Homework 2
CS-4033
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Assume: • $X \in \mathbb{R}^p$, $Y \in \mathbb{R}$, $\beta \in \mathbb{R}$, $w_i \in \mathbb{R}$ and $\sigma_i \in \mathbb{R}$ $\forall i$

• Observations are independent and identically distributed

• Interested on finding $f(Y=y \mid X=x)$

• Assume $y_i = w_i \cdot x_i^T \beta + \varepsilon_i$, where $\varepsilon_i \sim N(0, \sigma_i^2)$

Then we have that $\varepsilon_i = y_i - x_i^T \beta \sim N(0, \sigma_i^2)$

$\therefore Y \mid X = x_i \sim N(x_i^T \beta, \sigma_i^2)$

Observations: $(y_1, x_1), \ldots, (y_n, x_n)$

From assumptions:

$y_i = x_i \beta + \varepsilon_i$, then the likelihood funcion for one observation i is:

$$f(x_i \mid \beta, \sigma_i^2) = \frac{1}{\sqrt{2\pi\sigma_i^2}} \cdot e^{\frac{-(y_i - \beta x_i)^2}{2\sigma_i^2}}$$, from PMF of Gaussian Distribution $N(0, \sigma_i^2)$

For all observations:

$$\mathcal{L}(\beta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_i^2}} \cdot e^{\frac{-(y_i - \beta x_i)^2}{2\sigma_i^2}}$$

$$= \underbrace{\frac{1}{\sqrt{2\pi\sigma_i^2}}}_{\text{constant}} \prod_{i=1}^{n} e^{\frac{-(y_i - \beta x_i)^2}{2\sigma_i^2}}$$

Now if we take the log of $\mathcal{L}(\beta)$:

$$\log(\mathcal{L}(\beta)) = \log\left[\frac{1}{\sqrt{2\pi\sigma_i^2}} \prod_{i=1}^{n} e^{\frac{-(y_i - \beta x_i)^2}{2\sigma_i^2}}\right]$$

$$= n \log\left[\frac{1}{\sqrt{2\pi\sigma^2}}\right] + \log\left[\prod_{i=1}^{n} e^{\frac{-(y_i - \beta x_i)^2}{2\sigma_i^2}}\right]$$

$$= n \log\left[\frac{1}{\sqrt{2\pi\sigma^2}}\right] - \underbrace{\sum_{i=1}^{n} \frac{1}{2\sigma_i^2}(y_i - \beta x_i)^2}_{\log \prod_{i=1}^{n} f(x_i) = \sum_{i=1}^{n} \log f(x_i)}$$

Thus,

$$\boxed{\mathcal{L}(\beta) = \sum_{i=1}^{n} \underbrace{\frac{1}{2\sigma_i^2}}_{w_i} \cdot (y_i - \beta x_i)^2}$$

$\therefore w_i = \frac{1}{2\sigma_i^2}$ in our derived $\mathcal{L}(\beta)$, which is the same as weighted least square

$J(\beta) = \mathcal{L}(\beta) = \sum_{i=1}^{n} w_i \cdot (x_i^T \beta - y_i)^2$, where $w_i = \frac{1}{2\sigma_i^2}$