Pricing kernels and their dependence on the implied volatility index

Roman Lykhnenko

Ladislaus von Bortkiewicz Chair of Statistics
Humboldt–Universität zu Berlin
http://lvb.wiwi.hu-berlin.de/

Motivation

Consider risky security with the price process $\{S_t, t \in [0, T]\}$. Price at time t of any derivative with payoff $\psi(S_T)$ is

$$P_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-r_{t,\tau}}\psi(S_T)\right] = \mathbb{E}\left[e^{-r_{t,\tau}}\psi(S_T)\mathcal{K}(S_T)\right],$$

with a pricing kernel $\mathcal{K} = \dfrac{q}{p}$, where p and q are the pdf of the historical measure and the risk neutral density of S_T, respectively.

Pricing Kernel Puzzle

In standard models PK must be nonincreasing in the stock market index returns. In practice, PKs are often locally increasing.

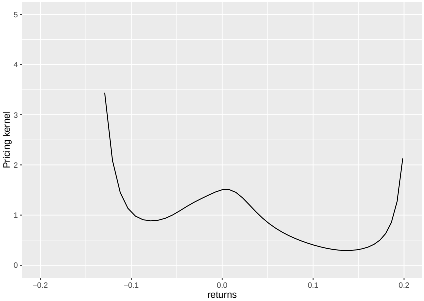

Figure 1: Unconditional pricing kernel of DAX 30 index return at maturity 1 month, sample period is January 4,2012 - December 31,2012

But if properly conditioned ...

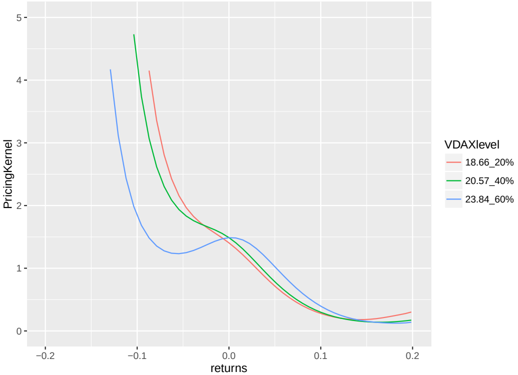

Figure 2: Pricing kernel of DAX 30 index return at maturity 1 month conditioned on $20\%, 40\%, 60\%$ quantiles of VDAX index, sample period is January 4,2012 - December 31,2012

Outline

Basic notation

- S_t : underlying asset price at time t,
- K : strike price,
- T : expiration date,
- $\tau = T - t$: time to maturity,
- p_{S_T} and q_{S_T} are physical and risk-neutral densities respectively,
- Z_t : value of VDAX index at time t,
- V_t : volatility value at time t,
- $C(S_t, K, \tau, r_{t,\tau}, V_t)$: the European call-option price at time t.

Identification of state price density

Due to *Breeden and Litzenberger(1978)*

$$q(S_T|S_t, K, \tau, r_{t,\tau}, V_t) = e^{r_{t,\tau}} \frac{\partial^2 C(S_t, K, \tau, r_{t,\tau}, V_t)}{\partial K^2}\bigg|_{K=S_T}$$

Since V_t is not observable in practice, it is replaced by the observable Z_t

$$q(S_T|S_t, K, \tau, r_{t,\tau}, Z_t) = e^{r_{t,\tau}} \frac{\partial^2 C(S_t, K, \tau, r_{t,\tau}, Z_t)}{\partial K^2}\bigg|_{K=S_T}$$

Dimension reduction

Assume:

$$C(K,\tau,Z_t,S_t) = S_t C(\frac{K}{S_t},\tau,Z_t,1),$$

define a new function

$$\overline{C}(m,\tau,Z_t) = \frac{1}{S_t} C(K,\tau,Z_t,S_t)$$

where $m = \frac{K}{S_t}$ is the moneyness of an option.

