# The elementary excitations of the BCS model in the canonical ensemble

Germán Sierra,[1*] José María Román[1] and Jorge Dukelsky[2]
[1]*Instituto de Física Teórica, CSIC/UAM, Madrid, Spain.*
[2]*Instituto de Estructura de la Materia, CSIC, Madrid, Spain*
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We summarize previous works on the exact ground state and the elementary excitations of the exactly solvable BCS model in the canonical ensemble. The BCS model is solved by Richardson equations, and, in the large coupling limit, by Gaudin equations. The relationship between this two kinds of solutions are used to classiffy the excitations.

## I. INTRODUCTION

One of the most fundamental problems in a many body system is to know which are the ground state (GS) and the low energy excited states, which determine the thermodynamic properties and the response to external fields. In most cases the excited states can be understood in terms of a collection of elementary excitations characterized by their statistics, discrete quantum numbers as spin, charge and dispersion relation. In the pairing model of superconductivity, proposed by Bardeen, Cooper and Schrieffer in 1957, this problem was solved long ago in the grand canonical ensemble for a large number of particles [1, 2]. However recent studies, motivated by the fabrication of ultrasmall metallic grains, show that the grand canonical BCS solution deviates strongly from the exact numerical or analytical solution for systems with a fixed and small number of particles (for a review see [3]). Most of the previous studies have focused on the ground state of the BCS system, and some of its excitations. In this contribution we shall review further progress concerning the understanding of the full excitation spectrum [4].

## II. THE GRAND CANONICAL BCS ANSATZ

The BCS model of superconductivity is characterized by the energy levels of the electrons and the scattering potential between them. When the latter is a constant, the BCS model is exactly solvable à la Bethe [5] and integrable [6]. This is the model we shall used to study the GS and excitations. The Hamiltonian of the reduced BCS is defined by [3]

$$H_{BCS} = \frac{1}{2} \sum_{j,\sigma=\pm} \varepsilon_{j\sigma} c^\dagger_{j\sigma} c_{j\sigma} - G \sum_{j,j'} c^\dagger_{j+} c^\dagger_{j-} c_{j'-} c_{j'+} \ , \tag{1}$$

where $c_{j,\pm}$ (resp. $c^\dagger_{j,\pm}$) is an electron annihilation (resp. creation) operator in the time-reversed states $|j,\pm\rangle$ with energies $\varepsilon_j/2$, and $G$ is the BCS dimensionful coupling constant. The sums in (1) run over a set of $N$ doubly degenerate energy levels $\varepsilon_j/2$ $(j = 1, \ldots, N)$. We adopt Gaudin's notation, according to which $\varepsilon_j$ denotes the energy of a pair occupying the level $j$ [7]. To fix ideas we shall use the so called equally space or picket fence model, that is the one employed in the study of ultrasmall superconducting grains [3]. This model is given by the choice $\varepsilon_j = d(2j - N - 1)$, where $d = \omega/N$ is the single particle energy level spacing and $\omega/2$ is the Debye energy. The coupling $G$ can be written as $G = gd$, where $g$ is dimensionless.

The BCS ansatz for the GS of this Hamiltonian in the grand canonical ensemble is given by [1, 2]

$$|BCS\rangle = \prod_j (u_j + v_j \ c^\dagger_{j,+} c^\dagger_{j,-}) \ |0\rangle, \tag{2}$$

where $|0\rangle$ is the Fock vacuum of the electron operators and $u_j, v_j$ are the BCS variational parameters given by

$$u_j^2 = \frac{1}{2}\left(1 + \frac{\xi_j}{E_j}\right), \qquad v_j^2 = \frac{1}{2}\left(1 - \frac{\xi_j}{E_j}\right), \tag{3}$$

$$\xi_j = \varepsilon_j - \varepsilon_0 - G, \qquad E_j = \sqrt{\xi_j^2 + \Delta^2}. \tag{4}$$

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In these eqs. $\varepsilon_0$ and $\Delta$ are twice the chemical potential and the BCS gap, which are found by solving the following equations:

$$\frac{1}{G} = \sum_j \frac{1}{E_j}, \qquad M = \frac{1}{2} \sum_j \left( 1 - \frac{\xi_j}{E_j} \right), \tag{5}$$

where $M$ is the number of electron pairs. The most studied case in the literature corresponds to the half-filled situation, where the number of electrons, $N_e = 2M$, equals the number of levels $N$ [3]. In this case the solution of eqs. (5) in the large $N$ limit is given by $\Delta = \omega / \sinh(1/g)$ and $\varepsilon_0 = 0$.

The excited states in the grand canonical ensemble can be obtained acting on the GS ansatz (2) with the Bogoliubov operators $\gamma_{j,\sigma}$ ($\sigma = \pm$)

$$\gamma_{j_1,\sigma_1} \dots \gamma_{j_n,\sigma_n} |BCS\rangle, \qquad \gamma_{j,\pm} = u_j \, c_{j\pm} \mp v_j \, c^\dagger_{j\mp}, \tag{6}$$

and have an energy $\frac{1}{2}(E_{j_1} + \dots + E_{j_2})$. Recall that in our conventions $E_j$ is twice the quasiparticle energy, thus the factor 1/2 in the energy of the states (6).

