Cyclic RG theories and c-function behavior in finite size systems

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Outline:

1. Introduction: RG fixed points and limit cycles
2. Historical motivation and recent developments
3. The Glazek-Wilson model
4. Russian doll superconductors
5. Cyclic Kosterlitz-Thouless flows
6. Log-periodic behavior of c-function
7. Conclusions and prospects

References:

A. LeClair, J. M. Román, and G. Sierra,

- Log-periodic behavior of finite size effects in field theories with RG limit cycles, (hep-th/0312141)

- Russian doll renormalization group, Kosterlitz-Thouless flows, and the cyclic sine-Gordon model, (hep-th/0301142, to appear in NPB)

- Russian doll renormalization group and superconductivity, (cond-mat/0211338, to appear in PRB)

1. Introduction: RG fixed points and limit cycles

Renormalization Group (RG):

Central role in Particle Physics, Statistical Mechanics, and Condensed Matter

Explains the universal properties near critical points

Example: Specific heat C(T) of a spin system around T_c:

$$C(T) \sim (T - T_c)^{-\alpha}$$

α depends on the universality class

$\alpha = 0$ for 2D-Ising
$\alpha \sim 0.133$ for 3D-Ising

Scaling behavior is associated to fixed points of the RG transformations:

A Hamiltonian with a spacing $\sim 1/\Lambda \rightarrow H = H(\Lambda)$

Under a scale transformation $\Lambda \rightarrow \Lambda'$ with $b = \Lambda/\Lambda'$ the Hamiltonian is transformed:

$$H' = R_b(H)$$

The Hamiltonian remains invariant at fixed points:

$$H^* = R_b(H^*)$$

Around fixed points the Hamiltonian can be expanded as:

$$H = H^* + \sum_k g_k O_k$$

O_k are the scaling fields

g_k are the coupling constants

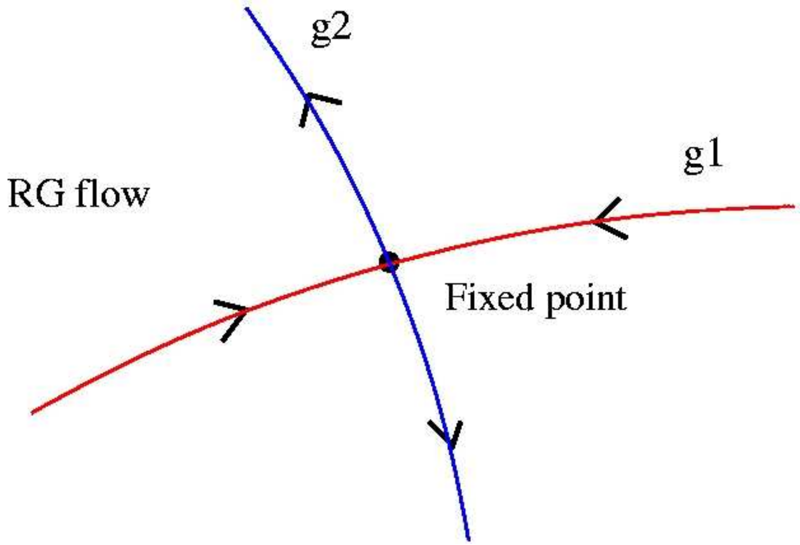
g_k transform under RG:

$$g'_k = b^{y_k} g_k$$

$\Rightarrow$ Classification of scaling fields O_k

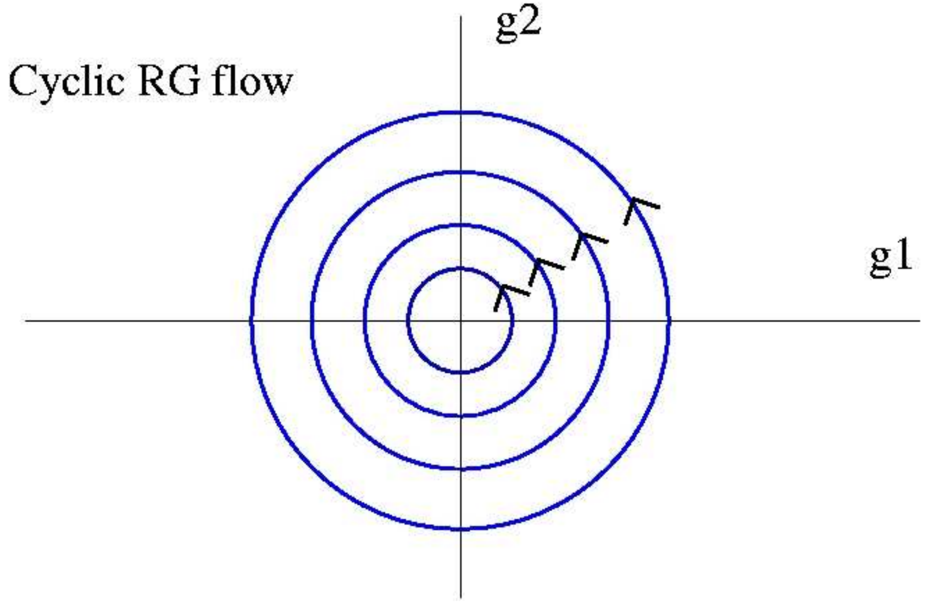
$$O_k : \begin{cases} y_k < 0 & \text{irrelevant} \\ y_k > 0 & \text{relevant} \\ y_k = 0 & \text{marginal} \end{cases}$$

