Options on volatility index

Roman Lykhnenko

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Abstract

After introduction in 2006, by the Chicago Board Options Exchange, of options written on its implied volatility index VIX, volatility options became crucial in modern finance. Using volatilty derivatives one can significantly improve diversification of portfolio since volatility and equity returns tend to be strongly negatively correlated. Thus, problem of valuation of volatility options is of a particular interest. In this paper Gruenbichler and Longstaff (1996) model is presented. The model is calibrated to the options on VSTOXX (volatility index offered by Eurex Exchange). Also some basic concepts of option management are discussed with regard to VSTOXX.

Contents

1 Introduction

In this paper Gruenbichler and Longstaff (1996) model (or GL (1996) for convenience) for valuation of volatility derivatives is presented. This model assumes diffusion processes for the dynamics of the underlying's behavior, and, based on the general approach of risk-neutral evaluation, simple closed-form expressions for valuation of options and futures on volatility index are derived. In this paper analysis of VSTOXX options has been done, and GL (1996) model has been calibrated to call options on VSTOXX traded on 2014-03-31.

The paper is constructed as follows. Section 1 presents basic concepts of option management. Section 2 describes Gruenbichler and Longstaff (1996) model. Section 3 starts with application of basic concepts discussed in Section 1 for analysis of options on VSTOXX, at the end of this section calibration of GL (1996) to the real data is presented. In Section 5 main conclusions are briefly represented.

2 Basic concepts of option management

An option is a contract which gives the buyer(the owner or holder) the right, but not the obligation, to buy or sell an underlying asset at specified price on or before a specified date, depending on the form of the option. The buyer of the call option has the right to buy an agreed quantity of a particular commodity(the underlying) from the seller of the option at a certain time(the expiration date) for a certain price (the strike price)

Hence **the pay-off**, i.e. the value of the call option at expiry, is

$$\max\{S_T - K, 0\} \tag{1}$$

where S_T is a price of the underlying at maturity, and K is a strike. In this paper only call options have been considered.

Under some assumptions about dynamics of the underlying asset, current value of a call option $C(S,t)$ can be calculated using Black-Scholes formula

$$\begin{aligned} C(S,t) &= N(d_1)S - N(d_2)Ke^{-r(T-t)}, \\ d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right] \\ d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned} \tag{2}$$

where

- N is the cumulative distribution function of the standard normal distribution
- $T - t$ is the time to maturity
- S is the spot price of the underlying asset at time t
- K is the strike price
- r is the risk free rate
- σ is the volatility of returns of the underlying asset

Also for such an option on defines moneyness as $\frac{K}{S_t}$.

A notion of implied volatility plays significant role in the option management. Implied volatility of an option contract is that value of the volatility of the underlying instrument which, when input in an option pricing model will return a theoretical value equal to the current market price of the option. In general, the value of an option depends on an estimate of the future realized price volatility, σ, of the underlying. Or, mathematically:

$$C = f(\sigma, \cdot) \tag{3}$$

where C is the theoretical value of an option, and f is a pricing model that depends on σ, along with other inputs. Function f is monotonically increasing in σ, so a unique solution exists. In general, a pricing model function f does not have a closed-form solution for its inverse. Instead, a root finding technique is used to solve the equation:

$$f(\sigma_{\bar{C}}, \cdot) - \bar{C} = 0 \tag{4}$$

Newton-Raphson method was used for such a purpose in this paper.

It is worth mentioning that if Black-Scholes formula were a correct approach for option pricing then implied volatilities must be the same for all strikes, but it is hardly the case in practice, and this phenomenon is known as volatility smile.

3 Valuation of volatility derivatives based on Gruenbichler and Longstaff (1996) model

3.1 The valuation framework

This section presents general valuation framework for pricing of volatility derivatives using Gruenbichler and Longstaff (1996) model.

