Available online at www.sciencedirect.com
SCIENCE DIRECT®
Nuclear Physics B 683 [FS] (2004) 387–422
NUCLEAR PHYSICS B
www.elsevier.com/locate/npe

Charge and spin quantum fluids generated by many-electron interactions

J.M.P. Carmelo [a], J.M. Román [a], K. Penc [b]

[a] *GCEP-Center of Physics, University of Minho, Campus Gualtar, P-4710-057 Braga, Portugal*
[b] *Research Institute for Solid State Physics and Optics, POB 49, H-1525 Budapest, Hungary*

Received 24 November 2003; accepted 29 January 2004

Abstract

In this paper we describe the electrons of the non-perturbative one-dimensional (1D) Hubbard model by a fluid of unpaired rotated electrons and a fluid of zero-spin rotated-electron pairs. The rotated electrons are related to the original electrons by a mere unitary transformation. For *all* finite values of energy and for the whole parameter space of the model this two-fluid picture leads to a description of the energy eigenstates in terms of occupancy configurations of η-spin $1/2$ holons, spin $1/2$ spinons, and c pseudoparticles only. The electronic degrees of freedom couple to external charge (and spin) probes through the holons and c pseudoparticles (and spinons). Our results refer to very large values of the number of lattice sites N_a. The holon (and spinon) charge (and spin) transport is made by 2ν-holon (and 2ν-spinon) composite pseudoparticles such that $\nu = 1, 2, \ldots$. For electronic numbers obeying the inequalities $N \leqslant N_a$ and $N_\downarrow \leqslant N_\uparrow$ there are no zero-spin rotated-electron pairs in the ground state and the unpaired-rotated-electron fluid is described by a charge c pseudoparticle fluid and a spin $\nu = 1$ two-spinon pseudoparticle fluid. The spin two-spinon pseudoparticle fluid is the 1D realization of the two-dimensional *resonating valence bond* spin fluid.

PACS: 70

1. Introduction

The relation for the whole Hilbert space of the non-perturbative one-dimensional (1D) Hubbard model [1,2] of the original electrons to the quantum objects whose occupancy configurations describe its energy eigenstates is an interesting and important open problem. Such a non-perturbative relation is needed for the study of the finite-energy few-electron spectral properties of the many-electron quantum problem. The main goal of this paper

E-mail address: carmelo@fisica.uminho.pt (J.M.P. Carmelo).

0550-3213/$ – see front matter
doi:10.1016/j.nuclphysb.2004.01.036

is the study of such a relation. Our study is motivated by both the unusual finite-energy spectral properties observed in quasi-1D materials [3,4] and the relation of these materials to two-dimensional (2D) quantum problems [5–7].

In a chemical potential μ and magnetic field h the 1D Hubbard Hamiltonian can be written as

$$\hat{H} = \hat{H}_{SO(4)} + \sum_{\alpha=c,s} \mu_\alpha 2\hat{S}^z_\alpha, \tag{1}$$

where $\mu_c = \mu$ is the chemical potential, $\mu_s = \mu_0 h$, μ_0 is the Bohr magneton, and the number operators

$$\hat{S}^z_c = -\frac{1}{2}[N_a - \hat{N}], \qquad \hat{S}^z_s = -\frac{1}{2}[\hat{N}_\uparrow - \hat{N}_\downarrow], \tag{2}$$

are the diagonal generators of the η-spin and spin $SU(2)$ algebras [8,9], respectively. The Hamiltonian $\hat{H}_{SO(4)}$ on the right-hand side of Eq. (1) reads

$$\hat{H}_{SO(4)} = \hat{H}_H - (U/2)\hat{N} + (U/4)N_a, \tag{3}$$

where

$$\hat{H}_H = \hat{T} + U\hat{D}, \tag{4}$$

is the Hubbard model in standard notation. In the latter expression

$$\hat{T} = -t\sum_{j=1}^{N_a}\sum_{\sigma=\uparrow,\downarrow}\sum_{\delta=-1,+1} c^\dagger_{j,\sigma} c_{j+\delta,\sigma}, \tag{5}$$

is the *kinetic-energy* operator and

$$\hat{D} = \sum_j c^\dagger_{j,\uparrow} c_{j,\uparrow} c^\dagger_{j,\downarrow} c_{j,\downarrow} = \sum_j \hat{n}_{j,\uparrow}\hat{n}_{j,\downarrow}, \tag{6}$$

is the electron double-occupation operator. We call the kinetic energy T and number of electron doubly occupied sites D the expectations values of the operators (5) and (6), respectively.

The Hamiltonian given in Eq. (3) has $SO(4)$ symmetry [8–10] and commutes with the six generators of the η-spin and spin algebras. The expressions of the two corresponding diagonal generators are given in Eq. (2) and the off-diagonal generators of these two $SU(2)$ algebras read

$$\hat{S}^\dagger_c = \sum_j (-1)^j c^\dagger_{j,\downarrow} c^\dagger_{j,\uparrow}, \qquad \hat{S}_c = \sum_j (-1)^j c_{j,\uparrow} c_{j,\downarrow}, \tag{7}$$

and

$$\hat{S}^\dagger_s = \sum_j c^\dagger_{j,\downarrow} c_{j,\uparrow}, \qquad \hat{S}_s = \sum_j c^\dagger_{j,\uparrow} c_{j,\downarrow}, \tag{8}$$

respectively. The η-spin and spin square operators have eigenvalue $S_\alpha[S_\alpha + 1]$ with $\alpha = c$ and $\alpha = s$, where S_c and S_s denote the η-spin and spin values, respectively.

We consider that the number of lattice sites N_a is large and even and $N_a/2$ is odd. The electronic number operators of Eq. (2) read $\hat{N} = \sum_\sigma \hat{N}_\sigma$ and $\hat{N}_\sigma = \sum_j \hat{n}_{j,\sigma}$ where $\hat{n}_{j,\sigma} = c^\dagger_{j,\sigma} c_{j,\sigma}$. Here $c^\dagger_{j,\sigma}$ and $c_{j,\sigma}$ are spin σ electron operators at site $j = 1, 2, \ldots, N_a$. We denote the lattice constant by a and the system length by $L = aN_a$. Throughout this paper we use units of Planck constant one. We consider electronic densities $n = N/L$ and spin densities $m = [N_\uparrow - N_\downarrow]/L$ in the domains $0 \leqslant n \leqslant 1/a$; $1/a \leqslant n \leqslant 2/a$ and $-n \leqslant m \leqslant n$; $-(2/a - n) \leqslant m \leqslant (2/a - n)$, respectively.

The Bethe-ansatz solvability of the 1D Hubbard model (1) is restricted to the Hilbert subspace spanned by the lowest-weight states (LWSs) or highest-weight states (HWSs) of the η-spin and spin algebras, i.e., by the states whose S_α and S^z_α numbers are such that $S_\alpha = -S^z_\alpha$ or $S_\alpha = S^z_\alpha$, respectively, where $\alpha = c$ for η-spin and $\alpha = s$ for spin. In this paper we choose the η-spin and spin LWSs description of the Bethe-ansatz solution. In this case, that solution describes energy eigenstates associated with densities and spin densities in the domains $0 \leqslant n \leqslant 1/a$ and $0 \leqslant m \leqslant n$, respectively. The description of the states corresponding to the extended n and m domains mentioned above is achieved by application onto the latter states of off-diagonal generators of the η-spin and spin $SU(2)$ algebras [11]. Recently, universal properties of the model (1) entropy were considered [12].

It is well known that the low-energy eigenstates of the model (1) can be described by occupancy configurations of holons and spinons [13–17]. On the other hand, in Ref. [18] it was found that all energy eigenstates associated with the 1D Hubbard model Bethe-ansatz solution [1,2] can be described in terms of occupancy configurations of pseudoparticles. According to the studies of Ref. [18] there is an infinite number of pseudoparticle branches: the c pseudoparticles and the $\alpha\nu$ pseudoparticles such that $\alpha = c, s$ and $\nu = 1, 2, \ldots$. (The α, γ pseudoparticle notation of Ref. [18] is such that $\gamma = \nu - \delta_{\alpha,s}$.) The thermodynamic coupled non-linear equations introduced by Takahashi [2] can be understood as describing a Landau liquid of such pseudoparticles. In Appendix A we relate the quantum numbers of the latter equations to the discrete pseudoparticle momentum values. In Appendix B we provide some aspects of the pseudoparticle description that are useful for the studies of this paper. The physical quantities can be expressed as functionals of the pseudoparticle momentum distribution functions whose occupancies describe the energy eigenstates. For instance, the η-spin value S_c, spin value S_s, and momentum P read

$$S_c = \frac{1}{2}\frac{L}{2\pi}\left[\int_{q_c^-}^{q_c^+} dq\,\left[1 - N_c(q)\right] - 2\sum_{\nu=1}^{\infty}\int_{-q_{c\nu}}^{q_{c\nu}} dq\,\nu N_{c\nu}(q)\right], \tag{9}$$

$$S_s = \frac{1}{2}\frac{L}{2\pi}\left[\int_{q_c^-}^{q_c^+} dq\,N_c(q) - 2\sum_{\nu=1}^{\infty}\int_{-q_{s\nu}}^{q_{s\nu}} dq\,\nu N_{s\nu}(q)\right], \tag{10}$$

and

$$P = \frac{L}{2\pi}\left\{\int_{q_c^-}^{q_c^+} dq\, N_c(q) q + \sum_{\nu=1}^{\infty}\int_{-q_{s\nu}}^{q_{s\nu}} dq\, N_{s\nu}(q) q + \sum_{\nu=1}^{\infty}\int_{-q_{c\nu}}^{q_{c\nu}} dq\, N_{c\nu}(q)\left[\frac{\pi}{a}(1+\nu) - q\right]\right\}, \tag{11}$$

respectively, where when $|P| > \pi/a$ the value of the momentum should be brought to the first Brillouin zone. The limiting pseudoparticle momentum values of these equations are given in Appendix B. Moreover, the energy spectrum associated with the Hamiltonian (3) can be written as

$$E_{SO(4)} = -2t\frac{L}{2\pi}\int_{q_c^-}^{q_c^+} dq\, N_c(q)\cos(k(q)a) + 4t\frac{L}{2\pi}\sum_{\nu=1}^{\infty}\int_{-q_{c\nu}}^{q_{c\nu}} dq\, N_{c\nu}(q)\,\mathrm{Re}\Big\{\sqrt{1-\big(\Lambda_{c\nu}(q) + i\nu U/4t\big)^2}\Big\} + \frac{L}{2\pi}\frac{U}{2}\left[\int_{q_c^-}^{q_c^+} dq\left[\frac{1}{2} - N_c(q)\right] - 2\sum_{\nu=1}^{\infty}\int_{-q_{c\nu}}^{q_{c\nu}} dq\, \nu N_{c\nu}(q)\right]. \tag{12}$$

Here $k(q)$ and $\Lambda_{\alpha\nu}(q)$ with $\alpha = c$ are the rapidity-momentum functional and the $\alpha\nu$ rapidity functional, respectively. These functionals are defined by the Takahashi's thermodynamic equations which can be rewritten in functional form as follows

$$k(q) = q - \frac{1}{\pi a}\sum_{\nu=1}^{\infty}\int_{-q_{s\nu}}^{q_{s\nu}} dq'\, N_{s\nu}(q')\arctan\left(\frac{\sin(k(q)a) - \Lambda_{s\nu}(q')}{\nu U/4t}\right) - \frac{1}{\pi a}\sum_{\nu=1}^{\infty}\int_{-q_{c\nu}}^{q_{c\nu}} dq'\, N_{c\nu}(q')\arctan\left(\frac{\sin(k(q)a) - \Lambda_{c\nu}(q')}{\nu U/4t}\right), \tag{13}$$

$$k_{c\nu}(q) = q + \frac{1}{\pi a}\int_{q_c^-}^{q_c^+} dq'\, N_c(q')\arctan\left(\frac{\Lambda_{c\nu}(q) - \sin(k(q')a)}{\nu U/4t}\right) + \frac{1}{2\pi a}\sum_{\nu'=1}^{\infty}\int_{-q_{c\nu'}}^{q_{c\nu'}} dq'\, N_{c\nu'}(q')\Theta_{\nu,\nu'}\left(\frac{\Lambda_{c\nu}(q) - \Lambda_{c\nu'}(q')}{U/4t}\right), \tag{14}$$

and

$$0 = q - \frac{1}{\pi a}\int_{q_c^-}^{q_c^+} dq'\, N_c(q')\arctan\left(\frac{\Lambda_{s\nu}(q) - \sin(k(q')a)}{\nu U/4t}\right) + \frac{1}{2\pi a}\sum_{\nu'=1}^{\infty}\int_{-q_{s\nu'}}^{q_{s\nu'}} dq'\, N_{s\nu'}(q')\Theta_{\nu,\nu'}\left(\frac{\Lambda_{s\nu}(q) - \Lambda_{s\nu'}(q')}{U/4t}\right). \tag{15}$$

Here

$$k_{c\nu}(q) = \frac{2}{a}\,\mathrm{Re}\big\{\arcsin(\Lambda_{c\nu}(q) + i\nu U/4t)\big\}, \tag{16}$$

is the $c\nu$ rapidity-momentum functional and the function $\Theta_{\nu,\nu'}(x)$ is defined in Eq. (B.5) of Appendix B. Eqs. (13)–(15) apply to all regular energy eigenstates. The ground state and the low-energy eigenstates involve occupancy configurations of the c and $s1$ pseudoparticle branches only [19]. The pseudoparticles have energy dispersions and residual interactions controlled by f functions, as the Fermi-liquid quasi-particles [10,18,19]. However, in contrast to such quasi-particles, the pseudoparticles do not carry the same charge and spin as the original electrons. The ground-state pseudoparticle momentum distribution functions and the pseudoparticle energy dispersions are presented in Appendix C. Often the low-energy excitations generated by the occupancy configurations of the c and $s1$ pseudoparticle branches are identified with holons and spinons, respectively [3,13,15].

In this paper we find that the missing link between the original electrons and the pseudoparticles whose occupancy configurations describe the energy eigenstates are the rotated electrons. Rotated electrons are related to the original electrons by a unitary transformation. Such a transformation was considered for large values of the on-site Coulombian repulsion U in Ref. [20]. Interestingly, we find that the above $c\nu$ and $s\nu$ pseudoparticles are composite 2ν-holon and 2ν-spinon quantum objects, respectively. This leads to a description where all energy eigenstates of the 1D Hubbard model can be described in terms of occupancy configurations of three elementary quantum objects: the spinon, the holon, and the c pseudoparticle. We find that such elementary objects and the rotated electrons are related. Within our general description the energy eigenstates which are not associated with the Bethe-ansatz solution, are described by holons and spinons that are not part of the composite $c\nu$ and $s\nu$ pseudoparticles, respectively. Interestingly, the latter holons and spinons are invariant under the electron–rotated-electron unitary transformation.

As the quarks are confined within the composite nucleons [21], the elementary spin $1/2$ spinons, η-spin $1/2$ holons, and c pseudoparticles are confined within the present many-electron system. Indeed, only electrons can be created to or annihilated from such a system. However, within the non-perturbative many-electron system the electronic degrees of freedom are organized in terms of the elementary spinons, holons, and c pseudoparticles. All energy eigenstates can be described as occupancy configurations of these three elementary quantum objects only.

In this paper we also address the important problem of the transport of charge and spin. We find that the electronic degrees of freedom couple to charge (and spin) probes through the holons and c pseudoparticles (and spinons). Those are the carriers of charge (and spin) of the model for *all* energy scales. The quantum-object description introduced in this paper is used in Ref. [22] in the construction of a theory for evaluation of finite-energy few-electron spectral functions. That theory is applied in Refs. [3,23] to the study of the finite-energy one-electron spectral properties of low-dimensional materials, as further discussed in Section 5.

The paper is organized as follows: in Section 2 we describe the non-perturbative organization of the electronic degrees of freedom. The confirmation of the validity of the holon, spinon, and c pseudoparticle description is the subject of Section 3. In Section 4 we study the problem of charge and spin transport. Finally, the summary and concluding remarks are presented in Section 5.

