# Russian Doll Renormalization Group and Kosterlitz-Thouless Flows

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(Dated: January 24, 2003)

We investigate the previously proposed cyclic regime of the Kosterlitz-Thouless renormalization group (RG) flows. The period of one cycle is computed in terms of the RG invariant. Using bosonization, we show that the theory has $U_q(\hat{sl(2)})$ quantum affine symmetry, with *q real*. Based on this symmetry, we study two possible S-matrices for the theory, differing only by overall scalar factors. We argue that one S-matrix corresponds to a continuum limit of the XXZ spin chain in the anti-ferromagnetic domain $\Delta < -1$. The latter S-matrix has a periodicity in energy consistent with the cyclicity of the RG. We conjecture that this S-matrix describes the cyclic regime of the Kosterlitz-Thouless flows. The other S-matrix we investigate is an analytic continuation of the usual sine-Gordon one. It has an infinite number of resonances with masses that have a Russian doll scaling behavior that is also consistent with the period of the RG cycles computed from the beta-function. Closure of the bootstrap for this S-matrix leads to an infinite number of particles of higher spin with a mass formula suggestive of a string theory.

PACS numbers: 11.10.Hi, 11.55.Ds, 75.10.Jm



## I. INTRODUCTION

The Renormalization Group (RG) continues to be one of the most important tools for studying the qualitative and quantitative properties of quantum field theories and many-body problems in condensed matter physics. A widely important class of theories are those with RG fixed points in the ultraviolet (UV) or infrared (IR), and our intuitive understanding of the generic behavior of quantum field theory is largely based on theories with these properties. The notion that massive states decouple in the flow toward the IR is an example of such an intuition. However fixed point behavior is not the only possibility, and physically sound examples with other kinds of behavior are important to explore. This paper is concerned with *cyclic* RG flows, whose possibility was considered as early as 1971 by Wilson[1]. However at the time no interesting models were known that exhibited this behavior.

Recently a cyclic RG behavior has been found in a number of models, wherein the couplings return to their initial values after a *finite* RG time $\lambda$:

$$g(e^{\lambda}L) = g(L), \tag{1}$$

where $L$ is the RG length scale [2, 3, 4, 5]. The models in [2, 3] are problems in zero-dimensional quantum mechanics. The model in [5] is a natural extension of the BCS model of superconductivity and thus a many-body problem.

The cyclic RG property eq. (1) has some important implications for the spectrum of the hamiltonian. Namely, if $\{E_n, g, L\}$ is the spectrum of eigenvalues of the hamiltonian for a system of size $L$, then

$$\{E_n, g, e^{\lambda}L\} = \{E_n, g, L\}, \tag{2}$$

i.e. the energy spectrum at fixed $g$ should reveal a periodicity as a function of $L$. A nice feature of the model in [5] is that in addition to the beta-function with the cyclic property one could obtain analytic results for the spectrum using the standard BCS mean field treatment. In this way one could study the interplay between the cycles of the RG flow and the spectrum. The manner in which the spectrum reproduces itself after one RG cycle is dependent on the existence of an infinite number of eigenstates with the 'Russian doll' scaling behavior in an appropriate limit

$$E_{n+1} \approx e^{\lambda}E_n. \tag{3}$$

In each cycle these eigenstates reshuffle themselves such that the $(n+1)$'th state plays the same role as the $n$'th state of the previous cycle.

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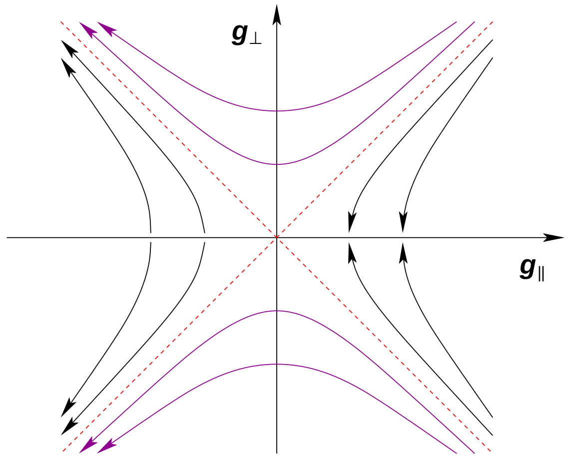


FIG. 1: Kosterlitz-Thouless flows.

Another possible signature of a cyclic RG is a cyclicity of the S-matrix itself. Applying RG equations to the S-matrix $S(E)$, where $E$ is the energy, one expects:

$$S(e^{\lambda}E) = S(E) \tag{4}$$

The above equation was anticipated in [1].

The models considered in [4] are relativistic models of quantum field theory in $2d$. What is surprising is that this theory is in fact a well-known theory that arises in a multitude of physical problems: an anisotropic left-right current-current interaction that gives rise to the Kosterlitz-Thouless (KT) flows. The standard picture of these flows, based on the one-loop approximation, is shown in figure 1. The $g_\parallel$ axis is a line of IR or UV fixed points. The flows in the region $|g_\perp| > |g_\parallel|$ on the other hand are already peculiar in that they both originate and terminate at no clearly identifiable fixed point. In [4] it was proposed that this region has a cyclic RG flow based on the all-orders beta-function conjectured in [6]. However since the flows extended beyond the perturbative domain $|g_{\parallel,\perp}| < 1$, it remained unclear whether other non-perturbative effects would spoil the cyclicity. Furthermore, at the time little was known about the spectrum. In any case, if this proposal is correct it implies that an important feature of the KT flows has been overlooked.

The main purpose of this article is to provide further evidence for the cyclicity of the KT flows by proposing an S-matrix. Let us summarize our main results. In theories with UV or IR fixed points the universal properties encoded in the beta-functions are anomalous dimensions of fields which are related to the slope of the beta-function at the fixed point. On the other hand, for a cyclic flow, the only universal, coordinate independent property is the period of one cycle. We compute the period $\lambda$ in section II in terms of the RG invariant $Q \equiv -h^2/16$ (at small coupling $Q \approx g_\parallel^2 - g_\perp^2$), and find the result $\lambda = 2\pi/h$.

In section III we bosonize the currents and obtain a sine-Gordon theory with a special form of the sine-Gordon coupling with both real and imaginary parts. (See eq. (23).) Other regimes of the KT flows are known to have an exact S-matrix description corresponding to the sinh-Gordon and massive/massless sine-Gordon theories. In section IV we argue that the theory possesses $\mathcal{U}_q(\widehat{sl(2)})$ symmetry with $q$ *real*. (The usual sine-Gordon model corresponds to $q$ a phase.) This symmetry fixes the S-matrix up to an overall scalar factor. The only free parameter of this S-matrix is $h$ which governs the period of the RG cycles, and we find $q = -e^{-\pi h/2}$.

For the simplest choice of the overall scalar factor, our S-matrix can be understood as a certain relativistic continuum limit of the XXZ spin chain with $\Delta < -1$. In terms of $\Delta$, the period of a cycle is

$$\lambda = \frac{\pi^2}{\cosh^{-1}(-\Delta)}. \tag{5}$$

In this case, the signature of the cylic RG is the periodicity of the S-matrix eq. (4), where the period agrees precisely with the beta function computation in section II.

The other possible overall scalar factor we investigate is an analytic continuation of the usual sine-Gordon one. The resulting S-matrix is not real analytic, and for this reason we believe the S-matrix related to the XXZ spin chain is the correct one physically. Nevertheless this second S-matrix has some interesting properties we explore in the last section of the paper. The soliton S-matrix possesses poles which imply the existence of an infinite number of resonances with mass

$$m_n = 2M_s \cosh \frac{n\pi}{h}, \qquad n = 1, 2, 3, ..., \infty, \tag{6}$$

where $M_s$ is the mass of the soliton. We show that these resonances are actually a spin 1 triplet and a singlet. The above spectrum, eq. (6), which is rather novel in the subject of integrable quantum field theory, is the Russian doll spectrum that was anticipated based on the analogy with the results in [5]. Namely,

$$m_n \sim M_s \, e^{n\pi/h} \quad \Longrightarrow \quad m_{n+2} \approx e^{\lambda} m_n, \qquad n \gg \frac{h}{\pi}, \tag{7}$$

to be compared with eq. (3). This property of the spectrum is a strong indication of the cyclicity of the RG. Furthermore, we find that closing the S-matrix bootstrap leads to resonances of higher spin with a mass formula that suggests a string theory description. Theories with an infinite number of resonance poles were also found in [7, 8]; however these theories appear to be rather different, the S-matrices being built out of elliptic functions.

## II. CYCLIC RG FLOWS IN ANISOTROPIC $SU(2)$ CURRENT PERTURBATIONS

We define the couplings $g_{\|}$, $g_{\perp}$ as the following current-current perturbation of the $su(2)$ level $k = 1$ Wess-Zumino-Witten (WZW) model:

$$S = S_{WZW} + \int \frac{d^2x}{2\pi} \left( 4g_{\perp}(J^+\overline{J}^- + J^-\overline{J}^+) - 4g_{\|} J_3 \overline{J}_3 \right), \tag{8}$$

where $J^a$ ($\overline{J}^a$) are the left (right) -moving currents, normalized as in [6].