A model with one irrelevant, g_1 , and one relevant, g_2 couplings presents a RG flow



y_k are real eigenvalues of the RG transformation

Complex eigenvalues $y_k = ih \Rightarrow \begin{cases} \text{Complex couplings } g = g_1 + ig_2 \\ \text{are periodic in RG scale s} \end{cases}$



$$g(s) = e^{ihs} g_0, \quad (b = e^s)$$

Period: $\lambda = \dfrac{2\pi}{h}$

$$\boxed{g(s+\lambda) = g(s)}$$

The renormalized Hamiltonian repeats itself after a finite scale transformation

$$b = \exp\left(\frac{2\pi}{h}\right)$$

RG limit cycles $\Longrightarrow$ self-similarity

2. Historical motivation and recent developments

In 1971 K. G. Wilson suggested the possible existence of limit cycles in strong interactions involving two or more coupling constants:

“The $e^+ - e^-$ annihilation experiments above 1 GeV may distinguish a fixed point from a limit cycle or other asymptotic behavior”

For limit cycles:

“one will see perpetual oscillations in the $e^+ - e^-$ total hadronic cross section in the limit of large momentum transfer q^2

This effect has not been seen, but it remains as a possibility for other Physical systems

Recent developments:

- In 1993 Bedaque, Hammer and van Kolck studied a QM Hamiltonian in Nuclear Physics with two and three body delta function potentials that exhibits limit cycle behavior.

- In 2001 Bernard and LeClair found cyclic Kosterlitz-Thouless flows in an anisotropic current-current WZW model.

- In 2002 Glazek and Wilson defined a discrete QM Hamiltonian with two couplings whose RG has limit cycles and chaotic behavior.

- In 2002/03 LeClair, Román and Sierra proposed an extension of the BCS model of superconductivity with limit cycle behavior and a S-matrix theory for the "cyclic sine-Gordon model".

3. The Glazek-Wilson model

A regularized QM discrete model, with IR and UV cut-offs:

$$H_{n,m}(g_N, h_N) = b^{n+m}\left[\delta_{n,m} - g_N - i h_N \,\mathrm{sgn}(n-m)\right] \qquad (-M \le n, m \le N)$$

The RG analysis starts from the Schrödinger equation

$$\sum_{m=-M}^{N} H_{n,m}\psi_m = E\psi_n \quad \rightarrow \quad \psi_N = \psi_N\left(\psi_{n<N}\right) \quad \Rightarrow \quad H'_{n,m}(g'_{N-1}, h'_{N-1})$$

The elimination of the highest energy component defines a new Hamiltonian in terms of the new couplings:

$$g_{N-1} = g_N + \frac{g_N^2 + h_N^2}{1 - g_N}, \qquad h_{N-1} = h_N \equiv h \quad \Longrightarrow \quad \text{RG invariant}$$

After p iterations: $$g_{N-p} = h\tan\left(\tan^{-1}\left(\frac{g_N}{h}\right) + p\tan^{-1}h\right)$$

Choosing h such that $p\tan^{-1}h = \pi$ $\longrightarrow$ limit cycle

If $\pi/\tan^{-1}h$ the flow of g_N is chaotic

Main physical consequence:

Existence of an infinite number of bound states related by the period of the flow

$$\boxed{E_{n+1}(g) = e^{-\lambda}E_n(g), \qquad e^{\lambda} = b^p}$$

which accumulate at the origin, $E_0 < E_1 < \ldots < E_\infty = 0$

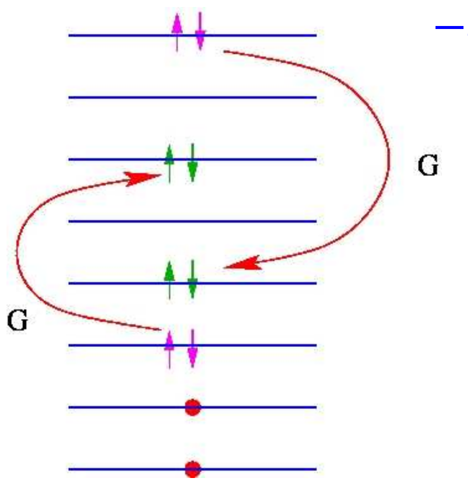
This is a generic feature of models with limit cycles and bound states
(Russian doll scaling)

4. Russian doll superconductors

The standard BCS model of superconductivity

$$H = \sum_{i=1\,\sigma=\uparrow,\downarrow}^{N} \varepsilon_i b_i^\dagger b_i - \sum_{i,j=1}^{N} V_{ij} b_i^\dagger b_j \qquad V_{ij} = V_{ji}^*$$

$b_i^\dagger = c_{i,\uparrow}^\dagger c_{i,\downarrow}^\dagger \quad b_i = c_{i,\downarrow} c_{i,\uparrow}$ COOPER PAIR OPERATORS



$-\omega < \varepsilon_i < \omega$ are equally spaced energy levels with level spacing 2δ

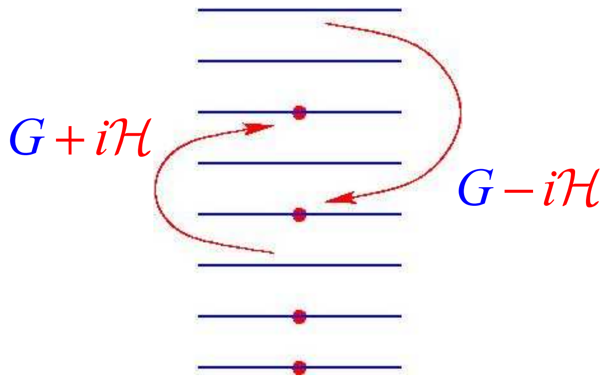
The standard choice is $V_{ij} = G > 0$

A single condensate characterized by a gap

$$\Delta_0 \sim 2\omega \exp(-1/g), \qquad g = G/\delta$$

Extended BCS model (by LeClair, Román and Sierra, cond-mat/0211338)

$$\boxed{V_{ij} = \begin{cases} G + i\mathcal{H} & \text{if} \quad \varepsilon_i > \varepsilon_j \\ G & \text{if} \quad \varepsilon_i = \varepsilon_j \\ G - i\mathcal{H} & \text{if} \quad \varepsilon_i < \varepsilon_j \end{cases}}$$



$$G = g\delta$$
$$\mathcal{H} = h\delta$$

Mean-field formulation leads to the gap equation:

$$\tilde{\Delta}_i = \sum_j \frac{V_{ij}\tilde{\Delta}_j}{\sqrt{\varepsilon_j^2 + \Delta_j^2}} \qquad \tilde{\Delta}_i \equiv \Delta_i e^{i\phi_i}$$