2. Non-perturbative organization of the electronic degrees of freedom

Each lattice site $j = 1, 2, \ldots, N_a$ of the model (1) can either be doubly occupied, empty, or singly occupied by a spin-down or spin-up electron. The maximum number of electrons is $2N_a$ and corresponds to density $n = 2/a$. Besides the N electrons, it is useful to consider $[2N_a - N]$ *electronic holes*. (Here we use the designation *electronic hole* instead of *hole*, in order to distinguish this type of hole from the *pseudoparticle hole*, which appears later in this paper.) Our definition of electronic hole is such that when a lattice site is empty, we say that it is occupied by two electronic holes. If a lattice site is singly occupied, we say that it is occupied by an electron and an electronic hole. If a lattice site is doubly occupied, it is unoccupied by electronic holes. Thus, in our language an empty site corresponds to an occupancy of two electronic holes at the same site j. We denote by N^h the total number of electronic holes

$$N^h = [2N_a - N]. \tag{17}$$

One can describe charge and spin transport either in terms of electrons or of electronic holes. In Section 4 we discuss these two alternative schemes. The studies of this paper use often the spin σ electron description. The eigenvalues of the diagonal generators (2) can be written in terms of the numbers N^h, N, $N_\uparrow$, and $N_\downarrow$ as follows

$$S_c^z = -\frac{1}{4}\left[N^h - N\right], \qquad S_s^z = -\frac{1}{2}[N_\uparrow - N_\downarrow]. \tag{18}$$

The pseudoparticle representation is extracted from the Bethe-ansatz solution, yet that solution provides no explicit information about the relation of these quantum objects to the original electrons. In this section we introduce a holon and spinon description which emerges from the electron–rotated-electron unitary transformation. In the ensuing section, we confirm the relation of rotated electrons to holons, spinons, and c pseudoparticles provided here.

2.1. The electron–rotated-electron unitary transformation

We start by presenting a brief description of the concept of rotated electron. The electron–rotated-electron unitary transformation maps the electrons onto rotated electrons such that rotated-electron double occupation, no occupation, and spin-up and spin-down single occupation are good quantum numbers for *all* values of U/t. Thus, there is at least a complete set of common energy and rotated-electron double occupation, no occupation, and spin-up and spin-down single occupation eigenstates. As mentioned in Section 1, until now a complete set of such states for all values of U/t was not found and this issue remains an interesting open question.

Let us introduce the electron–rotated-electron unitary transformation. We call $c_{j,\sigma}^{\dagger}$ the electrons that occur in the 1D Hubbard model (1) and (3), while the operator $\tilde{c}_{j,\sigma}^{\dagger}$ such that

$$\tilde{c}_{j,\sigma}^{\dagger} = \hat{V}^{\dagger}(U/t) c_{j,\sigma}^{\dagger} \hat{V}(U/t), \tag{19}$$

represents the rotated electrons, where the electron–rotated-electron unitary operator $\hat{V}(U/t)$ is defined below. Inversion of the relation (19) leads to $c_{j,\sigma}^{\dagger} = \hat{V}(U/t)\tilde{c}_{j,\sigma}^{\dagger} \times \hat{V}^{\dagger}(U/t)$. The rotated-electron double occupation operator is given by

$$\tilde{D} \equiv \hat{V}^{\dagger}(U/t)\hat{D}\hat{V}(U/t) = \sum_{j} \tilde{c}_{j,\uparrow}^{\dagger}\tilde{c}_{j,\uparrow}\tilde{c}_{j,\downarrow}^{\dagger}\tilde{c}_{j,\downarrow}, \tag{20}$$

where $\hat{D}$ is the electron double occupation operator given in Eq. (6). Note that $c_{j,\sigma}^{\dagger}$ and $\tilde{c}_{j,\sigma}^{\dagger}$ are only identical in the $U/t \to \infty$ limit where electron double occupation becomes a good quantum number.

The expression of any rotated operator $\tilde{O} = \hat{V}^{\dagger}(U/t)\hat{O}\hat{V}(U/t)$ in terms of the rotated electron operators $\tilde{c}_{j,\sigma}^{\dagger}$ and $\tilde{c}_{j,\sigma}$ is the same as the expression for the corresponding general operator $\hat{O} = \hat{V}(U/t)\tilde{O}\hat{V}^{\dagger}(U/t)$ in terms of the electron operators $c_{j,\sigma}^{\dagger}$ and $c_{j,\sigma}$, respectively. The operators $\hat{V}^{\dagger}(U/t)$ and $\hat{V}(U/t)$ associated with the electron–rotated-electron unitary transformation can be written as

$$\hat{V}^{\dagger}(U/t) = e^{-\hat{S}}, \qquad \hat{V}(U/t) = e^{\hat{S}}. \tag{21}$$

The operator $\hat{S}$ of Eq. (21) is uniquely defined by the following two equations

$$\tilde{H}_H = \hat{V}^{\dagger}(U/t)\hat{H}_H\hat{V}(U/t) = \hat{H}_H + [\hat{H}_H, \hat{S}] + \frac{1}{2}\big[[\hat{H}_H, \hat{S}], \hat{S}\big] + \cdots, \tag{22}$$

and

$$\big[\hat{H}_H, \hat{V}^{\dagger}(U/t)\hat{D}\hat{V}(U/t)\big] = [\hat{H}_H, \tilde{D}] = 0, \tag{23}$$

where the Hamiltonian $\hat{H}_H$ and the rotated-electron double occupation operator $\tilde{D}$ are given in Eqs. (3) and (20), respectively.

The transformation associated with the electron–rotated-electron unitary operator $\hat{V}(U/t)$ was introduced in Ref. [20]. The studies of that reference referred to large values of U/t and did not clarify for arbitrary values of U/t the relation of rotated-electron double occupation to the quantum numbers provided by the Bethe-ansatz solution. However, this transformation is uniquely defined for all values of U/t by Eqs. (21)–(23). Eqs. (22) and (23) can be used to derive an expression for the unitary operator order by order in t/U. The authors of the second paper of Ref. [20] carried out this expansion up to eighth order (see Ref. [12]).

2.2. The holon, spinon, and c pseudoparticle fluids

One should distinguish the total η spin (and spin) value, which we denote by S_c (and S_s) and the corresponding η-spin (and spin) projection, which we denote by S_c^z (and S_s^z) from the η-spin (and spin) carried by the elementary quantum objects. We call s_c (and s_s) the η-spin (and spin) carried by the holons, spinons, and other elementary objects and σ_c (and σ_s) their η-spin (and spin) projection. The operators $\hat{M}_{c,\sigma_c}$ and $\hat{M}_{s,\sigma_s}$, which count the number of the $\sigma_c = \pm 1/2$ holons and $\sigma_s = \pm 1/2$ spinons, have the following form:

$$\hat{M}_{c,-1/2} = \tilde{R}_{c,-1} = \hat{V}^\dagger(U/t) \sum_j c_{j\uparrow}^\dagger c_{j\uparrow} c_{j\downarrow}^\dagger c_{j\downarrow} \hat{V}(U/t),$$
$$\hat{M}_{c,+1/2} = \tilde{R}_{c,+1} = \hat{V}^\dagger(U/t) \sum_j c_{j\uparrow} c_{j\uparrow}^\dagger c_{j\downarrow} c_{j\downarrow}^\dagger \hat{V}(U/t),$$
$$\hat{M}_{s,-1/2} = \tilde{R}_{s,-1} = \hat{V}^\dagger(U/t) \sum_j c_{j\downarrow}^\dagger c_{j\uparrow} c_{j\uparrow}^\dagger c_{j\downarrow} \hat{V}(U/t),$$
$$\hat{M}_{s,+1/2} = \tilde{R}_{s,+1} = \hat{V}^\dagger(U/t) \sum_j c_{j\uparrow}^\dagger c_{j\downarrow} c_{j\downarrow}^\dagger c_{j\uparrow} \hat{V}(U/t). \tag{24}$$

Here $\hat{V}(U/t)$ is the electron–rotated-electron unitary operator uniquely defined for all values of U/t by Eqs. (21)–(23) and the operators

$$\hat{R}_{c,-1} \equiv \hat{D} = \sum_j c_{j,\uparrow}^\dagger c_{j,\uparrow} c_{j,\downarrow}^\dagger c_{j,\downarrow}, \qquad \hat{R}_{c,+1} = \sum_j c_{j,\uparrow} c_{j,\uparrow}^\dagger c_{j,\downarrow} c_{j,\downarrow}^\dagger, \tag{25}$$

and

$$\hat{R}_{s,-1} = \sum_j c_{j,\downarrow}^\dagger c_{j\uparrow} c_{j,\uparrow}^\dagger c_{j,\downarrow}, \qquad \hat{R}_{s,+1} = \sum_j c_{j,\uparrow}^\dagger c_{j,\downarrow} c_{j,\downarrow}^\dagger c_{j,\uparrow}, \tag{26}$$

are such that $\hat{R}_{c,-1}$ counts the number of electron doubly-occupied sites, $\hat{R}_{c,+1}$ counts the number of electron empty sites, $\hat{R}_{s,-1}$ counts the number of electron down-spin singly-occupied sites, and $\hat{R}_{s,+1}$ counts the number of electron up-spin singly-occupied sites.

The new physics brought about by the relations of Eq. (24) is that the Hilbert-space rotation performed by the unitary operator $\hat{V}(U/t)$ produces new exotic quantum objects whose occupancy configurations describe the exact energy eigenstates. The unitary rotation is such that for all values of U/t the number of emerging $-1/2$ holons, $+1/2$ holons, $-1/2$ spinons, and $+1/2$ spinons equals precisely the number of rotated-electron doubly occupied sites, empty sites, spin-down singly occupied sites, and spin-up singly occupied sites, respectively. In Section 3 we confirm that the $\pm 1/2$ holon and $\pm 1/2$ spinon numbers introduced in Eq. (24) are consistent with the numbers of c pseudoparticles and $\alpha\nu$ pseudoparticles where $\alpha = c, s$ and $\nu = 1, 2, \ldots$ obtained from the Bethe-ansatz solution.

The first step of the organization of the electronic degrees of freedom which arises from the non-perturbative effects of the many-electron interactions is the Hilbert-space rotation that maps electrons onto rotated electrons. This involves a decoupling of the N-electron system in a N_c-rotated-electron and a $[N - N_c]$-rotated-electron fluids. (The number of electrons equals the number of rotated electrons.) On the one hand, N_c is nothing but the number of c pseudoparticles introduced in Appendix B from Takahashi's thermodynamic Bethe-ansatz equations. On the other hand, we identify the number N_c with the number of rotated-electron singly occupied sites. It follows that the number $[N - N_c]/2$ equals the number of rotated-electron doubly occupied sites. This is consistent with the fact that $[N - N_c]$ is always an even number. Moreover, the N^h-electronic hole system decouples into a N_c-rotated-electronic hole and a $[N^h - N_c]$-rotated-electronic hole fluids, where the number $[N^h - N_c]$ is also even. The number N_c also equals the number of rotated-electronic holes associated with the N_c rotated-electron singly occupied sites. Furthermore, the above $[N^h - N_c]$-rotated-electronic hole fluid corresponds to the $[N^h - N_c]/2$ rotated-electron empty sites. The N_c-rotated-electron (and N_c-rotated-electronic hole) fluid corresponds to N_c *unpaired* rotated electrons (and rotated electronic holes), whereas the $[N - N_c]$-rotated-electron fluid (and $[N^h - N_c]$-rotated-electronic hole fluid) is described by $[N - N_c]/2$ (and $[N^h - N_c]/2$) quantum objects described by on-site pairs of rotated electrons (and on-site pairs of rotated-electronic holes).

The $[N - N_c]$-rotated-electron fluid involves $[N - N_c]/2$ spin-down rotated electrons and an equal number of spin-up rotated electrons which *pair* and form $[N - N_c]/2$ spin zero singlet rotated-electron on-site pairs associated with the doubly occupied sites. Each of these rotated-electron on-site pairs has $s_c = 1/2$, $s_s = 0$, and $\sigma_c = -1/2$ and is identified with a $\sigma_c = -1/2$ holon or, simply, a $-1/2$ holon. Thus, the $-1/2$ holons correspond to the rotated-electron doubly occupied sites, consistently to Eqs. (24)–(26). Moreover, the two rotated-electronic holes of the $[N^h - N_c]$-rotated-electronic hole fluid are also *paired* in the same empty side. Each of these $[N^h - N_c]/2$ rotated-electronic hole pairs has $s_c = 1/2$, $s_s = 0$, and $\sigma_c = +1/2$ and is called a $\sigma_c = +1/2$ holon or, simply, a $+1/2$ holon. Such $\sigma_c = +1/2$ holons correspond to the rotated-electron empty sites, again consistently with the relations given in Eqs. (24)–(26). We emphasize that the total number $[N - N_c]/2 + [N^h - N_c]/2 = [N_a - N_c]$ of holons equals the number $N_c^h = [N_a - N_c]$ of c pseudoparticle holes, as discussed below.

The N_c-rotated-electron fluid is characterized by an exotic separation of the spin and charge degrees of freedom. Such a decoupling leads to a N_c spinon fluid and a N_c c pseudoparticle fluid. The c pseudoparticle has no spin degrees of freedom. It corresponds to the charge part of a rotated-electron singly occupied site. There is a *local c pseudoparticle* description [22] associated with the momentum c pseudoparticle representation obtained from the Bethe-ansatz solution in Appendix B. It is the local c pseudoparticle which corresponds to the charge part of a rotated-electron singly occupied site. The local and momentum pseudoparticle descriptions are related by a simple Fourier transform [22]. We call the charge part of the rotated electron of such a site a *chargeon*. In case of charge transport in terms of electronic holes, one should introduce the *antichargeon*. This is the charge part of the rotated-electronic hole of a singly occupied site. Furthermore, the spin-projection $\pm 1/2$ spinon corresponds to the spin part of a spin-projection $\pm 1/2$ rotated-electron singly occupied site. The c pseudoparticle excitations describe the charge motion of the rotated-electron singly occupied sites relative to the rotated-electron doubly-occupied and empty sites. Moreover, the N_c-rotated-electronic hole fluid corresponds to the N_c antichargeons. Thus, the c pseudoparticle includes a chargeon and an antichargeon. However, we emphasize that in what charge transport is concerned, the chargeon and antichargeon correspond to alternative descriptions of the c pseudoparticle, as discussed in Section 4. Also spin transport can either be described in terms of electrons or electronic holes, as discussed in the same section. In general we consider in this paper the description of charge and spin transport in terms of electrons. For electron spin transport, the number of spinons with spin projections $\sigma_s = +1/2$ and $\sigma_s = -1/2$ equals the number of spin-up rotated electrons $[N_c + N_\uparrow - N_\downarrow]/2$ and the number of spin-down rotated electrons $[N_c + N_\downarrow - N_\uparrow]/2$ in the N_c-rotated-electron fluid, respectively. The numbers $[N_c + N_\uparrow - N_\downarrow]/2$ and $[N_c + N_\downarrow - N_\uparrow]/2$ also equal the numbers of singly occupied sites occupied by spin-up and spin-down rotated electrons, respectively.

Let us summarize the above expressions for the numbers of the elementary quantum objects in a few equations. Consistently with the notation used in the relations given in Eq. (24), we call M_{c,σ_c} the number of σ_c holons and the total number of holons is denoted by $M_c = \sum_{\sigma_c} M_{c,\sigma_c}$. These values read

$$M_{c,-1/2} = \frac{1}{2}[N - N_c], \qquad M_{c,+1/2} = \frac{1}{2}\big[N^h - N_c\big],$$
$$M_c = \frac{1}{2}\big[N^h + N\big] - N_c = [N_a - N_c]. \tag{27}$$

On the other hand, we call M_{s,σ_s} the number of σ_s spinons and $M_s = \sum_{\sigma_s} M_{s,\sigma_s}$ the total number of spinons. These numbers are given by

$$M_{s,-1/2} = \frac{1}{2}[N_c - N_\uparrow + N_\downarrow], \qquad M_{s,-1/2} = \frac{1}{2}[N_c + N_\uparrow - N_\downarrow],$$
$$M_s = N_c. \tag{28}$$

The numbers M_{c,σ_c}, M_{s,σ_s}, and N_c of the elementary quantum objects are good quantum numbers.