## III. PAIR-HOLE REPRESENTATION OF BCS MODEL

Every energy level $j = 1, \dots, N$ has four possible states given by

$$\begin{aligned} |0\rangle : &\quad \text{empty}, \\ c^\dagger_{j\sigma} |0\rangle : &\quad \text{singly occupied}, \\ c^\dagger_{j+} c^\dagger_{j-} |0\rangle : &\quad \text{doubly occupied}. \end{aligned} \tag{7}$$

An important property of the Hamiltonian (1) is the "blocking" of levels which are singly occupied, which means that these levels decouple from the rest of the system. This is a consequence of the equation

$$H_{BCS} \, c^\dagger_{j\sigma} |\psi\rangle = \frac{\varepsilon_j}{2} \, c^\dagger_{j\sigma} \, |\psi\rangle + c^\dagger_{j\sigma} \, H_{BCS} |\psi\rangle, \tag{8}$$

where the state $|\psi\rangle$ does not contain the operators $c^\dagger_{j\sigma}$. Thus the singly occupied levels only contribute with their kinetic energy and one can study the dynamics on those levels which are either empty or doubly occupied.

Let us call $\mathcal{H}_{N,M}$ the Hilbert space of states of $M$ pairs distributed among $N$ different energy levels. To describe $\mathcal{H}_{N,M}$ let us define the hard-core boson operators

$$b_j = c_{j,-} c_{j,+}, \qquad b^\dagger_j = c^\dagger_{j,+} c^\dagger_{j,-}, \qquad N_j = b^\dagger_j b_j, \tag{9}$$

which satisfy the commutation relations

$$[b_j, b^\dagger_{j'}] = \delta_{j,j'} \, (1 - 2N_j). \tag{10}$$

The Hamiltonian (1) restricted to the Hilbert space $\mathcal{H}_{N,M}$ can then be written as

$$H_{BCS} = \sum_j \varepsilon_j b^\dagger_j b_j - G \sum_{j,j'} b^\dagger_j b_{j'} \; . \tag{11}$$

A basis of states of $\mathcal{H}_{N,M}$ is given by

$$|I\rangle = \prod_{j \in I} b^\dagger_j \, |0\rangle, \tag{12}$$

where $I$ denotes a set of $M$ different integers ranging from 1 to $N$. Thus the dimension of $\mathcal{H}_{N,M}$ is given by the combinatorial number $C^N_M$. A convenient pictorial representation of the singly and occupied states is given by [4]

$$\circ \leftrightarrow |0\rangle, \qquad \bullet \leftrightarrow b^\dagger_j \, |0\rangle. \tag{13}$$

At $G = 0$ the ground state of (11) is the Fermi sea obtained filling all the levels below the Fermi energy $\varepsilon_F$. The set $I_0$ is given in this case by $I_0 = \{1, 2, \dots, M\}$ (see fig. 1). Similarly the lowest energy excited state at $G = 0$ corresponds to the choice $I_1 = \{1, 2, \dots, M-1, M+1\}$ (see fig. 2).

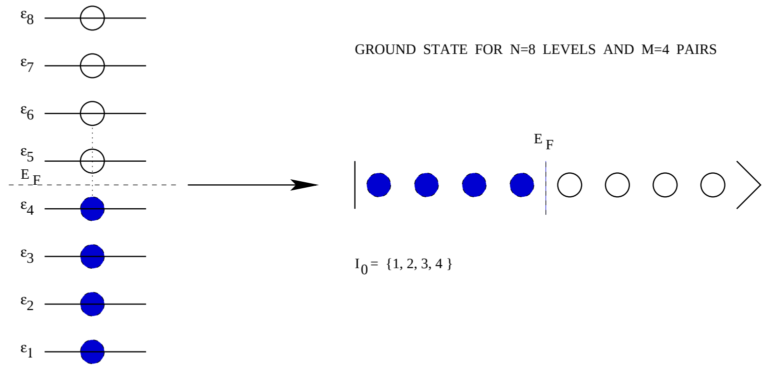


FIG. 1: Pair-hole representation of the ground state at $G = 0$ for $N = 2M = 8$.

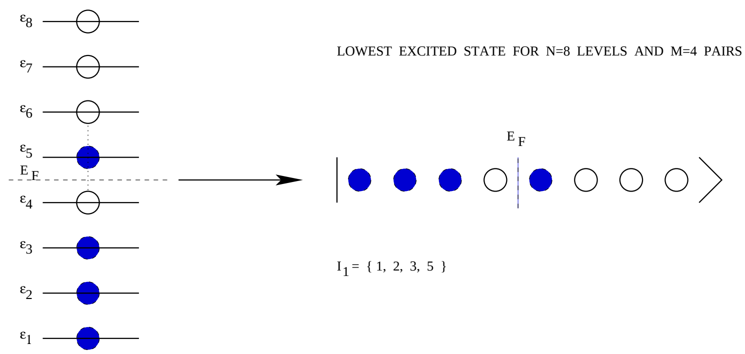


FIG. 2: Pair-hole representation of the lowest energy excited state at $G = 0$ for $N = 2M = 8$.

