Simulation of the Lévy processes

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Abstract

Algorithms for simulation of such types of Lévy processes as square-root diffusion processes, compound Poisson processes, jump diffusion processes, stable Lévy processes have been investigated in this paper. These algorithms were implemented as classes using S3 class system in R. Obtained simulation classes have been enriched by methods to plot trajectory, returns, kernel density estimator of the log returns for corresponding Lévy processes. In the case of square-root diffusion process Monte Carlo procedure for pricing european call option has been performed, and obtained option prices have been compared with their counterparts calculated using exact formula for call option valuation, that is available in this case.

Contents

1 Introduction

Derivative valuation is a rather well studied field in the modern mathematical finance. One of the mostly used tools in this area is risk-neutral pricing. Main idea behind risk-neutral valuation principle is that value of european call (put) option may be computed as discounted expectation of the terminal payoff with respect to some martingale measure. This martingale measure is induced by the dynamics of the underlying asset. In other words, if one can model behavior of the underlying security than calculation of the price of contingent claim can be done by computing corresponding expected value, that is rather technical task. The well-known Black-Scholes model for pricing derivatives benefits exactly from this approach, in particular, geometric Brownian motion has been taken as a model for underlying asset, and, as a consequence, value of an option has been derived. But geometric Brownian motion is very bad model of reality. So if one replaces Brownian motion by a more comprehensive stochastic process to model evolution of underlying asset then a tractable class of models generalizing Black-Scholes can be obtained. In this place the notion of Lévy process comes in play, since this family of stochastic processes is rather rich and can be used to model a wide range of real-world dynamics. That is why techniques of simulation of Lévy processes are very important.

The paper is constructed as follows. Section 2 briefly describes general definition of the Lévy process. Section 3 covers general valuation framework for derivative pricing as discussed in [5]. Closed-form formula for pricing call options when underlying asset follows square-root diffusion process is presented. Also Monte Carlo procedure for option valuation is discussed, and Monte Carlo estimators for option prices are compared with their conterparts obtained using exact formula. Sections 4 discusses compoud Poisson process. Section 5 devoted to jump diffusion process, and algorithm for its simulation. Section 6 investigates stable Lévy process. In Section 7 main conclusions are briefly represented.

2 Lévy process

Lévy processes are the simplest type of processes whose path contains continuous motion as well as jump increments of random size arising at random time points.

Definition.

A stochastic process $X = \{X_t : t \geq 0\}$ is said to be a Lévy process if it satisfies the following properties:

(i) $X_0 \overset{\text{a.s.}}{=} 0$*;*

(ii) For any $0 \leq t_1 < t_2 < \cdots < t_n < \infty, X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \ldots, X_{t_n} - X_{t_{n-1}}$ *are independent;*

(iii) $X_t - X_s \overset{\mathscr{L}}{=} X_{t-s}$*, for any* $s < t$*;*

(iv) $\lim\limits_{h \to 0} P(|X_{t+h} - X_t| > \varepsilon) = 0$*, for any* $\varepsilon > 0$ *and* $t \geq 0$*.*

Such types of the Lévy process have been considered in this paper

- Square-root diffusion processes
- Compound Poisson processes
- Jump diffusion processes
- Stable processes

3 Square-root diffusion process

Square-root diffusion process was introduced to finance world in 1985 by Cox, Ingersoll, Ross in their Cox-Ingersoll-Ross model (or CIR model) describing the evolution of interest rates [3]. Afterwards this process has gained significant popularity in mathematical finance.

Having made assumtion that underlying asset follows square-root diffusion process, one can derive closed-form expressions for the price of european call option as it is shown in [5]. Also one can use Monte Carlo approach to price call options if dynamics of the underlying security is known. In this sections these two methods are presented.

3.1 The valuation framework

Let denote the current value of the underlying by V_t (or by V if time is obvious), and assume that the dynamics of V_t is given by the square-root diffusion process

$$dV_t = \kappa_V(\theta_V - V_t)dt + \sigma_V \sqrt{V_t} dZ_t \tag{1}$$

where

- V_t is the time t value of the underlying asset

- θ_V is a mean of V_t, assumed positive
- κ_V is the rate at which V_t reverts, assumed positive
- σ_V is the volatility of V_t, assumed positive
- Z_t is a standard Brownian motion

Typical path of the square-root diffusion process is presented in the Figure 1.