The following relations between the numbers of $\pm 1/2$ holons, $\pm 1/2$ spinons, and c pseudoparticles are valid for the whole Hilbert space of the model

$$M_c = \sum_{\sigma_c = \pm 1/2} M_{c,\sigma_c} = N_c^h = [N_a - N_c],$$
$$M_s = \sum_{\sigma_s = \pm 1/2} M_{s,\sigma_s} = N_c, \qquad \sum_{\alpha = c,s} M_\alpha = N_a. \tag{29}$$

The total number of holons plus the total number of spinons equals the number of lattice sites N_a. This result follows from the one-to-one correspondence between the holon and spinon numbers and the numbers of rotated electrons. Since N_a is even, such a sum rule implies that the symmetry of the 1D Hubbard model is $SO(4)$, consistently with the results of Refs. [8,9]. Indeed, half of the irreducible representations of $SU(2) \times SU(2)$ are excluded by the constraint imposed by such a sum rule: states where the total number of holons plus spinons is odd are excluded.

2.3. *The 2ν-holon and 2ν-spinon composite pseudoparticles*

The $c\nu$ pseudoparticles (and $s\nu$ pseudoparticles) obtained from the Bethe-ansatz solution in Appendix B and whose energy bands are given in Appendix C are 2ν-holon (and

2ν-spinon) composite objects. Such composite pseudoparticles involve an equal number $\nu = 1, 2, \ldots$ of $+1/2$ holons and $-1/2$ holons (and $+1/2$ spinons and $-1/2$ spinons). The composite $c\nu$ pseudoparticle has $s_c = 0$ and $s_s = 0$ and the composite $s\nu$ pseudoparticle has $s_s = 0$ and no charge degrees of freedom.

As for the case of the c pseudoparticles, there is a *local $\alpha\nu$ pseudoparticle* description associated with the momentum $\alpha\nu$ pseudoparticle representation obtained from the Bethe-ansatz solution in Appendix B [22]. The local $c\nu$ pseudoparticle (and $s\nu$ pseudoparticle) is a 2ν-holon (and 2ν-spinon) composite quantum object of ν $-1/2$ holons and ν $+1/2$ holons (and ν $-1/2$ spinons and ν $+1/2$ spinons) associated with ν rotated-electron doubly-occupied and empty sites (and ν spin-down and spin-up rotated-electron singly occupied sites), respectively. The local and momentum $\alpha\nu$ pseudoparticle descriptions are related by a simple Fourier transform [22]. The momentum $c\nu$ pseudoparticle (and $s\nu$ pseudoparticle) excitations are associated with hoping rotated-electron processes within the doubly-occupied and empty site (and singly-occupied site) subsystem. The momentum of these excitations corresponds to the third term (and second term) of Eq. (11). On the other hand, the momentum c pseudoparticle excitations correspond to relative movements of the rotated electrons associated with these two subsystems. These excitations conserve both the rotated-electron internal doubly-occupied and empty site occupancy configurations and the singly-occupied site occupancy configurations. The momentum of these excitations corresponds to the first term of Eq. (11).

The numbers of $\pm 1/2$ holons ($\alpha = c$) and of $\pm 1/2$ spinons ($\alpha = s$) can be written as

$$M_{\alpha,\pm 1/2} = L_{\alpha,\pm 1/2} + \sum_{\nu=1}^{\infty} \nu N_{\alpha\nu}, \quad \alpha = c, s, \tag{30}$$

where the term $\sum_{\nu=1}^{\infty} \nu N_{\alpha\nu}$ on the right-hand side of this equation gives the number of $\pm 1/2$ holons ($\alpha = c$) or of $\pm 1/2$ spinons ($\alpha = s$) that are part of the 2ν-holon composite $c\nu$ pseudoparticles or 2ν-spinon composite $s\nu$ pseudoparticles, respectively. On the right-hand side of Eq. (30) $N_{\alpha\nu}$ gives the corresponding number of $\alpha\nu$ pseudoparticles and

$$L_{\alpha,\sigma_\alpha} = S_\alpha - 2\sigma_\alpha S_\alpha^z = L_\alpha/2 - 2\sigma_\alpha S_\alpha^z, \quad \alpha = c, s, \tag{31}$$

is the number of $\sigma_c = \pm 1/2$ holons ($\alpha = c$) or of $\sigma_s = \pm 1/2$ spinons ($\alpha = s$) which are not part of composite $c\nu$ or $s\nu$ pseudoparticles, respectively.

Interestingly, the spin singlet and composite two-spinon character of the $s1$ pseudoparticles reveals that the quantum spin fluid described by these quantum objects is the 1D realization of the two-dimensional *resonating valence bond* (RVB) spin fluid [13]. In Section 4 we find that the charge (and spin) quantum fluid described by the c pseudoparticles and $c\nu$ pseudoparticles (and $s\nu$ pseudoparticles) plays a central role in the charge (and spin) transport properties. Indeed, the holons (and spinons) which are not part of $c\nu$ pseudoparticles (and $s\nu$ pseudoparticles) are localized quantum objects and do not contribute to charge (and spin) transport [10]. In the limit $U/t \to \infty$ both the $c\nu$ pseudoparticles and $s\nu$ pseudoparticles also become localized quantum objects and only the c pseudoparticles contribute to transport of charge [10]. In this limit the c pseudoparticles are related to the spin-less fermions of Ref. [15].

2.4. The Yang holons, HL spinons, and the corresponding two $SU(2)$ algebras

We call $\pm 1/2$ *Yang holons* (and $\pm 1/2$ HL *spinons*) the $\pm 1/2$ holons (and $\pm 1/2$ spinons) which are not part of composite $c\nu$ pseudoparticles (and $s\nu$ pseudoparticles) and whose numbers are given by Eq. (31) with $\alpha = c$ (and $\alpha = s$). In the case of the HL spinons, HL stands for Heilmann and Lieb who are the authors of Ref. [8], whereas in the case of the Yang holons, Yang stands for Yang who is the author of Ref. [9]. In these references it was found that in addition to the $SU(2)$ spin symmetry, the Hubbard model has a $SU(2)$ η-spin symmetry.

An important physical property of the $\pm 1/2$ Yang holons and $\pm 1/2$ HL spinons is that these quantum objects remain invariant under the electron–rotated-electron unitary transformation for all values of U/t. This is confirmed by the studies of Ref. [10], where it is found that $\pm 1/2$ Yang holons (and $\pm 1/2$ HL spinons) refer to electron doubly-occupied sites (and spin $\pm 1/2$ electron singly occupied sites). Although for finite values of U/t electron double occupation is not a good quantum number, these particular quantum objects have a local character such that these concepts apply to their energy-eigenstate site occupancy configurations. It follows that the Yang holon (and HL spinon) and the rotated Yang holon (and rotated HL spinon) are the same quantum object for all values of U/t. This is in contrast to the pseudoparticles which become invariant under the electron–rotated-electron unitary transformation as $U/t \to \infty$ only [10]. Otherwise, the pseudoparticle and the rotated pseudoparticle are in general different quantum objects.

The Yang holons (and HL spinons) also have a very different behavior relative to holons (and spinons) which are part of composite $c\nu$ pseudoparticles (and $s\nu$ pseudoparticles) in what the η-spin (and spin) $SU(2)$ algebra is concerned. Indeed, the designations of Yang holons and HL spinons is justified by the fact that out of a total number $M_{\alpha,\pm 1/2}$ of $\pm 1/2$ holons ($\alpha = c$) or $\pm 1/2$ spinons ($\alpha = s$), only the $L_{c,\pm 1/2}$ $\pm 1/2$ holons or $L_{s,\pm 1/2}$ $\pm 1/2$ spinons which are not part of $\alpha\nu$ pseudoparticles, are sensitive to the application of the η-spin ($\alpha = c$) and spin ($\alpha = s$) off-diagonal generators, respectively. This results from the fact that the composite $\alpha\nu$ pseudoparticles are η-spin ($\alpha = c$) and spin ($\alpha = s$) singlet combinations of 2ν holons ($\alpha = c$) and 2ν spinons ($\alpha = s$) and thus have $s_\alpha = 0$, i.e., have zero η spin ($\alpha = c$) or zero spin ($\alpha = s$).

The off-diagonal generators of the η-spin algebra, Eq. (7), transform $-1/2$ Yang holons into $+1/2$ Yang holons and vice versa. Also the off-diagonal generators of the spin algebra, Eq. (8), transform $-1/2$ HL spinons into $+1/2$ HL spinons and vice versa. Thus, these generators produce η-spin and spin flips, respectively. Application of $\hat{S}_c^\dagger$ (or $\hat{S}_s^\dagger$) produces a η-spin flip (or spin flip) which transforms a $+1/2$ Yang holon (or $+1/2$ HL spinon) into a $-1/2$ Yang holon (or $-1/2$ HL spinon). Application of $\hat{S}_c$ (or $\hat{S}_s$) produces a η-spin flip (or spin flip) which transforms a $-1/2$ Yang holon (or $-1/2$ HL spinon) into a $+1/2$ Yang holon (or $+1/2$ HL spinon). The LWSs of the η-spin (or spin) algebra, have zero occupancies of $-1/2$ Yang holons (or of $-1/2$ HL spinons). Similarly, the HWSs of the η-spin (or spin) algebra have zero occupancies of $+1/2$ Yang holons (of $+1/2$ HL spinons). Our description corresponds to the whole Hilbert space of the 1D Hubbard model, and includes states with both $-1/2$ and $+1/2$ Yang holon finite occupancies and with both $-1/2$ and $+1/2$ HL spinon finite occupancies.

The electron–rotated-electron unitary operator $\hat{V}(U/t)$ commutes with all three generators of the η-spin (and spin) algebra. It is this symmetry that is behind the invariance under the same transformation of the corresponding holons (and spinons) which are sensitive to these generators. However, there is no such a symmetry requirement for the holons (and spinons) whose arrangements are insensitive to the application of the same generators. This analysis is consistent with the transformation laws of the associated composite $c\nu$ pseudoparticles (and $s\nu$ pseudoparticles) which are not invariant under the electron–rotated-electron unitary transformation. Such an interplay of the electron–rotated-electron unitary transformation and η-spin and spin $SU(2)$ symmetries is important for the spectral properties [22]. Creation and annihilation elementary operators can be introduced for the $\alpha\nu$ pseudoparticles [22]. Consistently with the above separation, these operators commute with the off-diagonal generators of the η-spin and spin algebras.

From Eqs. (30) and (31), the total number of holons ($\alpha = c$) and spinons ($\alpha = s$) given in Eq. (29) can be expressed as

$$M_\alpha = 2S_\alpha + \sum_{\nu=1}^{\infty} 2\nu N_{\alpha\nu} = L_\alpha + \sum_{\nu=1}^{\infty} 2\nu N_{\alpha\nu}, \quad \alpha = c, s, \tag{32}$$

where

$$L_\alpha = \sum_{\sigma_\alpha=\pm 1/2} L_{\alpha,\sigma_\alpha} = 2S_\alpha, \quad \alpha = c, s, \tag{33}$$

is for $\alpha = c$ (and $\alpha = s$) the total number of Yang holons (and HL spinons). Note that Eq. (33) is consistent with the application of the off-diagonal generators of the η-spin and spin algebras producing Yang holon η-spin flips and HL spinon flips, respectively. Let us consider a regular energy eigenstate. Since in such a state there are $2S_c$ $+1/2$ Yang holons and $2S_s$ $+1/2$ HL spinons, one can apply the generator $\hat{S}_c^\dagger$ $2S_c$ times and the generator $\hat{S}_s^\dagger$ $2S_s$ times. This leads to η-spin and spin towers with $2S_c + 1$ and $2S_s + 1$ states, respectively. Therefore, from each LWS of the S_α algebra, one can generate $2S_\alpha$ non-LWSs of that algebra, which is the correct result.

We emphasize that the property that the generators of the η-spin and spin $SU(2)$ algebra commute with the c pseudoparticle and composite $\alpha\nu$ pseudoparticle creation and annihilation operators has the following important effect: all $2S_\alpha$ energy eigenstates obtained from a given regular energy eigenstate have the same pseudoparticle momentum distribution functions $N_c(q)$ and $N_{\alpha\nu}(q)$ for all branches $\alpha = c, s$ and $\nu = 1, 2, \ldots$. Thus, these states are described by similar pseudoparticle occupancy configurations and only differ in the relative numbers of $+1/2$ Yang holons and $-1/2$ Yang holons ($\alpha = c$) or/and in the relative numbers of $+1/2$ HL spinons and $-1/2$ HL spinons ($\alpha = s$). Thus, the set of $2S_\alpha$ non-regular energy eigenstates belonging to the same tower of states differ only in the values of the numbers $L_{\alpha,-1/2}$ and $L_{\alpha,+1/2}$ such that $L_{\alpha,-1/2} + L_{\alpha,+1/2} = 2S_\alpha$, yet the total number of holons and spinons is the same for all of them. Importantly, this confirms that the coupled functional equations (13)–(15), which involve the pseudoparticle momentum distribution functions and do not depend on the L_{α,σ_α} numbers, describe both regular and non-regular energy eigenstates. The values of the rapidity-momentum functional $k(q)$ and rapidity functionals $\Lambda_{\alpha\nu}(q)$ are the same for all $2S_\alpha + 1$ states in the same tower. Such functionals are eigenvalues of operators which commute with the off-diagonal generators of η-spin and spin algebras. This is consistent with the η-spin and spin $SU(2)$ symmetries, which imply that the Hamiltonian commutes with these generators and thus the energy (12) is the same for the set of $2S_\alpha + 1$ states belonging to the same η-spin ($\alpha = c$) or spin ($\alpha = s$) tower. We recall that such an energy expression is fully determined by the values of the rapidity-momentum and rapidity functionals. These results reveal that the holon and spinon description introduced in this paper permits the extension of the functional equations (13)–(15) to the whole Hilbert space of the 1D Hubbard model.

From the use of Eqs. (30) and (31) we find that the η-spin value S_c, spin value S_s, and the corresponding projections S_c^z and S_s^z, respectively, can be expressed in terms of the numbers of $\pm 1/2$ Yang holons and of $\pm 1/2$ HL spinons as follows

$$S_\alpha = \frac{1}{2}[L_{\alpha,+1/2} + L_{\alpha,-1/2}] = \frac{1}{2}M_\alpha - \sum_{\nu=1}^{\infty} \nu N_{\alpha\nu}, \quad \alpha = c, s, \tag{34}$$

$$\begin{aligned} S_c^z &= -\frac{1}{2}[L_{c,+1/2} - L_{c,-1/2}] = -\frac{1}{2}[M_{c,+1/2} - M_{c,-1/2}], \\ S_s^z &= -\frac{1}{2}[L_{s,+1/2} - L_{s,-1/2}] = -\frac{1}{2}[M_{s,+1/2} - M_{s,-1/2}]. \end{aligned} \tag{35}$$

On the right-hand side of Eqs. (34) and (35) S_c, S_s, S_c^z, and S_s^z were also expressed in terms of the numbers of holons, spinons, and pseudoparticles.

The η-spin and spin value expressions (9), (10), and (34) and the energy expression (12) are valid for the whole Hilbert space. In contrast, we note that the momentum expression (11) refers only to LWSs of the η-spin and spin $SU(2)$ algebras. Generalization of that momentum expression to the whole Hilbert space leads to

$$\begin{aligned} P &= \frac{L}{2\pi}\int_{q_c^-}^{q_c^+} dq\, N_c(q)q + \frac{L}{2\pi}\sum_{\nu=1}^{\infty}\int_{-q_{s\nu}}^{q_{s\nu}} dq\, N_{s\nu}(q)q \\ &\quad + \frac{L}{2\pi}\sum_{\nu=1}^{\infty}\int_{-q_{c\nu}}^{q_{c\nu}} dq\, N_{c\nu}(q)\left[\frac{\pi}{a} - q\right] + \frac{\pi}{a}M_{c,-1/2}. \end{aligned} \tag{36}$$

Here the contributions from the c pseudoparticles, spinons (in terms of composite $s\nu$ pseudoparticles), and holons lead to different terms. Note that the term $[\pi/a]M_{c,-1/2}$ is consistent with creation of a $-1/2$ Yang holon requiring momentum π/a. The value of the $-1/2$ Yang holon momentum follows from the form of the η-spin off-diagonal generators defined in Eq. (7). There is an additional contribution of momentum $[\pi/a]\nu$ to the term $[\pi/a]M_{c,-1/2}$ and of momentum $[\pi/a - q]$ to the third term of Eq. (36) for each 2ν-holon composite $c\nu$ pseudoparticle.