## IV. EXACT SOLUTION OF THE BCS MODEL IN THE CANONICAL ENSEMBLE

In 1963 Richardson showed that the eigenstates of the Hamiltonian (11) with $M$ pairs have the (unnormalized) product form [5]

$$|M\rangle = \prod_{\nu=1}^{M} B_\nu^\dagger |\text{vac}\rangle, \qquad B_\nu^\dagger = \sum_{j=1}^{N} \frac{1}{\varepsilon_j - E_\nu}\, b_j^\dagger \,, \tag{14}$$

where the parameters $E_\nu$ ($\nu = 1, \ldots, M$) are, in general, complex solutions of the $M$ coupled algebraic equations

$$\frac{1}{G} = \sum_{j=1}^{N} \frac{1}{\varepsilon_j - E_\nu} - \sum_{\mu=1(\neq\nu)}^{M} \frac{2}{E_\mu - E_\nu} \,, \qquad \nu = 1, \ldots, M, \tag{15}$$

which play the role of Bethe ansatz equations for this problem [8, 9, 10, 11, 12, 13, 14, 15]. The energy of these states is given by the sum of the auxiliary parameters $E_\nu$, i.e.

$$E(M) = \sum_{\nu=1}^{M} E_\nu. \tag{16}$$

The ground state of $H_{BCS}$ is given by the solution of eqs. (15) which gives the lowest value of $E(M)$. The (normalized) states (14) can also be written as [16]

$$|M\rangle = \frac{C}{\sqrt{M!}} \sum_{j_1,\cdots,j_M} \psi(j_1, \ldots, j_M) b_{j_1}^\dagger \cdots b_{j_M}^\dagger |\text{vac}\rangle, \tag{17}$$

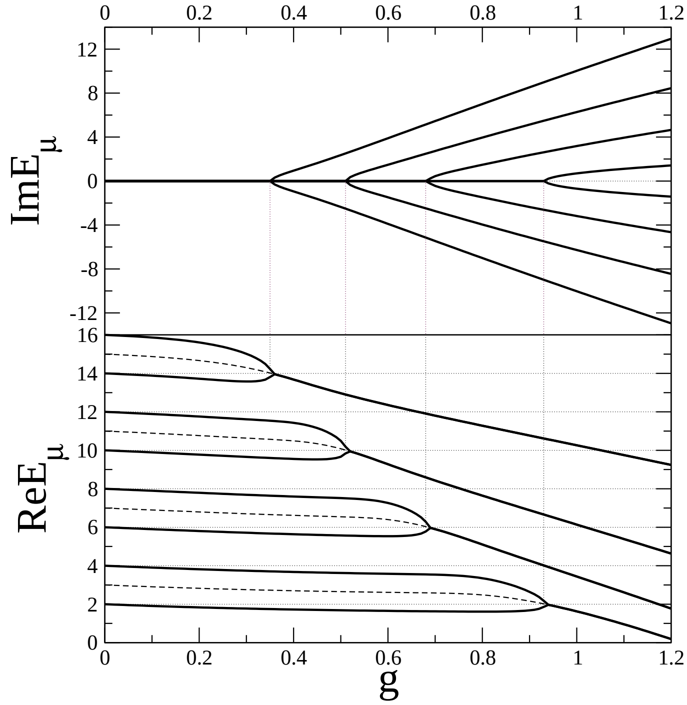


FIG. 3: Evolution of the real and imaginary parts of $E_\mu(g)$, in units of $d = \omega/N$, for the equally spaced model with $M = N/2 = 8$, as a function of the coupling constant $g = G/d$ [17, 18]. For convenience the energy levels are choosen in this figure as $\varepsilon_j = 2j$.

where the sum excludes double occupancy of pair states and the wave function $\psi$ takes the form

$$\psi(j_1, \cdots, j_M) = \sum_{\mathcal{P}} \prod_{k=1}^{M} \frac{1}{\varepsilon_{j_k} - E_{\mathcal{P}k}} \,. \tag{18}$$

The sum in (18) runs over all the permutations, $\mathcal{P}$, of $1, \cdots, M$. The constant $C$ in (17) guarantees the normalization of the state [16] (i.e. $\langle M|M\rangle = 1$).

The number of solutions of the Richardson's eqs. (15) is equal to the dimension of the Hilbert space $\mathcal{H}_{N,M}$, namely $C_M^N$ [7]. For finite and small values of $N$, $M$ the solutions $\{E_\mu(G)\}_{\mu=1}^M$ have to be found numerically. These solutions and the corresponding eigenstates can be classified according to the values taken by $E_\mu(G)$ at $G = 0$, namely [17]

$$\lim_{G\to 0} E_\mu(G) = \varepsilon_j, \;\; j \in I, \tag{19}$$

where $I$ is the set $I$ that label the basis (12) of $\mathcal{H}_{N,M}$. This means that the spectrum of the BCS Hamiltonian follows an adiabatic evolution as a function of $G$ [17, 18] For very small values of $G$ all the roots $E_\mu(G)$ of eq. (15) are real and close to their $G = 0$ value given in eq. (19). This happens for all the states labelled by $I$.

