Comment on "Polynomial-Time Simulation of Pairing Models on a Quantum Computer"

In a recent Letter[1], Wu *et al.* proposed a polynomial-time algorithm (PTA) for simulations of a wide class of pairing hamiltonians on a NMR quantum computer. While this is an interesting result we want to recall in this Comment the existence of classical algorithms for attacking this problem to *any* required precision. There are two main sort of algorithms. If the pairing model (PM) turns out to be integrable then it can be solved à la Bethe. This happens in the reduced BCS model characterized by a single coupling constant, but there are more general cases reviewed below. The other algorithm is the density matrix renormalization group (DMRG), which is applicable to both integrable and nonintegrable models.

The basic idea of the Letter is that within the next generation of quantum computers (QC) with up to 10-100 qubits, it will be possible to solve hard quantum many body problems in Hilbert spaces with $N \sim 10^2$ single particle levels and $M \sim 10^2$ number of particles. However the authors seem to have overlooked the aforementioned facts concerning the power of analytical and numerical methods as the DMRG. While the analytical methods are only applicable to integrable models, the DMRG is of general applicability.

As mentioned in [1] the PM is of fundamental importance for nuclear structure studies in medium and heavy nuclei and it was introduced in nuclear physics [2] soon after the work of Bardeen, Cooper and Schieffer. Even though the BCS theory explains qualitatively the superconducting phenomenon in finite nuclei, it suffers from the strong number fluctuations.

More recently the (PM) Hamiltonian has been extensively used to describe the physics of ultrasmall superconducting grains. It was shown in this context that only by resorting to the exact numerical solution [3], using the DMRG, was it possible to describe the disappearance of superconducting correlations as a function of the grain size. The ground state energy for systems as large as $N \sim 400$ with Hilbert space dimensions of $\sim 10^{130}$ were treated with high numerical accuracy (5 significant figures). In particular, the case of $N = 100$ was selected to study the convergence properties of the method showing that the exact ground state energy can be obtained to seven significant figures with a moderate computational effort (see Table I of Ref. [3]). The DMRG is a highly accurate iterative method based on the Wilson renormalization group. In the infinite algorithm procedure used in [3] the number of iterations needed to obtain the low lying states of a pairing Hamiltonian is $N/2$ which has to be compared with the $\sim N^4$ computational steps required by PTA of Ref. [1]. At each iteration de DMRG procedure stores the matrix elements of all relevant operators in an optimum subspace of m selected states. Due to the exponential convergence of the method m is tuned

to obtain the desired precision. For a general pairing hamiltonian there are three operators per level, therefore the total memory usage is $O(3m^2N)$ which for reasonable values of $m \sim 10^2$ allows the treatment of systems with $N \sim 10^3$. In [3] we used a constant pairing interaction but the DMRG approach can easily accommodate arbitrary pairing matrix elements, and it is specially suited to give the ground state and the low lying excited states.

Immediately after this work appeared, it was realized that the PM has been solved exactly by Richardson in the 1960's [4]. Recently [5], the exact solution of the PM has been generalize to a large class of pairing Hamiltonians. The most general pairing Hamiltonian can be written as

$$H_p = \sum_i \varepsilon_i n_i + \sum_{i \neq j} V_{ij}^1 \, c_{i+}^\dagger c_{i-}^\dagger c_{j-} c_{j+} + \sum_{i \neq j} V_{ij}^2 \, n_i n_j \quad (1)$$

where $n_i = c_{i+}^\dagger c_{i+} + c_{i-}^\dagger c_{i-}$ and i runs over the N double degenerate levels. Note that the third term corresponds to a monopole interaction absent in the Hamiltonian treated in the Letter. As shown in ref. [5] there is a wide subset of exactly solvable Hamiltonians parametrizing the interaction as $\varepsilon_i = \epsilon_i - g \sum_{k(\neq i)} \gamma \left(\epsilon_i - \epsilon_j \right) \cot \gamma \left(\eta_i - \eta_j \right)$, $V_{ij}^1 = 2g \frac{\gamma(\epsilon_i - \epsilon_j)}{\sin \gamma(\eta_i - \eta_j)}$ and $V_{ij}^2 = \frac{g}{2} \gamma \left(\epsilon_i - \epsilon_j \right) \cot \gamma \left(\eta_i - \eta_j \right)$. Where g, ϵ_i and η_i are a set of $2N + 1$ free parameters and γ can take the values 0, 1 and $-i$ defining the three families of exactly solvable models, the rational, the trigonometric, and the hyperbolic models respectively. As we see, among the $2N^2 - N$ set of free parameters of the most general and non-integrable Hamiltonian (1), a restriction to a subset of $6N + 3$ free parameters leads to an exactly solvable Hamiltonian. We believe that most of the physical problems can be modelled with a pairing Hamiltonian within the integrable subset, but if some particular case requires a non-integrable pairing Hamiltonian, it can then be solved numerically to the desired precision using the DMRG.

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[1] L. A. Wu, M. S. Byrd and D. A. Lidar, Phys. Rev. Lett. **89**, 057904 (2002).
[2] A. Bhor, B. Mottelson and D. Pines, Phys. Rev. **110**, 936 (1958).
[3] J. Dukelsky and G. Sierra, Phys. Rev. Lett. 83, 172 (1999)
[4] R.W. Richardson and N. Sherman, Nucl. Phys. **52**, 221 (1964).
[5] J. Dukelsky, C. Esebbag and P. Schuck, Phys. Rev. Lett. **87**, 066403 (2001).