3. Confirmation of the holon, spinon, and *c* pseudoparticle description

The goals of this section are to confirm: (1) the validity of the holon and spinon number operator expressions given in Eq. (24); (2) that the $\alpha\nu$ pseudoparticle representation provided by the Bethe-ansatz solution leads to the same complete description of the LWSs as the $\pm 1/2$ holons and $\pm 1/2$ spinons that are not invariant under the electron–rotated-electron unitary transformation.

The concept of CPHS ensemble subspace where CPHS stands for c pseudoparticle, holon, and spinon [10] is used in this section. This is a Hilbert subspace spanned by all states with fixed values for the $-1/2$ Yang holon number $L_{c,-1/2}$, $-1/2$ HL spinon number $L_{s,-1/2}$, c pseudoparticle number N_c, and for the sets of $\alpha\nu$ pseudoparticle numbers $\{N_{c\nu}\}$ and $\{N_{s\nu}\}$ corresponding to the $\nu = 1, 2, \ldots$ branches.

3.1. Relation of the holons, spinons, and c pseudoparticles to rotated electrons

Let us consider the four expectation values $R_{\alpha,l_\alpha} = \langle \hat{R}_{\alpha,l_\alpha} \rangle$, where $\alpha = c, s$ and $l_\alpha = -1, +1$. The corresponding operator $\hat{R}_{c,-1}$ counts the number of electron doubly-occupied sites, $\hat{R}_{c,+1}$ counts the number of electron empty sites, $\hat{R}_{s,-1}$ counts the number of spin-down electron singly-occupied sites, and $\hat{R}_{s,+1}$ counts the number of spin-up electron singly-occupied sites. These operators are given in Eqs. (25) and (26) and obey the relations (56) and (57) of Ref. [10]. These operational relations are valid for the whole parameter space and reveal that in a given electronic ensemble space, out of the four expectation values of the operators of Eqs. (25) and (26), only one is independent. The electron double-occupation expectation value $D \equiv R_{c,-1}$ was studied in Ref. [10]. The use of the operational Eqs. (56) and (57) of the same reference provides the corresponding values for $R_{c,+1}$, $R_{s,-1}$, and $R_{s,+1}$.

To start with we consider that the $\sigma_c = \pm 1/2$ holon and $\sigma_s = \pm 1/2$ spinon numbers M_{α,σ_α} are unrelated to the operators given in Eq. (24). These numbers are here defined by Eq. (30) in terms of the Bethe-ansatz pseudoparticle numbers $N_{\alpha\nu}$ and η-spin ($\alpha = c$) and spin ($\alpha = s$) numbers S_α and S_α^z (31). Since the four numbers $\{M_{\alpha,\sigma_\alpha}\}$ where $\alpha = c, s$ and $\sigma_\alpha = \pm 1/2$ are good quantum numbers, they are considered as eigenvalues of corresponding $\sigma_c = \pm 1/2$ holon and $\sigma_s = \pm 1/2$ spinon number operators $\hat{M}_{\alpha,\sigma_\alpha}$ which commute with the Hamiltonian $\hat{H}$ of Eq. (1) and whose expression in terms of electronic operators is unknown. Our goal is to show that such number operators are given by the expressions of Eq. (24).

Also the pseudoparticle numbers are good quantum numbers and can be associated with corresponding pseudoparticle number operators [22]. Thus, the $\sigma_c = \pm 1/2$ holon and $\sigma_s = \pm 1/2$ spinon number M_{α,σ_α}, the c-pseudoparticle number $N_c(q)$ at momentum q, and the set of $\alpha\nu$ pseudoparticle numbers $\{N_{\alpha\nu}(q)\}$ at momentum q where $\alpha = c, s$ and $\nu = 1, 2, \ldots$ are eigenvalues of corresponding number operators $\hat{M}_{\alpha,\sigma_\alpha}$, $\hat{N}_c(q)$, and set of operators $\{\hat{N}_{\alpha\nu}(q)\}$, respectively. All these number operators have unknown expressions in terms of creation and annihilation electronic operators. Together with the Hamiltonian $\hat{H}$ of Eq. (1), the set of number operators $\{\hat{M}_{\alpha,-1/2}\}$, $\hat{N}_c(q)$, and $\{\hat{N}_{\alpha\nu}(q)\}$ where $\alpha = c, s$ and $\nu = 1, 2, \ldots$ constitute a complete set of compatible and commuting hermitian operators. An alternative complete set of compatible operators results from replacing the two $-1/2$ holon and $-1/2$ spinon operators $\hat{M}_{c,-1/2}$ and $\hat{M}_{s,-1/2}$, respectively, by the two $-1/2$ Yang-holon and $-1/2$ HL-spinon operators $\hat{L}_{c,-1/2}$ and $\hat{L}_{s,-1/2}$, respectively, in the above set. The eigenvalues of the operators belonging to these complete sets label each of the 4^{N_a} energy eigenstates which form a complete set of states for the 1D Hubbard model. The set of operators $\{\hat{L}_{\alpha,-1/2}\}$, $\hat{N}_c(q)$, and $\{\hat{N}_{\alpha\nu}(q)\}$ can be expressed in terms of elementary creation and annihilation operators for the c pseudoparticles, $\alpha\nu$ pseudoparticles, $-1/2$ Yang holons, and $-1/2$ HL spinons [22]. However, the expression for the set of pseudoparticle momentum distribution operators $\hat{N}_c(q)$ and $\hat{N}_{\alpha\nu}(q)$ in terms of creation and annihilation electronic operators is a complex problem. Indeed, the results of Ref. [10] reveal that the corresponding pseudoparticle excitations are described by complex U/t dependent electronic occupancy configurations. Also the expression of the holon ($\alpha = c$) and spinon ($\alpha = s$) number operators $\{\hat{M}_{\alpha,\sigma_\alpha}\}$ in terms of creation and annihilation electronic operators until now has been an unsolved problem. Below, we solve this latter problem by showing that such holon and spinon number operator expressions are those given in Eq. (24).

From Eq. (29), we find that the $M_{c,+1/2}$ and $M_{s,+1/2}$ holon numbers are exclusive functions of the N_c c-pseudoparticle number and $M_{c,-1/2}$ holon and $M_{s,-1/2}$ spinon numbers and can be written as $M_{c,+1/2} = N_a - N_c - M_{c,-1/2}$ and $M_{s,+1/2} = N_c - M_{s,-1/2}$. By construction, the number of spin σ electrons can be expressed in terms of the numbers on the right-hand side of these equations as $N_\uparrow = N_c + M_{c,-1/2} - M_{s,-1/2}$ and $N_\downarrow = M_{c,-1/2} + M_{s,-1/2}$. Based on these expressions we find that the $\pm 1/2$ holon and $\pm 1/2$ spinon number operators must obey the following relations

$$\begin{aligned} &\hat{M}_{c,+1/2} = N_a - \hat{N} + \hat{M}_{c,-1/2}, \qquad \hat{M}_{s,-1/2} = \hat{N}_\downarrow - \hat{M}_{c,-1/2}, \\ &\hat{M}_{s,+1/2} = \hat{N}_\uparrow - \hat{M}_{c,-1/2}, \qquad \sum_{\alpha=c,s} \sum_{\sigma_\alpha=\pm 1/2} \hat{M}_{\alpha,\sigma_\alpha} = N_a. \end{aligned} \tag{37}$$

The relations (37) reveal that for given spin σ electron numbers, out of the four eigenvalues $M_{c,-1/2}$, $M_{c,+1/2}$, $M_{s,-1/2}$, and $M_{s,+1/2}$ only one is independent. Usually we consider the value of the number $M_{c,-1/2}$ of $-1/2$ holons and evaluate the corresponding values of $M_{c,+1/2}$, $M_{s,-1/2}$, and $M_{s,+1/2}$ from the relations of Eq. (37). Moreover, since for given N_c, $M_{c,-1/2}$, and $M_{s,-1/2}$, the values of the numbers $M_{c,+1/2}$ and $M_{s,+1/2}$ are dependent, in general we do not consider the latter two numbers nor corresponding number operators.

From the comparison of the three operational relations of Eq. (56) of Ref. [10] with the first three operational relations of Eq. (37), we find that there is a one-to-one correspondence between the four operators $\{\hat{R}_{\alpha,l_\alpha}\}$ and the four operators $\{\hat{M}_{\alpha,\sigma_\alpha}\}$. The similarity of relation (56) of Ref. [10] and relation (37) is a clear indication that the four operators $\{\hat{R}_{\alpha,l_\alpha}\}$ and the corresponding four operators $\{\hat{M}_{\alpha,\sigma_\alpha}\}$ might be related by a canonical unitary transformation. It follows both from these relations and from Eqs. (57) of Ref. [10] and Eq. (37) that the hermitian operators $\hat{R}_{\alpha,l_\alpha}$ and $\hat{M}_{\alpha,\sigma_\alpha}$ have the same set of eigenvalues which are the integer numbers $0, 1, 2, \ldots, N_a$. Moreover, the two Hilbert subspaces spanned by the two sets of corresponding orthonormal eigenstates of the operators $\hat{R}_{\alpha,l_\alpha}$ and $\hat{M}_{\alpha,\sigma_\alpha}$ have the same dimension.

Let us denote the energy eigenstates of the 1D Hubbard model at a given value of U/t by $|\psi_l(U/t)\rangle$ where $l = 1, 2, \ldots, 4^{N_a}$. For spin densities $|m| < n$ and electronic densities $0 < n \leqslant 1/a$ and for spin densities $|m| < 2/a - n$ and electronic densities $1/a \leqslant n < 2/a$, there is a unitary transformation associated with an operator $\hat{V}(U/t)$ which transforms each of the energy eigenstates $|\psi_l(U/t)\rangle$ onto a new state,

$$|\phi_l\rangle = \hat{V}(U/t)\big|\psi_l(U/t)\big\rangle, \quad l = 1, 2, \ldots, 4^{N_a}. \tag{38}$$

Although here we call $\hat{V}(U/t)$ the operator appearing in Eq. (38) and above we used the same designation for the electron–rotated-electron unitary operator of Eq. (21), to start with these are considered unrelated operators. Below we confirm they are the same operator. The 4^{N_a} orthonormal states $|\phi_l\rangle$ of Eq. (38) are the eigenstates of the 1D Hubbard model in the limit $U/t \to \infty$ and the transformation is such that $\hat{V}(U/t)$ becomes the unit operator as $U/t \to \infty$ and the two states $|\psi_l(\infty)\rangle$ and $|\phi_l\rangle = \hat{V}(\infty)|\psi_l(\infty)\rangle$ become the same state in that limit. Since both of the above set of states are complete and correspond to orthonormal states, the operator $\hat{V}(U/t)$ is indeed unitary.

In the limit $U/t \to \infty$, there is a huge degeneracy of η-spin and spin occupancy configurations and thus there are several choices for complete sets of energy eigenstates with the same energy spectrum. Among these several choices only the above set of 4^{N_a} orthonormal states $|\phi_l\rangle$ can be generated from the corresponding energy eigenstates of the finite-U/t 1D Hubbard model by adiabatically turning off the parameter t/U. The states $|\phi_l\rangle$ have unique expressions in terms of electron doubly-occupied, empty, spin-down singly-occupied, and spin-up singly-occupied site distribution configurations.

Operators whose expressions in terms of both the elementary creation and annihilation electronic and rotated-electron operators are independent of the on-site repulsion U, commute with the unitary operator $\hat{V}(U/t)$. This is because the expressions of these operators in terms of electronic operators and rotated-electron operators are identical and independent of U. As further discussed below, this is the case of the momentum operator and of the $\pm 1/2$ Yang-holon ($\alpha = c$) and $\pm 1/2$ HL-spinon ($\alpha = s$) number operators whose expression is given by

$$\hat{L}_{\alpha,\pm 1/2} = \sqrt{\vec{\hat{S}}_\alpha . \vec{\hat{S}}_\alpha + 1/4} - 1/2 \mp \hat{S}^z_\alpha, \quad \alpha = c, s. \tag{39}$$

On the right-hand side of this equation $\hat{S}^z_\alpha$ is the diagonal generator of the η-spin ($\alpha = c$) and spin ($\alpha = s$) algebras whose expression is provided in Eq. (2). The remaining components of the operator $\vec{\hat{S}}_\alpha$ of Eq. (39) are given by the usual expressions $\hat{S}^x_\alpha = (\hat{S}^\dagger_\alpha + \hat{S}_\alpha)/2$ and $\hat{S}^y_\alpha = (\hat{S}^\dagger_\alpha - \hat{S}_\alpha)/2i$ where the off-diagonal generators $\hat{S}^\dagger_\alpha$ and $\hat{S}_\alpha$ are given in Eqs. (7) and (8) for $\alpha = c$ and $\alpha = s$, respectively. Let us consider the operational expression obtained by replacing the electronic operators $c^\dagger_{j,\sigma}$ and $c_{j,\sigma}$ by the corresponding rotated-electron operators $\tilde{c}^\dagger_{j,\sigma}$ and $\tilde{c}_{j,\sigma}$ in any operator expression. For the momentum operator, η-spin and spin operators (2), (7), and (8), and associated Yang-holon and HL-spinon number operator (39) that expression describes precisely the same operator as the original expression. In contrast, for finite values of U/t the Hamiltonian (1) and the above sets of operators $\{\hat{R}_{\alpha,l_\alpha}\}$, $\{\hat{M}_{\alpha,\sigma_\alpha}\}$, $\hat{N}_c(q)$, and $\{\hat{N}_{\alpha\nu}(q)\}$ do not commute with the operator $\hat{V}(U/t)$. It follows that the expressions of these operators are not invariant under the replacement of the electronic operators $c^\dagger_{j,\sigma}$ and $c_{j,\sigma}$ by the corresponding rotated-electron operators.

An important point is that $|\psi_l(U/t)\rangle$ are eigenstates of the operators $\hat{M}_{\alpha,\sigma_\alpha}$ whereas $|\phi_l\rangle$ of Eq. (38) are eigenstates of the operators $\hat{R}_{\alpha,l_\alpha}$. For any value of U/t the following set of hermitian operators $\hat{V}(U/t)\hat{H}\hat{V}^\dagger(U/t)$, $\{\hat{R}_{\alpha,-1}\}$, $\hat{V}(U/t)\hat{N}_c(q)\hat{V}^\dagger(U/t)$, and $\{\hat{V}(U/t)\hat{N}_{\alpha\nu}(q)\hat{V}^\dagger(U/t)\}$ where $\alpha = c, s$, $\sigma_\alpha = \pm 1/2$, and $\nu = 1, 2, \ldots$ form a complete set of compatible hermitian operators. The expression for the operators $\hat{V}(U/t)\hat{H}\hat{V}^\dagger(U/t)$, $\hat{V}(U/t)\hat{N}_c(q)\hat{V}^\dagger(U/t)$, and $\hat{V}(U/t)\hat{N}_{\alpha\nu}(q)\hat{V}^\dagger(U/t)$ in terms of elementary creation and annihilation electronic operators equals the expression in terms of these elementary operators for the operators $\hat{H}$, $\hat{N}_c(q)$, and $\hat{N}_{\alpha\nu}(q)$ in the limit $U/t \to \infty$, respectively. Moreover, the canonical unitary transformation associated with the operator $\hat{V}(U/t)$ has the following property: if the state $|\psi_l(U/t)\rangle$ is eigenstate of the operators $\hat{M}_{\alpha,\sigma_\alpha}$, $\hat{N}_c(q)$, and $\hat{N}_{\alpha\nu}(q)$ with given eigenvalues M_{α,σ_α}, $N_c(q)$, and $N_{\alpha\nu}(q)$, respectively, then the corresponding state $|\phi_l\rangle = \hat{V}(U/t)|\psi_l(U/t)\rangle$ is eigenstate of the operators $\hat{R}_{\alpha,l_\alpha}$, $\hat{V}(U/t)\hat{N}_c(q)\hat{V}^\dagger(U/t)$, and $\hat{V}(U/t)\hat{N}_{\alpha\nu}(q)\hat{V}^\dagger(U/t)$ with the same eigenvalues $R_{\alpha,l_\alpha} = M_{\alpha,\sigma_\alpha}$, $N_c(q)$, and $N_{\alpha\nu}(q)$, respectively. In contrast, the energy eigenvalues of the Hamiltonians $\hat{H}$ and $\hat{V}(U/t)\hat{H}\hat{V}^\dagger(U/t)$ relative to the states $|\psi_l(U/t)\rangle$ and $|\phi_l\rangle = \hat{V}(U/t)|\psi_l(U/t)\rangle$, respectively, are in general different. Thus, if the state $|\psi_l(U/t)\rangle$ is eigenstate of the operator $\hat{M}_{c,-1/2}$ of eigenvalue $M_{c,-1/2} = 0, 1, 2, \ldots$, then the corresponding state $|\phi_l\rangle = \hat{V}(U/t)|\psi_l(U/t)\rangle$ is eigenstate of the operator $\hat{R}_{c,-1} \equiv \hat{D}$ with the same eigenvalue $R_{c,-1} \equiv D = M_{c,-1/2} = 0, 1, 2, \ldots$. Note that $\hat{R}_{c,-1} \equiv \hat{D}$ is the electron double-occupation operator (6) and that the relation $\hat{R}_{c,-1} \equiv \hat{D} = \hat{V}(U/t)\hat{M}_{c,-1/2}\hat{V}^\dagger(U/t)$ implies that

$$\hat{M}_{c,-1/2} = \hat{V}^\dagger(U/t)\hat{D}\hat{V}(U/t). \tag{40}$$

In addition, the eigenvalues of the operator $\hat{M}_{c,-1/2}$ are good quantum numbers and thus it is such that

$$[\hat{H}_H, \hat{M}_{c,-1/2}] = 0. \tag{41}$$

The above properties together with Eqs. (40) and (41), which have precisely the same form as Eqs. (20) and (23), respectively, confirm that

$$\hat{M}_{c,-1/2} = \tilde{D} = \sum_j \tilde{c}^\dagger_{j,\uparrow}\tilde{c}_{j,\uparrow}\tilde{c}^\dagger_{j,\downarrow}\tilde{c}_{j,\downarrow}. \tag{42}$$