Let denote the current value of the volatility index by V_t (or by V if time is obvious), and assume that the dynamics of V_t is given by

$$dV_t = \kappa_V(\theta_V - V_t)dt + \sigma_V\sqrt{V_t}dZ_t \tag{5}$$

where

- V_t is the time t value of the volatility index, for example the VSTOXX
- θ_V is a mean of the volatility index, assumed positive
- κ_V is the rate at which V_t reverts, assumed positive
- σ_V is the volatility of the volatility, assumed positive
- Z_t is a standard Brownian motion

It should be mentioned that such a specification of volatility dynamics is consistent with properties observed from real volatility indexes, i.e. mean-reversion, positivity of V_t, and conditional heteroscedasticity increasing with the level of V_t

Now consider the valuation of a contingent claim on volatility index. Especially, examine contingent claim with a pay off $B(V_T)$, which at time T depends only on V_T. Denote current value of such a claim by $A(V,T)$, and T-maturity riskless rate as $D(T)$. Then value of the given contigent claim can be calculated as

$$A(V,T) = D(T)E[B(V_T)], \tag{6}$$

where the expectation is taken with respect to the risk-neutral process for V_t

Thus, valuation of volatility derivative can be done by directly evaluating the expactation in (6)

3.2 Valuation of options on volatility index

Let $C(V, K, T)$ denote the current price of a call option on V, where K is the strike price, and T is time to maturity. Based on (6), one can express call price as

$$C(V, K, T) = D(T)E[\max\{V_T - K, 0\}] \tag{7}$$

Having evaluated this expectation (with respect to the risk-neutral measure), one obtains closed-form expression for the value of the volatility call option

$$\begin{aligned} C(V_0, K, T) = D(T) \cdot e^{-\beta T} \cdot V_0 \cdot Q(\gamma \cdot K|\nu + 4, \lambda) \\ +D(T) \cdot \left(\frac{\alpha}{\beta}\right) \cdot \left(1 - e^{-\beta T}\right) \cdot Q(\gamma \cdot K|\nu + 2, \lambda) \\ -D(T) \cdot K \cdot Q(\gamma \cdot K|\nu, \lambda)) \end{aligned} \tag{8}$$

where

- $\alpha = \kappa\theta$
- $\beta = \kappa + \zeta$
- $\gamma = \frac{4\beta}{\sigma^2(1-e^{-\beta T})}$
- $\nu = \frac{4\alpha}{\sigma^2}$
- $\lambda = \gamma \cdot e^{-\beta T} \cdot V$
- $D(T)$ is the appropriate discount factor
- Q is the complementary non-central χ^2 distribution

It is worth mentioning that α, β, and σ are parameters of the risk-neutral (risk-adjusted) process of the underlying security(volatility index in this case)

3.3 Valuation of volatility futures

Let $F(V, T)$ denote the futures price for a futures contract on volatility index V with maturity T. General formula for the futures price can be expressed as

$$F(V, T) = E(V_T) \tag{9}$$

where the expectation is taken with respect to the risk-neutral process for V.

Evalution of the expectation above gives following formula for the futures price

$$F(V_0, T) = \left(1 - e^{-\kappa_V T}\right) \cdot \theta_V + e^{-\kappa_V T} \cdot V_0 \tag{10}$$

4 Empirical Study

4.1 Data Description for Empirical Analysis

The goal of empirical study presneted in this section is calibration of Rosenbichler and Longstaff (1996) model to real data in order to assess how accurate this model can fit real data. For the purpose of calibration, it was decided to use VSTOXX index as a proxy for volatility index. The VSTOXX is traded by Eurex Exchange, and reflects the market expectations about volatility by measuring the square root of the implied variance across all options of a given time to expiration. VSTOXX is calculated every five seconds using real time EURO STOXX 50 index options (The EURO STOXX 50 is a stock index of Eurozone also traded by Eurex Exchange). Simultaneous dynamics of VSTOXX and EURO STOXX 50 is represented in Figure 1.