Let us first consider the well-known one-loop result. The beta functions are

$$\frac{dg_{\|}}{dl} = -4g_{\perp}^2, \qquad \frac{dg_{\perp}}{dl} = -4g_{\perp}g_{\|}, \tag{9}$$

where the length scale is $L = a\, e^l$, with $a$ a microscopic distance. We will refer to $l$ as the RG 'time'. The flows possess the RG invariant:

$$Q = g_{\|}^2 - g_{\perp}^2, \tag{10}$$

so that the RG trajectories are hyperbolas. We are interested in the region $g_{\perp}^2 > g_{\|}^2$ where $Q$ is negative. Let us parametrize $Q$ as follows:

$$Q \equiv -\frac{h^2}{16}, \qquad \sqrt{Q} \equiv i\frac{h}{4}, \tag{11}$$

where $h > 0$. Eliminating $g_{\perp}$, one finds

$$\frac{dg_{\|}}{dl} = -4\left( g_{\|}^2 + \frac{h^2}{16} \right). \tag{12}$$

This is the same beta-function as in [5]. The latter is easily integrated:

$$g_{\|}(l) = -\frac{h}{4} \tan\left( h(l - l_0) \right), \tag{13}$$

where $l_0$ is a constant. One sees from eq. (13) that $g_{\|}$ flows to $-\infty$ then jumps to $+\infty$ and begins a new cycle. The periodicity of the RG flow is $g_{\|}(l + \lambda_{1-\text{loop}}) = g_{\|}(l)$ where $\lambda_{1-\text{loop}} = \pi/h$.

Though the 1-loop RG already indicates a cyclicity, since the flows extend outside of the perturbative domain $|g_{\|}| < 1$ one cannot conclude the flows are indeed cyclic based on the 1-loop approximation. However it was shown in [4] that the cyclicity persists to higher orders in perturbation theory, as we now review.

The all-orders beta function proposed in [6] is

$$\frac{dg_{\|}}{dl} = \frac{-4g_{\perp}^2(1 + g_{\|})^2}{(1 - g_{\perp}^2)^2}, \qquad \frac{dg_{\perp}}{dl} = \frac{-4g_{\perp}(g_{\|} + g_{\perp}^2)}{(1 - g_{\perp}^2)(1 - g_{\|})}. \tag{14}$$

A few remarks on the status of the conjectured beta-function eq. (14) are in order. A number of important checks were performed in [4]. The most sensitive check was of the massless flows that arise in the imaginary sine-Gordon theory defined by $g_\perp \to i g_\perp$. The above beta-function correctly predicts the known non-perturbative relation between the anomalous dimensions in the UV and IR. An all orders beta-function was also proposed by Al. Zamolodchikov [9], and his result was quoted in [10]. Zamolodchikov's argument was global in nature and did not rely on summing up perturbation theory; the main input was the known properties of the massless flows, thus his beta-function appears to be the unique one up to a change of coordinates that captures the non-perturbative aspects of the flows. It can be shown that the beta-function in [10] is equivalent to (14) under a change of coordinates. Recently it has been argued that there are additional contributions that first arise at 4-loops [11]. Though this would seem to spoil the properties of the massless flows, arguments were given in [11] that the new contributions would not. This is a rather technical issue and consequently we will assume the above beta-function is correct. Clearly, the present work is a further check of the global properties of the conjectured all-orders beta function.

The RG flows were analyzed in detail in [4]. It was shown there that though the beta function possesses poles at $\pm 1$, the flows approach the poles along non-singular trajectories and can be extended to all length scales. The analysis of the flows is simplified by recognizing that they possess a non-trivial RG invariant:

$$Q = \frac{g_\parallel^2 - g_\perp^2}{(1+g_\parallel)^2 (1 - g_\perp^2)}. \tag{15}$$

(We have rescaled $Q$ by a factor of 16 in comparison to [4].)

One can use $Q$ to eliminate $g_\perp$, obtaining

$$\frac{dg_\parallel}{dl} = -4 \frac{\left(1 - (1+g_\parallel)^2 Q\right)\left(g_\parallel^2 - (1+g_\parallel)^2 Q\right)}{(1-g_\parallel)^2}. \tag{16}$$

This is readily integrated:

$$l - l_0 = \frac{1}{4\sqrt{Q}} \left[ \tanh^{-1}\left( \frac{g_\parallel(1-Q) - Q}{\sqrt{Q}} \right) - \tanh^{-1}\left( \sqrt{Q}(g_\parallel + 1) \right) \right] + \log\left( \frac{g_\parallel^2 - (1+g_\parallel)^2 Q}{1 - (1+g_\parallel)^2 Q} \right), \tag{17}$$

where $l_0$ is a constant. Note that there are no singularities at the location of the original poles of the beta function.

As described in [4], the RG flows possess a number of phases depending on the value of $Q$. For the flows with UV or IR fixed points the perturbing operators away from the fixed point are scaling fields, i.e. their scaling dimension is constant along the flow. Thus for the flows with fixed points, the RG invariant $Q$ encodes these scaling dimensions. The flows with fixed points all correspond to $Q > 0$ and are reviewed in more detail in the next section.

The cyclic flows on the other hand correspond to $Q < 0$. Here, since there are no UV or IR fixed points, $Q$ does not encode anomalous dimensions, but rather the only universal feature of the flow, namely the period of the cycles, which we now calculate. The coupling $g_\parallel$ flows toward $-\infty$, where it jumps to $+\infty$ and eventually returns to its initial value. One can see explicitly this jump from the solution as follows. Parameterizing $Q$ as in eq. (11), for $h$ large, and $|g_\parallel| \approx \infty$, the solution is

$$g_\parallel \approx -\frac{4}{h} \tan\left[\frac{1}{2} h (l - l_0)\right]. \tag{18}$$

Thus when $l - l_0 = \pi/h$ the coupling jumps from $-\infty$ to $+\infty$.

As before, define $\lambda$ as the period in $l$ for one cycle, eq. (1). Evidently $\lambda = l_{-\infty} - l_\infty$ where $l_{\pm\infty}$ are the RG times when $g_\parallel = \pm\infty$. From the exact solution eq. (17), and the fact that $\tanh^{-1}(i\infty) = i\pi/2$, one finds the simple result

$$\lambda = \frac{2\pi}{h}. \tag{19}$$

Note that this is twice the 1-loop result.

The RG results presented so far also give an indication of the mass gaps of this theory. Indeed from eq. (18) we see that when $l - l_0 = \pi/h$ the coupling $g_\parallel$ becomes $-\infty$, which implies a mass gap $M_0 \sim \frac{1}{a} e^{-\pi/h}$ associated to the length scale $a\, e^{\pi/h}$ (we are assuming for simplicity $l_0 \ll \pi/h$ and $h \ll 1$). This mass is usually associated with the existence of two particles, the spinons, in the application of this model to the XXZ spin chain in the antiferromagnetic regime (see section V).

## III. BOSONIZATION

The level-1 $su(2)$ current algebra can be bosonized as follows:

$$J^{\pm} = \frac{1}{\sqrt{2}} e^{\pm i\sqrt{2}\varphi}, \qquad J_3 = \frac{i}{\sqrt{2}} \partial_z \varphi, \tag{20}$$

where $\varphi(z)$ is the $z = t + ix$-dependent part of a free massless scalar field $\phi = \varphi(z) + \overline{\varphi}(\overline{z})$. Viewing the $g_\parallel$ coupling as a perturbation of the kinetic energy term and rescaling the field $\phi$ one obtains the action

$$S = \frac{1}{4\pi} \int d^2x \left( \frac{1}{2} (\partial\phi)^2 + \Lambda \cos b\phi \right), \tag{21}$$

where to lowest order $\Lambda \propto g_\perp$. The field $\phi$ is normalized such that when $\Lambda = 0$, $\langle \phi(z, \overline{z})\phi(0) \rangle = -\log z\overline{z}$ and the scaling dimension of $\cos b\phi$ is $b^2$. (The coupling $b$ is related to the conventional sine-Gordon coupling $\beta$ as $b^2 = \beta^2/4\pi$ so that the free fermion point corresponds to $b = 1$.) It was shown that the all-orders beta-function eq. (14) is consistent with the known two-loop beta-function of the sine-Gordon theory in [12].