### Ground State solution

In particular for the GS, i.e. $I_0$, as we increase $G$ the real roots $E_M$ and $E_{M-1}$, nearest to the Fermi level, approach from above and below the energy $\varepsilon_{M-1}$, and become equal to it at some critical value $G = G_{c1}$. For $G > G_{c1}$ the two roots $E_M$ and $E_{M-1}$ become a complex conjugate pair. Increasing $G$ further one encounters a similar phenomena for the roots $E_{M-2}$ and $E_{M-3}$, and so on, until all the roots become complex (see fig. 3).

In the limit where $N$ is large, while $M/N$ and $\omega = dN$ remain finite the complex roots $E_\mu$ form an open arc $\Gamma$ with end points $\varepsilon_0 \pm i\,\Delta$ (see fig. 4) [7, 18, 19]. This gives an interesting geometrical meaning to the chemical potential ($\varepsilon_0/2$) and the BCS gap ($\Delta/2$) for the exactly solvable BCS model. There are also roots which stay real staying in the segment $(-\omega, \varepsilon_1)$ where $\varepsilon_1$ is the intersection point of $\Gamma$ with the real axis.

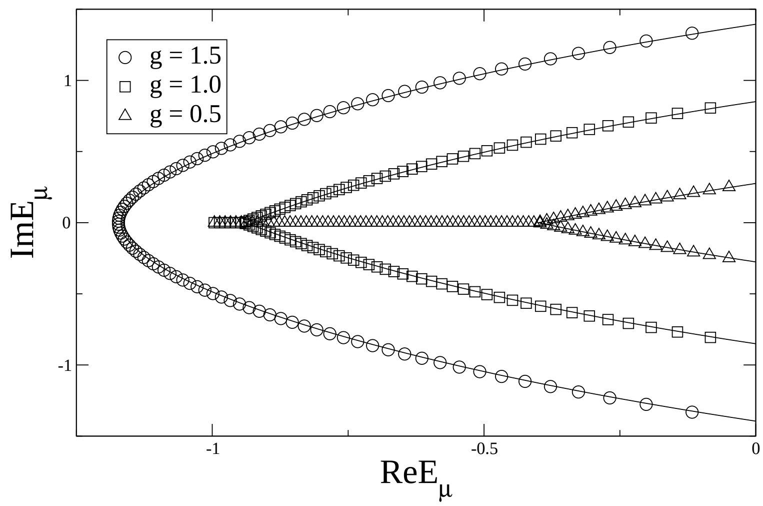


FIG. 4: Plot of the roots $E_\mu$ for the equally spaced model in the complex plane [18]. The discrete symbols denote the numerical values for $M = 100$. The continuous lines are the analytical curves obtained in [18]. All the energies are in units of $\omega$.

## V. EXCITATIONS OF THE CANONICAL BCS MODEL

In the canonical ensemble, where the number of electrons is fixed, the excited states can be obtained from the GS in two ways: by *breaking Cooper pairs* or by *pair-hole excitations* [4].

### Breaking Cooper pairs

In fig. 5 we show at $G = 0$ the excited state obtained by breaking the electron pair nearest to the Fermi level. The spin up electron remains in the same energy level, while the spin down goes one level up, producing two blocked levels. As shown in fig. 5 the problem becomes identical to that of two decoupled levels and a system with two less levels active for pairing interactions. Hence for $G > 0$ the energy of this excitation is the sum of the single particle energy of the blocked levels plus the GS energy of the system with these levels being removed. This situation is similar to what happens when the system has an odd number of electrons or when it is under the effect of an external magnetic field [3]. In both cases there are singly occupied levels, which only contribute with their non interacting energy. This sort of excitations are the ones that have been mainly studied in the literature.

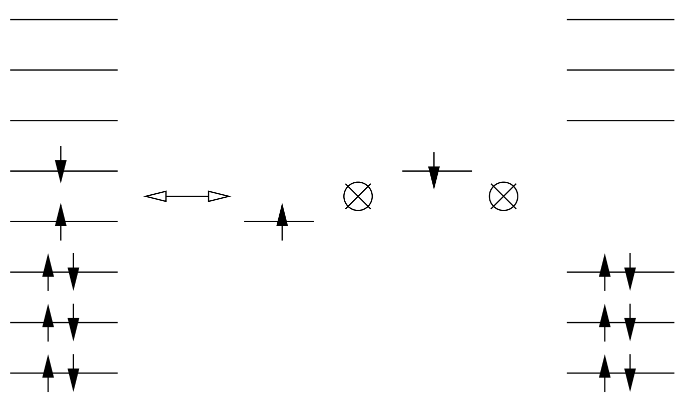

FIG. 5: Excited state obtained by breaking a Cooper pair at $G = 0$. On the rhs the singly occupied levels are blocked and decouple from the rest of the system.

**Pair-hole excitations**

The pair-hole excitations consist in promoting a pair below the Fermi level to a level above it. At $G = 0$ the resulting state can be viewed as a pair-hole excitation of the Fermi sea (see fig. 6). This sort of excitations belong to the Hilbert space $\mathcal{H}_{N,M}$ and were first consider in reference [4]. It is clear that a general excitation consists in the combination of broken Cooper pairs and pair hole-excitations. The rest of this review will focuse on the latter ones.

