}} \\ a_3^+ a_3^- E^z E^{\bar{z}} \\ a_z^+ a_z^- E^z E^z + a_{\bar{z}}^- a_{\bar{z}}^+ E^{\bar{z}} E^{\bar{z}} \\ (a_z^+ a_3^- + a_z^- a_3^+) E^{\bar{z}} E^{\bar{z}} + (a_{\bar{z}}^- a_3^+ + a_{\bar{z}}^+ a_3^-) E^z E^z \\ (a_z^+ a_{\bar{z}}^- + a_z^- a_{\bar{z}}^+) B^3 B^3 \\ a_3^+ a_3^- B^3 B^3, \end{array} \tag{E.1}$$

which together with the contributions of the leading order terms (4.1) and (4.2) lead to

$$\begin{aligned} \omega \;=\; & \mu B^3 + \frac{2}{m}[1 + \epsilon_1 E^z E^{\bar{z}} + \beta_1 B^3 B^3] k^z k^{\bar{z}} \\ & + \frac{1}{2\gamma m}[1 + \epsilon_2 E^z E^{\bar{z}} + \beta_2 B^3 B^3](k^3)^2 \\ & + \epsilon_3[(E^z k^{\bar{z}})^2 + (E^{\bar{z}} k^z)^2] \\ & + 2\epsilon_4[(E^z E^z k^z) + (E^{\bar{z}} E^{\bar{z}} k^{\bar{z}})] k^3. \end{aligned} \tag{E.2}$$

The effect of our particular constant electric and magnetic fields is, apart from creating the well known energy gap proportional to the magnetic field, (i) renormalising $1/m$ and γ and (ii) producing producing further anisotropies in the dispersion relation (due to ϵ_3 and ϵ_4), which vanish if the electric field vanishes.

For the antiferromagnetic case the leading corrections due to higher order terms arise

from the same terms as in the ferromagnetic case (E.2) plus the following extra terms,

$$\begin{array}{l} a_0^+ a_0^- E^z E^{\bar{z}} \\ a_0^+ a_0^- B^3 B^3 \\ i[(a_0^+ a_z^- + a_0^- a_z^+) E^z B^3 - (a_0^- a_{\bar{z}}^+ + a_0^+ a_{\bar{z}}^-) E^{\bar{z}} B^3, \end{array} \tag{E.3}$$

which together with the contributions of the leading order terms (5.1) and (5.2) lead to

$$\begin{aligned} \omega_\pm \;=\; & \pm\mu B^3 \pm i\eta[(E^z k^{\bar{z}}) - (E^{\bar{z}} k^z)] B^3 \\ & + \Big[4v^2[1 + \epsilon_1 E^z E^{\bar{z}} + \beta_1 B^3 B^3] k^z k^{\bar{z}} \\ & \quad + (\gamma v)^2 [1 + \epsilon_2 E^z E^{\bar{z}} + \beta_2 B^3 B^3] (k^3)^2 \\ & \quad + \epsilon_3 [(E^z k^{\bar{z}})^2 + (E^{\bar{z}} k^z)^2] \\ & \quad + 2\epsilon_4 [(E^z E^z k^z) + (E^{\bar{z}} E^{\bar{z}} k^{\bar{z}})] k^3 \Big]^{1/2}. \end{aligned} \tag{E.4}$$

The effect of our particular constant electric and magnetic fields is, apart from providing the well known "chemical potential" term proportional to the magnetic field, (i) renormalising v and γ, (ii) producing further anisotropies in the dispersion relation (due to ϵ_3 and ϵ_4), which vanish if the electric field vanishes, and (iii) producing a momentum dependent distinction between the two components of the spin wave (due to η). We find the latter effect particularly interesting. Notice that it requires both the electric and magnetic fields be differnt from zero.

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