Such a relation also confirms that the transformation associated with Eq. (38) is indeed the electron–rotated-electron unitary transformation. Thus, the operator $\hat{V}(U/t)$ appearing in Eqs. (38) and (40) is uniquely defined for all values of U/t by Eqs. (21)–(23). It follows that for all values of U/t the number of $-1/2$ holons equals the rotated-electron double occupation. Importantly, the number of $-1/2$ holons is a good quantum number which labels all energy eigenstates $|\psi_l(U/t)\rangle$ of the finite-U/t 1D Hubbard model, such that $l = 1, 2, \ldots, 4^{N_a}$. For all values of U/t we have found the relation between the quantum numbers of the Bethe-ansatz solution and $SO(4)$ symmetry of the 1D Hubbard model and the eigenvalues of the rotated electron double occupation associated with the electron–rotated-electron unitary transformation. According to our results, the number of $-1/2$

holon operators can be written in terms of the rotated electron operators as given in Eq. (42).

There is one and only one canonical unitary transformation which maps each of the 4^{N_a} finite-U/t energy eigenstates $|\psi_l(U/t)\rangle$ into each of the corresponding 4^{N_a} energy eigenstates $|\phi_l\rangle = \hat{V}(U/t)|\psi_l(U/t)\rangle$ of the $U/t \to \infty$ Hubbard model. It is the electron–rotated-electron unitary transformation. The energy eigenstates $|\psi_l(U/t)\rangle$ are uniquely defined by the sets of eigenvalues $\{M_{\alpha,\sigma_\alpha}\}$, $\{N_c(q)\}$, and $\{N_{\alpha\nu}(q)\}$ where $\alpha = c, s$, $\sigma_\alpha = \pm 1/2$, and $\nu = 1, 2, \ldots$ associated with the corresponding set of operators $\{\hat{M}_{\alpha,\sigma_\alpha}\}$, $\{\hat{N}_c(q)\}$, and $\{\hat{N}_{\alpha\nu}(q)\}$ where again $\alpha = c, s$, $\sigma_\alpha = \pm 1/2$, and $\nu = 1, 2, \ldots,$ respectively. The state $|\phi_l\rangle$ is also uniquely defined by the same set of eigenvalues $\{R_{\alpha,l_\alpha} = M_{\alpha,\sigma_\alpha}\}$, $\{N_c(q)\}$, and $\{N_{\alpha\nu}(q)\}$ where $\alpha = c, s$, $\sigma_\alpha = \pm 1/2$, and $\nu = 1, 2, \ldots$ but now related to the different set of operators $\{\hat{R}_{\alpha,l_\alpha}\}$, $\{\hat{V}(U/t)\hat{N}_c(q)\hat{V}^\dagger(U/t)\}$, and $\{\hat{V}(U/t)\hat{N}_{\alpha\nu}(q)\hat{V}^\dagger(U/t)\}$ where again $\alpha = c, s$, $\sigma_\alpha = \pm 1/2$, and $\nu = 1, 2, \ldots,$ respectively. Since for finite values of U/t, the hermitian operators $\hat{M}_{\alpha,\sigma_\alpha}$, $\hat{N}_c(q)$, and $\hat{N}_{\alpha\nu}(q)$ do not commute with the electron–rotated-electron unitary operator $\hat{V}(U/t)$, they also do not commute with the operators $\hat{R}_{\alpha,l_\alpha}$, $\hat{V}(U/t)\hat{N}_c(q)\hat{V}^\dagger(U/t)$, and $\hat{V}(U/t)\hat{N}_{\alpha\nu}(q)\hat{V}^\dagger(U/t)$.

The four operators $\hat{R}_{\alpha,l_\alpha}$ of Eqs. (25) and (26) and the four operators $\hat{M}_{\alpha,\sigma_\alpha}$ of Eq. (24) commute with the six generators of the η-spin and spin algebras given in Eqs. (2), (7), and (8). It is straightforward to confirm that the electron–rotated-electron unitary operator commutes with these six generators and, therefore, the commutators $[\hat{V}(U/t), \hat{\vec{S}}_\alpha.\hat{\vec{S}}_\alpha] = [\hat{V}(U/t), \hat{N}_\sigma] = 0$ vanish. Here $\hat{\vec{S}}_c.\hat{\vec{S}}_c \equiv \hat{\vec{\eta}}.\hat{\vec{\eta}}$ and $\hat{\vec{S}}_s.\hat{\vec{S}}_s \equiv \hat{\vec{S}}.\hat{\vec{S}}$ are the square η-spin and spin operators, respectively, that appear in expression (39), and $\hat{N}_\sigma$ is the spin σ electron number operator. We recall that both the energy eigenstates $|\psi_l(U/t)\rangle$ and the corresponding states $|\phi_l\rangle = \hat{V}(U/t)|\psi_l(U/t)\rangle$ given in Eq. (38) are eigenstates of the spin σ electron number operator. The above commutation relations are associated with the invariance of the operators $\hat{\vec{S}}_\alpha.\hat{\vec{S}}_\alpha$ and $\hat{N}_\sigma$ under the replacement of the electronic operators $c_{j,\sigma}^\dagger$ and $c_{j,\sigma}$ by the corresponding rotated-electron operators $\tilde{c}_{j,\sigma}^\dagger$ and $\tilde{c}_{j,\sigma}$. Thus, the $\pm 1/2$ Yang-holon ($\alpha = c$) and $\pm 1/2$ HL-spinon ($\alpha = s$) number operators (39) and momentum operator also commute with the unitary operator $\hat{V}(U/t)$, $[\hat{V}(U/t), \hat{L}_{\alpha,\sigma_\alpha}] = 0$ and $[\hat{V}(U/t), \hat{P}] = 0$.

The commutation relation $[\hat{V}(U/t), \hat{L}_{\alpha,\sigma_\alpha}] = 0$, is consistent with the following two findings of Ref. [10]: for all values of U/t creation (annihilation) of a $-1/2$ Yang holon creates (annihilates) precisely one electron doubly occupied site; creation or annihilation of a $-1/2$ HL spinon at fixed electronic numbers generates an on-site electronic spin flip. Note that according to the general definition of a $-1/2$ holon and a $-1/2$ spinon, for all values of U/t creation (annihilation) of a $-1/2$ Yang holon also creates (annihilates) precisely one rotated-electron doubly occupied site. Moreover, creation or annihilation of a $-1/2$ HL spinon at fixed electronic numbers also generates an on-site rotated-electron spin flip. This is consistent with the property that the expression of the $-1/2$ Yang holon and $-1/2$ HL spinon generator has the same expression in terms of electron and rotated-electron creation and annihilation operators. Moreover, the commutation of the unitary operator $\hat{V}(U/t)$ with the momentum operator is related to the invariance of the electronic lattice and associated lattice constant a and length $L = aN_a$ under the electron–rotated-electron transformation. We note that the momentum operator is the generator of the translations performed in the electronic lattice. The above invariance of such a lattice is built in, by construction, in the electron–rotated-electron unitary transformation. Indeed, the concepts of rotated-electron double occupation, no occupation, and spin-down and spin-up single occupation imply, by construction, that invariance. Although for finite values of U/t the electron and rotated-electron site distribution configurations that describe the same energy eigenstate are different, electrons and rotated electrons occupy a lattice with the same lattice constant a and length $L = aN_a$ [22].

The physics behind the invariance of the momentum operator and charge and spin operators under the unitary electron–rotated-electron transformation, and of the transformation laws of other operators under such a transformation is important for the transport of the charge and spin. (This issue is studied in Section 4.) Indeed, in spite of carrying a finite charge, if a quantum object is invariant under the electron–rotated-electron transformation it is localized and thus does not contribute to the charge transport. The same situation occurs in the case of spin transport.

3.2. Consistency with the quantum numbers and counting of states obtained from the Bethe-ansatz solution

Let us confirm that the $\alpha\nu$ pseudoparticle representation provided by the Bethe-ansatz solution leads to the same complete LWS description as the $\pm 1/2$ holons and $\pm 1/2$ spinons that are not invariant under the electron-rotated electron unitary transition. By construction, the $\pm 1/2$ holons (and $\pm 1/2$ spinons) transform according the representations of the η-spin (and spin) $SU(2)$ algebra for the whole Hilbert space.

The rotated-electron and rotated-electronic hole contents of the $\pm 1/2$ holons, $\pm 1/2$ spinons, and the c pseudoparticles and the associated $c\nu$ pseudoparticles, $\pm 1/2$ Yang holons, $s\nu$ pseudoparticles, and $\pm 1/2$ HL spinons can be confirmed by relating the changes in the electronic hole and electron numbers to the corresponding changes in the number of these quantum objects. It follows from Eqs. (27) and (28) that the changes ΔN^h, ΔN, $\Delta N_\uparrow$, and $\Delta N_\downarrow$ in the values of the numbers of rotated electronic holes (and electronic holes) and rotated electrons (and electrons) can be related to the corresponding changes in the values of the numbers of these quantum objects as follows

$$\Delta N^h = \Delta N_c + 2\Delta M_{c,+1/2} = \Delta N_c + 2\Delta L_{c,+1/2} + \sum_{\nu=1}^{\infty} 2\nu \Delta N_{c\nu}, \tag{43}$$

$$\Delta N = \Delta N_c + 2\Delta M_{c,-1/2} = \Delta N_c + 2\Delta L_{c,-1/2} + \sum_{\nu=1}^{\infty} 2\nu \Delta N_{c\nu}, \tag{44}$$

$$\Delta N_\uparrow = \Delta M_{s,+1/2} + \Delta M_{c,-1/2} = \Delta L_{s,+1/2} + \Delta L_{c,-1/2} + \sum_{\alpha=c,s} \sum_{\nu=1}^{\infty} \nu \Delta N_{\alpha\nu}, \tag{45}$$

and

$$\Delta N_{\downarrow} = \Delta M_{s,-1/2} + \Delta M_{c,-1/2} = \Delta L_{s,-1/2} + \Delta L_{c,-1/2} + \sum_{\alpha=c,s} \sum_{\nu=1}^{\infty} \nu \Delta N_{\alpha\nu}. \quad (46)$$

Since Eqs. (43)–(46) are valid for all electronic and spin densities and finite values of U, these equations are consistent with the rotated-electron and rotated-electronic hole contents found above for the corresponding quantum objects. However, such a consistency follows directly from our definitions and Eqs. (43)–(46) are a necessary, but not sufficient condition for the validity of the results introduced in Section 2. In order to confirm their full consistency with the exact Bethe-ansatz solution and $SO(4)$ symmetry of the 1D Hubbard model, some stronger requirements must be fulfilled.

Although in our counting of states we use some techniques similar to those used in Ref. [11], we note that our goal is different. Here we are not checking the completeness of the Hilbert space. Instead, we are interested in the dimension of the subspaces spanned by states with fixed values of holon and spinon numbers. Thus, we consider a partition of the Hilbert space into the set of such subspaces. Analysis of the results of Ref. [11] reveals that the Hilbert-space subspace partition used in the studies of that reference corresponds to a different choice of subspaces. Obviously, in the end we sum the dimensions of all the subspaces and check whether this gives the correct value for the total number of energy eigenstates.

Let us consider the set of states with fixed η-spin value S_c or spin value S_s representative of a collection of a number M_α of $s_c = 1/2$ holons ($\alpha = c$) or $s_s = 1/2$ spinons ($\alpha = s$). The number of such states, all with the same fix values of M_α and S_α, is given by

$$\mathcal{N}(S_\alpha, M_\alpha) = (2S_\alpha + 1)\left\{\binom{M_\alpha}{M_\alpha/2 - S_\alpha} - \binom{M_\alpha}{M_\alpha/2 - S_\alpha - 1}\right\}. \quad (47)$$

This counting of states follows from the η-spin or spin rules of summation in the case of M_α quantum objects of spin (or η spin) 1/2 and whose total spin (or η spin), $S_\alpha \leqslant M_\alpha/2$, is fixed. Thus for $\alpha = c$ or $\alpha = s$ Eq. (47) gives the number of $SU(2)$ representation states which span the Hilbert subspace associated with a system of total η-spin value S_c constituted by M_c η-spin 1/2 holons or of total spin value S_s and constituted by a number M_s of spin 1/2 spinons, respectively. On the other hand, at constant values of M_c and M_s, the number of states associated with the c pseudoparticle excitations of the 1D Hubbard model corresponds to the different possible occupancy configurations of a number $N_c = M_s$ of c pseudoparticles and $N_c^h = M_c$ of c pseudoparticle holes such that $N_c + N_c^h = M_s + M_c = N_a$. Thus, the number of different c pseudoparticle configurations reads $\binom{N_a}{N_c} = \binom{N_a}{N_c^h}$ and can also be expressed as $\binom{N_a}{M_s} = \binom{N_a}{M_c}$. Therefore, if the assumptions of Section 2 are correct, the number of energy eigenstates which span the Hilbert subspaces of the 1D Hubbard model with fixed values of S_c, S_s, M_c, and M_s, where the numbers M_c and M_s are given by Eq. (32), must read

$$\binom{N_a}{M_c} \mathcal{N}(S_c, M_c) \mathcal{N}(S_s, M_s) = \binom{N_a}{M_s} \mathcal{N}(S_c, M_c) \mathcal{N}(S_s, M_s), \quad (48)$$

where $\mathcal{N}(x, y)$ is the function given in Eq. (47).

The requirement that the dimension of all Hilbert subspaces of the 1D Hubbard model with fixed values of S_c, S_s, M_c, and M_s must have the form (48), that is they must have the form of a product of three independent factors corresponding to the c pseudoparticle, η-spin-holon, and spin-spinon excitations, is a strong requirement. For instance, the validity of Eq. (48) would require that the total number of energy eigenstates of the 1D Hubbard model is written in the following form

$$\mathcal{N}_{\text{tot}} = \sum_{S_c=0}^{[N_a/2]} \sum_{M_c/2=S_c}^{[N_a/2]} \sum_{S_s=0}^{[N_a/2]} \sum_{M_s/2=S_s}^{[N_a/2]} \delta_{N_a, M_c+M_s} \times \binom{N_a}{M_c} . \mathcal{N}_M(S_c, M_c) . \mathcal{N}_M(S_s, M_s), \tag{49}$$

where $\delta_{i,j}$ is the Krönecker delta, such that $\delta_{i,j} = 1$ for $i = j$ and $\delta_{i,j} = 0$ for $i \neq j$. We emphasize that the validity of Eqs. (48) and (49), together with Eqs. (43)–(46) would be a necessary and sufficient condition for the consistency of the results of Section 2 with the exact Bethe-ansatz solution and $SO(4)$ symmetry of the 1D Hubbard model.