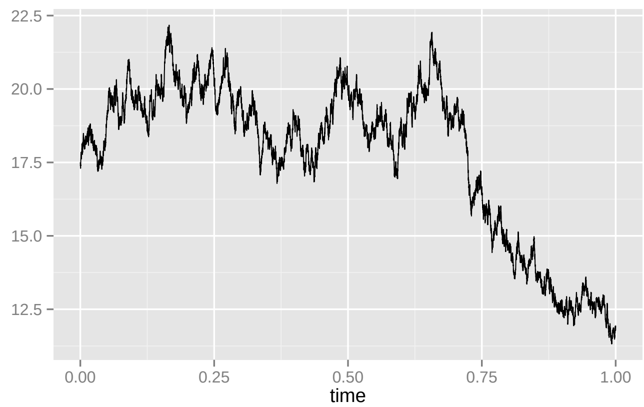

Figure 1: Square-root diffusion process starting from 17.5, $\kappa_V = 0.1$, $\theta_V = 20.0$, $\sigma_V = 2.0$

Now consider the valuation of a contingent claim on V_t. Especially, examine contingent claim with a pay off $B(V_T)$, which at time T depends only on V_T. Denote current value of such a claim by $A(V, T)$, and T-maturity riskless rate as $D(T)$. Then value of the given contigent claim can be calculated as

$$A(V, T) = D(T)E[B(V_T)], \tag{2}$$

where the expectation is taken with respect to the risk-neutral process for V_t.

Thus, valuation of a derivative can be done by directly evaluating the expactation in (2)

3.2 Valuation of options

Let $C(V, K, T)$ denote the current price of a call option on V, where K is the strike price, and T is time to maturity. Based on (2), one can express call price as

$$C(V, K, T) = D(T)E[\max\{V_T - K, 0\}] \tag{3}$$

Having evaluated this expectation (with respect to the risk-neutral measure), one obtains closed-form expression for the value of the volatility call option

$$\begin{aligned} C(V_0, K, T) = D(T) \cdot e^{-\beta T} \cdot V_0 \cdot Q(\gamma \cdot K|\nu + 4, \lambda) \\ +D(T) \cdot \left(\frac{\alpha}{\beta}\right) \cdot \left(1 - e^{-\beta T}\right) \cdot Q(\gamma \cdot K|\nu + 2, \lambda) \\ -D(T) \cdot K \cdot Q(\gamma \cdot K|\nu, \lambda)) \end{aligned} \tag{4}$$

where

- $\alpha = \kappa\theta$
- $\beta = \kappa + \zeta$
- $\gamma = \frac{4\beta}{\sigma^2(1-e^{-\beta T})}$
- $\nu = \frac{4\alpha}{\sigma^2}$
- $\lambda = \gamma \cdot e^{-\beta T} \cdot V$
- $D(T)$ is the appropriate discount factor
- Q is the complementary non-central χ^2 distribution

It is worth mentioning that α, β, and σ are parameters of the risk-neutral (risk-adjusted) process of the underlying security V_t

$$dV_t = (\alpha - \beta V_t)dt + \sigma_V \sqrt{V_t} dZ_t \tag{5}$$

3.3 Valuation of futures

Let $F(V,T)$ denote the futures price for a futures contract on V with maturity T. General formula for the futures price can be expressed as

$$F(V,T) = E(V_T) \tag{6}$$

where the expectation is taken with respect to the risk-neutral process for V.

Evalution of the expectation above gives following formula for the futures price

$$F(V_0,T) = \left(1 - e^{-\kappa_V T}\right) \cdot \theta_V + e^{-\kappa_V T} \cdot V_0 \tag{7}$$

3.4 Monte Carlo Simulation

Let assume that underlying asset follows square-root diffusion process

$$dV_t = (\alpha - \kappa V_t)dt + \sigma_V \sqrt{V_t} dZ_t \tag{8}$$

then underlying instrument is ruled by such risk-neutral dynamics

$$dV_t = (\alpha - \beta V_t)dt + \sigma_V \sqrt{V_t} dZ_t \tag{9}$$

where $\beta = \kappa + \zeta$, ζ denotes the expected premium for volatility risk, and is a crucial factor for determination of risk-neutral equation. Having simulated a sufficient number of risk-neutral pathes of the process, one can obtain Monte Carlo estimator for the price of the call option:

$$C(V_0, K, T) = \frac{D(T)}{I} \sum_{i=1}^{I} \max(V_T^i - K, 0) \tag{10}$$

where V_T^i is value at maturity of the i-th simulated path of the process, I is a number of simulatioons, $D(T) = e^{-rT}$ is discount factor.