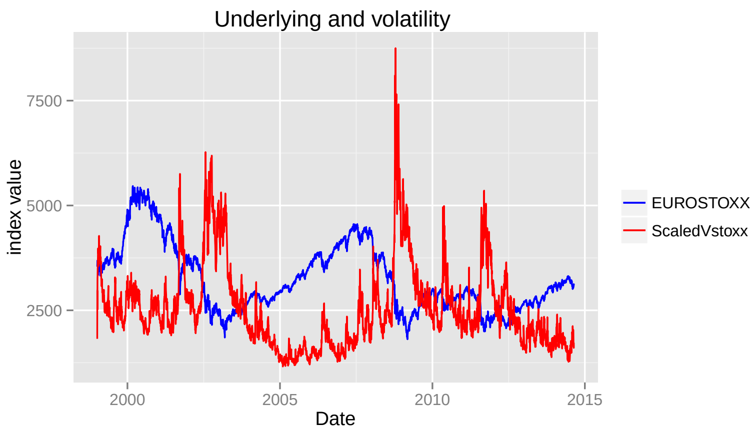

Figure 1: VSTOXX and ESTOXX 50

It should be mentioned that volatility and equity returns are strongly negatively correlated (correlation equals -0.72) for the time interval from 2000-01-01 till 2014-10-01, which allows for significant diversification benefits by adding a volatility security to portfolio.

In this paper only call options on VSTOXX traded on 2014-03-31 have been considered, and calibration of the model was done with respect to this option data. Considered options have eight different maturity dates. Relation between maturity, strike and market price of the considered options is represented in Figure 2.

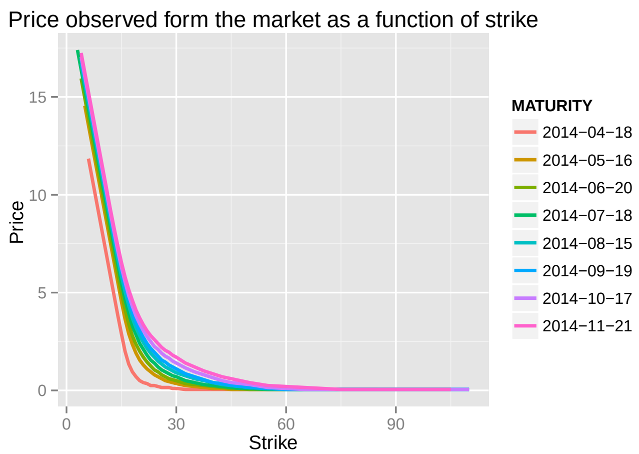

Figure 2: VSTOXX call options traded on 2014-03-31

From Figure 1 one can see that options with later maturity (in other words, with bigger time to maturity) have larger price for a given strike than options maturing earlier. That is because there is more time to have an event (change of the price of underlying asset) that can occur to make options appropriate for execution.

In order to assess quality of the option market, one needs to consider aggregated characteristics. For such a reason, firstly, average values of options have been calculated for all available strikes (Figure 3). This plot shows are well-known property of call options: if strike increases than price of an option should decrease.