In the continuum limit

$$\begin{cases} \Delta_i \to \Delta(\varepsilon) \\ \phi_i \to \phi(\varepsilon) \end{cases}$$

$$\Delta(\varepsilon) = \Delta = const$$

$$\phi(\varepsilon) = h \sinh^{-1}\frac{\varepsilon}{\Delta}$$

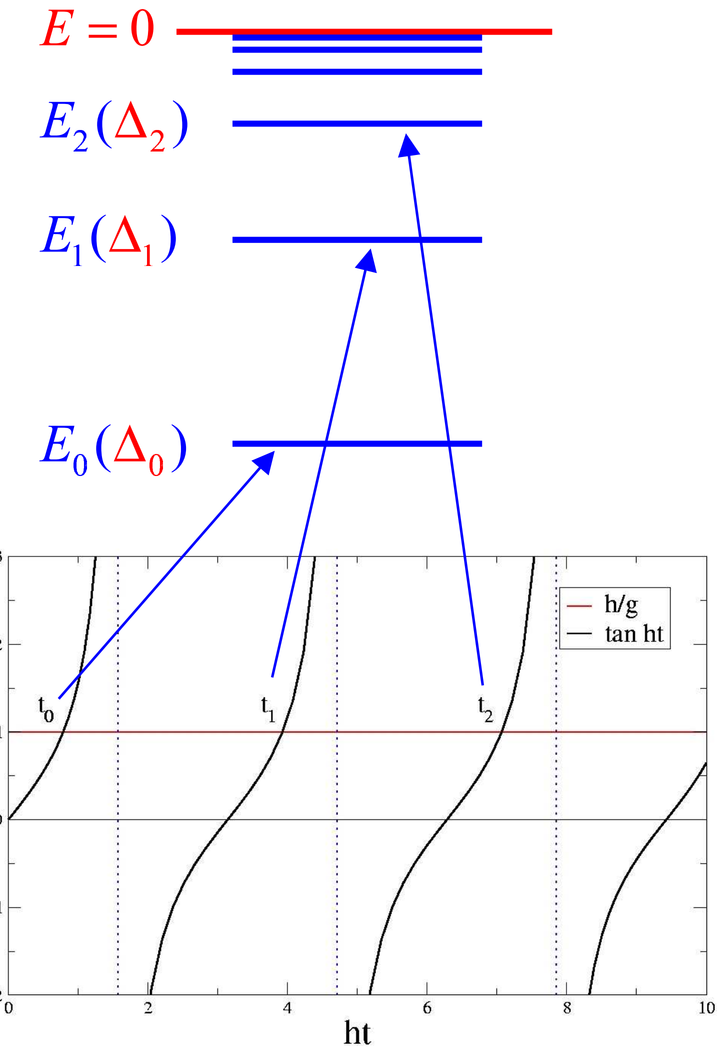
General solution to the gap equation:

$$\Delta_n = \frac{\omega}{\sinh t_n} \qquad t_n = t_0 + \frac{n\pi}{h} \qquad n = 0, 1, 2, \ldots, \infty$$

$$\tan(ht_n) = \frac{h}{g}$$

Infinite number of solutions

Spectrum



In the low energy regime:

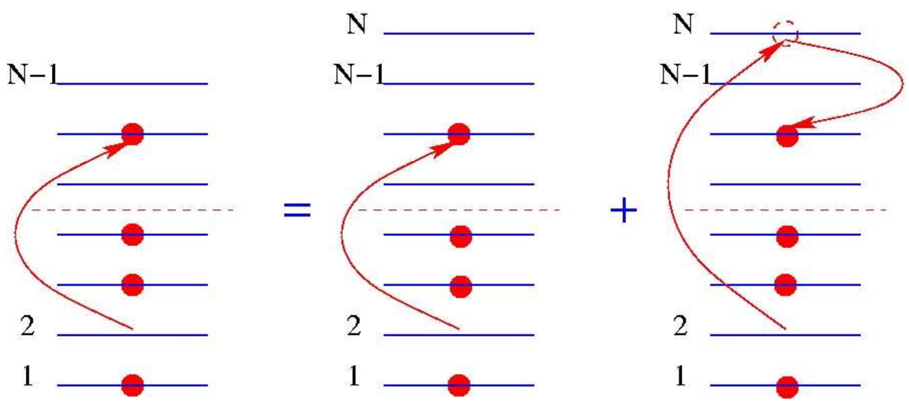
$\Delta_n \ll \omega \quad \Rightarrow$ Russian doll scaling

$$\Delta_n \sim 2N\delta \exp(-t_0 - n\pi / h)$$
$$\Delta_{n+1} = e^{-\lambda} \Delta_n$$

$$\lambda = \frac{\pi}{h}$$

RG with limit cycles

Renormalization group of BCS $H(G_N, \mathcal{H}_N) \longrightarrow H(G_{N-1}, \mathcal{H}_{N-1})$



$$G_{N-1} + i\mathcal{H}_{N-1} = G_N + i\mathcal{H}_N + \frac{1}{N\delta}(G_N + i\mathcal{H}_N)(G_N - i\mathcal{H}_N) \quad \rightarrow \quad \mathcal{H}_N = \mathcal{H}_{N-1}$$