In order to verify how accurate Monte Carlo approach can price options, simulation study has been conducted. For such a purpose, a set of strike prices has been selected, and current stock price was fixed. Then one can calculate prices of the obtained options using either Monte Carlo procedure or exact formula (4). To calculate Monte Carlo $I = 50000$ trajectories of the square-root diffusion process have been simulated for a given set of strike prices, and corresponding Monte Carlo prices of the options have been calculated using 10. It should mentioned that exact discretization scheme (see e.g.[4]) has been used for simulation of the square-root diffusion process. Mean Absolute Error between option values obtained by Monte Carlo and exact formula is 0.015, and more comprehensive results of the simulation study are presented in Figure 2.

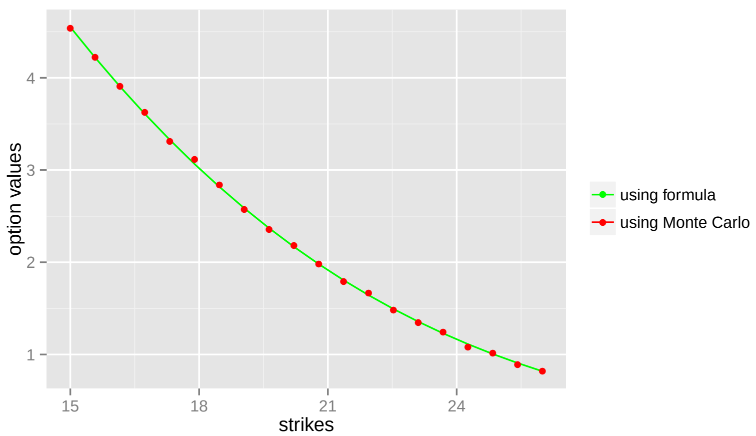

Figure 2: current stock value 17.5, $\kappa_V = 0.1$, $\theta_V = 20.0$, $\sigma_V = 2.0$, $\zeta = 0$, $K = 20.0$, $r = 0.01$, Mean Absolute Error is 0.015

Results shown in Figure 2 give the right to conclude that Monte Carlo simulations could provide with rather accurate option values.

4 Compound Poisson process

A compound Poisson process, parameterised by a rate λ and jump size distribution G, is a process $\{ Y(t) : t \geq 0 \}$ given by

$$Y(t) = \sum_{i=1}^{N(t)} D_i, \tag{11}$$

where $\{ N(t) : t \geq 0 \}$ is a Poisson process with rate λ, and $\{ D_i : i \geq 1 \}$ are independent and identically distributed random variables, with distribution function G, which are also independent of $\{ N(t) : t \geq 0 \}$.

For greater generality drift b can be also added

$$Y(t) = bt + \sum_{i=1}^{N(t)} D_i \tag{12}$$

Algorithm for simulation

1. Simulate a random variable N from Poisson distribution with parameter λT. Then N gives the total number of jumps on the interval $[0, T]$

2. Simulate N independent r.v. U_i, uniformely distributed on the interval $[0, T]$. These variables correspond to the jump times.

3. Simulate jump sizes: N independent r.v. Y_i with given law. The trajectory is given by

$$Y(t) = bt + \sum_{i=1}^{N} \mathbf{1}_{U_i<t} Y_i \tag{13}$$

Typical pathes of the compound Poisson process are presented in Figure 3 and Figure 4.