For the flows with fixed points, i.e. $Q > 0$, the coupling $b$ can be related to $Q$ by properly matching the slope of the beta function at the fixed points where $g_\perp = 0$. The result is [4]

$$b^2 = \frac{2}{1 + 2\sqrt{Q}} \qquad \Longleftrightarrow \qquad \frac{1}{\sqrt{Q}} = \frac{2b^2}{2 - b^2}. \tag{22}$$

The region $0 < \sqrt{Q} < \infty$ is the massive sine-Gordon phase with $0 < b^2 < 2$. If $0 < b^2 < 1$, i.e. $\sqrt{Q} > 1/2$, the spectrum contains bound states of the solitons and antisolitons called breathers, which are absent if $1 < b^2 < 2$, i.e. $0 < \sqrt{Q} < 1/2$. When $\sqrt{Q} < -1/2$, $b^2 < 0$, and $b$ is purely imaginary, this corresponds to the massive sinh-Gordon theory. Finally when $-1/2 < \sqrt{Q} < 0$, the perturbation is irrelevant since $b^2 > 2$ and this corresponds to the massless regime with an IR fixed point. All these regions are plotted in figure 2.

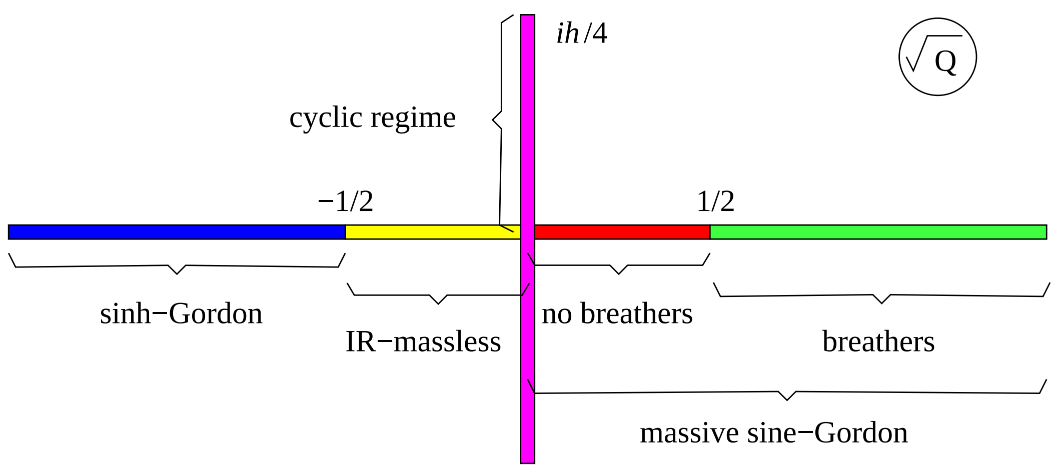


FIG. 2: Description of the different regimes of the sine-Gordon model as a function of $\sqrt{Q}$.

Parameterizing $Q$ as in eq. (11), and making an analytic extension of eq. (22), we see that the cyclic regime corresponds to

$$b^2 = \frac{2}{1 + ih/2} = \frac{4}{(4 + h^2)^{1/2}}\, e^{-i \tan^{-1}(h/2)}. \tag{23}$$

Since $b$ has both real and imaginary parts, this theory is neither the sine-Gordon nor the sinh-Gordon theories. In this bosonized description the hamiltonian does not appear to be hermitian however this is an artifact of the bosonized description since in the current algebra description the action is real. To lowest order, the origin of the imaginary part of $b$ is the square-root in the rescaling of the field involved in going from eq. (20) to eq. (21). Furthermore the spin-chain realization in section VI also has a hermitian hamiltonian.

## IV. QUANTUM AFFINE SYMMETRY AND S-MATRICES

In this section we will use some properties of the bosonized description to put forward a conjecture concerning the spectrum and S-matrices. The sine-Gordon theory is known to possess the $\mathcal{U}_q(\widehat{sl(2)})$ quantum affine symmetry which commutes with the hamiltonian and the S-matrix [13], where

$$q = e^{2\pi i/b^2} = -e^{2\pi i\sqrt{Q}}. \tag{24}$$

The construction in [13] is valid regardless of value of $b$. For the cyclic regime, though $b$ has real and imaginary parts, $q$ is real:

$$q = -e^{-\pi h/2}. \tag{25}$$

The form of $b$ in eq. (23) is rather special in that it leads to the real value of $q$ in eq. (25); a more general complex $b$ would imply a complex $q$.

We point out that the sine-Gordon theory is also known to possess another, dual, quantum group structure $\mathcal{U}_{\widetilde{q}}(\widehat{sl(2)})$ which encodes the commutation relations of the monodromy matrix [14, 15]. The dual quantum group is the algebraic structure underlying the Bethe ansatz in the quantum inverse scattering method. The latter is closely tied to the hamiltonian itself, since the trace of the monodromy matrix yields the integrals of motion. Generally, if $q = e^{\pi a}$ then the dual is $\widetilde{q} = e^{\pi/a}$. Let us flip the sign of $q$, which in the field theory is nothing more than a Klein transformation modifying the statistics of the soliton fields. Then the dual-$q$ is precisely the inverse of the finite RG transformation of one cycle: $\widetilde{q}^{-1} = e^{2\pi/h} = e^{\lambda}$. Though this is an intriguing observation, we will make no further use of it in this paper.

As for the usual sine-Gordon theory, we assume the theory contains solitons of topological charge $\pm 1$. In the XXZ spin chain, these solitons are the well-known spinons. (See the next section.) It was shown in [13] that the S-matrix is fixed up to an overall scalar factor by requiring it to commute with the quantum affine symmetry. We now describe the resulting soliton S-matrix in detail. Let $\beta$ denote the usual rapidity that parameterizes the energy and momentum of the soliton:

$$E = M_s \cosh\beta, \qquad p = M_s \sinh\beta. \tag{26}$$

Let $A_a(\beta)$ denote the creation operators for the solitons, where $a = \pm$ denotes the topological charge. The S-matrix may be viewed as encoding the exchange relation of these operators:

$$A_a(\beta_1)A_b(\beta_2) = S_{ab}^{cd}(\beta_1 - \beta_2)\ A_d(\beta_2)A_c(\beta_1). \tag{27}$$

Requiring that the S-matrix commute with the quantum affine symmetry leads to the following structure:

$$S(\beta) = \begin{pmatrix} S_{++}^{++} & 0 & 0 & 0 \\ 0 & S_{+-}^{+-} & S_{+-}^{-+} & 0 \\ 0 & S_{-+}^{+-} & S_{-+}^{-+} & 0 \\ 0 & 0 & 0 & S_{--}^{--} \end{pmatrix} = -\frac{\rho(\beta)}{2i} \begin{pmatrix} q\zeta - q^{-1}\zeta^{-1} & 0 & 0 & 0 \\ 0 & \zeta - \zeta^{-1} & q - q^{-1} & 0 \\ 0 & q - q^{-1} & \zeta - \zeta^{-1} & 0 \\ 0 & 0 & 0 & q\zeta - q^{-1}\zeta^{-1} \end{pmatrix}, \tag{28}$$

where

$$q = -e^{-\pi h/2}, \qquad \zeta = e^{-i\beta h/2}. \tag{29}$$

($\beta = \beta_1 - \beta_2$.) The above S-matrix automatically satisfies the Yang-Baxter equation.

The quantum affine symmetry does not constrain the overall scalar factor $\rho$, however there are additional constraints coming from crossing symmetry and unitarity. Crossing-symmetry requires:

$$S_{ab}^{cd}(\beta) = S_{b\overline{c}}^{d\overline{a}}(i\pi - \beta), \Longrightarrow \rho(\zeta) = \rho(-\frac{1}{q\zeta}) \tag{30}$$

Unitarity corresponds to $S^\dagger(\beta)S(\beta) = 1$. If the S-matrix is real analytic, i.e.

$$S^\dagger(\beta) = S(-\beta) \tag{31}$$

then unitary requires

$$S(-\beta)S(\beta) = 1 \tag{32}$$

The above equation leads to the following constraint on the overall scalar factor $\rho$:

$$\rho(\zeta)\rho(\zeta^{-1}) = \frac{-4q^2}{(1-\zeta^2 q^2)(1-\zeta^{-2}q^2)} \tag{33}$$

A minimal solution to the above equations is easily found iteratively. (See e.g. [13].) One starts with a solution to eq. (33), $\rho_0 = 2iq/(1-q^2\zeta^2)$. This is made crossing symmetric as follows: $\rho_1 = \rho_0(\zeta)\rho_0(-1/q\zeta)$. This spoils unitarity, which can be corrected by multiplying by $\rho_1(\zeta^{-1})$. Then one starts over again correcting crossing symmetry. The process never terminates and leads to the infinite product:

$$\rho(q,\zeta) = \frac{2iq}{1-q^2\zeta^2}\prod_{n=0}^{\infty}\frac{(1-q^{4+4n}\zeta^{-2})(1-q^{2+4n}\zeta^2)}{(1-q^{4+4n}\zeta^2)(1-q^{2+4n}\zeta^{-2})} \tag{34}$$

This infinite product is convergent since $|q|<1$.