Dimension reduction

Thus,

$$q(S_T|K,\tau,V_t) = \frac{1}{S_t} e^{r_{t,\tau}} \frac{\partial^2 \overline{C}}{\partial m^2}\bigg|_{m=\frac{S_T}{S_t}}$$

The density estimates are defined on the scale of S_T. To define the density on the scale of $R_T = log(S_T/S_t)$ such a transformation must be applied:

$$q(R_T) = q(S_T)S_T$$

Local linear estimation of risk neutral densities

$$\min_{\alpha,\beta} \sum_{i=1}^{n} \left\{ \overline{C}_i - \alpha - \beta^T (u_i - u) \right\}^2 K_h (u_i - u)$$

where $u_i = (\tau_i, z_i, m_i)^T$ is the characteristic of the *ith* option, $\overline{C}_i$ is the scaled option price, $u = (\tau, z, m)^T$ is fixed

- α is an estimate of $\overline{C}(u)$
- β is an estimate of $\frac{\partial \overline{C}}{\partial u}$

$$\frac{\partial^2 \overline{C}}{\partial^2 m} = \frac{\partial \beta_4}{\partial m}$$

last derivative can be calculated numerically.

Bandwidth selection

Bandwidth can be choosen as $h_j = c_j \sigma_j n^{(-1/(4+d))}$, where σ_j is unconditional standard deviation of regressor j, and c_j is chossen by

- Leave-one-out cross validation

$$\min_h \frac{1}{n} \sum_{i=1}^{n} \left\{ \overline{C}_i - \widehat{\overline{C}}_{h,-i} (\tau_i, z_i, m_i) \right\}^2 \omega (\tau_i, z_i, m_i)$$

  where $-i$ means leaving the ith observation out, ω is the weighting function.

- K-fold cross validation, which is faster than leave-one-out method (*Hastie et al.,2001*)

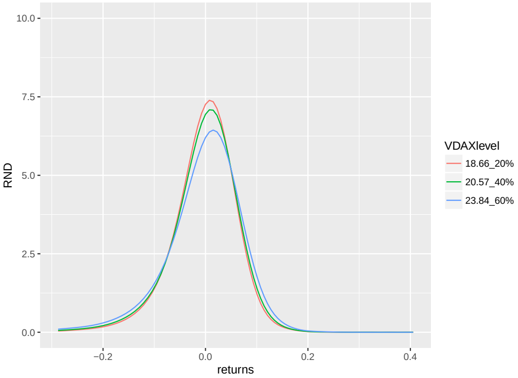

Figure 3: Risk neutral density of DAX 30 index return at maturity 1 month conditioned on 20%, 40%, 60% quantiles of VDAX index, sample period is Jan,2012 - Dec,2012

Asymptotic theory

$$
\begin{aligned}
& n^{1/2} h_m^2 (h_\tau h_z h_m)^{1/2} \left(\widehat{q}\,(r \mid \tau, z, m) - q\,(r \mid \tau, z, m)\right) \\
& \xrightarrow{d} N\left(0, m^2 \left[\int K^2(c) dc\right]^3 \right. \\
& \times \left[\int \left(c\dot{K}^2(c) + K(c)\right)^2 dc\right] \Big/ \left[\int K(c) c^2 dc\right]^2 \\
& \times s^2\,(\tau, z, m) \,/\, \pi\,(\tau, z, m)
\end{aligned}
$$

where $s^2\,(\tau, z, m)$ is the conditional variance for the local linear regression of C on the state variables, and $\pi\,(\tau, z, m)$ is the joint density of these variables

Conditional variance estimation

Based on *Fan and Yao (1998)*.
Let $m(u) = \mathbb{E}\,(C \mid u)$ and $\sigma^2\,(u) = Var\,(C \mid u)$.
Estimate $m(u)$ by the local linear technique, i.e., $\widehat{m}(u) = \widehat{a}$ if

$$(\widehat{a}, \widehat{b}) = \arg \min_{a,b} \sum_{i=1}^{n} \left\{ C_i - a - b^T\,(u_i - u) \right\}^2 K_h\,(u_i - u)$$

and then estimate $\sigma^2(u)$ by $\widehat{\sigma}^2(u) = \widehat{\alpha_1}$ if

