

## 6.864 Advanced Natural Language Processing<sup>1</sup>

### Lecture 4: EM Algorithm and Topic Model

24 September 2015

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Earlier, we saw cases where the full data was observable and we could apply a straightforward maximum-likelihood estimator for the parameters. In Lecture 4, we introduce a Hidden Variable Model where data is partially observable. Then, we provide basic intuitions for the EM algorithm which can be used to optimize the likelihood on such Hidden Variable Models instead of directly solving it analytically. We explore further by demonstrating the EM algorithm on a toy example with biased coins. Later, we connect our concept to more relevant topics in NLP such as topic modeling.

### Hidden Variable Models

#### Motivation

To gain intuition about Hidden Variable Models, see the following examples.

1. *Topic model*: Consider the following unigram model; A topic is denoted by  $z$  where  $z \in \mathcal{Z}$ , and  $\mathcal{Z}$  is our pre-defined universe of topics of  $k$  dimension. The document  $D$  is comprised of  $N$  words such that  $D = \{w_1, w_2, \dots, w_N\}$ . Each word  $w_i$  is sampled from a corresponding topic  $z_i$ . If the entire data had been observed, we could simply estimate the conditional distribution as the empirical count,

$$\hat{p}(w|z) = \frac{\text{count}(w, z)}{\text{count}(z)} \quad (1)$$

However, such straightforward approach seen in (1) is no longer applicable if the variables  $z_i$  are hidden. Since it is unlikely to have each word in a document explicitly tagged with its related topics, topic models naturally arise as Hidden Variable Models.

2. *Machine Translation Alignment*: In Machine Translation, one is given two sentences in different languages with the same meaning. As seen in Figure 2, when the word "I" is paired together many times with the corresponding Korean word, then it is probably a good indicator that "I" should be translated to that word. Therefore, the first step in Machine Translation is to align corresponding translations. However, the alignments are not explicitly given in real life applications and should be treated as hidden variables one has to estimate.

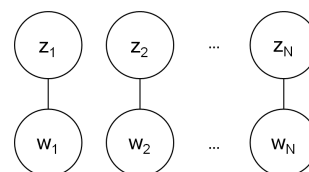


Figure 1: Topic model is where each word is associated with a conditional topic. For example, it is more likely to observe word  $w_i = \text{"Cat"}$  if the topic is  $z_i = \text{"Pets"}$ .



Figure 2: In machine translation, each word from one language has a hidden alignment to its counterpart.

3. *Part-of-Speech Tagging*: Part-of-speech tagging, which will be covered in the next lecture, matches each word in a sentence with its part of speech in English. For the example in Figure 2, "I" is tagged with "pronoun", "love" with "verb", "big" with "adjective", and "dogs" with "noun". When the full observed data observable, one can apply a supervised learning algorithm. However when the parts of speech are not observed, then they are treated as hidden variables.

### Observed Case vs. Unobserved Case

In previous lectures we looked at models where the full data was observed. We use

$$D = \{(x_1, y_1), (x_2, y_2), \dots, (x_N, y_N)\}$$

to denote the given dataset in such cases, which is represented as a list of sampled tuples. Each data point  $(x, y)$  is independently sampled from some distribution  $p_{X,Y}(x, y)$  parameterized by  $\theta$ . The objective function we are trying to maximize is given by the following

$$\theta^* = \operatorname{argmax}_{\theta} \sum_{i=1}^N \log p_{X,Y}(x_i, y_i | \theta) \quad (2)$$

Now, let's look at the unobserved case where  $y$  is unseen. In other words,  $x$  is the observed variable and  $y$  is the hidden variable. In this case, the observed data is given by

$$D = \{x_1, x_2, \dots, x_N\}$$

In this case, we replace the objective function with the log-likelihood of the *partially observed data*, which is

$$\begin{aligned} \theta^* &= \operatorname{argmax}_{\theta} \sum_{i=1}^N \log p_X(x_i | \theta) \\ &= \operatorname{argmax}_{\theta} \sum_{i=1}^N \log \sum_{y \in \mathcal{Y}} p_{X,Y}(x_i, y | \theta) \end{aligned} \quad (3)$$

where the second equality is due to marginalization over  $y$  over  $p_{X,Y}(x, y)$ . The key point is to note that the criteria for the observed case and the unobserved case is different, and we should use a different objective function when we are trying to optimize for the parameter  $\theta$ . In the following sections we will demonstrate how the EM algorithm finds the optimum parameter for (3).