The validity of Eqs. (48) and (49) requires the fulfillment of the following two conditions:

(a) The sum of states of Eq. (49) equals 4^{N_a}, which is the total number of energy eigenstates of the 1D Hubbard model [11].

(b) The value of Eq. (48) equals the corresponding subspace dimension derived by counting the possible c pseudoparticle and $\alpha\nu$ pseudoparticle occupancy configurations which generate all LWSs of the η-spin and spin $SU(2)$ algebras in that subspace as well as the states generated by application of the suitable η-spin and spin off-diagonal generators onto these LWSs, i.e., equals the following number

$$\binom{N_a}{M_c} . \tilde{\mathcal{N}}(S_c, M_c) . \tilde{\mathcal{N}}(S_s, M_s). \tag{50}$$

Here

$$\tilde{\mathcal{N}}(S_\alpha, M_\alpha) = (2S_\alpha + 1) \sum_{\{N_{\alpha\nu}\}} \prod_{\nu=1}^{\infty} \binom{N^*_{\alpha\nu}}{N_{\alpha\nu}}, \quad \alpha = c, s, \tag{51}$$

where the $\{N_{\alpha\nu}\}$ summation is over all $\alpha\nu$ pseudoparticle occupancy configurations which satisfy the constraints $M_\alpha - 2S_\alpha = 2\sum_{\nu=1}^{\infty} \nu N_{\alpha\nu}$ for $\alpha = c, s$, and $N^*_{\alpha\nu}$ is the number of available $\alpha\nu$ pseudoparticle discrete momentum values defined in Eqs. (B.6) and (B.7) of Appendix B. The value of the quantity $\binom{N^*_{\alpha\nu}}{N_{\alpha\nu}}$ is provided by the Bethe-ansatz solution, whereas the factor $(2S_\alpha + 1)$ on the right-hand side of Eq. (51) is associated with the η-spin ($\alpha = c$) or spin ($\alpha = s$) $SU(2)$ algebra.

Eq. (51) gives the number of energy eigenstates which span the Hilbert subspaces with constant values of S_c, S_s, M_c, and M_s, where M_α are the numbers defined in Eq. (32). The total number of energy eigenstates of the 1D Hubbard model can be obtained by summation of the number of states of each of these subspaces as follows

$$\tilde{\mathcal{N}}_{\mathrm{tot}} = \sum_{S_c=0}^{[N_a/2]} \sum_{M_c/2=S_c}^{[N_a/2]} \sum_{S_s=0}^{[N_a/2]} \sum_{M_s/2=S_s}^{[N_a/2]} \delta_{N_a, M_c+M_s} \binom{N_a}{M_c} . \tilde{\mathcal{N}}(S_c, M_c) . \tilde{\mathcal{N}}(S_s, M_s)$$
$$= 4^{N_a}, \tag{52}$$

where the numbers $\tilde{\mathcal{N}}(S_\alpha, M_\alpha)$ with $\alpha = c, s$ are defined in Eq. (51).

It is straightforward to show that the sums and products in Eqs. (51) and (52) are equivalent to the ones evaluated in Ref. [11] and, therefore, the sum of states equals 4^{N_a}. However, whether the number (49) also equals 4^{N_a} must be checked. Let us rewrite the number of Eq. (49) as

$$\mathcal{N}_{\mathrm{tot}} = \sum_{S_c=0}^{[N_a/2]} \sum_{S_s=0}^{[N_a/2]} (2S_c+1)(2S_s+1) \sum_{M_c/2=S_c}^{[N_a/2]} \sum_{M_s/2=S_s}^{[N_a/2]} \delta_{N_a, M_c+M_s} \binom{N_a}{M_c}$$
$$\times \left\{ \binom{M_c}{M_c/2 - S_c} - \binom{M_c}{M_c/2 - S_c - 1} \right\}$$
$$\times \left\{ \binom{M_s}{M_s/2 - S_s} - \binom{M_s}{M_s/2 - S_s - 1} \right\}. \tag{53}$$

Evaluation of the M_c and M_s sums leads then to

$$\mathcal{N}_{\mathrm{tot}} = \sum_{S_c=0}^{[N_a/2]} \sum_{S_s=0}^{[N_a/2]} (2S_c+1)(2S_s+1)$$
$$\times \left[\binom{N_a}{N_a/2 - S_c + S_s} \left\{ \binom{N_a}{N_a/2 - S_c - S_s} - \binom{N_a}{N_a/2 - S_c - S_s - 2} \right\} \right.$$
$$- \binom{N_a}{N_a/2 - S_c - S_s - 1}$$
$$\left. \times \left\{ \binom{N_a}{N_a/2 - S_c + S_s + 1} - \binom{N_a}{N_a/2 - S_c + S_s - 1} \right\} \right]. \tag{54}$$

Performing the S_c and S_s sums leads finally to

$$\mathcal{N}_{\mathrm{tot}} = 4^{N_a}. \tag{55}$$

This result confirms that the above condition (a) is met. However, the fact that both the Eqs. (49) and (52) equal 4^{N_a} does not necessarily imply that the values of Eqs. (48) and (50) are also equal for all subspaces spanned by states with fixed holon and spinon numbers.

To check condition (b), one has to perform the sums and products on the right-hand side of Eq. (51). This is a difficult technical task. Fortunately, a similar problem was solved in the case of the isotropic antiferromagnetic Heisenberg chain in Appendix A of Ref. [24] and can be generalized to the more complex case of the 1D Hubbard model. Provided that one handles carefully the interplay between the holon, spinon, and electronic numbers, this leads to the following result

$$\tilde{\mathcal{N}}(S_\alpha, M_\alpha) = \mathcal{N}(S_\alpha, M_\alpha) = (2S_\alpha+1) \left\{ \binom{M_\alpha}{M_\alpha - 2S_\alpha} - \binom{M_\alpha}{M_\alpha - 2S_\alpha - 1} \right\}. \tag{56}$$

From the form of this expression, we find that the function $\tilde{\mathcal{N}}(x, y)$ of Eq. (51) equals the function $\mathcal{N}(x, y)$ of Eq. (47).

Thus, the conditions (a) and (b) are met and hence the assumptions of Section 2 concerning the relation of holons and spinons to the c pseudoparticles and $\alpha\nu$ pseudoparticles whose occupancy configurations describe the energy eigenstates are consistent with the Bethe-ansatz solution and $SO(4)$ symmetry of the 1D Hubbard model. This implies that the total number of energy eigenstates of the 1D Hubbard model can be written as in Eq. (49).

The subspace dimension of Eq. (48) is a product of three numbers. Two of these numbers are nothing but the value of Eq. (47) of different states with the same value of S_α that, following the counting rules of η-spin and spin summation, one can generate from M_α $s_\alpha = 1/2$ holons ($\alpha = c$) and spinons ($\alpha = s$). These two values are uniquely defined by the fixed values of the total η-spin and spin of the subspace and by the fixed numbers of η-spin $1/2$ holons and spin $1/2$ spinons in that subspace. The third number corresponds to the c pseudoparticle excitations. This is the number of states associated with the possible occupancy configurations of $N_c = M_s$ c pseudoparticles and $N_c^h = M_c$ c pseudoparticle holes where $N_a = N_c + N_c^h = M_c + M_s$. These charge excitations describe the translational motion of the rotated-electron singly-occupied sites relative to the rotated-electron doubly-occupied and empty sites.

This confirms that for all densities and finite values of U, there is a separation of the spin-spinon, η-spin-holon, c pseudoparticle excitations for the set of Hilbert subspaces of the 1D Hubbard model with fixed values of η-spin, spin, holon, and spinon number. The charge excitations correspond to both the c pseudoparticle and η-spin-holon excitations. Since this holds true for all these Hilbert subspaces, such a separation occurs at all energy scales of the model. Our results confirm that the c pseudoparticle excitations do not refer to the η-spin or spin degrees of freedom.

We have just confirmed that the relation of the holons and spinons to the c pseudoparticles, Yang holons, $c\nu$ pseudoparticles, HL spinons, and $s\nu$ pseudoparticles conjectured in Section 2 is correct and refers to the whole Hilbert space of the 1D Hubbard model. This complemented with the confirmation presented in the previous subsection of the relation of holons, spinons, and c pseudoparticles to rotated electrons and electrons justifies the validity and consistency of our η-spin $1/2$ holon, spin $1/2$ spinon, and c pseudoparticle description for the whole Hilbert space of the 1D Hubbard model.

Our results also confirm that the energy states described by Takahashi's charge rapidity functional $\Lambda_{c\nu}(q)$ and spin rapidity functional $\Lambda_{s\nu}(q)$ are, for the whole Hilbert space, associated with 2ν-holon and 2ν-spinon composite excitations, respectively. The excitations associated with the rapidity-momentum functional $k(q)$ correspond to the c pseudoparticle branch of excitations. The separation of the spin-spinon, η-spin-holon, c pseudoparticle excitations also occurs in the momentum expression (36), each of these three branches of elementary excitations giving rise to independent momentum contributions. That Takahashi's thermodynamic Bethe-ansatz equations refer to three types of independent excitations follows directly from inspection of the form of these equations. However, that the η-spin (and spin) excitations described by these equations correspond to occupancy configurations of η-spin $1/2$ (and spin $1/2$) holons (and spinons) only for *all* energy eigenstates is an useful result. The studies of Refs. [22,23] confirm that such a holon and spinon description can be used as a valuable tool for extraction of important physical information about the spectral properties of the model.

4. Transport of charge and spin

The study of the transport of charge and spin includes the characterization of the quantum objects which carry charge or spin. The value of the elementary charge or spin carried by these objects is an issue of interest for the description of the spectral properties associated with charge and spin transport and the study of charge–charge and spin–spin correlations. The spin of the different quantum objects was already given in Section 2. However, here we show that some of the objects which carry spin do not contribute to the transport of spin. The study of the transport of charge also requires the definition of the elementary charge carriers which as a result of the non-perturbative organization of the electronic degrees of freedom couple to external charge probes.

We start by studying the transport of charge. According to the results obtained in the previous sections, the spinons and composite $s\nu$ pseudoparticles have no charge degrees of freedom and thus do not contribute to charge transport. There are two alternative descriptions for charge transport: in terms of electrons and in terms of electronic holes. If we describe charge transport in terms of the N electrons or N^h electronic holes, we find by use of Eq. (30) that the value of the charge deviation associated with the transition between two energy eigenstates can be expressed as follows

$$\begin{aligned}
-e\Delta N &= -e\frac{L}{2\pi}\int_{q_c^-}^{q_c^+} dq\,\Delta N_c(q) \\
&\quad + \sum_{\nu=1}^{\infty}(-2e\nu)\frac{L}{2\pi}\sum_{\nu=1}^{\infty}\int_{-q_{c\nu}}^{q_{c\nu}} dq\,\Delta N_{c\nu}(q) - 2e\Delta L_{c,-1/2}, \\
+e\Delta N^h &= +e\frac{L}{2\pi}\int_{q_c^-}^{q_c^+} dq\,\Delta N_c(q) \\
&\quad + \sum_{\nu=1}^{\infty}(+2e\nu)\frac{L}{2\pi}\sum_{\nu=1}^{\infty}\int_{-q_{c\nu}}^{q_{c\nu}} dq\,\Delta N_{c\nu}(q) + 2e\Delta L_{c,+1/2}.
\end{aligned} \tag{57}$$

The form of the first expression of Eq. (57) for any pair of energy eigenstates confirms that in the case of description of charge transport in terms of electrons, the charge carriers are the chargeons and the $-1/2$ holons. For finite values of U/t only the $-1/2$ holons which are part of 2ν-holon composite $c\nu$ pseudoparticles contribute to charge transport. Analysis of the form of the above expression reveals that the charge carried by a $-1/2$ holon is $-2e$ and equals twice the charge of the electron $-e$. This is consistent with the $-1/2$ holon corresponding to a rotated-electron doubly occupied site. Within the electron charge transport description, a $c\nu$ pseudoparticle carries charge $-2\nu e$. That charge corresponds to the 2ν rotated electrons associated with the ν $-1/2$ holons contained in the $c\nu$ pseudoparticle. In this case the charge of the c pseudoparticle corresponds to that of the chargeon. The form of the first expression of Eq. (57) confirms that such a quantum object carries the charge of the electron $-e$.

If instead we describe charge transport in terms of the N^h electronic holes, the form of the second expression of Eq. (57) for any pair of energy eigenstates confirms that in this case the charge carriers are the antichargeons and the $+1/2$ holons. Again, for finite values of U/t only the $+1/2$ holons which are part of 2ν-holon composite $c\nu$ pseudoparticles contribute to charge transport, as we discuss below. The form of the above expression reveals that the charge carried by a $+1/2$ holon is given by $+2e$. This is consistent with the $+1/2$ holon corresponding to a rotated-electron empty site. Within the electronic hole charge transport description, a $c\nu$ pseudoparticle carries charge $+2\nu e$. Such a charge corresponds to the 2ν rotated-electronic holes associated with the ν $+1/2$ holons contained in the $c\nu$ pseudoparticle. In this case, the charge of the c pseudoparticle corresponds to the charge $+e$ of the antichargeon. We recall that the number N_c of c pseudoparticles equals both the numbers of rotated electrons (chargeons) and rotated-electronic holes (antichargeons) of the singly occupied sites.

In Section 2 we learn that a c pseudoparticle is a composite quantum object made out of a chargeon and an antichargeon. Moreover, the $c\nu$ pseudoparticle is a η-spin singlet composite quantum object made out of ν $-1/2$ holons and ν $+1/2$ holons. The description of charge transport in terms of c pseudoparticles and composite $c\nu$ pseudoparticles is possible but requires a careful handling of the problem. Indeed in the case of the description of the charge transport in terms of electrons (and electronic holes) the c pseudoparticles behave as chargeons (and antichargeons) and the composite $c\nu$ pseudoparticles behave as being made out of ν $-1/2$ holons (and ν $+1/2$ holons) only. This justifies why these quantum objects have opposite charges in the case of transport in terms of electrons and electronic holes, respectively. This relative character of the value of the c pseudoparticle and composite $c\nu$ pseudoparticle charges reveals that the real elementary charge carriers are the chargeons and $-1/2$ holons or the antichargeons and the $+1/2$ holons. In contrast to the charge of the c and composite $c\nu$ pseudoparticles, the value of the charge of the former quantum objects is uniquely defined.

An important property of the physics of the charge transport of the 1D Hubbard model is that at finite values of U/t only the holons which are part of $c\nu$ pseudoparticles contribute to charge transport. In the case of charge transport in terms of electrons (and electronic holes), note that in spite of the charge $-2e$ (and $+2e$) of the $-1/2$ (and $+1/2$) Yang holons, these quantum objects do not contribute to charge transport. The reason is that they are the same quantum object as the corresponding rotated $-1/2$ (and $+1/2$) Yang holons and thus refer to localized electron doubly-occupied (and empty) sites for all values of U/t. In contrast, for finite values of U/t both the c pseudoparticles and composite $c\nu$ pseudoparticles are in general not invariant under the electron–rotated-electron unitary transformation and have a momentum dependent energy dispersion, whose expression is given in Eqs. (C.15)–(C.18) of Appendix C. According to the electron double-occupation studies of Ref. [10], these quantum objects correspond to complex electron site distribution configurations. Thus, both the associated chargeons (or antichargeons) and ν $-1/2$ holons (or $+1/2$ holons) contribute to charge transport. On the other hand, it is found in Ref. [10] that in the limit $U/t \to \infty$ the ν $-1/2$ holons (and $+1/2$ holons) contained in a $c\nu$ pseudoparticle correspond to localized electron doubly occupied sites (and empty sites). This behavior is associated with the fact that as $U/t \to \infty$ the $c\nu$ pseudoparticle becomes the same quantum object as the rotated $c\nu$ pseudoparticle. Thus, in spite of the charge $-2\nu e$ (and $+2\nu e$) of the ν $-1/2$ holons (and $+1/2$ holons) which are part of the $c\nu$ pseudoparticle, in the limit $U/t \to \infty$ such a quantum object does not contribute to charge transport. This effect is also associated with the fact that all η-spin holon occupancy configurations become degenerated in that limit. In that limit, the only charge carriers are the chargeons or antichargeons associated with the c pseudoparticles. As $U/t \to \infty$, the energy dispersion of these quantum objects becomes a free spinless-fermion spectrum [15].