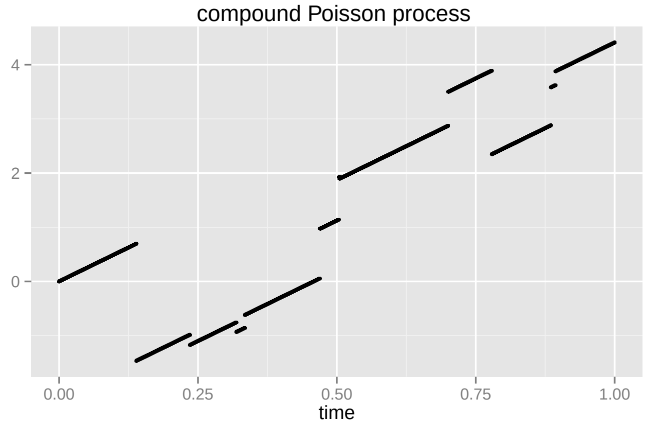

Figure 3: compound Poisson process with drift 5, and jump intensity 10

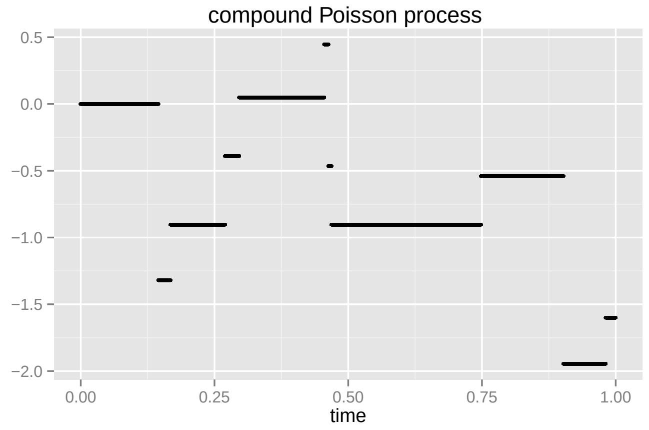

Figure 4: compound Poisson process with drift 0, and jump intensity 20

5 Jump diffusion process

A Levy process of jump-diffusion type has the following form:

$$X(t) = bt + cW_t + \sum_{i=1}^{N(t)} Y_i, \tag{14}$$

where $\{\, N(t) : t \geq 0 \,\}$ is a Poisson process counting the jumps of X, and Y_i are jump sizes(i.i.d. variables). It is especially important to specify the distribution of jump sizes. In the **Merton model**, jumps in the log-price X_t are assumed to have a Gaussian distribution. In the **Kou model**, the distribution of jump sizes is asymetric exponential. In this paper Gaussian distribution of jump sizes has been used for simulation of the jump diffusion process.

Algorithm for simulation

Simulation of $(X(t_1), \ldots, X(t_n))$ for n fixed times $t_1, \ldots, t_n$

1. Simulate n independent centered Gaussian random variables G_i with variance $(t_i - t_{i-1})\sigma^2$
2. Simulate the compound Poisson part. The discretized trajectory of the jump diffusion process is given by

$$X(t_i) = bt_i + \sum_{i=1}^{N} G_k + \sum_{i=1}^{N} \mathbf{1}_{U_i < t} Y_i \tag{15}$$

Typical path of the jump diffusion process is presented in Figure 5.

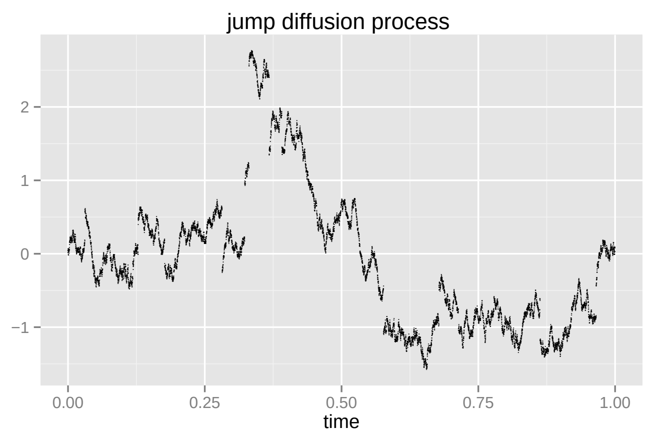

Figure 5: Jump diffusion process with drift 0, jump intensity 20.

As discussed above, jump diffusion process can be represented as a sum of jump component and gaussian component. Jump component (see Figure 6) is responsible for discontinuity points of jump diffusion.