### EM Algorithm

**Example 1.** We will set up a simple example to walk through how the EM algorithm works. Suppose we have a card and two different coins,

where all probabilities are biased. We will take the following steps to sample a sequence of coin flips.

1. Flip the card, which either returns side  $A$  or side  $B$ .
2. If  $A$  was drawn from step 1, toss coin 1 three times. Otherwise if  $B$  was drawn, toss coin 2 three times.
3. repeat steps 1-2

Expressing this more formally,  $y$  represents the side of the card and  $x$  represents the outcome of three consecutive coin tosses.  $H$  and  $T$  denotes the head and tail of a coin, respectively.

$$y \in \mathcal{Y} = \{A, B\}$$

$$x \in \mathcal{X} = \{HHH, HTH, \dots, TTT\}$$

There are three free parameters in this setup, given by  $\theta = \{\alpha, \pi_1, \pi_2\}$ . The parameters are described as follows

$$\alpha = p(\text{Card} = A)$$

$$\pi_1 = p(\text{Coin1} = H)$$

$$\pi_2 = p(\text{Coin2} = H)$$

It is easy to see that

$$p_{X,Y}(x, y | \theta) = p_Y(y | \theta) p_{X|Y}(x | y, \theta)$$

$$p_Y(y | \theta) = \begin{cases} \alpha & \text{if } y = A \\ 1 - \alpha & \text{if } y = B \end{cases}$$

$$p_{X|Y}(x | y, \theta) = \begin{cases} \pi_1^{h(x)} (1 - \pi_1)^{t(x)} & \text{if } y = A \\ \pi_2^{h(x)} (1 - \pi_2)^{t(x)} & \text{if } y = B \end{cases}$$

Where we define  $h(x)$  to be a function that is equal to the number of heads in sample  $x$ , and  $t(x)$  to be the number of tails. For example, if  $x_0 = HHT$ , then  $h(x_0) = 2$  and  $t(x_0) = 1$ .

### Observed Case

In the observed case, the card flips and the coin tosses are entirely observable. Each card flip is associated with three coin tosses. Suppose after three iterations, the sampled data was given as follows

$$D = \{(A, HHH), (A, HHH), (B, TTT)\}$$

where  $(y, x) = (A, HHH)$  denotes that the observed card flip was  $A$ , followed by three heads. We had seen in earlier lectures that applying partial derivative to the log-likelihood given in (2), the MLE

parameters are simply given by the empirical counts. Using this fact for granted, the parameter values for these specific samples are given by

$$\begin{aligned}\alpha^* &= \frac{\text{count}(A)}{3} = \frac{2}{3} \\ \pi_1^* &= \frac{\text{count}(A, H)}{\text{count}(A, H) + \text{count}(A, T)} = \frac{6}{6} \\ \pi_2^* &= \frac{\text{count}(B, H)}{\text{count}(B, H) + \text{count}(B, T)} = \frac{0}{3}\end{aligned}$$

Shortly we will see that this analytic solution has a very similar form to the solution for each iteration in the EM algorithm, which is covered in the next section.