Note that when the initial state is a ground state, the deviations $\Delta L_{\alpha,\pm 1/2}$ and the pseudoparticle momentum distribution function deviations $\Delta N_c(q)$ and $\Delta N_{\alpha\nu}(q)$ which appear in Eq. (57) for $\alpha = c$ are relative to the ground-state values given in Eqs. (C.2), (C.3) and (C.24), (C.25) of Appendix C and read

$$\Delta L_{\alpha,\pm 1/2} = L_{\alpha,\pm 1/2}, \qquad \Delta N_c(q) = N_c(q) - N_c^0(q),$$
$$\Delta N_{\alpha\nu}(q) = N_{\alpha\nu}(q). \tag{58}$$

The charge carried by the holons was found here from analysis of Eq. (57). The charge carried by a c pseudoparticle and a composite $c\nu$ pseudoparticle can be obtained by other methods. With the use of periodic boundary conditions, the charge transport properties of the model can be studied by threading the N_a-site ring by a flux [25]. That scheme leads to a generalization of the Takahshi's thermodynamic equations (13)–(15), including the dependence on the flux. From the use of that method one finds for the charge transport in terms of electrons that the charge carried by a c pseudoparticle is $-e$, whereas the charge carried by a composite $c\nu$ pseudoparticle is $-2e\nu$. These results are consistent with the charges found here for the holons.

Spin transport can also be described in terms of electrons or electronic holes. For the electronic description the N_c spinons correspond to the N_c rotated electrons of the singly occupied sites. On the other hand, the electronic hole description of spin transport corresponds to identifying the N_c spinons with the N_c rotated-electronic holes which correspond to rotated-electron singly occupied sites. The spin projection of a electronic hole is the opposite of the corresponding electron. However, we are reminded that the composite $s\nu$ pseudoparticles contain an equal number ν of spin-down and spin-up spinons. Thus, their number of spin-down and spin-up spinons remains invariant under the change from the electron to the electronic hole description of spin transport.

As for the transport of charge, quantum objects which have spin but are invariant under the electron–rotated-electron unitary transformation are localized and do not contribute to the transport of spin. This is the case of the HL spinons for all values of U/t and of the composite $s\nu$ pseudoparticles for $U/t \to \infty$. Furthermore, according to the results of Ref. [10], composite $s\nu$ pseudoparticles with momentum $q = \pm q_{s\nu}$ are localized for all values of U/t and do not contribute to the transport of spin. Importantly, for zero spin density the momentum values reduce to $q = \pm q_{s\nu} = 0$ for $s\nu$ pseudoparticles belonging to branches such that $\nu > 1$. It follows that for zero spin density the $s\nu$ pseudoparticles such that $\nu > 1$ are localized and do not contribute to the transport of spin for all values of U/t. Thus, for zero spin density the only quantum objects which contribute to the transport of spin are the spinons contained in the two-spinon $s1$ pseudoparticles.

A similar situation occurs for the transport of charge at half filling. For such a density the momentum values reduce to $q = \pm q_{c\nu} = 0$ for the $c\nu$ pseudoparticle branches. It follows that the composite $c\nu$ pseudoparticles are localized and do not contribute to the transport of charge for all values of U/t. For values of excitation energy/frequency smaller than the Mott–Hubbard gap [1], the c pseudoparticle band is full and there is no transport of charge for the half-filling insulator phase. For values of excitation energy/frequency larger than the Mott–Hubbard gap, the only half-filling carriers of charge are the chargeons or antichargeons contained in the c pseudoparticles. In summary, these results confirm the importance for the transport of charge and spin of the transformation laws under the electron–rotated-electron unitary rotation.

5. Summary and concluding remarks

In this paper we have shown that the rotated electrons play a central role in the relation of the electrons to the exotic quantum objects whose occupancy configurations describe all energy eigenstates of the 1D Hubbard model. Such a clarification revealed that as a result of the Hilbert-space electron–rotated-electron unitary rotation, the η-spin $1/2$ holons, spin $1/2$ spinons, and c pseudoparticles emerge. The energy eigenstates of the 1D Hubbard model can be described in terms of occupancy configurations of these three elementary quantum objects only, for *all* finite values of energy and for the whole parameter space of the model.

Our results also reveal the relation of the excitations associated with Takahashi's ideal charge and spin strings of length $\nu = 1, 2, \ldots,$ obtained by means of the Bethe-ansatz solution to the $\pm 1/2$ holons and $\pm 1/2$ spinons, respectively. These charge and spin excitations can be described in terms of occupancy configurations of 2ν-holon composite $c\nu$ pseudoparticles and 2ν-spinon composite $s\nu$ pseudoparticles, respectively. In contrast to the holons (and spinons) which are part of composite quantum objects, the $\pm 1/2$ Yang holons (and $\pm 1/2$ HL spinons) are invariant under the electron–rotated-electron transformation for all values of U/t and thus have a localized character.

We found that the transformation laws under the electron–rotated-electron unitary rotation of the quantum objects whose occupancy configurations describe the energy eigenstates play a crucial role in the transport of charge and spin. Indeed, quantum objects which have charge (or spin), but are invariant under such a unitary rotation are localized and do not contribute to the transport of charge (or spin). Moreover, we found that for finite values of U/t and zero magnetization, the transport of charge (or spin) is described by the charge c pseudoparticle and $c\nu$ pseudoparticle quantum fluids (or the spin $s1$ pseudoparticle quantum fluid). Within the electronic description of transport, the elementary carriers of charge were found to be the charge $-e$ chargeon and the charge $-2e$ and η-spin projection $-1/2$ holon. Interestingly, the spin singlet and composite two-spinon character of the $s1$ pseudoparticles reveals that the quantum spin fluid described by these quantum objects is the 1D realization of the two-dimensional resonating valence bond spin fluid [13].

Several authors described the low-energy excitations of the 1D Hubbard model in terms of holons and spinons [3,13–18]. However, often such holons and spinons correspond to different definitions. While the pseudoparticle description of Ref. [18] is valid for the LWSs, the α, β hole representation used for the non-LWSs in that reference only applies to towers whose LWSs have no charge strings and no spin strings of length $\nu > 1$. Our study reveals that the concept of holon and spinon is not limited to low energy. Indeed, it refers to *all* energy scales of the model. In this paper we found that the charge carriers associated with the distribution of k's of the Bethe-ansatz solution are the chargeons or antichargeons. However, many authors identified the energy spectrum of the distribution of k's with the holon spectrum. Such an interpretation is behind the value of charge obtained for the $-1/2$ holon and $+1/2$ holon in Ref. [16] (called antiholon and holon in that reference), which is half of the value found here. The chargeons and antichargeons carry indeed charge $-e$ and $+e$, respectively, but are not related to the η-spin $SU(2)$ representation, as confirmed in Section 3. Only the $-1/2$ holons and $+1/2$ holons of charge $-2e$ and $+2e$, respectively, defined in this paper correspond to the η-spin $SU(2)$ representation. For large values of U/t the $-1/2$ holons and $+1/2$ holons introduced here become the dublons and holons, respectively, considered in Ref. [17]. Thus, for large values of U/t our holon definition is the same as in that reference.

The holon, spinon, and c pseudoparticle description introduced in this paper is useful for the evaluation of few-electron spectral function expressions for finite values of energy. In this paper we have not addressed the pseudoparticle problem in terms of an operator algebra. However, the concepts introduced here are used in Refs. [22,23] in the construction of a suitable operator algebra for the study of matrix elements of few-electron operators between energy eigenstates. That algebra involves the c pseudofermion and composite $\alpha\nu$ pseudofermions, which carry momentum $q + Q_c(q)/L$ and $q + Q_{\alpha\nu}(q)/L$, respectively. Here the values of the momenta $Q_c(q)/L$ and $Q_{\alpha\nu}(q)/L$ are determined by phase shifts. These objects carry the same charge or spin and, in case of having a composite character, have the same holon or spinon contents as the corresponding c pseudoparticle and composite $\alpha\nu$ pseudoparticles. They are related to the latter objects by a mere momentum unitary transformation such that $q \to q + Q_c(q)/L$ and $q \to q + Q_{\alpha\nu}(q)$. While the results obtained in this paper play a major role in the studies of Refs. [22,23], the expression of the problem in terms of pseudofermion operators is most suitable for the evaluation of the above-mentioned matrix elements. The motivation of the investigations of both the present paper and Refs. [22,23] is the study of the unconventional finite-energy spectral properties observed in low-dimensional materials [3,4]. Applications of the holon and spinon description introduced in this paper and in Ref. [22] to the study of the one-electron removal spectral properties of the organic conductor TTF-TCNQ are presented in Refs. [3,23].

Acknowledgements

We thank Katrina Hibberd, Luís Miguel Martelo, João Lopes dos Santos, and Pedro Sacramento for stimulating discussions. We also thank the support of EU money Center of Excellence ICA1-CT-2000-70029, K.P. thanks the support of OTKA grant T037451, and J.M.R. thanks the financial support of FCT (Portugal) under the fellowship No. SFRH/BPD/3636/2000.

Appendix A. Relation to Takahashi's notation

By using his string hypothesis in the general Bethe-ansatz equations for the model and taking logarithms, Takahashi arrived to a system of coupled non-linear equations [2]. These equations introduce the quantum numbers I_j, J'^n_α, and J^n_α whose occupancy configurations describe the energy eigenstates. The occupied and unoccupied values of these numbers are integers or half-odd integers. The above-mentioned non-linear equations involve *rapidities* Λ'^n_α and Λ^n_α. These are the real parts of the $N_a \to \infty$ ideal charge, Λ', and spin, Λ, strings of length n, respectively.

It is convenient for our description to replace the above Takahashi indices α and $n = 1, 2, \ldots$ by j and $\nu = 1, 2, \ldots,$ respectively. In Appendix B we express the pseudoparticle discrete momentum values in terms of the above Takahashi's quantum numbers which within our notation read

$$I^c_j \equiv I_j, \qquad I^{c\nu}_j \equiv J'^{\nu}_j, \qquad I^{s\nu}_j \equiv J^{\nu}_j. \tag{A.1}$$

By use of the notation relation given in this equation all expressions introduced in Appendix B can be directly obtained from the expressions provided in Ref. [2].

Appendix B. The pseudoparticle description

The pseudoparticle description corresponds to associating the quantum numbers

$$q_j = \frac{2\pi}{L} I^c_j, \qquad q_j = \frac{2\pi}{L} I^{c\nu}_j, \qquad q_j = \frac{2\pi}{L} I^{s\nu}_j, \tag{B.1}$$

such that

$$q_{j+1} - q_j = \frac{2\pi}{L}, \tag{B.2}$$

with the discrete momentum values of c pseudoparticles, $c\nu$ pseudoparticles, and $s\nu$ pseudoparticles. From the properties of Takahashi's equations and associated quantum numbers, we find that the pseudoparticles obey a Pauli principle relative to the momentum occupancies, i.e., one discrete momentum value q_j can either be unoccupied or singly occupied by a pseudoparticle. Within such a pseudoparticle description the set of N_c and $N_{\alpha\nu}$ occupied q_j discrete momentum values correspond to the number of pseudoparticles in the c and $\alpha\nu$ bands, respectively.

Since the energy eigenstates are uniquely defined by the occupation numbers of the discrete momentum values q_j of the c and $\alpha\nu$ pseudoparticle bands, it is useful to introduce c and $\alpha\nu$ momentum distribution functions $N_c(q)$ and $N_{\alpha\nu}(q)$. These distributions read $N_c(q_j) = 1$ and $N_{\alpha\nu}(q_j) = 1$ for occupied values of the discrete momentum values q_j and $N_c(q_j) = 0$ and $N_{\alpha\nu}(q_j) = 0$ for unoccupied values of q_j. We also introduce the following pseudoparticle-hole momentum distribution functions

$$N^h_c(q_j) \equiv 1 - N_c(q_j), \qquad N^h_{\alpha\nu}(q_j) \equiv 1 - N_{\alpha\nu}(q_j). \tag{B.3}$$

All LWSs of the η-spin and spin $SU(2)$ algebras are uniquely defined by the set of infinite momentum distribution functions $N_c(q)$ and $\{N_{\alpha\nu}(q)\}$ with $\alpha = c, s$ and $\nu = 1, 2, \ldots.$

Moreover, Takahashi's rapidity momentum k_j, charge rapidity Λ'^{ν}_j, and spin rapidity Λ^{ν}_j can be written as follows

$$k(q_j) \equiv k_j, \qquad \Lambda_{c\nu}(q_j) \equiv \Lambda'^{\nu}_j, \qquad \Lambda_{s\nu}(q_j) \equiv \Lambda^{\nu}_j. \tag{B.4}$$

The value of the rapidity momentum functional $k(q)$ and rapidity functional $\Lambda_{c\nu}(q)$ is defined by Eqs. (13)–(15) where the function $\Theta_{\nu,\nu'}(x)$ is given by

$$\begin{aligned}\Theta_{\nu,\nu'}(x) &= \Theta_{\nu',\nu}(x) \\ &= \delta_{\nu,\nu'}\left\{2\arctan\left(\frac{x}{2\nu}\right) + \sum_{l=1}^{\nu-1} 4\arctan\left(\frac{x}{2l}\right)\right\} \\ &\quad + (1-\delta_{\nu,\nu'})\left\{2\arctan\left(\frac{x}{\nu-\nu'|}\right) + 2\arctan\left(\frac{x}{\nu+\nu'}\right)\right. \\ &\quad \left. + \sum_{l=1}^{\frac{\nu+\nu'-|\nu-\nu'|}{2}-1} 4\arctan\left(\frac{x}{|\nu-\nu'|+2l}\right)\right\}.\end{aligned} \tag{B.5}$$

The discrete momentum values q_j of the $\alpha\nu$ (or c) pseudoparticles are such that $j = 1, 2, \ldots, N^*_{\alpha\nu}$ (or $j = 1, 2, \ldots, N_a$), where

$$N^*_{\alpha\nu} = N_{\alpha\nu} + N^h_{\alpha\nu}, \tag{B.6}$$

(or $N_a = N_c + N^h_c$) and $N^h_{\alpha\nu}$ (or N^h_c) is the number of unoccupied momentum values in the $\alpha\nu$ (or c) pseudoparticle band. Both the numbers $N^*_{\alpha\nu}$ and the associate numbers $N^h_{\alpha\nu}$ are dependent and fully determined by the values of the set of numbers N_c and $\{N_{\alpha\nu'}\}$, where $\alpha = c, s$ and $\nu' = \nu+1, \nu+2, \nu+3, \ldots,$ of occupied c and $\alpha\nu'$ pseudoparticle momentum values, respectively. The numbers $N^h_{\alpha\nu}$ of $\alpha\nu$ pseudoparticles and N^h_c of c pseudoparticle holes are given by

$$N^h_{\alpha\nu} = 2S_\alpha + 2\sum_{\nu'=\nu+1}^{\infty}(\nu'-\nu)N_{\alpha\nu'} = L_\alpha + 2\sum_{\nu'=\nu+1}^{\infty}(\nu'-\nu)N_{\alpha\nu'}, \tag{B.7}$$

and

$$N^h_c = N_a - N_c, \tag{B.8}$$

respectively. Here L_c and L_s are the numbers of Yang holons and HL spinons, respectively, and the η-spin and spin values $S_c = L_c/2$ and $S_s = L_s/2$, respectively, are also determined by the values of the pseudoparticle numbers and are given in Eqs. (9) and (10). The pseudoparticle and pseudoparticle–hole momentum distribution functions are such that

$$\sum_{j=1}^{N_a} N_c(q_j) = N_c, \qquad \sum_{j=1}^{N_a} N^h_c(q_j) = N_a - N_c, \tag{B.9}$$

$$\sum_{j=1}^{N^*_{\alpha\nu}} N_{\alpha\nu}(q_j) = N_{\alpha\nu}, \qquad \sum_{j=1}^{N^*_{\alpha\nu}} N^h_{\alpha\nu}(q_j) = N^h_{\alpha\nu}. \tag{B.10}$$

From the combination of Eqs. (9), (10), (B.7), (B.9), and (B.10) the number of $c\nu$ and $s\nu$ pseudoparticle holes (B.7) can be rewritten as follows

$$N^h_{c\nu} = N_a - N_c - \sum_{\nu'=1}^{\infty} \big(\nu + \nu' - |\nu - \nu'|\big) N_{c\nu'},$$
$$N^h_{s\nu} = N_c - \sum_{\nu'=1}^{\infty} \big(\nu + \nu' - |\nu - \nu'|\big) N_{s\nu'}. \tag{B.11}$$

For the ground-state CPHS ensemble subspace of densities in the ranges $0 \leqslant n \leqslant 1$ and $0 \leqslant m \leqslant n$ considered in Appendix C, the expression of the number $N^*_{\alpha\nu}$ given in Eqs. (B.6), (B.7), and (B.11) simplifies to

$$N^*_{c\nu} = (N_a - N), \qquad N^*_{s1} = N_\uparrow, \qquad N^*_{s\nu} = (N_\uparrow - N_\downarrow), \quad \nu > 1, \tag{B.12}$$

whereas N^*_c is given by $N^*_c = N_a$ for all energy eigenstates.