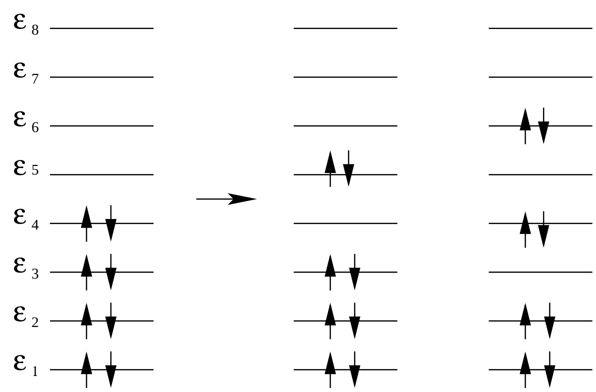


FIG. 6: Two possible excited states obtained by moving one and two pairs above the Fermi level.

In fig. 7 we plot the real part of $\{E_\mu(g)\}_{\mu=1}^{M}$ for three states, labelled $I_0, I_1, I_2$ and a system with $N = 40$ energy levels and $M = 20$ pairs as a function of the BCS dimensionless coupling $g$ from 0 to 1.5. The states are given by the choices

$$I_0 = \{1, \ldots, 18, 19, 20\}, \quad I_1 = \{1, \ldots, 18, 19, 21\}, \quad I_2 = \{1, \ldots, 18, 20, 22\}. \tag{20}$$

The state $I_0$ is the GS of the system and the pattern that follows $\mathrm{Re}E_\mu(g)$ is the same as in fig. 3. Notice that as $g \to \infty$ all the roots become complex and escape to infinity. The state $I_1$ is the lowest energy state of the system, and as we see in fig. 7, in the limit $g \to \infty$, the root $E_{20}(g)$ stays real and finite while the remaining $M - 1 = 19$ roots escape to $\infty$. Finally, the state $I_2$ has two roots $E_{20}(g)$ and $E_{19}(g)$ that stay real and finite in the $g \to \infty$ limit. This is a general feature of the numerical solutions of the Richardson's equations (15), namely in the limit $g \to \infty$ there are $N_G$ roots $E_\mu(g)$ that remain finite, not necessarily real, while the $M - N_G$ remaining ones go to $\infty$. As we shall see below the number $N_G$ of finite roots can be interpreted as the number of elementary excitations of a given excited state.

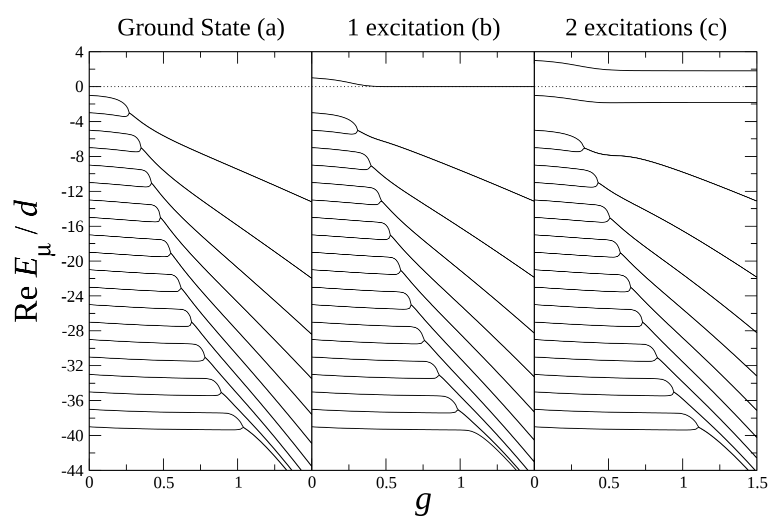


FIG. 7: Real part of $E_\mu$ for the equally spaced model with $M = N/2 = 20$ pairs and $N_G = 0, 1, 2$ excitations [4].

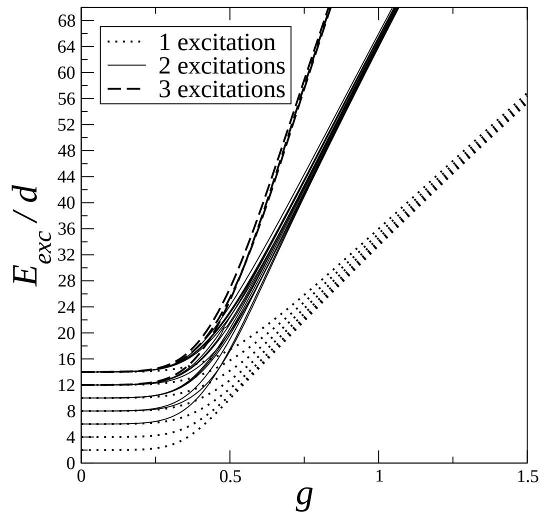


FIG. 8: Excitation energies $E_{exc} = E - E_{GS} \leq 14d$ for $M = 20$ pairs at half filling [4]. There are 44 =13+26+5 states corresponding to $N_G = 1, 2, 3$ respectively. The particle-hole symmetry reduces these numbers to 25 = 7+15+3.