$$(\widehat{\alpha_1}, \widehat{\beta}_1) = \arg \min_{\alpha_1, \beta_1} \sum_{i=1}^{n} \left\{ \widehat{r}_i - \alpha_1 - \beta_1^T\,(u_i - u) \right\}^2 W_h\,(u_i - u)$$

where $\widehat{r}_i = (C_i - \widehat{m}(u_i))^2$

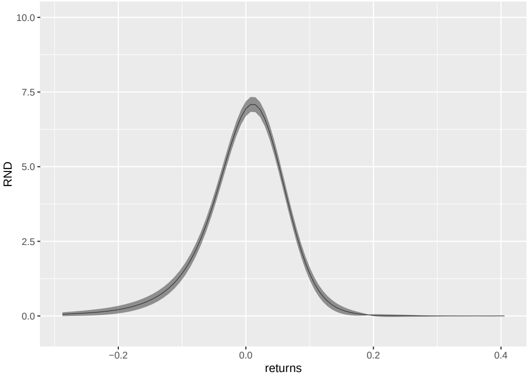

Figure 4: Risk neutral density of DAX 30 index return at maturity 1 month conditioned on 40% quantile of VDAX index.The grey areas are 95% confidence intervals. Sample period is Jan,2012 - Dec,2012

Estimation of physical density

Need to estimate conditional physical density of returns, i.e

$$p(r|\tau, z)$$

Collect time series of

$$\left(log(\frac{S_{t-i}}{S_{t-i-\tau}}), z_{t-i} \right), i = 1, 2, \dots, k$$

The problem boils down to estimation of the conditional density

$$p(r_\tau|z)$$

Fan et al.(1996) observed:

$$\mathbb{E}\left(K_h(R_\tau - r_\tau) \mid Z = z\right) \approx p\left(r_\tau \mid z\right)$$

Estimation of the conditional density can be regarded as a nonparametric regression problem

Local linear smoother

$$\min_{\alpha,\beta} \sum_{i=1}^{k} \left\{ K_h(r_i - r) - \alpha - \beta\,(z_i - z) \right\}^2 K_{h_2}(z_i - z)$$

The density estimator is given by

$$\widehat{p}\,(r \mid \tau, z) = \widehat{\alpha}$$

The asymptotic behaviour is given by

$$k^{1/2}\,(h_r h_z)^{1/2}\,(\widehat{p}\,(r \mid \tau, z) - p\,(r \mid \tau, z))$$
$$\xrightarrow{d} N\left(0, \left[\int K^2(c)dc\right]^3 \widehat{p}\,(r \mid \tau, z) \Big/ \pi\,(z)\right),$$

as $kh_r h_z \to \infty$

Local linear smoother

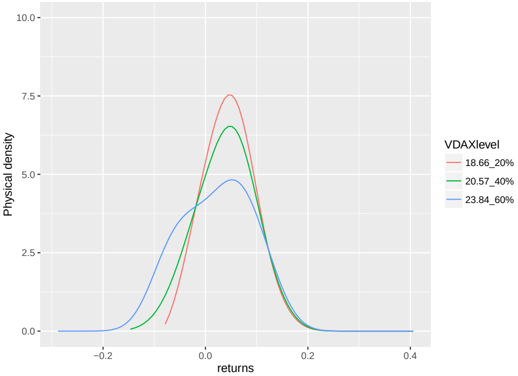

Figure 5: Physical density of DAX 30 index return at maturity 1 month conditioned on $20\%, 40\%, 60\%$ quantiles of VDAX index, sample period is Jan,2012 - Dec,2012

Local linear smoother

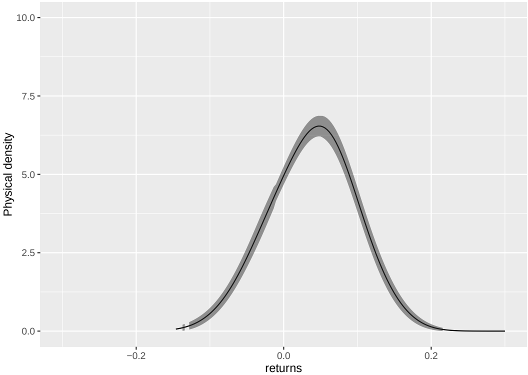

Figure 6: Physical density of DAX 30 index return at maturity 1 month conditioned on 40% quantile of VDAX index.The grey areas are 95% confidence intervals. Sample period is Jan,2012 - Dec,2012