The $I^{\alpha\nu}_j$ numbers of Eq. (B.1) are integers (half-odd integers), if $N^*_{\alpha\nu}$ is odd (even) and thus the q_j discrete momentum values obey Eq. (B.2), are distributed symmetrically about zero, and are such that

$$|q_j| \leqslant q_{\alpha\nu}, \tag{B.13}$$

where

$$q_{\alpha\nu} = \frac{\pi}{L}\big[N^*_{\alpha\nu} - 1\big] \approx \frac{\pi N^*_{\alpha\nu}}{L}. \tag{B.14}$$

On the other hand, the I^c_j numbers of Eq. (B.1) are integers (half-odd integers), if $\frac{N_a}{2} - \sum_{\alpha=c,s}\sum_{\nu=1}^{\infty} N_{\alpha\nu}$ is odd (even) and the momentum values q_j obey Eq. (B.2) and are such that

$$q^-_c \leqslant q_j \leqslant q^+_c, \tag{B.15}$$

where

$$q^+_c = -q^-_c = \frac{\pi}{a}\left[1 - \frac{1}{N_a}\right], \tag{B.16}$$

for $\frac{N_a}{2} - \sum_{\alpha=c,s}\sum_{\nu=1}^{\infty} N_{\alpha\nu}$ even and

$$q^+_c = \frac{\pi}{a}, \qquad q^-_c = -\frac{\pi}{a}\left[1 - \frac{2}{N_a}\right], \tag{B.17}$$

for $\frac{N_a}{2} - \sum_{\alpha=c,s}\sum_{\nu=1}^{\infty} N_{\alpha\nu}$ odd.

Appendix C. The ground-state distributions and pseudoparticle energy bands

In this appendix we provide both the ground-state pseudoparticle momentum distribution functions and the expressions which define the pseudoparticle energy bands. The specific c and $\alpha\nu$ pseudoparticle momentum distribution functions which describe a ground state corresponding to values of the densities in the ranges $0 \leqslant n \leqslant 1/a$ and $0 \leqslant m \leqslant n$ read [18]

$$N_c^0(q) = \Theta\big(q_{Fc}^+ - q\big), \quad 0 \leqslant q \leqslant q_c^+,$$
$$N_c^0(q) = \Theta\big(q - q_{Fc}^-\big), \quad q_c^- \leqslant q \leqslant 0, \tag{C.1}$$

$$N_{s1}^0(q) = \Theta(q_{Fs1} - q), \quad 0 \leqslant q \leqslant q_{s1},$$
$$N_{s1}^0(q) = \Theta(q + q_{Fs1}), \quad -q_{s1} \leqslant q \leqslant 0 \tag{C.2}$$

and

$$N_{\alpha\nu}^0(q) = 0, \quad -q_{\alpha\nu} \leqslant q \leqslant q_{\alpha\nu}, \quad \alpha = c, s, \quad \nu \geqslant 1 + \delta_{\alpha,s}. \tag{C.3}$$

In these equations, the momentum limiting values q_{s1}, $q_{c\nu}$, and $q_{s\nu}$ for $\nu > 1$ are given in Eq. (B.14) of Appendix B and $q_c^{\pm}$ is defined in Eqs. (B.15)–(B.17) of the same appendix. Moreover, the expressions of the c pseudoparticle ground-state *Fermi* momentum values are given by

$$q_{Fc}^{\pm} = \pm q_{Fc}, \tag{C.4}$$

for both N and $\frac{N_a}{2} - \sum_{\alpha=c,s}\sum_{\nu=1}^{\infty} N_{\alpha\nu}$ even or odd and by

$$q_{Fc}^{+} = q_{Fc} + \frac{\pi}{L}, \qquad q_{Fc}^{-} = -q_{Fc} + \frac{\pi}{L}, \tag{C.5}$$

or by

$$q_{Fc}^{+} = q_{Fc} - \frac{\pi}{L}, \qquad q_{Fc}^{-} = -q_{Fc} - \frac{\pi}{L}, \tag{C.6}$$

for $\frac{N_a}{2} - \sum_{\alpha=c,s}\sum_{\nu=1}^{\infty} N_{\alpha\nu}$ even or odd and N odd or even, respectively, where

$$q_{Fc} = \frac{\pi}{a}\left[n - \frac{1}{N_a}\right] = 2k_F + O(1/L). \tag{C.7}$$

Furthermore, the $s1$ pseudoparticle ground-state *Fermi* momentum values read

$$q_{Fs1}^{\pm} = \pm q_{Fs1}, \tag{C.8}$$

for both $N_\uparrow$ and $N_\downarrow$ even or odd and

$$q_{Fs1}^{+} = q_{Fs1} + \frac{\pi}{L}, \qquad q_{Fs1}^{-} = -q_{Fs1} + \frac{\pi}{L}, \tag{C.9}$$

or

$$q_{Fs1}^{+} = q_{Fs1} - \frac{\pi}{L}, \qquad q_{Fs1}^{-} = -q_{Fs1} - \frac{\pi}{L}, \tag{C.10}$$

for $N_\uparrow$ even or odd and $N_\downarrow$ odd or even, respectively, where

$$q_{Fs1} = \frac{\pi}{a}\left[n_\downarrow - \frac{1}{N_a}\right] = k_{F\downarrow} + O(1/L). \tag{C.11}$$

In the particular case of the ground-state momentum function distributions (C.1)–(C.3), the momentum limiting values given by Eqs. (B.14)–(B.17) of Appendix B, including $1/L$ contributions, simplify to

$$q_c = \frac{\pi}{a}\left[1 - \frac{1}{N_a}\right] = \pi + O(1/L),$$
$$q_{s1} = \frac{\pi}{L}[N_\uparrow - 1] = k_{F\uparrow} + O(1/L), \tag{C.12}$$

and

$$q_{c\nu} = \frac{\pi}{L}[N_a - N - 1] = (\pi - 2k_F) + O(1/L), \tag{C.13}$$

$$q_{s\nu} = \frac{\pi}{L}[N_\uparrow - N_\downarrow - 1] = (k_{F\uparrow} - k_{F\downarrow}) + O(1/L), \quad \nu > 1. \tag{C.14}$$

The pseudoparticle energy bands read [18]

$$\epsilon_c(q) = a \int_Q^{k^0(q)} dk\, 2t\eta_c(k), \tag{C.15}$$

$$\epsilon_{s1}(q) = \int_\infty^{\Lambda^0_{s1}(q)} d\Lambda\, 2t\eta_{s,1}(\Lambda) + 2\mu_0 h, \tag{C.16}$$

$$\epsilon^0_{c\nu}(q) = \int_\infty^{\Lambda^{(0)}_{c\nu}(q)} d\Lambda\, 2t\eta_{c\nu}(\Lambda), \tag{C.17}$$

$$\epsilon^0_{s\nu}(q) = \int_\infty^{\Lambda^0_{s\nu}(q)} d\Lambda\, 2t\eta_{s\nu}(\Lambda). \tag{C.18}$$

Here $k^0(q)$, $\Lambda^0_{s1}(q)$, $\Lambda^{(0)}_{c\nu}(q)$, and $\Lambda^0_{s\nu}(q)$ are the values specific to the ground-state. These values are obtained by solution of the integral equations (13)–(15) in the particular case when the pseudoparticle momentum distribution functions are given by Eqs. (C.1)–(C.3). Moreover, the functions $2t\eta_c(k)$, $2t\eta_{c\nu}(\Lambda)$, and $2t\eta_{s\nu}(\Lambda)$ are defined by the following integral equations

$$2t\eta_c(k) = 2t\sin(ka) + \frac{4t\cos(ka)}{\pi} \int_{-B}^{B} d\Lambda\, \frac{U}{U^2 + (4t)^2(\Lambda - \sin(ka))^2} 2t\eta_{s1}(\Lambda), \tag{C.19}$$

$$2t\eta_{c\nu}(\Lambda) = -4t\,\mathrm{Re}\left(\frac{4t\Lambda - i\nu U}{\sqrt{(4t)^2 - (4t\Lambda - i\nu U)^2}}\right) + \frac{4t}{\pi}\int_{-Q}^{Q} dk\,\frac{\nu U}{(\nu U)^2 + (4t)^2(\sin(ka) - \Lambda)^2}2t\eta_c(k), \quad \nu \neq 0, \tag{C.20}$$

and

$$2t\eta_{s\nu}(\Lambda) = \frac{4t}{\pi}\int_{-Q}^{Q} dk\,\frac{\nu U}{(\nu U)^2 + (4t)^2(\sin(ka) - \Lambda)^2}2t\eta_c(k) - \frac{2t}{\pi U}\int_{-B}^{B} d\Lambda'\,\Theta^{[1]}_{1,\nu}\left(\frac{4t(\Lambda' - \Lambda)}{U}\right)2t\eta_{s1}(\Lambda'). \tag{C.21}$$

Here

$$\begin{aligned}\Theta^{[1]}_{\nu,\nu'}(x) &= \frac{d\Theta_{\nu,\nu'}(x)}{dx} \\ &= \delta_{\nu,\nu'}\left\{\frac{1}{\nu[1 + (\frac{x}{2\nu})^2]} + \sum_{l=1}^{\nu-1}\frac{2}{l[1 + (\frac{x}{2l})^2]}\right\} \\ &\quad + (1 - \delta_{\nu,\nu'})\left\{\frac{2}{|\nu - \nu'|[1 + (\frac{x}{|\nu-\nu'|})^2]} + \frac{2}{(\nu + \nu')[1 + (\frac{x}{\nu+\nu'})^2]}\right. \\ &\quad \left. + \sum_{l=1}^{\frac{\nu+\nu'-|\nu-\nu'|}{2}-1}\frac{4}{(|\nu - \nu'| + 2l)[1 + (\frac{x}{|\nu-\nu'|+2l})^2]}\right\},\end{aligned} \tag{C.22}$$

is the x derivative of the function given in Eq. (B.5) of Appendix B and the parameters Q and B are defined by

$$Q = k^0(2k_F), \qquad B = \Lambda^0_{s1}(k_{F\downarrow}). \tag{C.23}$$

The pseudoparticle momentum distribution functions (C.1)–(C.3) correspond to the ground state. We note that the pseudoparticle occupancies of such a state were studied in Refs. [18,19]. Combining the pseudoparticle language of these references with the holon and spinon description introduced in this paper, we find that the ground state is characterized by the following values for the pseudoparticle, $\pm 1/2$ holon, and $\pm 1/2$ spinon numbers

$$\begin{aligned}&M^0_{c,-1/2} = L^0_{c,-1/2} = 0, \qquad M^0_{c,+1/2} = L^0_{c,+1/2} = N_a - N, \\ &N^0_c = N, \qquad N^0_{c\nu} = 0,\end{aligned} \tag{C.24}$$

in the charge sector and

$$\begin{aligned}&M^0_{s,-1/2} = N^0_{s1} = N_\downarrow, \qquad M^0_{s,+1/2} = N_\uparrow, \\ &L^0_{s,-1/2} = N^0_{s\nu} = 0, \quad \nu \geqslant 2, \qquad L^0_{s,+1/2} = N_\uparrow - N_\downarrow,\end{aligned} \tag{C.25}$$

in the spin sector.

References

[1] E.H. Lieb, F.Y. Wu, Phys. Rev. Lett. 20 (1968) 1445;
P.B. Ramos, M.J. Martins, J. Phys. A 30 (1997) L195;
M.J. Martins, P.B. Ramos, Nucl. Phys. B 522 (1998) 413.
[2] M. Takahashi, Prog. Theor. Phys. 47 (1972) 69.
[3] M. Sing, U. Schwingenschlögl, R. Claessen, P. Blaha, J.M.P. Carmelo, L.M. Martelo, P.D. Sacramento, M. Dressel, C.S. Jacobsen, Phys. Rev. B 68 (2003) 125111;
R. Claessen, M. Sing, U. Schwingenschlögl, P. Blaha, M. Dressel, C.S. Jacobsen, Phys. Rev. Lett. 88 (2002) 096402.
[4] G.-H. Gweon, J.D. Denlinger, J.W. Allen, R. Claessen, C.G. Olson, H. Hochst, J. Marcus, C. Schlenker, L.F. Schneemeyer, J. Electron Spectrosc. Relat. Phenom. 117–118 (2001) 481.
[5] M. Granath, V. Oganesyan, D. Orgad, S.A. Kivelson, Phys. Rev. B 65 (2002) 184501;
E.W. Carlson, D. Orgad, S.A. Kivelson, V.J. Emery, Phys. Rev. B 62 (2000) 3422.
[6] J. Zaanen, O.Y. Osman, H.V. Kruis, Z. Nussinov, J. Tworzydlo, Philos. Mag. B 81 (2001) 1485.
[7] A.L. Chernyshev, S.R. White, A.H. Castro Neto, Phys. Rev. B 65 (2002) 214527.
[8] O.J. Heilmann, E.H. Lieb, Ann. N.Y. Acad. Sci. 172 (1971) 583;
E.H. Lieb, Phys. Rev. Lett. 62 (1989) 1201.
[9] C.N. Yang, Phys. Rev. Lett. 63 (1989) 2144.
[10] J.M.P. Carmelo, P.D. Sacramento, Phys. Rev. B 68 (2003) 085104.
[11] F.H.L. Essler, V.E. Korepin, K. Schoutens, Phys. Rev. Lett. 67 (1991) 3848.
[12] V.E. Korepin, cond-mat/0311056, Phys. Rev. Lett. (2004), in press.
[13] P.W. Anderson, in: The Theory of Superconductivity in the High-T_c Cuprates, Princeton Univ. Press, Princeton, NJ, 1997.
[14] J.C. Talstra, S.P. Strong, P.W. Anderson, Phys. Rev. Lett. 74 (1995) 5256;
J.C. Talstra, S.P. Strong, Phys. Rev. B 56 (1997) 6094.
[15] K. Penc, K. Hallberg, F. Mila, H. Shiba, Phys. Rev. Lett. 77 (1996) 1390.
[16] F.H.L. Essler, V.E. Korepin, Phys. Rev. Lett. 72 (1994) 908.
[17] Y. Mizuno, K. Tsutsui, T. Tohyama, S. Maekawa, Phys. Rev. B 62 (2000) R4769.
[18] J.M.P. Carmelo, N.M.R. Peres, Phys. Rev. B 56 (1997) 3717.
[19] J.M.P. Carmelo, P. Horsch, A.A. Ovchinnikov, Phys. Rev. B 45 (1992) 7899.
[20] A.B. Harris, R.V. Lange, Phys. Rev. 157 (1967) 295;
A.H. MacDonald, S.M. Girvin, D. Yoshioka, Phys. Rev. B 37 (1988) 9753.
[21] R.M. Barnett, et al., Phys. Rev. D 50 (1994) 1173.
[22] J.M.P. Carmelo, K. Penc, cond-mat/0311075.
[23] J.M.P. Carmelo, K. Penc, L.M. Martelo, P.D. Sacramento, J.M.B. Lopes dos Santos, R. Claessen, M. Sing, U. Schwingenschlögl, cond-mat/0307602.
[24] M. Takahashi, Prog. Theor. Phys. 46 (1971) 401.
[25] N.M.R. Peres, J.M.P. Carmelo, D.K. Campbell, A. Sandvik, Z. Phys. B 103 (1997) 217.