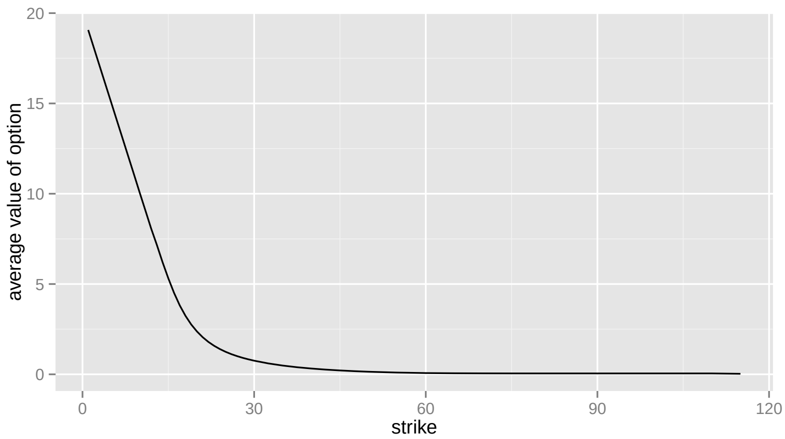

Figure 3: VSTOXX call options traded on 2014-03-31

Secondly, average values of options have been calculated for all available times to maturity (Figure 4). We can observe a small tendency to price long-term options more than short-term, but one should pay attention to the fact that there are different sets of strikes available for each maturity on the market.

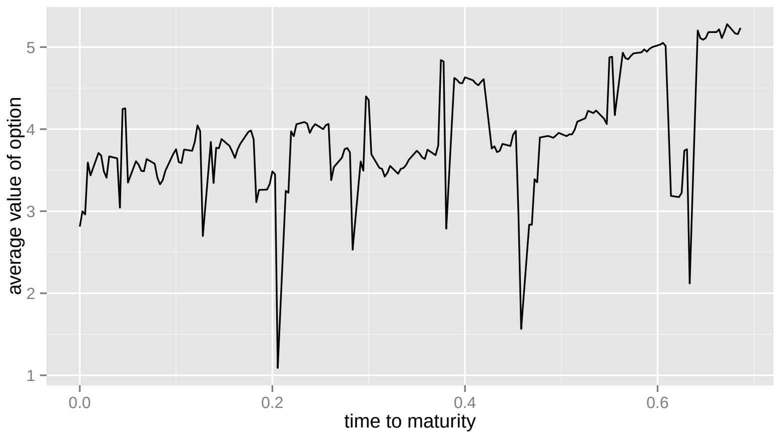

Figure 4: VSTOXX call options traded on 2014-03-31

Analysis of implied volatilities plays important role in risk manegement. Structure of implied volatilities of options on VSTOXX is shown in Figure 5. Black-Scholes framework was used in order to calculate implied volatilities. One can see that there is a tendency to have smaller volatility for options with larger time to maturity.

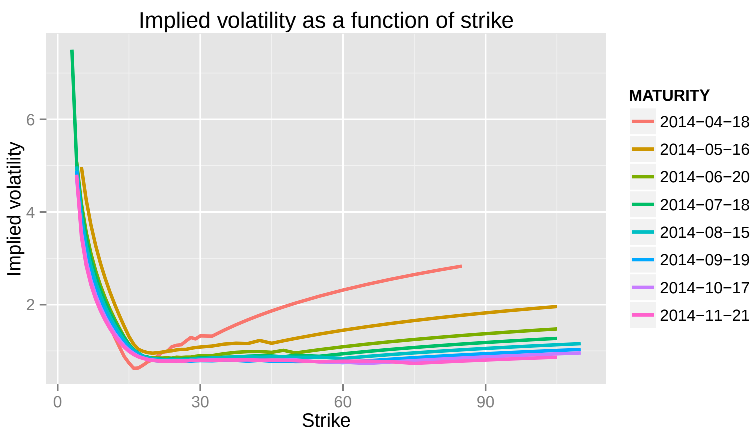

Figure 5: Implied volatility of VSTOXX call options traded on 2014-03-31

4.2 Calibration of Gruenbichler and Longstaff (1996) model

Calibration procedure used in this paper is essentialy a searching for such parameters of Gruenbichler and Longstaff (1996) model, that when input in the model, will return predicted option values as close as possible to the option prices observed from the market. In order to find such a set of optimal parameters, one must define some error function that is minimized under these parameters. There are different possibilities of selection of such a function. In this paper Mean Squared Error (MSE $= \frac{1}{n}\sum_{i=1}^{n}(\hat{C}_i - C_i)^2$) was used as a measure of descrepancy between option prices predicted based on the model, and observed from the market. Thus, optimization problem can be expressed in such a way

$$\min_{\kappa_V,\theta_V,\sigma_V} \frac{1}{N}\sum_{n=1}^{N}\left(C_n^* - C_n^{GL96}(\kappa_V,\theta_V,\sigma_V)\right)^2 \tag{11}$$

where C_n^* are option prices observed from the market, and C_n^{GL96} option prices predicted by Gruenbichler and Longstaff (1996) model.