**Excitation energies**

To support this conjecture we have computed the excitation energy $E_{\rm exc} = E - E_{GS}$ of some low lying excited states. Fig.8 shows the energies $E_{\rm exc}$ for a system with $M = N/2 = 20$ as a function of $g$. At $g = 0$ $E_{\rm exc}$ is simply given by the energy needed to lift the pairs from the Fermi sea up to the unoccupied energy levels, but as $g$ increases the excitation energies quickly converge to asymptotas whose slope is given essentially by $N_G$,

$$\lim_{g\to\infty} E_{\rm exc} = N_G\,\Delta, \qquad \Delta \sim g\omega. \tag{21}$$

Moreover using the methods of Refs. 7, 18 one can show in the large $N$ limit that the excitation energy of a Richardson state is given by

$$E_{exc} = \sum_{\alpha=1}^{N_G} \sqrt{(E_\alpha - \varepsilon_0)^2 + \Delta^2}\ , \tag{22}$$

where $\varepsilon_0$ is twice the chemical potential, and the energies $\{E_\alpha\}_{\alpha=1}^{N_G}$ are the ones that remain finite in the $g \to \infty$ limit. In the latter limit one has $\Delta \sim g\omega$ and eq. (22) becomes eq. (21). The excitation energy given by eq. (22) fits quite well the excitation energies of our prototype example ($N = 40$, $M = 20$), as shown in fig. 8.

In order to compare our results with the BCS standard solution let us consider the excitation energy of a *real Cooper pair*, $\gamma^\dagger_{j+}\gamma^\dagger_{j-}$ in the Bogoliubov approach, which is given by $\sqrt{\varepsilon_j^2 + \Delta^2}$ (notice that $\Delta \equiv 2\Delta_{BCS}$). The standard Bogoliubov quasiparticle with an energy $\frac{1}{2}\sqrt{\varepsilon_j^2 + \Delta^2}$ would have to be compared with excitations involving broken Cooper pairs. Since $E_\alpha$ in eq. (22) lies between two energy levels, with $\varepsilon_{j+1} - \varepsilon_j = 2d \sim 1/N$ (e.g. in fig. 7b $E_{20}(\infty) = 0$ with $\varepsilon_{20} < E_{20} < \varepsilon_{21}$), $E_\alpha = \varepsilon_j + O(1/N)$. Therefore, our theory is consistent within $O(1/N)$ corrections, as it is well known from the existing relation between the canonical and grand canonical ensembles.

## VI. COUNTING THE NUMBER OF STATES WITH $N_G$ ELEMENTARY EXCITATIONS

We said above that the dimension of the Hilbert space $\mathcal{H}_{N,M}$ of the states with $M$ pairs distributed into $N$ different energy levels is given by the combinatorial number $C_M^N$, which is also equal to the number of solutions of the Richardson's equations (15). The next question is: how many of these solutions contain $N_G$ roots that remain finite in the $g \to \infty$ limit? Let us call $d_{N,M,N_G}$ that number which must obviously satisfy $C_M^N = \sum_{N_G=0}^{M} d_{N,M,N_G}$.

In reference [7] Gaudin made the following conjecture

$$d_{N,M,N_G} = \begin{cases} C^N_{N_G} - C^N_{N_G-1} & 0 \leq N_G \leq M \\ 0 & N_G > M \end{cases} . \tag{23}$$

In table 1 we give some examples of this formula. Since $d_0 = 1$, there is only one state where all the energies $E_\mu$ go to infinity as $g \to \infty$, which is nothing but the GS of the system.

| N$^o$ of solutions | $d_M$ | $d_{M-1}$ | $\cdots$ | $d_1 = N-1$ | $d_0 = 1$ |
|---|---|---|---|---|---|
| $E_\mu$ finite | $M$ | $M-1$ | $\cdots$ | 1 | 0 |
| $E_\mu$ infinite | 0 | 1 | $\cdots$ | $M-1$ | $M$ |

Table 1.- Classification of roots in the $g \to \infty$ limit. Here $d_{N_G}$ stands for $d_{N,M,N_G}$.

This conjecture can be motivated as follows. Suppose that in the limit $g \to \infty$ there is a set $\{E_\alpha(\infty)\}_{\alpha=1}^{N_G}$ of $N_G$ roots that remain finite while the remaining $M - N_G$ ones escape to infinity. Then from eqs. (15) one derives that the roots $E_\alpha = E_\alpha(\infty)$ satisfy

$$0 = \sum_{j=1}^{N} \frac{1}{\varepsilon_j - E_\alpha} - \sum_{\beta \neq \alpha}^{N_G} \frac{2}{E_\beta - E_\alpha} , \qquad \alpha = 1, \dots, N_G. \tag{24}$$

These equations are due to Gaudin and they appear in the diagonalization of a spin chain model known as Gaudin magnets [20]. The number of solutions of eqs. (24) was shown by Gaudin to be given by (23) [7]. In reference [4] we checked numerically this Gaudin's conjecture for small systems. Furthermore we were able to show, in the large $N$ limit, that for finite $g$ the roots $E_\alpha = E_\alpha(g)$ satisfy an "effective" Gaudin equations

$$0 = \sum_{j=1}^{N} \frac{1}{R(\varepsilon_j)(\varepsilon_j - E_\alpha)} - \sum_{\beta \neq \alpha}^{N_G} \frac{2}{R(E_\beta)(E_\beta - E_\alpha)} , \tag{25}$$

with $R(E) = \sqrt{(E - \varepsilon_0)^2 + \Delta^2}$. As $g \to \infty$ one has $\Delta \sim g\omega$ and eqs. (25) become eqs. (24).