The above S-matrix is real analytic eq. (31) and thus unitary in the usual sense. Most remarkably, it possesses a periodicity that is a clear signature of a cyclic renormalization group. Up to some minus signs, the S-matrix satisfies:

$$S(\beta + \frac{2\pi}{h}) = S(\beta) \tag{35}$$

In the limit of large energies, where $E \approx me^{\beta}/2$, the above equation implies eq. (4).

That the S-matrix presented in this section correctly describes the current-current perturbation in the cyclic regime is the main conjecture of this paper. The lagrangian defines a hermitian theory, and our S-matrix is unitary as it should be. It also has a periodicity that is correctly predicted by the RG analysis of section II. In the next section we provide further support of our conjecture based on the XXZ spin chain.

## V. REALIZATION IN THE XXZ HEISENBERG CHAIN

In this section we give further support of the S-matrix proposed in the last section by taking a particular continuum limit of the XXZ spin chain. It is well-known that in the continuum limit the spin 1/2 Heisenberg chain is well described by the anisotropic current-current perturbation of section II. (See for instance [10, 30].)

The spin chain has the hamiltonian

$$H = -J\sum_{i=1}^{N}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta\ \sigma_i^z\sigma_{i+1}^z\right). \tag{36}$$

Let us first relate $\Delta$ to the couplings $g_\parallel$, $g_\perp$ in the critical (massless) regime, $-1<\Delta<1$. In the current-current RG description, here the flows terminate in the IR along the critical line $0<g_\parallel<1$ with $g_\perp = 0$. In this regime, $-1/2 < \sqrt{Q} < 0$. The spin-chain may be mapped onto a theory of fermions via the Jordan-Wigner transformation. Bosonizing these Jordan-Wigner fermions, one obtains the sine-Gordon theory. (See for instance [10, 30, 31].) In this bosonized description, the flows arrive at the fixed point via the irrelevant operator $\cos(b\phi)$ with scaling dimension $b^2$. The parameter $b$ is known from the Bethe ansatz to be related to $\Delta$ as follows: $\Delta = \cos 2\pi/b^2$. In terms of $Q$:

$$\Delta = -\cos 2\pi\sqrt{Q}. \tag{37}$$

Let us turn now to the cyclic regime, where $-\infty < Q < 0$. Let us assume that the formula (37) is valid in this regime also. Although $\sqrt{Q}$ is imaginary, $\Delta$ is still real. Parameterizing $Q$ as in eq. (11), $\sqrt{Q} = ih/4$, one finds

$$\Delta = -\cosh\frac{\pi h}{2}. \tag{38}$$

From eq. (38) one sees that the cyclic regime of the RG should correspond to the antiferromagnetic domain $\Delta < -1$ in the spin chain. In terms of $\Delta$, the period $\lambda$ of the RG flows is given in eq. (5), namely $\lambda = \pi^2/\cosh^{-1}(-\Delta)$.

There is one additional check of the above series of mappings. The spin chain is known to directly possess the $\mathcal{U}_q(\widehat{sl(2)})$ symmetry on the lattice [16], where $q$ is related to $\Delta$ as follows:

$$\Delta = \frac{1}{2}(q+q^{-1}). \tag{39}$$

Observe that the above equation is consistent with eqs. (38) and (25).

The XXZ spin chain with $\Delta < -1$ was studied in [16] using the underlying $\mathcal{U}_q(\widehat{sl(2)})$ symmetry on the lattice. One feature of this algebraic approach is the construction of so-called type II vertex operators $Z_a$, $a = \pm 1$, which can be interpreted as creation operators for the fundamental spinons. These operators satisfy the exchange relation

$$Z_a(\zeta_1) Z_b(\zeta_2) = R_{ab}^{cd}(\zeta_1/\zeta_2)\ Z_d(\zeta_2) Z_c(\zeta_1). \tag{40}$$

(Compare with eq. (27)). The matrix $R$ in the above equation can be interpreted as the S-matrix for the spinons on the lattice. An S-matrix for this regime of the spin chain was also obtained from the Bethe ansatz in [32]. The explicit form of $R$ is precisely as in eq. (28) where now $q$ parametrizes $\Delta$ as in eq. (39) and is thus the same as in eq. (25). The expression for $\rho$ on the lattice is the same as in eq. (34); in the algebraic construction of[16], the S-matrix is just the universal R-matrix for the quantum affine algebra.

On the lattice, the particles have energies expressed in terms of elliptic functions. Letting

$$\zeta = -i\, e^{i\alpha}, \tag{41}$$

then in terms of $\alpha$ the energy and momentum are

$$E(\alpha) = \frac{2K}{\pi} \sinh \frac{\pi K'}{K} \mathrm{dn}\left(\frac{2\mathrm{K}\alpha}{\pi}\right), \qquad \mathrm{p}(\alpha) = \mathrm{am}\left(\frac{2\mathrm{K}\alpha}{\pi}\right) - \frac{\pi}{2}. \tag{42}$$

(See [33, 34] for definitions.) In the above formulas the nome of the elliptic functions is $|q| = e^{-\pi h/2} = e^{-\pi K'/K}$ which is between 0 and 1.

Let us take a particular continuum limit which amounts to letting the momentum and also $h$ to be small. Let us first take the momentum to be small. From the definition of am(u), one finds that $p(\alpha = \pi/2) = 0$. Letting $\widetilde{\alpha} = \alpha - \pi/2$, from the definition of am(u) one has

$$K + \frac{2K\widetilde{\alpha}}{\pi} = \int_0^{p+\pi/2} \frac{d\psi}{\sqrt{1 - k^2 \sin^2 \psi}}, \tag{43}$$

which can be transformed to

$$\frac{2K\widetilde{\alpha}}{\pi} = \int_0^{p} \frac{d\psi'}{\sqrt{k'^2 + k^2 \sin^2 \psi'}}. \tag{44}$$

Above, $k$, $k'$ are the standard moduli of the elliptic functions. When $p$ is small, $\sin\psi'$ can be approximated by $\psi'$ and one finds

$$p = \frac{k'}{k} \sinh\left(\frac{2Kk\widetilde{\alpha}}{\pi}\right). \tag{45}$$

We now let $|q|$ approach one, i.e. $h$ small but not zero. Using known expressions in terms of the dual nome $\widetilde{q} = e^{-2\pi/h} = e^{-\lambda}$, which goes to zero in the limit [34], one has

$$K \approx \frac{\pi}{h}(1 + 4\widetilde{q}), \qquad K' \approx \frac{\pi}{2}(1 + 4\widetilde{q}), \qquad k \approx 1 - 8\widetilde{q}, \qquad k' \approx 4\sqrt{\widetilde{q}}. \tag{46}$$

Thus in this limit one finds

$$p \approx -k' \sinh\beta, \tag{47}$$

where the rapidity $\beta$ is identified as

$$\beta = -\frac{2\widetilde{\alpha}}{h}. \tag{48}$$

Turning to the energy, we need the identity

$$\mathrm{dn(u)} = \sqrt{1 - \mathrm{k}^2 \sin^2(\mathrm{am(u)})}. \tag{49}$$

Using the limits eq. (46) one finds

$$E = \pi k' \cosh\beta. \tag{50}$$

Thus the spinon velocity is $\pi$ and the mass goes with $k' = 4\sqrt{\widetilde{q}} = 4e^{-\pi/h}$. The latter result agrees with the RG estimate of the mass gap $M_0 \sim \frac{1}{a}e^{-\pi/h}$ made in section II. Identifying the rapidity $\beta$ from eq. (47) one finds that the spectral parameter $\zeta = e^{-i\beta h/2}$, which agrees with eq. (29).

To summarize this section, we have shown that a particular continuum limit of the XXZ spin chain exists which leads to the S-matrix conjectured in the last section. We also emphasize that in taking the particular continuum limit described above, we have implicitly provided an ultra-violet completion of the XXZ spin chain by defining a limit where the dispersion relation is relativistic.