Only options with moneyness between 0.75 and 1.25 (so called,close to the money options) have been selected to perform calibration of the model, although, moneyness threshold can be varied in the corresponding R code. There are only two maturities among options with such a moneyness. Results of calibration procedure for maturity 2014-05-16 are presented in Figure 6.

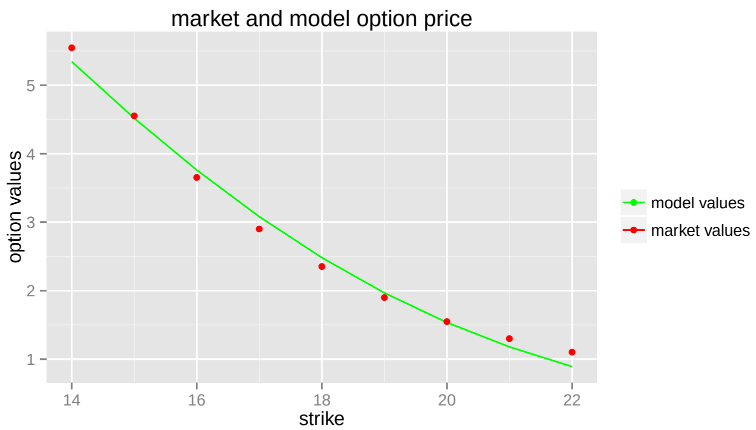

Figure 6: call option prices from model compared to observed from the market on 2014-03-31 (maturity 2014-05-16, MSE 0.056, moneyness between 0.75 and 1.25)

Results of calibration procedure for maturity 2014-07-18 are presented in Figure 7.

Figure 7: call option prices from model compared to observed from the market on 2014-03-31 (maturity 2014-07-18, MSE 0.037, moneyness between 0.75 and 1.25)

In these two cases one obtains that Mean Squared Error is 0.056 and 0.037, which comparing to the average price of the options 3.21, gives us the right to conclude that Gruenbichler and Longstaff (1996) model can provide us with rather accurate predictions of price of options on volatility index. On the other hand, one should always assess performance of the one model relative to the other model. That is why comparative analysis of models for pricing volatility options is a rather important field for further investigations.

5 Conclusion

This paper investigates mainly two things: options on VSTOXX and Gruenbichler and Longstaff (1996) model for valuation of volatility options. The negative correlation between volatility and equity has been observed, and implied volatility structure of options on VSTOXX has been investigated.

Gruenbichler and Longstaff (1996) model has been calibrated to the options on VSTOXX traded on 2014-03-31, and obtained in-sample-errors gives us the right to conclude that this model can accurately fit market data. On the other hand, investigation with regard to out-of-sample errors is still needed, as well, as comparision with other models for valuation of volatility derivatives.

References

[1] Borak, S., Härdle, W. K. and López-Cabrera, B. *Statistics of Financial Markets: Exercises and Solutions* Springer-Verlag, Heidelberg, 2010

[2] Franke, J., Härdle, W. K. and Hafner, C. M. *Statistics of Financial Markets: An Introduction* 3rd Edition, Springer-Verlag, Heidelberg, 2011

[3] Gruenbichler, A., and F. Longstaff. *Valuing Futures and Options on Volatility.* The Journal of Banking and Finance, 985-1001, 1996.

[4] Merton, R. *Theory of Rational Option Pricing.* The Journal of Economics and Management Science, 141-183, 1996.