Local constant smoother

$$\widehat{p}_{NW}(r \mid z) = \sum_{i=1}^{k} K_{h_z}(r - r_i)\omega_i^{NW}(z)$$

$$\omega_i^{NW}(z) = \frac{K_{h_z}(z - z_i)}{\sum_{i=1}^{k} K_{h_z}(z - z_i)}$$

Asymptotic variance:

$$Avar(\widehat{p}_{NW}(r \mid z)) = \left[\int K^2(c)dc\right]^2 (kh_z h_r)^{-1}(\widehat{p}(R \mid z)\pi(z)$$

as $kh_r h_z \to \infty$

Local constant smoother

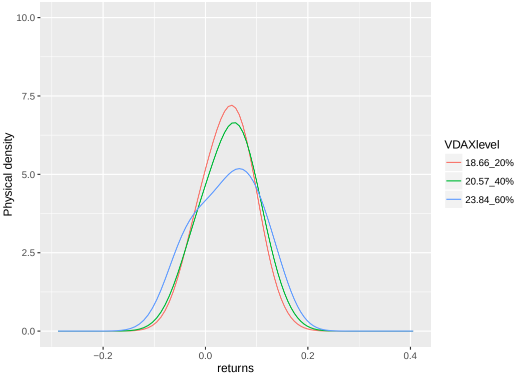

Figure 7: Physical density of DAX 30 index return at maturity 1 month conditioned on 20%, 40%, 60% quantiles of VDAX index, sample period is Jan,2012 - Dec,2012

Local constant smoother

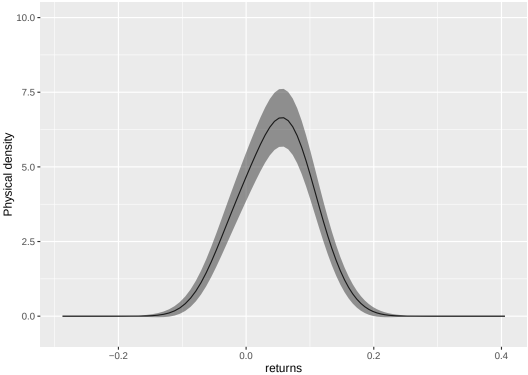

Figure 8: Physical density of DAX 30 index return at maturity 1 month conditioned on 40% quantile of VDAX index.The grey areas are 95% confidence intervals. Sample period is Jan,2012 - Dec,2012

Estimation of the pricing kernel

Pricing kernel can be estimated by

$$\widehat{\mathcal{K}} = \frac{\widehat{q}}{\widehat{p}}$$

Based on *Song, Xiu (2016)* asymptotic variance is given by

$$\begin{aligned}\widehat{Avar}\left(\widehat{\mathcal{K}}\left(r \mid \tau, z\right)\right) = & \frac{1}{\left(\widehat{p}\left(r \mid \tau, z\right)\right)^2}\widehat{Avar}\left(\widehat{q}\left(r \mid \tau, z\right)\right) \\ & + \frac{\left(\widehat{q}\left(r \mid \tau, z\right)\right)^2}{\left(\widehat{p}\left(r \mid \tau, z\right)\right)^4}\widehat{Avar}\left(\widehat{p}\left(r \mid \tau, z\right)\right)\end{aligned}$$

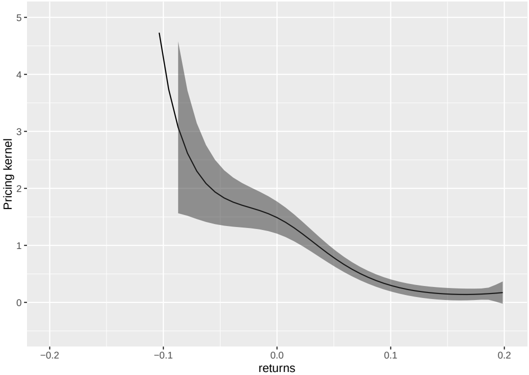

Figure 9: Pricing kernel of DAX 30 index return at maturity 1 month conditioned on 40% quantile of VDAX index.The grey areas are 95% confidence intervals. Sample period is Jan,2012 - Dec,2012