## VI. ANALYTIC CONTINUATION OF THE SINE-GORDON S-MATRIX

In this section we investigate another solution to the constraints in section IV on the S-matrix. Based on the bosonized description of section III, it is natural to consider the analytic extension of the sine-Gordon S-matrix to the complex value of the coupling $b$ given in eq. (23). We will refer to the resulting model as the cyclic sine-Gordon model. Since the usual sine-Gordon S-matrix also commutes with the quantum affine symmetry, the only difference between the cyclic sine-Gordon S-matrix and the one proposed in section IV is in the overall scalar factor $\rho$. Extending the formulas in [17] to the regime (23), one finds the S-matrix is given by eq. (28) but with $\rho$ given by $\rho_{SG}$, where:

$$\rho(\beta)_{SG} = -\frac{1}{\pi}\Gamma(\kappa)\,\Gamma(1-z)\,\Gamma(1-\kappa+z)\prod_{n=1}^{\infty}\frac{F_n(\beta)F_n(i\pi-\beta)}{F_n(0)F_n(i\pi)}, \tag{51}$$

with

$$F_n(\beta) = \frac{\Gamma(2n\kappa - z)\,\Gamma(1+2n\kappa - z)}{\Gamma((2n+1)\kappa - z)\,\Gamma(1+(2n-1)\kappa - z)}. \tag{52}$$

where we have defined

$$\kappa = \frac{ih}{2}, \qquad z = -i\frac{\beta\kappa}{\pi} = \frac{\beta h}{2\pi}. \tag{53}$$

The standard S-matrix for the massive sine-Gordon model is given by the same formulas (28,51,52) with the following substitutions:

$$\kappa = \frac{8\pi}{\gamma}, \qquad z = -\frac{8\beta i}{\gamma}, \qquad q = -e^{8\pi^2 i/\gamma}, \qquad \zeta = e^{-8\pi\beta/\gamma}, \tag{54}$$

where $\gamma$ is related to $b$ by the equation

$$\frac{\gamma}{8\pi} = \frac{b^2}{2-b^2} = \frac{1}{2\sqrt{Q}}, \qquad \sqrt{Q} > 0. \tag{55}$$

Hence the relationship between the massive sine-Gordon S-matrix and the S-matrix considered in this section amounts simply to the analytic continuation

$$h \quad \longleftrightarrow \quad -\frac{16\pi i}{\gamma}. \tag{56}$$

The above S-matrix satisfies the constraints of section IV, including algebraic unitarity eq. (32), however unlike the S-matrix in section IV it is not real analytic eq. (31), and thus cannot be considered unitary in the usual sense. As discussed in [20], in theories where parity is broken, real analyticity can be violated, and replaced by hermitian analyticity. For simplicity consider a diagonal scattering theory where the S-matrix for the scattering of particles $A_{a_1}(\beta_1)$ and $A_{a_2}(\beta_2)$ is given by $S_{a_1a_2}(\beta_1-\beta_2)$. For a parity non-invariant theory, $S_{a_1a_2}(\beta_1-\beta_2) \neq S_{a_2a_1}(\beta_1-\beta_2)$. Unitarity, $S^\dagger S = 1$, requires

$$S^*_{a_2a_1}(\beta_1-\beta_2)S_{a_1a_2}(\beta_1-\beta_2) = 1 \tag{57}$$

Imposing the requirement of hermitian analyticity

$$S_{a_2a_1}(\beta_1-\beta_2) = S^*_{a_1a_2}(\beta_2-\beta_1) \tag{58}$$

the constrain of unitarity becomes

$$S_{a_1a_2}(\beta)S_{a_1a_2}(-\beta) = 1 \tag{59}$$

Note that the latter corresponds to algebraic unitarity eq. (32) which is satisfied by the cyclic sine-Gordon S-matrix. We will henceforth assume the lack of real analyticity can be understood in this way.

The analytic extension eq. (23) is reminiscent of the staircase models[18, 19], however there are some important differences. The staircase model corresponds to $b^2 = -2(1/2 + ih_0/2\pi)/(1/2 - ih_0/2\pi)$. In terms of $\sqrt{Q}$, it corresponds to $\sqrt{Q} = -1/(1 + ih_0/\pi)$. For small $h_0$, this is a deformation of $\sqrt{Q} = -1$, which is in the sinh-Gordon regime. The S-matrix is thus an analytic extension of the **sinh**-Gordon S-matrix, with a single resonance, whereas ours is a deformation of the sine-Gordon one. Furthermore, for the staircase model, the c-function, though it roams, monotomically decreases. For a model with a cyclic RG we expect a cyclic c-function (see the conclusion).

### A. Resonances

The S-matrix for the cyclic sine-Gordon model has numerous poles which can be interpreted as resonances. Interestingly, other known models that violate parity real analyticity also have resonances [20]. More generally, consider a pole at $\beta = \mu - i\eta$ with $\mu,\ \eta > 0$, in the S-matrix for the scattering of two particles of masses $m_1$, $m_2$. As discussed in [21, 22, 23] this corresponds to a resonance of mass $M$ and inverse lifetime $\Gamma$ where

$$\left(M - \frac{i\Gamma}{2}\right)^2 = m_1^2 + m_2^2 + 2m_1m_2\cosh(\mu - i\eta). \tag{60}$$

Equivalently:

$$\begin{aligned} M^2 - \frac{\Gamma^2}{4} &= m_1^2 + m_2^2 + 2m_1m_2\cosh\mu\cos\eta, \\ M\Gamma &= 2m_1m_2\sinh\mu\sin\eta. \end{aligned} \tag{61}$$

The soliton S-matrix has an infinite number of poles at $\beta_n = 2\pi n/h$ and $\overline{\beta}_n = i\pi - 2\pi n/h$ with $n$ a positive integer. These poles are absent in the S-matrix proposed in section IV. The poles at $\beta_n$ yield a real mass and correspond to resonances with an infinitely long lifetime $\tau = 1/\Gamma = \infty$, while the poles at $\overline{\beta}_n$ yield an imaginary mass and are not physical (see table 1). It is interesting to compare this situation to what happens for the usual massive regime of the sine-Gordon theory where $0 < b^2 < 2$. In this case, when $\gamma < 8\pi$, the poles at $\overline{\beta}_n$ lead to the breather bound states, while those at $\beta_n$ are these same poles in the crossed channel.

| Cyclic sine-Gordon | Massive sine-Gordon |
|---|---|
| **Resonances**<br>$\beta_n = \frac{2\pi n}{h}$<br>$m_n = 2M_s\cosh\frac{n\pi}{h}\ \ (n = 1, \ldots, \infty)$ | Breathers in crossed channel<br>$\beta_n = \frac{in\gamma}{8}$ |
| Resonances in crossed channel<br>$\overline{\beta}_n = i\pi - \frac{2\pi n}{h}$ | **Breathers**<br>$\overline{\beta}_n = i(\pi - \frac{n\gamma}{8})$<br>$m_n = 2M_s\sin\frac{n\gamma}{16}\ \ (n = 1, \ldots < \frac{8\pi}{\gamma})$ |

Table 1.- Poles and masses of the cyclic and the massive sine-Gordon models.

Since the poles $\beta_n$ are right on the cut in the $s$-plane ($s = (p_1+p_2)^2$), it is desirable to incorporate a small imaginary part $-i\eta$ taking them off the cut and giving the particles a large finite lifetime. This can be viewed as a kind of "$i\epsilon$" prescription as follows. Letting

$$h \to h + i\epsilon, \tag{62}$$

with $\epsilon$ very small, the poles are now at

$$\beta_n \approx \frac{2\pi n}{h} - i\eta_n, \qquad \eta_n \approx \epsilon\frac{2\pi n}{h^2}. \tag{63}$$

The prescription eq. (62) does not spoil crossing-symmetry, algebraic unitarity, nor the Yang-Baxter equation. As $\epsilon \to 0$, from eq. (61) the mass $m_n$ and width $\Gamma_n$ of the n-th resonance is

$$m_n \approx 2M_s \cosh \frac{n\pi}{h}, \qquad \Gamma_n \approx \epsilon \, \frac{4\pi n}{h^2} M_s \sinh \frac{n\pi}{h}. \tag{64}$$

Each resonance of mass $m_n$, $n = 1, 2, ..\infty$, actually corresponds to four particles transforming in the direct sum of the q-deformed spin 1 and singlet representations. One can easily see this by noting that the entries $S^{++}_{++}, S^{-+}_{+-}$ and their charge conjugates are non-zero at the poles $\beta_n$, while $S^{+-}_{+-}$ and $S^{-+}_{-+}$ are zero. If the fundamental solitons have topological charge $\pm 1$, then each resonance is a triplet with topological charges $(-2, 0, 2)$ or a singlet with topological charge 0. This is in contrast to the usual breather bound state poles at $\overline{\beta}_n$ which occur in the scattering channel of a soliton with an anti-soliton and are thus singlets. ($S^{++}_{++}$ and $S^{--}_{--}$ are zero at the pole $\overline{\beta}_n$).

In the language of the fusion procedure [24], which is equivalent to the bootstrap, the poles $\overline{\beta}_n$ project onto the singlet whereas $\beta_n$ projects onto the ($q$-deformed) spin-1 and spin-0 representations. This can be seen explicitly by going to the so-called homogeneous gradation. The soliton S-matrix can be expressed as

$$S(\zeta, q) = -\frac{\rho}{2i} \, \sigma_{21} \, \widetilde{R} \, \sigma_{12}^{-1} \, P, \tag{65}$$

where $\sigma_{12} = \zeta_1^{H/2} \otimes \zeta_2^{H/2}$, $\sigma_{21} = \zeta_2^{H/2} \otimes \zeta_1^{H/2}$, with $H = \mathrm{diag}(1, -1)$ the topological charge and $P$ the permutation operator:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{66}$$

The matrix $\widetilde{R}$ can then be expressed in terms of the projectors $P_0$ and $P_1$ onto the q-deformed spin 0 and spin 1 representations:

$$\widetilde{R}(\zeta) = (\zeta q - \zeta^{-1} q^{-1}) P_1 + (\zeta^{-1} q - \zeta q^{-1}) P_0, \tag{67}$$

where

$$P_0 = \frac{1}{1 + q^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -q & 0 \\ 0 & -q & q^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{68}$$

and $P_1 = 1 - P_0$. The usual sine-Gordon breather poles at $\overline{\beta}_n$ correspond to $\zeta = (-)^{n+1} q^{-1}$ and one sees that at this $\zeta$, $\widetilde{R} \propto P_0$, showing that the breathers are indeed singlets. A pole at $\zeta = q$ projects onto the irreducible spin 1 representation. However our resonance poles at $\beta_n$ correspond to $\zeta = (-)^n$ where $\widetilde{R} \propto P_0 + P_1$. The resonances, as the sine-Gordon breathers [17], also carry a C-parity given by $(-1)^n$ for the triplets and $(-1)^{n+1}$ for the singlets.