## VII. A CLASSIFICATION PROBLEM

Gaudin's conjecture (23) gives the number $d_{N,M,N_G}$ of states with a given value of $N_G$, but says nothing of how to find them. Recall that the solutions of Richardson's eqs. (15) are obtained by starting from a given initial state at $g = 0$, labelled by the set $I$, and then increasing the value of $g$. Hence the problem is to find for each state $I$ which is the number of roots $N_G$, which remains finite in the $g \to \infty$ limit. This defines the function $N_G(I)$ which, in physical terms, gives the number of elementary excitations present in the state. The total number of states $I$ with $N_G = N_G(I)$ is nothing but $d_{N,M,N_G}$.

Finding $N_G(I)$ is a highly non trivial problem because it connects the two extreme cases $g = 0$ and $g = \infty$. From a numerical point of view it is also a challenging problem due to the existence of several sorts of singularities in the merging and splitting of roots with no obvious a priori pattern, except for the GS. Despite of these difficulties it is possible to give a simple algorithm yielding $N_G(I)$, which has an interesting combinatorial interpretation in terms of Young diagrams [4]. Before we give this algorithm we need some preliminary definitions.

### Path representation of states and Young diagrams

The first idea is to associate to every state $I$, in the Hilbert space $\mathcal{H}_{N,M}$, a path $\gamma_I$ in the lattice $\mathcal{Z}^2$. This is done, in the pair-hole representation of $I$, by associating to every empty level, $\circ$, a horizontal link in $\mathcal{Z}^2$, and to every occupied level, $\bullet$, a vertical link, starting from the lowest energy state. Placing the initial point of $\gamma_I$ at $(0,0)$, then its final point is at $(M, N-M)$. The number of up and right going paths from $(0,0)$ and $(N-M, M)$ is given by $C^N_M$, which equals the dimension of $\mathcal{H}_{N,M}$. This means that the path representation is faithful. In fig. 9 we show the paths associated to the ground state $I_0$ and an excited state $I$ at $g = 0$. The path $\gamma_{I_0}$ is given by $(0,0) \to (0,M) \to (N-M, M)$, while the path associated to any excited state $I$ lies always below it. This fact makes possible to associate a Young diagram $Y_I$ to every state $I$, whose boundary is given by the paths $\gamma_I$ and $\gamma_{I_0}$ with their common links being removed (see

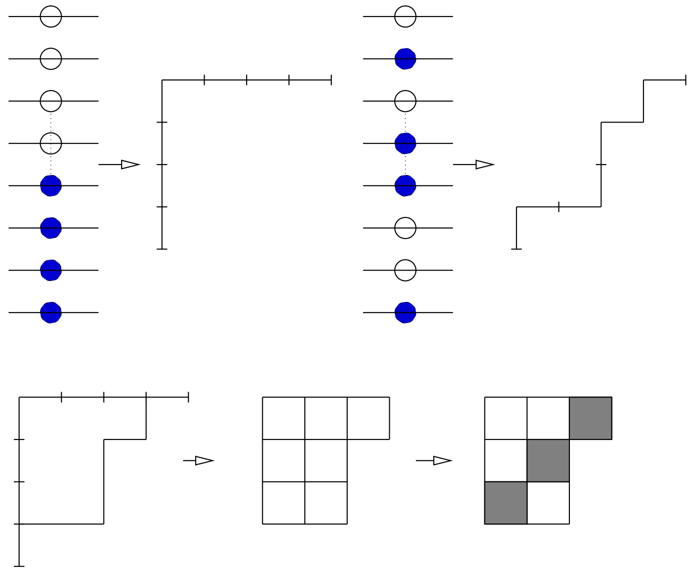

FIG. 9: Path representation of the ground state $I_0$ and an excited state $I$. Combining these two paths one gets the Young diagram $Y_I$. The number of elementary excitations of the state $I$ is given by $N_G = 3$, according to the rule (26).

fig. 9). The GS is associated to the empty Young diagram, i.e. $Y_{I_0} = \emptyset$. Using these definitions, the funtion $N_G(I)$ is given by the following algorithm [4]:

$$N_G(I) = \text{number of boxes on the longest SW} - \text{NE diagonal of } Y_I. \tag{26}$$

In the example shown in fig.9 we get $N_G = 3$. The algorithm (26) was proposed in reference [4] to explain a large amount of numerical simulations. As explained in [4] the physical mechanism underlying eq. (26) is the collective behavior of the holes and pairs that occupy the closest levels to the Fermi level. There are also combinatorial reasons to support it (see below), however an analytical proof is needed.