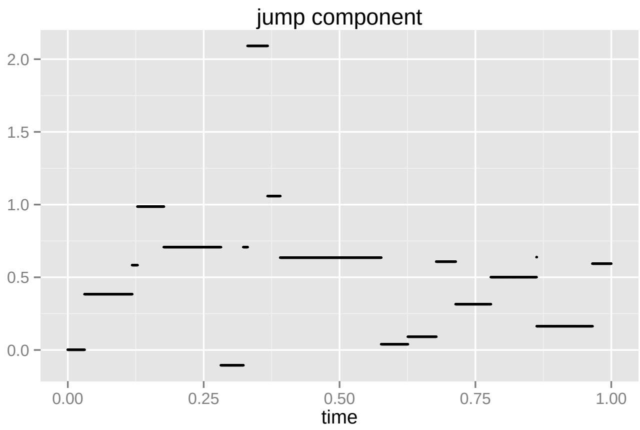

Figure 6: Jump component of the jump diffusion process with drift 0, jump intensity 20.

Gaussian component is represented in Figure 7.

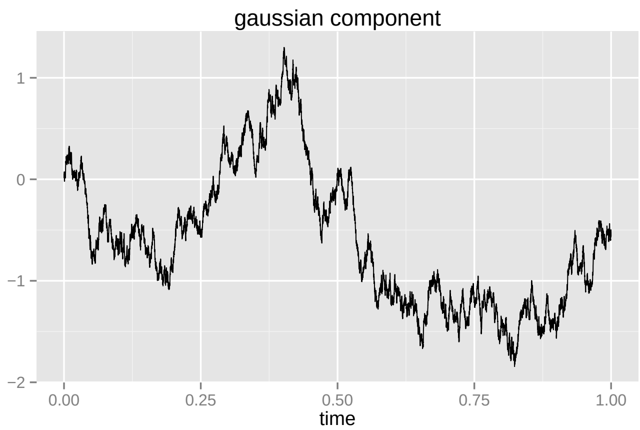

Figure 7: Gaussian component of the jump diffusion process with drift 0, jump intensity 20.

Density of log returns of the jump diffusion process is presented in Figure 8.

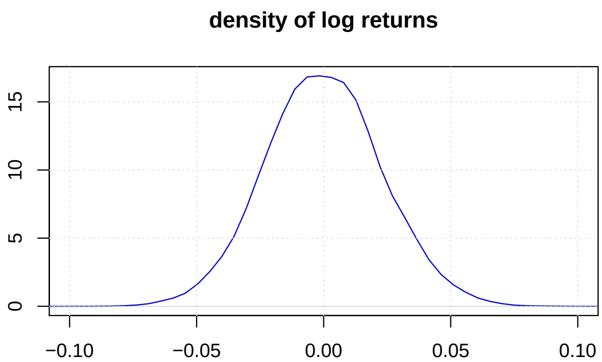

Figure 8: Density of log returns of the jump diffusion process with drift 0, jump intensity 20.

6 Stable Levy process

Stable Lévy process is a Lévy process that $\forall a > 0$ there is a constant c such that distribution of X_{at} coincides with the distribution of $a^{1/\alpha}X_t + ct$ for all $t \geq 0$. The constant α is called the *index of stability* and stable processes with index α are also referred to as α-stable processes.

Algorithm for simulation of the stable Levy process

Simulation of $(X_1, \dots, X_n)$ fot n fixed times $t_1, \dots, t_n$

1. Simulate n independent random variables γ_i uniformely distributed on $(-\pi/2, \pi/2)$ and n independent standart exponential random variables W_i

2. Compute ΔX_i for $i = 1 \dots n$ using

$$\Delta X_i = (t_i - t_{i-1})^{1/\alpha} \frac{\sin(\alpha\gamma_i)}{\cos(\gamma_i)^{1/\alpha}} \left(\frac{\cos((1-\alpha)\gamma_i)}{W_i} \right)^{(1-\alpha)/\alpha} \tag{16}$$

The discretized trajectory is given by

$$X(t_i) = \sum_{k=1}^{i} \Delta X_k$$

Typical pathes of the stable Lévy processes are presented in Figure 9 and Figure 10. Plots of log returns of the corresponding trajectories are shown in Figure 11 and Figure 12.