Data Description

Table 1: Summary statistics for options, DAX 30 and VDAX. Jan,2012 - Dec,2012

statistic	time to maturity(m)	moneyness	DAX	VDAX
Min.	0.23	0.06	5969	14.54
1st Qu.	1.64	0.75	6496	19.11
Median	3.35	0.92	6866	22.58
Mean	4.56	0.90	6854	22.42
3rd Qu.	7.43	1.08	7214	25.39
Max.	12	2.62	7654	33.25

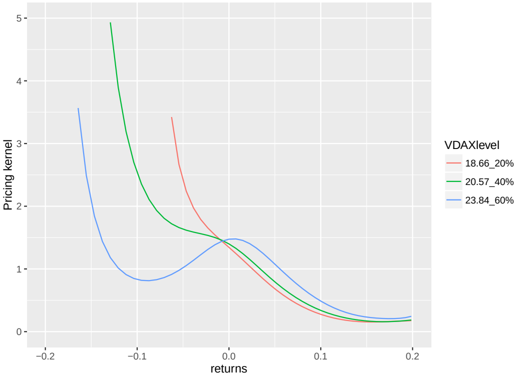

Figure 10: Pricing kernel of DAX 30 index return at maturity 1 month conditioned on $20\%, 40\%, 60\%$ quantiles of VDAX index, sample period is Jan,2012 - Dec,2012. RND and PD were calculated by local linear regression

Data Description

Table 2: Summary statistics for options, DAX 30 and VDAX. Jan,2009 - Dec,2009

	statistic	time to maturity(m)	moneyness	DAX	VDAX
1	Min.	0.26	0.08	3666	22.11
2	1st Qu.	1.87	0.72	4552	27.06
3	Median	4.20	0.96	4999	30.83
4	Mean	4.90	1.01	5022	32.75
5	3rd Qu.	7.82	1.23	5614	38.48
6	Max.	12	7.31	6012	50.02

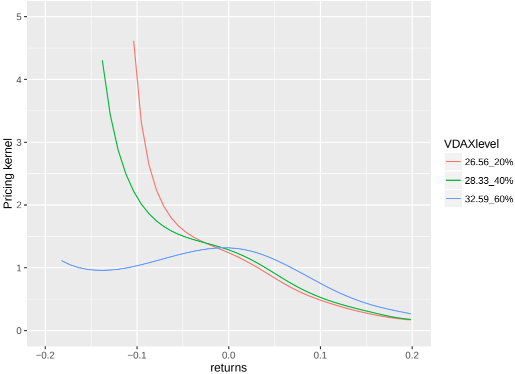

Figure 11: Pricing kernel of DAX 30 index return at maturity 1 month conditioned on $20\%, 40\%, 60\%$ quantiles of VDAX index, sample period is Jan,2009 - Dec,2009. RND and PD were calculated by local linear regression

Summary

- **Results**
  - Pricing kernels conditioned on volatility can be decreasing in the stock market index returns.
  - Hump-shape behaviour of PK near zero returns can be captured by conditioning on different volatility levels.
  - Dependence between curvature of PK near zero returns and volatility seems to be influenced by the global level of volatility(switching of preferences in periods of global high and low volatility)

References

M. Grith, W. K. Härdle, V. Krätschmer
Reference-Dependent Preferences and the Empirical Pricing Kernel Puzzle
Review of Finance, 2016

M. Grith, W. K. Härdle, J. Park
Shape invariant modelling of pricing kernels and risk aversion
Review of Financial Econometrics, 2013

Z.Song, D. Xiu
A tale of two option markets: Pricing Kernels and volatility risk
Journal of Econometrics, 2016