### Unobserved Case

Continuing with the problem setup in Example 1, suppose the card flips are no longer observable. In other words, we will treat the card flip  $y$  as a hidden variable. Consider the following sample data

$$D = \{HHH, HHH, TTT\} = \mathbf{x}$$

Let us make an initial guess on the parameter values. Assign arbitrary prior values such that

$$\alpha^{(0)} = 0.1, \quad \pi_1^{(0)} = 0.8, \quad \pi_2^{(0)} = 0.5 \quad (4)$$

The initial guess in (4) is denoted with a superscript 0 to mark that this is the belief of the parameters at time 0. We will see how the belief is updated over each iteration of the EM algorithm.

For each  $x_i$  in the dataset, we compute  $p_{Y|X, \theta}(y|x_i, \theta^{(0)})$  over all possible values of  $y \in \mathcal{Y} = \{A, B\}$ . For example, for our  $x_1 = HHH$  in the given example dataset,

$$\begin{aligned}p_{Y|X, \theta}(y_1 = A|x_1 = HHH, \theta^{(0)}) &= \frac{p_{X, Y|\theta}(x_1 = HHH, y_1 = A|\theta^{(0)})}{p_{X, Y|\theta}(x_1 = HHH, y_1 = A|\theta^{(0)}) + p_{X, Y|\theta}(x_1 = HHH, y_1 = B|\theta^{(0)})} \\ &= \frac{\alpha \pi_1^3}{\alpha \pi_1^3 + (1 - \alpha) \pi_2^3} \\ &\approx 0.3 \\ p_{Y|X, \theta}(y_1 = B|x_1 = HHH, \theta^{(0)}) &= \frac{p_{X, Y|\theta}(x_1 = HHH, y_1 = B|\theta^{(0)})}{p_{X, Y|\theta}(x_1 = HHH, y_1 = A|\theta^{(0)}) + p_{X, Y|\theta}(x_1 = HHH, y_1 = B|\theta^{(0)})} \\ &= \frac{(1 - \alpha) \pi_2^3}{\alpha \pi_1^3 + (1 - \alpha) \pi_2^3} \\ &\approx 0.7\end{aligned}$$

Recall from the previous section that in the fully observed case, each  $x_i$  was explicitly associated with a *single* count of either  $y_i = A$  or  $y_i = B$ . In the unobserved case, we assume that each  $x_i$  is associated with *fractional counts* over all possible values of  $y_i$ , to which we assign the *expected count* of  $y_i$  given  $x_i$  and our prior knowledge of  $\theta$ , i.e.  $\mathbb{E}[y|x = x_i, \theta^{(0)}]$ .

$x_i$	$y_i$	empirical count	$x_i$	$y_i$	fractional count
HHH	A	1	HHH	A	0.3
				B	0.7
HHH	A	1	HHH	A	0.3
				B	0.7
TTT	B	1	TTT	A	0.5
				B	0.5
(a) $y$ is observed			(b) $y$ is unobserved		

Table 1: In the fully observed case (a), a single value of  $y$  (which is the observed value itself) is associated with each  $i^{\text{th}}$  data sample. In the unobserved case (b), we assume that a entire list over the alphabet  $\mathcal{Y}$ , is associated with  $x_i$ . Each element in the spectrum is assigned a weight, or the *fractional count*, which corresponds to the expected value of  $y$  given  $x_i$ . The fractional count can also be understood as the confidence in the sample  $(x_i, y_i)$  given a prior belief over the parameters.

Hereafter the remaining steps are almost identical to the MLE maximization of the parameters. Again we will derive our distribution by counting, but since the actual counts are no longer given, we will count the fractional counts instead. The parameters for the next iteration, or time 1, will be computed as follows

$$\begin{aligned}
 \alpha^{(1)} &\leftarrow \frac{\text{fractional count of } A}{3} = \frac{0.3 + 0.3 + 0.5}{3} = \frac{1.1}{3} \\
 \pi_1^{(1)} &\leftarrow \frac{\text{fractional count of } (A, H)}{\text{fractional count of } (A, H) + \text{fractional count of } (A, T)} \\
 &= \frac{3 \times 0.3 + 3 \times 0.3}{3 \times 0.3 + 3 \times 0.3 + 3 \times 0.5} = \frac{1.8}{3.3} \\
 \pi_2^{(1)} &\leftarrow \frac{\text{fractional count of } (B, H)}{\text{fractional count of } (B, H) + \text{fractional count of } (B, T)} \\
 &= \frac{3 \times 0.7 + 3 \times 0.7}{3 \times 0.7 + 3 \times 0.7 + 3 \times 0.5} = \frac{4.2}{5.7}
 \end{aligned}$$