The above spectrum of masses has scaling properties that are consistent with a cyclic RG. As argued in section III the spinon mass $M_s$ must be proportional to $1/L$, where $L$ is the system size. In the limit $n \gg h/\pi$ or $h \ll 1$, eq. (64) becomes,

$$m_n(L) \sim \frac{1}{L} \, e^{n\pi/h}, \qquad n \gg h/\pi, \tag{69}$$

which exhibits the Russian doll scaling property

$$m_n(e^{-\lambda} L) \approx m_{n+2}(L), \tag{70}$$

similar to what was found in [5]. Thus the way the spectrum reproduces itself after one RG cycle is that $m_n$ at a given length $e^{-\lambda} L$ plays the same role as $m_{n+2}$ at the longer length $L$ (see figure 3). Notice that the jump by two in $n$ is consistent with the $C-$parity of the resonances. Eq. (70) also shows that after one RG cycle two new low energy masses appear in the spectrum. This gain of the lowest energy states in the spectrum is what allows for the reshuffling of resonances after one cycle and seems to be an essential ingredient of any cyclic RG.

In a finite size system we expect to have a finite number of resonances, $n_{\rm res}$, which can be estimated as follows. If in every RG cycle we get two more resonances, then after $p$ cycles with $L \sim a \, e^{p\lambda}$, we expect to have $n_{\rm res} = 2p$ resonances, with

$$n_{\rm res} \sim \frac{h}{\pi} \log \frac{L}{a}. \tag{71}$$

This kind of equation was checked numerically for the Russian Doll BCS model[5].

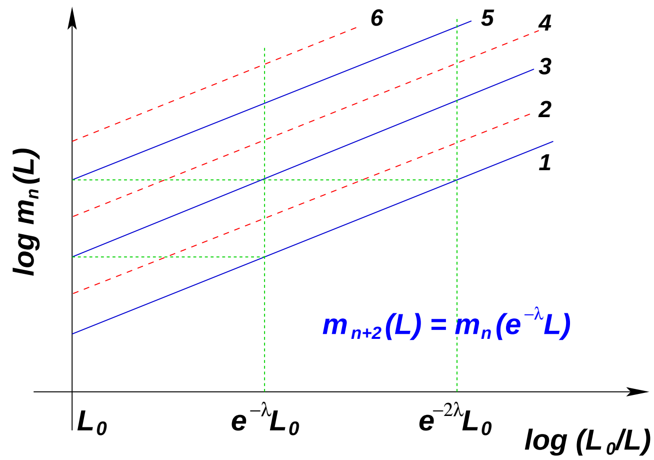


FIG. 3: Log-log plot of the resonance masses $m_n(L)$ versus $1/L$, where $L$ is the length scale of the system. $L_0$ denotes the maximal size of the system. The Russian doll scaling of eq. (70) is described by the dotted lines.

### B. Closing the bootstrap: stringy spectrum

In the limit that the regulator $\epsilon$ goes to zero, the resonances are stable particles and can exist as asymptotic states. Thus, one should attempt to close the bootstrap including the resonances in asymptotic states as proposed in [23]. Since the bootstrap is equivalent to the fusion construction in the theory of quantum affine algebras, it is clear for example that the resulting S-matrix for the scattering of the soliton with the triplets of resonances should be solution of the Yang-Baxter equation corresponding to the universal $\mathcal{R}$-matrix of $\mathcal{U}_q(\widehat{sl(2)})$ evaluated in the representations $V_{1/2} \otimes V_1$ where $V_j$ is the $2j+1$ dimensional $q$-deformed spin-$j$ representation.

Continuing to close the bootstrap leads to resonances in the q-deformed spin $j$ representation $V_j$ with $j = 1, 3/2, 2, ...\infty$. To simplify matters we will only consider the scattering of the particle in $V_j$ of maximal topological charge $+2j$. This avoids the complexities of non-diagonal scattering and will be sufficient to obtain a mass formula as a function of $j$.

Let $S^{(n)}_{1/2,j}(\beta)$ denote the diagonal scattering of the $n$-th spin $j$ particle of topological charge $2j$ with the soliton of topological charge $+1$. For $j = 1$, $n$ refers to the quantum number in eq. (64), and when $j = 1/2$, there is no $n$-dependence and $S^{(n)}_{1/2,1/2} = S^{++}_{++}$. If the particle of charge $2(j+1/2)$ appears as a resonance in the scattering of spin $1/2$ with spin $j$ particles, then the S-matrix has a simple pole:

$$S^{(n)}_{1/2,j}(\beta) \sim \frac{1}{\beta - \mu^{j+1/2}_{1/2,j}}, \tag{72}$$

and the masses are related by the formula

$$m_c^2 = m_a^2 + m_b^2 + 2 m_a m_b \cosh \mu^c_{ab}, \tag{73}$$

with $a, b, c = 1/2, 1, 3/2, ....$ The above equation implies the following relation between the fusion angles:

$$\mu^c_{ab} + \mu^a_{bc} + \mu^b_{ac} = 2\pi i. \tag{74}$$

In the equations (73,74) all the masses are implicitly at the same fixed $n$.

Given $S^{(n)}_{1/2,j}$ one can construct the S-matrix $S^{(n)}_{1/2,j+1/2}$ from the bootstrap procedure:

$$S^{(n)}_{1/2,j+1/2}(\beta) = S^{(n)}_{1/2,j}\left(\beta + \overline{\mu}^{1/2}_{j,j+1/2}\right)\, S_{1/2,1/2}\left(\beta - \overline{\mu}^{j}_{1/2,j+1/2}\right), \tag{75}$$

where $\overline{\mu} = i\pi - \mu$.

Let us first consider the scattering of the spin $1/2$ solitons with the spin 1 resonances discussed above. We already found that $\mu^1_{1/2,1/2} = 2\pi n/h$ and eq. (74) implies $\overline{\mu}^{1/2}_{1/2,1} = \pi n/h$. The bootstrap equation then reads

$$S^{(n)}_{1/2,1}(\beta) = S_{1/2,1/2}\left(\beta + \frac{n\pi}{h}\right)\, S_{1/2,1/2}\left(\beta - \frac{n\pi}{h}\right). \tag{76}$$

The factor $S_{1/2,1/2}(\beta)$ only has *real* poles at $z = \beta h/2\pi = p$ and zeros at $z = -p$ where $p = 1, 2, 3, ...$ Remarkably this leads to the fact that the only real pole of $S^{(n)}_{1/2,1}(\beta)$ is at $\beta = n\pi/h \equiv \mu^{3/2}_{1/2,1}$. The other poles of $S_{1/2,1/2}$ either lead to double poles or are canceled by the zeros in the other $S_{1/2,1/2}$ factor.

Let $m_n(j)$ denote the mass of the n-th resonance with spin $j$. The above value of $\mu^{3/2}_{1/2,1}$ leads to

$$m_n^2(j = 3/2) = 4M_s^2 \left( \frac{9}{4} + 2 \sinh^2 \frac{n\pi}{h} \right). \tag{77}$$

Moving on to the spin 2 resonances, one finds that the fusion angles are somewhat more complicated. Since the masses $m(j)$ are already determined for $j = 1/2, 1, 3/2$, the relevant fusion angles follow from eq. (74) and one obtains

$$\begin{aligned} \mu^{3/2}_{1/2,1} &= \frac{n\pi}{h}, \\ \exp \overline{\mu}^{1}_{1/2,3/2} &= \sqrt{\frac{2+H}{2+H^{-1}}}, \qquad\qquad H \equiv e^{2n\pi/h}, \\ \exp \overline{\mu}^{1/2}_{1,3/2} &= \sqrt{\frac{(1+H)(2+H^{-1})}{(1+H^{-1})(2+H)}}. \end{aligned} \tag{78}$$

The S-matrix is

$$S^{(n)}_{1/2,3/2}(\beta) = S^{(n)}_{1/2,1}\left(\beta + \overline{\mu}^{1/2}_{1,3/2}\right) S_{1/2,1/2}\left(\beta - \overline{\mu}^{1}_{1/2,3/2}\right). \tag{79}$$