$\boxed{\mathcal{H}_N \text{ is an RG invariant}}$

In the large-N limit we define a RG scale: $\boxed{s \equiv \log \frac{N_0}{N}}$

$$G = g\delta$$
$$\mathcal{H} = h\delta$$

Beta function: $\boxed{\frac{dg}{ds} = g^2 + h^2}$ $\boxed{h = const}$

RG solution: $$\boxed{g(s) = h\tan\left[hs + \tan^{-1}\frac{g_0}{h}\right]}$$ $g_0 = g(N_0)$

Cyclic behavior of the RG: $$\boxed{g(s+\lambda) = g(s) \quad \Leftrightarrow \quad g(e^{-\lambda}N) = g(N) \qquad \lambda \equiv \frac{\pi}{h}}$$

Synchronicity between the mean-field and the RG:

Using the MF solution: $\tan(ht_n) = \frac{h}{g}$ $\qquad \Delta_n = \frac{\omega}{\sinh t_n}$

we obtain a running coupling constant: $$\boxed{g(s) = h\tan\left[h\left(s - t_n\right) + \frac{\pi}{2}\right]}$$

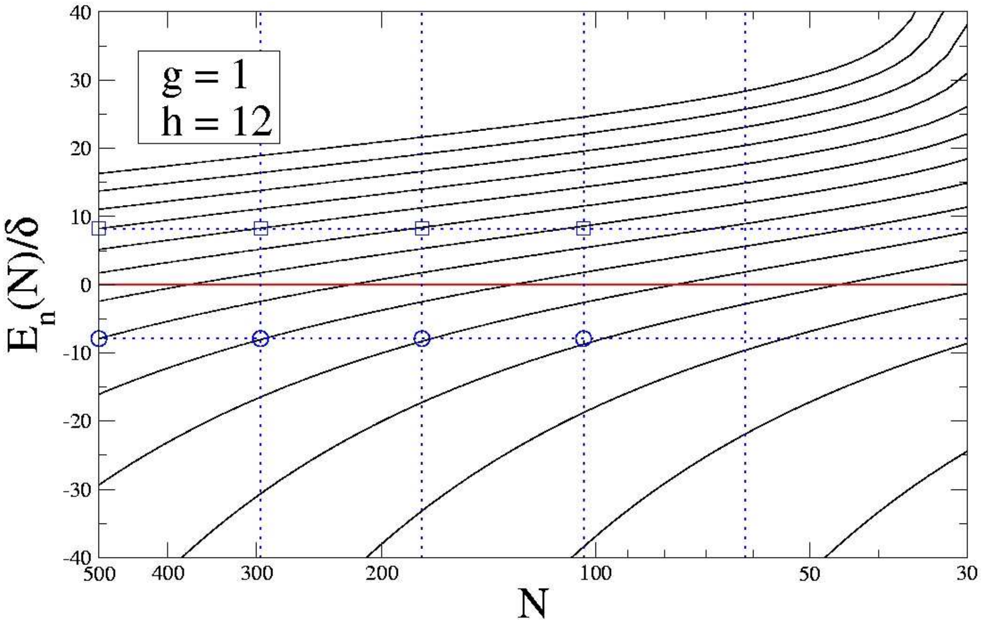
At $s = t_n$ the coupling $g(s)$ jumps from ∞ to $-\infty$.
Every jump eliminates a condensate from the spectrum.

Numerical work: One Cooper Pair problem

The BCS Hamiltonian becomes the N×N matrix

$$H_{jk} = \varepsilon_j \delta_{jk} - (G + i\mathcal{H}\,\mathrm{sgn}(j-k))$$

Exact eigenstates of one-pair Hamiltonian for N levels (N from 500 down to 30)



Cyclicity of the spectrum
(self-similarity)

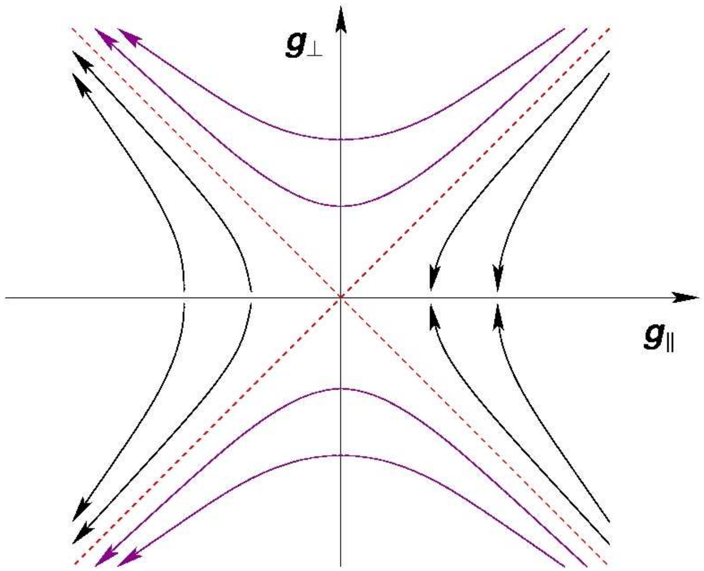
$$E_{n+1}(N) = E_n(e^{-\lambda_1} N)$$

$$E_{n+1}(N) = e^{-\lambda_1} E_n(N)$$

$$\boxed{\lambda_1 = \frac{2\pi}{h}}$$

5. Cyclic regime of Kosterlitz-Thouless flows

The Kosterlitz-Thouless flows arise in a multitude of systems



As a continuum field theory it corresponds to anisotropic current-current interactions for SU(2), given by the action

$$S = S_{wzw} + \int \frac{d^2x}{2\pi}\left[4g_\perp\left(J^+\bar{J}^- + J^-\bar{J}^+\right) - 4g_\parallel J_3\bar{J}_3\right]$$