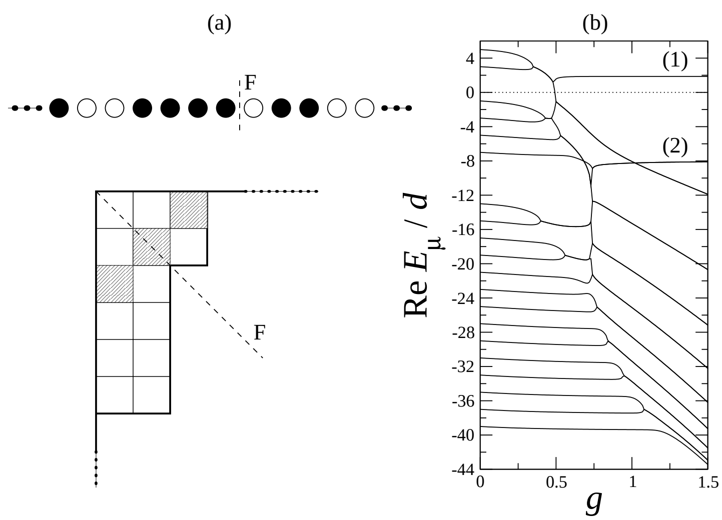


FIG. 10: a) The path and Young diagram of an excited state with $N_G = 3$. The dotted line denotes the position of the Fermi level. b) Real part of $E_\mu$. For $g$ large enough there is a real root (1) and a complex root (2) in agreement with the result $N_G = 3$.

The non-triviality of (26) can be seen in fig. 10, which shows the evolution of the real part of the roots $E_\mu$ as a function of $g$. Only for sufficiently large values of $g$, and after a complicated pattern of fusion and splitting of roots

(2 real roots $\leftrightarrow$ 1 complex root) one observes that, indeed, $N_G = 3$. It seems very difficult to explain this result on the basis of Bogoliubov's quasiparticle picture which only works properly for large number of particles in the grand canonical ensemble.

We have yet no proof of eq. (26), however there is the following consistency check based on Gaudin's conjecture (23): $d_{N,M,N_G}$ must be equal to the number of Young diagrams, $Y_I$, associated to the paths $\gamma_I$ going from $(0,0)$ to $(M, N-M)$, which have $N_G$ boxes on their longest SW-NE diagonal. The proof of this result uses the methods of Ref. 21. This result can also be proved using the RSOS counting formulas. (We thanks A. Berkovich for pointing out this fact). As an example we show in fig. 11 how the paths corresponding to the case $N = 6, M = N_G = 2$ can be mapped into RSOS paths.

We would like finally to mention the close relationship between the path representation and the associated Young diagrams of the BCS states with the crystal basis used in the Fock space representation of the affine quantum group $U_q(\widehat{Sl(2)})$ [22, 23, 24]. At the moment of writting this paper we do not know if there exists a quantum group structure underlying the exactly solvable BCS model. This may indeed be the case given that the integrability of the BCS model can be explained from an inhomogenous XXZ spin chain with boundary operators [11, 12, 13, 14].

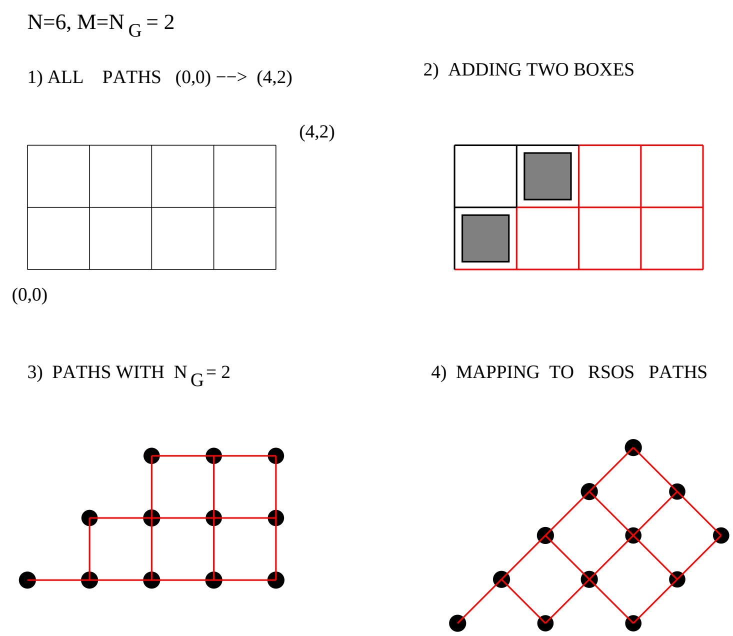


FIG. 11: 1) Shows the $C_2^6 = 9$ paths from $(0,0)$ to $(4,2)$, 2) The paths corresponding to $N_G = 2$ are those below the shaded boxes, 3) We select only those paths with $N_G = 2$ and 4) after a rotation the $N_G = 2$ paths become RSOS paths on a Bratelli diagram.

## VIII. CONCLUSIONS

We have shown in reference [4] that the pair-hole excited states of the exactly solvable BCS model in the canonical ensemble can be interpreted in terms of elementary excitations with peculiar counting properties related to the Gaudin model, which have no counterpart in the BCS-Bogoliubov's picture of quasiparticles.

The combinatorics involved in the counting of elementary excitations is similar to that of RSOS models, which suggests that these excitations are in fact solitons which were called "gaudinos" in [4].

Some open problems are 1) the relation between these gaudinos and the Bogoliubov quasiparticles, 2) analytic proof of the algorithm for computing the number of gaudinos $N_G(I)$ for each BCS state, 3) working out the physical consequences of these excitations in the canonical ensemble for small systems, 4) generalization of these results to higher spin representations and other Lie groups and supergroups.

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