Similarly to the $j = 3/2$ case, the above S-matrix has a *single* simple pole at $\beta = \mu^2_{1/2,3/2}$ where

$$\mu^2_{1/2,3/2} = \frac{n\pi}{h} - \overline{\mu}^{1/2}_{1,3/2}. \tag{80}$$

This leads to the mass

$$m_n^2(j = 2) = 4M_s^2 \left( 4 + 3 \sinh^2 \frac{n\pi}{h} \right). \tag{81}$$

Having understood the above cases we can readily extend the results to arbitrary $j$. Examining the poles in the S-matrix $S_{1/2,j}$ one finds a single, real, simple pole at

$$\mu^{j+1/2}_{1/2,j} = \mu^{j}_{1/2,j-1/2} - \overline{\mu}^{1/2}_{j-1/2,j}. \tag{82}$$

This leads to the mass formula

$$\begin{aligned} m_n^2(j) &= 4M_s^2 \left( j^2 + (2j-1) \sinh^2 \frac{n\pi}{h} \right) \\ &= M_s^2\, (2j - 1 + H)(2j - 1 + H^{-1}). \end{aligned} \tag{83}$$

From this mass formula one can determine all the fusion angles:

$$\begin{aligned} \exp \mu^{j+1/2}_{1/2,j} &= \sqrt{\frac{2j-1+H}{2j-1+H^{-1}}}, \\ \exp \overline{\mu}^{j}_{1/2,j+1/2} &= \sqrt{\frac{2j+H}{2j+H^{-1}}}, \\ \exp \overline{\mu}^{1/2}_{j,j+1/2} &= \sqrt{\frac{(2j-1+H)(2j+H^{-1})}{(2j-1+H^{-1})(2j+H)}}. \end{aligned} \tag{84}$$

The above mass spectrum is suggestive of a string theory. We remark that the higher spin particles that make up the Regge trajectory of the bosonic string also appear as resonances on the physical cut in the Veneziano amplitude [25]. Consider the “leading Regge trajectory”, i.e. the lowest mass at spin $j$. When $h$ is very small, the formula (83) becomes

$$m^2(j) \approx \frac{1}{\alpha'}(j - \alpha(0)), \tag{85}$$

where the Regge slope and intercept are

$$\frac{1}{\alpha'} = 2M_s^2 e^{2\pi/h}, \qquad \alpha(0) = 1/2. \tag{86}$$

(The leading Regge trajectory corresponds to $n = 1$.) The usual field theory limit in string theory is $\alpha' \to 0$ which corresponds to $h \to 0$, where all the resonances become infinitely massive. The fact that our theory has two coupling constants, namely $M_s$ and $h$, also suggests that the underlying string has another parameter such as a compactification radius. If there is indeed an underlying string description, much remains to be understood regarding the nature of this string theory.

### C. Jumping the $c_m = 1$ barrier?

We close this section with some speculative remarks regarding the possible nature of the string theory, if there is indeed such a description. Consider a non-critical bosonic string theory in $D$ spacetime dimensions, where $c_m = D-1$ is the Virasoro central charge of the matter content with $c_m < 26$. Because $c_m \neq 26$, there are some Liouville degrees of freedom with central charge $c_L$, where $c_m + c_L = 26$. The Liouville theory has the action

$$S_{\rm Liouville} = \int \frac{d^2x}{4\pi} \left( \frac{1}{2}(\partial\phi)^2 + \lambda e^{ib\phi} \right). \tag{87}$$

The central charge of the Liouville theory is related to $b$ as follows:

$$c_L = 1 + 6 \left( \frac{ib}{\sqrt{2}} + \frac{\sqrt{2}}{ib} \right)^2 . \tag{88}$$

The full range $1 < c_m < 25$ beyond the "$c_m = 1$" barrier, corresponds to

$$b^2 = 2e^{i\alpha}, \qquad \text{with} \qquad \cos\alpha = \frac{c_m - 13}{12} \tag{89}$$

Note that in the range we are considering, $-1 < \cos\alpha < 1$. Like the sine-Gordon theory, the Liouville theory has a finite $\mathcal{U}_q(sl(2))$ quantum group symmetry with $q$ the same as in eq. (24) [13, 26].

To relate our S-matrix theory to the string theory we would need to relate the Polyakov path integral over Riemann surfaces (summed over genus) for the matter coupled to 2d gravity, and it isn't at all clear how to even begin to do this. We point out however that this has been done for the $c_m = 1$ case leading to an exact 2D S-matrix [27]. (For a review, see[28, 29].) Here we merely examine the following. The $e^{ib\phi}$ interaction of the Liouville theory is one part of the $\cos b\phi$ interaction of the sine-Gordon theory, and we expect that the Liouville theory for $b$ given in eq. (89) shares some common features with the sine-Gordon theory at the same $b$.

For $\alpha = -h/2$, and $h$ small, eq. (89) matches eq. (23) and thus corresponds to our cyclic sine-Gordon model. One sees that this is just below the $c_m = 25$ barrier:

$$c_m \approx 25 - \frac{3}{2}h^2 \tag{90}$$

Interestingly, when $\alpha \approx \pi - h/2$ with $h$ small, $c_m$ is just above 1, $c_m \approx 1 + 3h^2/2$, and this corresponds to the deformation of the **sinh**-Gordon theory which is the staircase model[18]. It would be very interesting if the idea of a Russian doll RG plays a role in resolving the known long-standing difficulties in going beyond $c_m = 1$.

### D. Comparison between the cyclic sine-Gordon model and Russian doll BCS model

It is interesting to make a comparison between the cyclic sine-Gordon model we have studied in the previous sections and the Russian doll BCS model introduced in reference [5], which also possesses a cyclic RG. The model considered in the latter reference is a simple modification of the reduced BCS model used in the study of the superconducting properties of ultrasmall metallic grains. It describes the pairing interactions between $N/2$ pairs of electrons occupying $N$ energy levels $\varepsilon_j$, which are separated by an energy distance $2\delta$, i.e. $\varepsilon_{j+1} - \varepsilon_j = 2\delta$. The model is characterized by a scattering potential $V_{j,j'}$ equal to $g + i\,h$ for $\varepsilon_j > \varepsilon_{j'}$ and $g - i\,h$ for $\varepsilon_j < \varepsilon_{j'}$ in units of the energy spacing $\delta$.

The case $h = 0$ is equivalent to the usual BCS model which has a unique condensate with a superconducting gap given by $\Delta \sim 2N\delta e^{-1/g}$ for $g$ sufficiently small. When $h$ is non zero the BCS gap equation has an infinite number of solutions corresponding to condensates with gaps $\Delta_n(N) \sim NA\delta e^{-n\pi/h}$ $(n = 0, 1, \ldots, \infty)$, where $A$ depends on $g$ and $h$. This model can also be studied using a renormalization group which reduces the number of energy levels $N$, namely $N(s) = e^{-s}N$, by integrating out the high energy modes. The RG leaves invariant the coupling constant $h$, while $g(s)$ runs with an equation similar to the one loop result (12) and has a cyclic behavior with a period $\lambda_{BCS} = \pi/h$, which in turn is related to the cyclicity of the gaps $\Delta_n(e^{-\lambda_{BCS}}N) \approx \Delta_{n+1}(N)$. Notice that in this case, as compared to eq. (70), the jump in $n$ is one, due to the fact that there is no discrete symmetry associated to $n$. In a finite system the number of the condensates $n_c$ is given by $n_c \sim \frac{h}{\pi}\log N$, in close resemblance to eq. (71). All these results together with the corresponding ones for the cyclic sine-Gordon model are summarized in table 2.

| | Cyclic sine-Gordon | Russian doll BCS |
|---|---|---|
| RG-time | $L = e^l\, a$ | $N(s) = e^{-s}\, N$ |
| RG-period | $\lambda_{CSG} = \frac{2\pi}{h}$ | $\lambda_{BCS} = \frac{\pi}{h}$ |
| Energy scales | Resonances $m_n(L) \sim \frac{1}{L}e^{n\pi/h}$ | Condensates $\Delta_n(N) \sim NAe^{-n\pi/h}$ |
| Russian doll scaling | $m_n(e^{-\lambda_{CSG}}L) \approx m_{n+2}(L)$ | $\Delta_n(e^{-\lambda_{BCS}}N) \approx \Delta_{n+1}(N)$ |
| Finite systems | $n_{\rm res} \sim \frac{h}{\pi}\log(L/a)$ | $n_c \sim \frac{h}{\pi}\log N$ |

Table 2.- Comparison between the cyclic sine-Gordon model and the Russian doll BCS models.