Beta function at one-loop: $\frac{dg_{\parallel}}{dl} = -4g_{\perp}^2, \qquad \frac{dg_{\perp}}{dl} = -4g_{\perp}g_{\parallel} \qquad (L = \exp(l))$

There exists an RG invariant: $$\boxed{Q = g_{\parallel}^2 - g_{\perp}^2 \equiv -\frac{h^2}{16}}$$

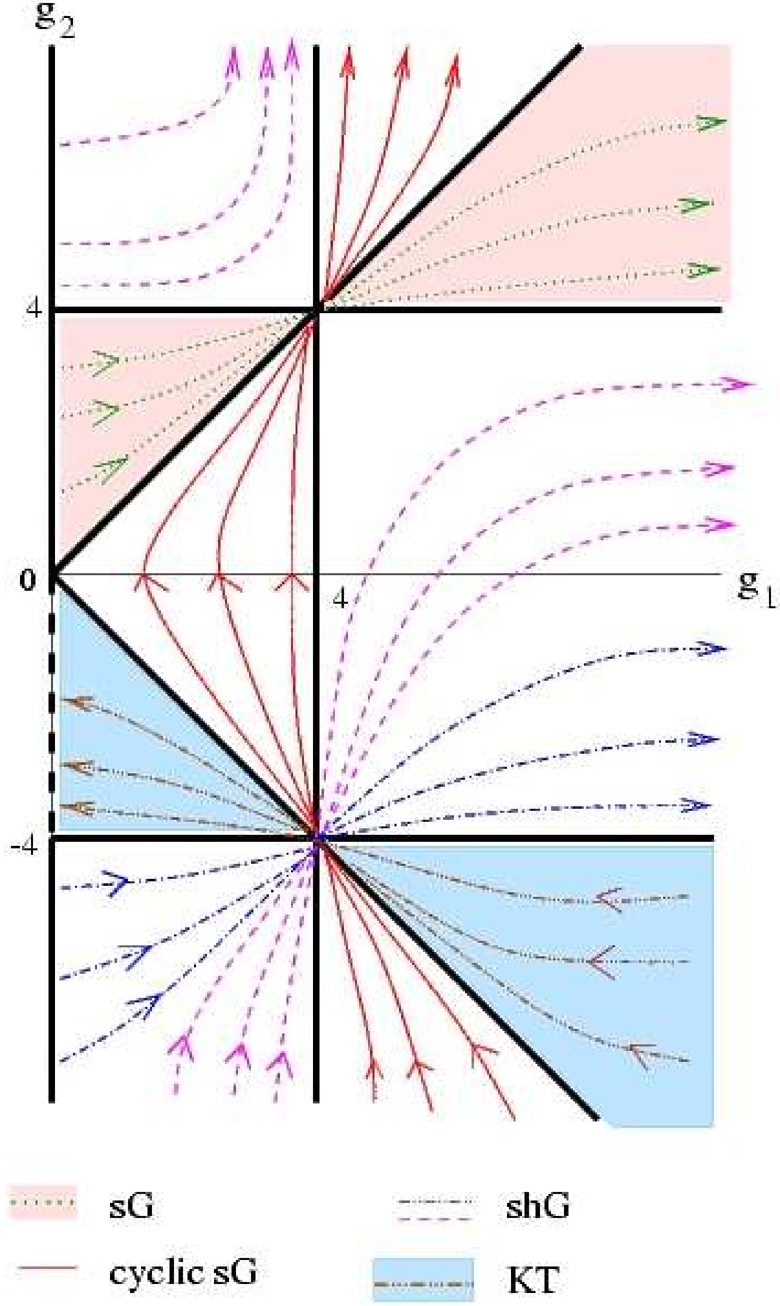
Eliminating $g_{\perp}$ and using the definition of h:

$$\frac{dg_{\parallel}}{dl} = -4\left(g_{\parallel}^2 + \frac{h^2}{16}\right) \quad \Longrightarrow \quad g_{\parallel} = -\frac{h}{4}\tan\left[h(l - l_0)\right]$$

This solution of the one-loop beta function shows periodic behavior:

$$\boxed{g_{\parallel}(e^{\lambda}L) = g_{\parallel}(L) \qquad \lambda_{1-loop} = \frac{\pi}{h}}$$

Does this behavior persist non-perturbatively?



(Bernard and LeClair)

5. Log-periodic behavior of c-function

The cyclic regime of the KT flows ca be mapped onto the sine-Gordon theory:

$$S = \int \frac{d^2x}{4\pi}\left[\frac{1}{2}(\partial\phi)^2 + \Lambda\cos b\phi\right] \qquad b^2 = \frac{2}{1+ih/2}$$

Benard and LeClair a quantum affine symmetry for the cyclic regime:

$$q = \exp(2\pi i/b^2) = -\exp(\pi h/2) \qquad \text{which is real}$$

Unitarity and crossing yield a unique S-matrix satisfying

$$\boxed{S\left(\beta + \frac{2\pi}{h}\right) = S(\beta)}$$

which is a signature of a cyclic RG.