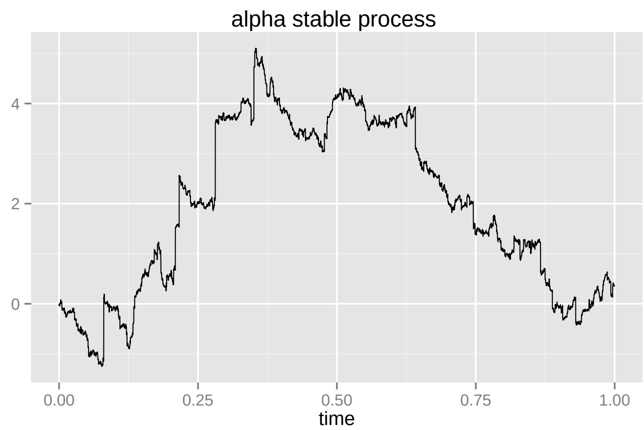

Figure 9: α-stable process with $\alpha = 1.7$

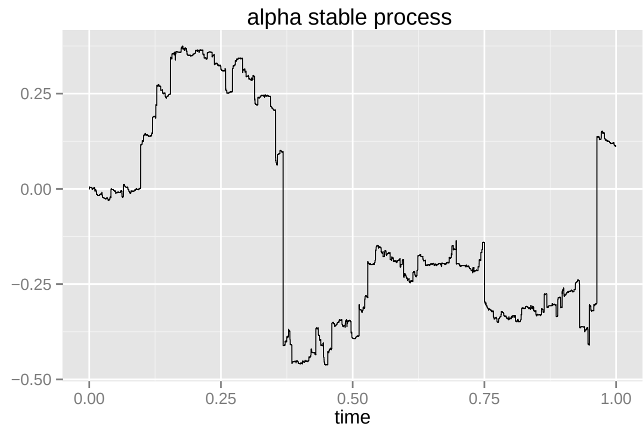

Figure 10: α-stable process with $\alpha = 1$

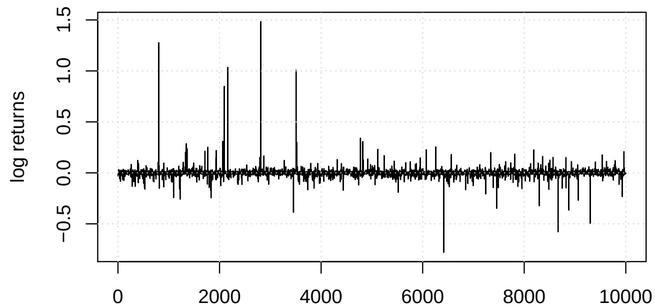

Figure 11: Log returns of α-stable process with $\alpha = 1.7$

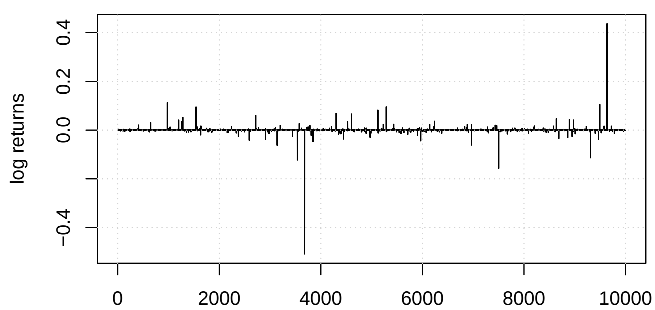

Figure 12: Log returns of α-stable process with $\alpha = 1$

It must be mentioned that parameter α determines the behavior of the stable Lévy process:

- When α is small, the process has very fat tails, and the trajectory is dominated by big jumps.

- When α is large, the behavior is determined by small jumps

7 Conclusion

In this paper such types of Lévy processes as square-root diffusion processes, compound Poisson processes, jump diffusion processes, stable Lévy processes have been considered. Simulation classes for these processes have been developed using S3 class system in R. Simulation class gives user a comfortable way of generating and plotting trajectories of the process, calculation of log returns and plotting corresponding kernel density estimators. In the case of square-root diffusion process, Monte Carlo estimators for option prices have been compared with their counterparts obtained using exact formula for option valuation.

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