Masterpraktikum - Scientific Computing, High Performance Computing

Laplace Equation

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Outline

- Solving Systems of Linear Equations
- CG Method
- Poisson/Laplace Equation

Solving Systems of Linear Equations

Given: $A\vec{x} = \vec{b}$, A regular, $\vec{b}$ known

- Direct Methods
  - Gauß
  - LU Decomposition
  - QR Decomposition
- Iterative Methods
  - Splitting Methods (Jacobi, Gauß-Seidl, SOR)
  - Projection Methods **CG**, GMRES, BiCGSTAB
  - QR Decomposition

CG Method

- Method to solve SLEs with a symmetric, positive definite system matrix
- Details: An Introduction to the Conjugate Gradient Method Without the Agonizing Pain by Jonathan Richard Shewchuk

Idea:

- Minimize: $F(x) = \frac{1}{2}(Ax, x) - (b, x)$
- $\nabla F(x) = \frac{1}{2}(A + A^T)x - b = Ax - b$

Definition

- residual: $r = b - Ax_i$

CG Method - Code

$i \Leftarrow 0$
$r \Leftarrow b - Ax$
$d \Leftarrow r$
$\delta_{new} \Leftarrow r^T r$
$\delta_0 \Leftarrow \delta_{new}$
While $i < i_{max}$ and $\delta_{new} > \varepsilon^2 \delta_0$ do
    $q \Leftarrow Ad$
    $\alpha \Leftarrow \frac{\delta_{new}}{d^T q}$
    $x \Leftarrow x + \alpha d$
    If i is divisible by 50
        $r \Leftarrow b - Ax$
    else
        $r \Leftarrow r - \alpha q$
    $\delta_{old} \Leftarrow \delta_{new}$
    $\delta_{new} \Leftarrow r^T r$
    $\beta \Leftarrow \frac{\delta_{new}}{\delta_{old}}$
    $d \Leftarrow r + \beta d$
    $i \Leftarrow i + 1$

Laplace Equation

$$\text{Solve: } \triangle f(\vec{x}) = 0$$

Here:

- $\vec{x} \in \mathbb{R}^2$
- we have Dirichlet boundary conditions
- we employ a regular full grid
- we do not construct the matrix, we use an implicitly given operator
- finite differences, e.g. in 1D: $f''(x) = \frac{f(x+h)-2f(x)+f(x-h)}{h^2}$