## VII. CONCLUSIONS

In this paper we have analyzed in detail the cyclic regime of the Kosterlitz-Thouless flows of a current-current perturbation of the $su(2)$ level $k = 1$ WZW model [4]. We have computed the RG time $\lambda$ of a complete cycle in terms of the RG invariant $Q = -h^2/16$ of the non perturbative RG equations [6], obtaining $\lambda = 2\pi/h$. Using standard bosonization techniques we have mapped the perturbed WZW model into the sine-Gordon model parametrized by a complex coupling $b^2 = 2/(1+ih/2)$. The latter model possesses a quantum affine algebra $\mathcal{U}_q(\widehat{sl(2)})$ with a real quantum deformation parameter $q = e^{-\pi h/2}$, together with a dual quantum group structure $\mathcal{U}_{\widetilde{q}}(\widehat{sl(2)})$ with $\widetilde{q}^{-1} = e^{2\pi/h} = e^{\lambda}$. Assuming the existence of solitons with topological charge $\pm 1$ with a factorized scattering S-matrix commuting with the $\mathcal{U}_q(\widehat{sl(2)})$ symmetry, we have examined two different S-matrix solutions differing by overall scalar factors.

The first S-matrix we considered in section IV is real analytic and thus strictly unitary in the usual sense. It can also be obtained from a certain limit of the XXZ spn chain. The S-matrix possesses a cyclicity as a function of energy, eq. (4) which is a clear signature of a cyclic RG. We conjectured that it describes the cyclic regime of the KT flows.

The other S-matrix we considered in section VI is an analytic extension of the usual massive sine-Gordon S-matrix. In this case the cyclic S-matrix has poles leading to an infinite number of resonances with the Russian doll scaling behavior also consistent with a cyclic RG flow. However since this S-matrix is not unitary in the usual sense because of the lack of real analyticity, it cannot describe the hermitian theory defined by the current-current perturbation. Thus it's physical meaning remains unclear. If there is a physically sensible underlying model, it probably corresponds to a parity violating theory as discussed in[20]. Perhaps such a theory can be obtained as a certain limit of the XYZ chain.

Remarkably we found that closure of the cyclic sine-Gordon S-matrix bootstrap leads to resonances of arbitrarily higher spin with a string-like mass formula. As discussed in section VI, it would be very interesting if the idea of a cyclic RG could shed new light on the problem of the $c = 1$ barrier.

A few remarks concerning the so-called c-theorem are warranted. The c-theorem is a statement regarding the irreversibility of the RG flow: as the scale is increased massive states decouple and $c$ decreases monotonically [35]. The c-function may be viewed as a function of the scale-dependent couplings, $c(L) = c(g(L))$. Thus, a cyclic RG would appear to give rise to a periodic c-function, violating the c-theorem. A thermodynamic Bethe ansatz analysis [36] of the c-function based on the S-matrix proposed in section IV indeed reveals a periodic c-function in the deep ultraviolet [37]. How can this violation of the fundamental c-theorem occur in a unitary theory as commonplace as the KT flows? One answer is that Zamolodchikov's proof of the c-theorem assumes that one can define the theory as a perturbation about a UV fixed point, however a cyclic RG by definition has no such fixed point.

This work further supports the idea that cyclic RG behavior may be more commonplace than originally anticipated, and may possibly represent a new paradigm for Physics. It is tempting, though probably foolhardy, to even speculate that the generations of fermions in the standard model are just the first few in an infinite sequence of Russian dolls.

**Acknowledgments**

We would like to thank D. Bernard, S. Lukyanov and L. Miramontes for discussions. This work has been supported by the Spanish grants SAB2001-0011 (AL), BFM2000-1320-C02-01 (JMR and GS), and by the NSF of the USA.

[1] K. G. Wilson, Phys. Rev. **D3** (1971) 1818.
[2] P. F. Bedaque, H.-W. Hammer, and U. van Kolck, "Renormalization of the Three-Body System with Short-Range Interactions", Phys. Rev. Lett. **82** (1999) 463, nucl-th/9809025.
[3] S. D. Glazek and K. G. Wilson, "Limit cycles in quantum theories", hep-th/0203088.
[4] D. Bernard and A. LeClair, " Strong-weak coupling duality in anisotropic current interactions", Phys. Lett. **B512** (2001) 78, hep-th/0103096.
[5] A. LeClair, J.M. Román and G. Sierra, "Russian Doll Renormalization Group and Superconductivity", cond-mat/0211338.
[6] B. Gerganov, A. LeClair and M. Moriconi, "On the Beta Function for Anisotropic Current Interactions in 2D" Phys.Rev.Lett. **86** (2001) 4753, hep-th/0011189.
[7] A. B. Zamolodchikov, Commun. Math. Phys. **69** (1979) 253.
[8] G. Mussardo and S. Penati, "A Quantum Field Theory with Infinite Resonance States", Nucl.Phys. B567 (2000) 454-492, hep-th/9907039.
[9] Al. Zamolodchikov, unpublished.
[10] S. Lukyanov, "Low energy effective Hamiltonian for the XXZ spin chain", cond-mat/9712314.
[11] A. W.W. Ludwig, K. J. Wiese, "The 4-loop beta-function in the 2D Non-Abelian Thirring model, and comparison with its conjectured "exact" form", cond-mat/0211531.
[12] A. LeClair, "Strong Coupling Fixed Points of Current Interactions and Disordered Fermions in 2D", Phys. Rev. **B64** (2001) 045329, cond-mat/0011413.
[13] D. Bernard and A. LeClair, "Quantum Group Symmetries and Non-Local Conserved Currents in 2D QFT", Comm. Math. Phys. **142** (1991) 99.
[14] E. K. Sklyanin, L. A. Takhtajan and L. D. Faddeev, "Quantum Inverse problem method I, Theor. Math. Phys. **40**, (1980) 688.
[15] V.E. Korepin, N.M. Bogoliubov and A.G. Izergin, "Quantum Inverse Scattering Method", Cambridge University Press (Cambridge, 1993).
[16] M. Jimbo and T. Miwa, "Algebraic Analysis of Solvable Lattice Models", CBMS n.85, American Mathematical Society (1985).
[17] A.B. Zamolodchikov and Al. B. Zamolodchikov, "Factorized S-Matrices in Two Dimensions as the Exact Solutions of Certain Relativistic Quantum Field Theory Models", Annals of Phys. **120** 253 (1979).
[18] Al. B. Zamolodchikov, "Resonance factorized scattering and roaming trajectories", ENS-LPS 335 (1991); hep-th/0005181.
[19] P. Dorey and F. Ravanini, "Staircase Models from Affine Toda Field Theory", Int.J.Mod.Phys. A8 (1993) 873, hep-th/9206052; "Generalising the staircase models", Nucl.Phys. B406 (1993) 708, hep-th/9211115.
[20] J.L. Miramontes, "Hermitian analyticity versus Real analyticity in two-dimensional factorised S-matrix theories", Phys.Lett. **B455** (1999) 231-238, hep-th/9901145.
[21] V. A. Brazhnikov, $\Phi^{(2)}$ Perturbations of WZW Model, Nucl. Phys. **B501** (1997) 685, hep-th/9612040.
[22] J. L. Miramontes and C. R. Fernández-Pousa, "Integrable quantum field theories with unstable particles", Phys. Lett. **B472** (2000) 392, hep-th/9910218.
[23] O.A. Castro-Alvaredo, J. Dreissig, A. Fring, "Integrable scattering theories with unstable particles", hep-th/0211168.
[24] P. P. Kulish, N. Yu. Reshetikhin and E. K. Sklyanin, Lett. Math. Phys. **5** (1981) 393.
[25] M.B. Green, J.H. Schwarz and E. Witten, "Superstring Theory", Vol.1, Cambridge University Press (Cambridge, 1987)
[26] C. Gomez and G. Sierra, Phys. Lett. **225B** (1991) 51.
[27] G. Moore, M. R. Plesser and S. Ramgoolam, "Exact S-Matrix for 2D String Theory", Nucl.Phys. B377 (1992) 143, hep-th/9111035.
[28] P. Ginsparg and G. Moore, "Lectures on 2D Gravity and 2D String Theory (TASI 1992)", hep-th/9304011.
[29] I. R. Klebanov, "String Theory in Two Dimensions", ICTP Spring School on String Theory and Quantum Gravity, Trieste, April 1991, hep-th/9108019.
[30] I. Affleck, in Les Houches 1988, North-Holland (Amsterdam, 1990).
[31] A.O. Gogolin, A.A. Nersesyan and A.M. Tsvelik, "Bosonization and Strongly Correlated Systems", Camdridge University Press (Cambridge, 1998).
[32] R. Nepomechi and L. Mezincescu, unpublished, private communication.
[33] R.J. Baxter, "Exactly Solved Models in Statistical Mechanics", Academic Press (London, 1982).
[34] E.T. Whittaker and G.N. Watson, "A Course on Modern Analysis", Cambridge University Press, (London, 1980).
[35] A.B. Zamolodchikov,"Irreversibility of the flux of the renormalization group in a two-dimensional field theory", JETP Lett. **43** (1986) 730.
[36] Al. Zamolodchikov, Nucl. Phys. B 342 (1990) 695; Nucl. Phys. **B358** (1991) 497.
[37] A. LeClair, J.M. Román and G. Sierra, in preparation.