The EM algorithm guarantees that the likelihood given by Equation (3) is increased over every iteration as we update our parameters  $\theta^{(0)}, \theta^{(1)}, \theta^{(2)}, \dots, \theta^{(T)}$ . In summary, the EM algorithm is given by the following steps

1. Randomly initialize  $\theta^{(0)}$
2. Based on  $\theta^{(t)}$ , compute  $p(y|x, \theta^{(t)})$  and count the fractionals
3. Re-estimate  $\theta^{(t+1)}$
4. Repeat until convergence

### Properties of EM Algorithms

We can make the following soft statements about the EM algorithm's behavior

1. The algorithm converges to a stationary point
2. The algorithm converges to a local maximum

On final note, when  $y$  is hidden, it is also likely that we do not have previous knowledge of the alphabet space  $\mathcal{Y}$ . The selection of the alphabet space is treated as a regularization problem, similar to selecting the number of units in a neural network, which will be optimized by cross-validation over a development data set.

### Topic Model

"Art"	"Budget"	"Children"	"Education"
new	money	care	state
film	congress	percent	president
show		family	schools
		welfare	Haiti

Table 2: For each given topic, some words are more likely to appear than others

### Introduction to Topic Model

In this section, we will briefly introduce Topic Model. Consider a simple unigram model where the probability of a word  $w_i$  is given by  $p(w_i) = \theta_i$ . We will also model a topic  $z$  such that  $z \in \mathcal{Z}$  and  $|\mathcal{Z}| = k$ , where  $\mathcal{Z}$  is our universe of topics and the dimension  $k$  is treated as a hyperparameter. Each word in the document will be generated from the distribution  $p(w_i|z) = \theta_{w_i|z}$ , according to the following steps

1. Sample a topic  $z$
2. Generate the entire document with the given topic

Then, the likelihood of the document  $d$  of size  $N$  can be expressed as

$$p(d|\theta) = \sum_{z \in \mathcal{Z}} \theta_z \prod_{i=1}^N \theta_{w_i|z}$$

However, this model is problematic because it assumes that the whole document comes from a single topic. In real-life, it is more likely that topics may change throughout the document. Motivated by this approach, we setup a new model where each word  $w_i$  is sampled from each corresponding topic  $z_i$ . The sampling procedure will now be

1. Sample a topic  $z_i$
2. Sample a word  $w_i$  conditional on  $z_i$
3. Sample the next topic  $z_{i+1}$
4. Repeat until end of document.

The likelihood for the new model is

$$p(d|\theta) = \prod_{i=1}^N \sum_{z \in \mathcal{Z}} \theta_{z|d} \theta_{w_i|z}$$

The distribution over the topics is unique for each document, represented by  $\theta_{z|d}$ . Across all documents, the word conditional on a given topic is sampled from a single "shared" distribution, represented by  $\theta_{w|z}$ . It is easy to see that if both the words and topics are fully observed, we can derive the MLE parameters by looking at the empirical distributions

$$\begin{aligned}\hat{\theta}_{w_i|z} &= \frac{\text{count}(w_i, z)}{\text{count}(z)} \\ \hat{\theta}_{z|d} &= \frac{\text{count}(z)}{N}\end{aligned}$$

In application, however, topic information are not given out in the training data and the problem should be treated as a Hidden Variable Model. We leave it to the readers to think about how the EM algorithm can be applied to solve this specific unigram topic model.