Finite size effects in free energy

The ground state energy in a box of size R: $$E(R) = -\frac{\pi}{6}\frac{c_{eff}(R)}{R}$$

where $c_{eff}(R)$ is the effective central charge: $\begin{cases} c=1 & \text{Boson} \\ c=1/2 & \text{Majorana fermion} \end{cases}$

Following Destri and de Vega (DdV):

$$c_{eff}(R) = \frac{6mR}{\pi^2}\operatorname{Im}\int_{-\infty}^{\infty} d\beta \sinh(\beta + i\alpha)\log\left(1+e^{iZ(\beta+i\alpha)}\right)$$

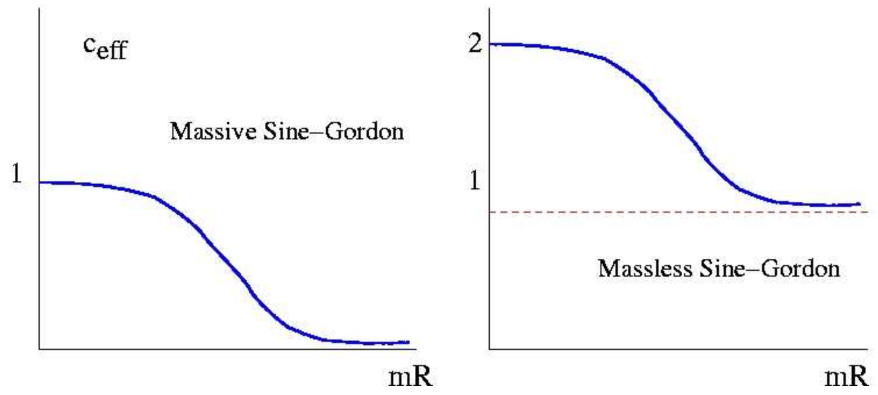
where $Z(\beta)$ satisfies the integral equation

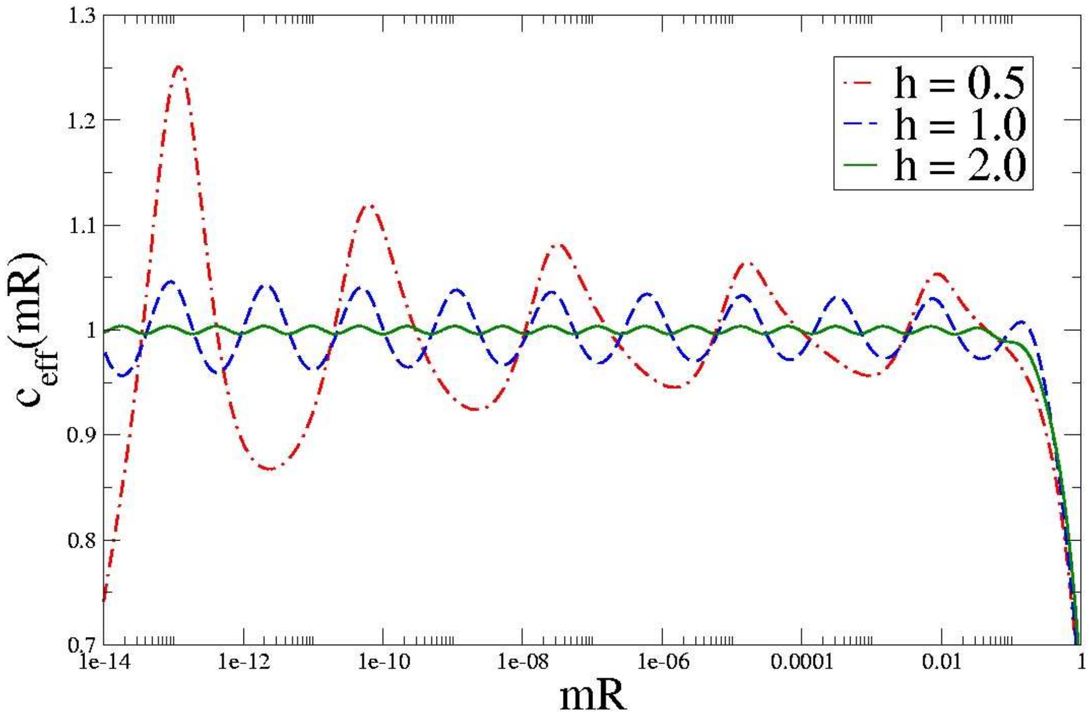
$$Z(\beta) = mR\sinh\beta + 2\operatorname{Im}\int d\beta' \, G(\beta-\beta'-i\alpha)\log\left(1+e^{iZ(\beta'+i\alpha)}\right)$$

with $$G(\beta) = \frac{1}{2\pi i}\frac{\partial}{\partial\beta}\log S_{++}^{++}(\beta)$$

DdV equations solutions

For regular sine-Gordon theory, either massive or massless





Cyclic sine-Gordon theory (massive regime)

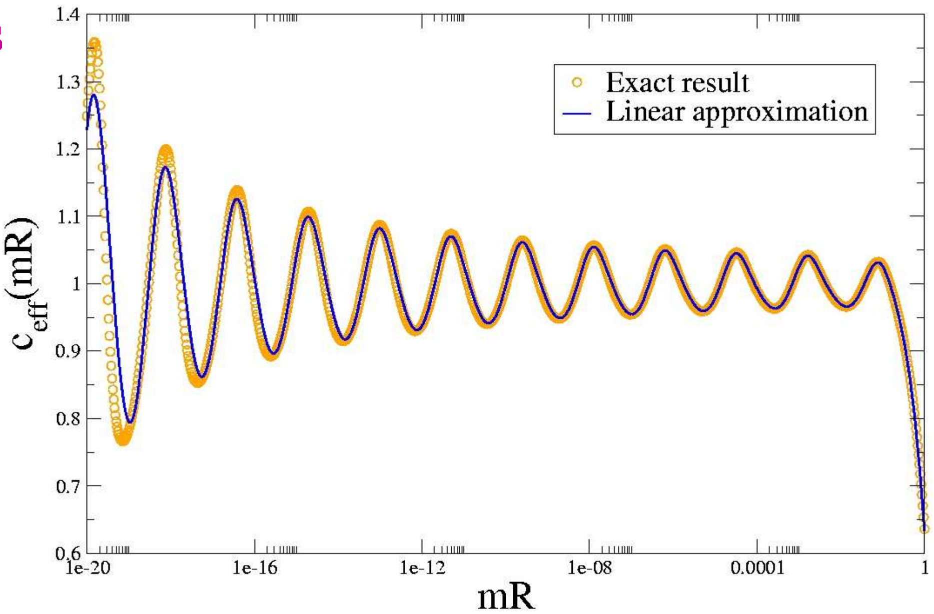
Main features:

•Oscillates in $s = \log(2\pi / mR)$ with a period π / h

•The amplitude of the oscillations around $c = 1$ increases from the IR to the UV

Linear approximation of DdV equations

$$c_1 = \frac{24}{\pi} \sum_{n=1}^{\infty} \frac{\mathrm{Im}\left[e^{2inhs} (1 - 2^{inh}) \zeta^2(-inh) \right]}{n\left(1 + e^{\pi nh} - \frac{h}{\pi}(s - \gamma - \log 2) \right)}$$

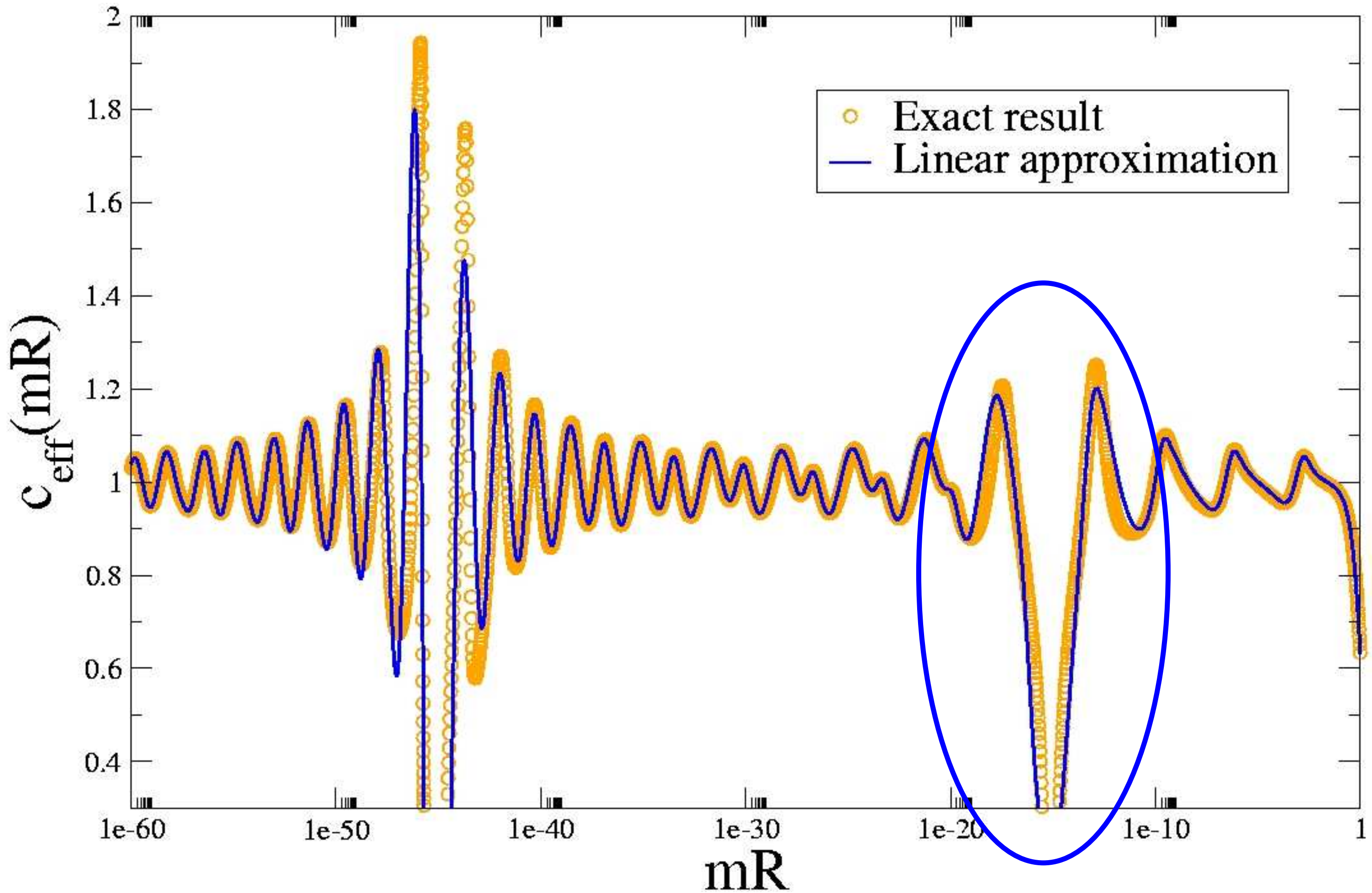


Blows up at critical distances:

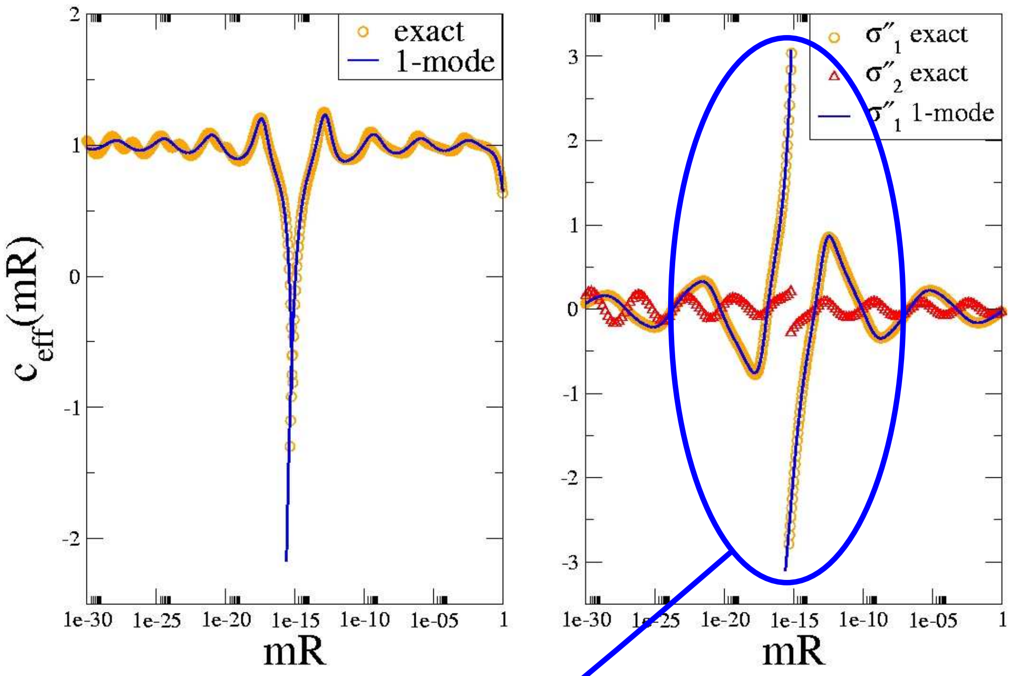
$$mR_{c,n} = \pi e^{-\gamma} \exp\left(-\frac{\pi}{h}(1 + e^{\pi nh}) \right) \qquad n = 1, 2, 3, \ldots \qquad (???)$$

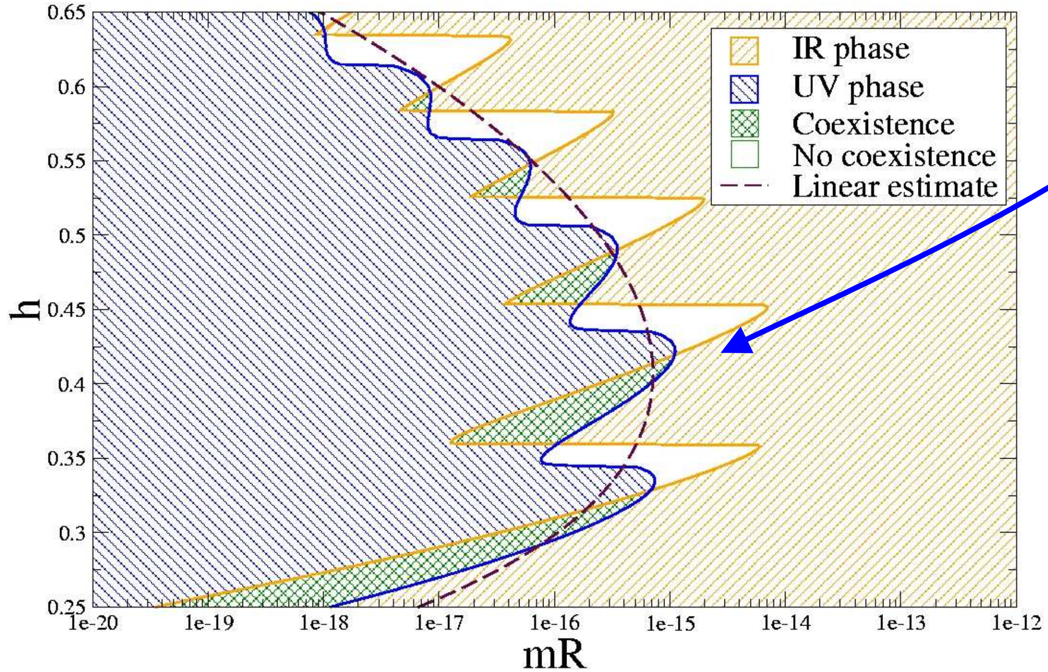
Cascade of critical distances

Multiple critical points deeper and deeper in the UV



Close-up of the
critical region





Coexistence regions

The complete solution
does not blow up

There is an overlap of
different solutions

7. Conclusions and prospects

We presented a range of models showing cycle limits in their RG flow:

1. The QM Glazek-Wilson model.
2. The Russian doll BCS model for superconductors.
3. WZW and sine-Gordon models.

The cyclic behavior of the RG flow affects the finite size properties of the systems:

1. The massive cyclic sine-Gordon model presents a limit cycle flow, which translate into a log-periodic behavior of its c-function.
2. The massless cyclic sine-Gordon model is expected flow in a perfect cycle, not just reach the cycle as a limit.

Prospects:

1. Is there a microscopic origin of the h-interaction in BCS?

2. For usual S-matrix models the UV is described by a CFT with a fix value of $c_{eff,UV} = c_{CFT}$.
   - What the "field theory" that describes the UV regime of the cyclic sine-Gordon model?
   - Continuous scale invariance must be replaced by discrete scale invariance, which is related to complex conformal weights and exponents.
   - This is reminiscent to self-similar fractals, which are characterized by log-periodic behavior.

3. What is the fate of the c-theorem in theories with limit cycles or chaotic RG?

4. Find new models with cyclic RG's, which may lead to new Physics.