Honors Linear Algebra **Fall 2024**
HW2: Vector Spaces **Due: 9/15/2024**

Problem 1. Let $\mathbb{F}$ be a finite field and let n be a positive integer ($n \geq 2$). Ket V be the vector space of all $n \times n$ matrices over $\mathbb{F}$. Which of the following sets of matrices A in V are subspaces of V.

(a) all invertible A;

A subspace of a vector space must be closed under both addition and scalar multiplication. The set of all invertible matrices is not closed under addition. For instance, the identity matrix I and $-I$ are both invertible, but their sum, $I + (-I) = 0$, is not invertible (it's the zero matrix, which is not invertible).

Thus, the set of all invertible matrices is not a subspace.

(b) all non-invertible A;

Similarly, the set of non-invertible matrices is also not closed under addition. For example, take two non-invertible matrices, their sum might be invertible, violating the closure under addition. Hence, this set does not form a subspace.

Thus, the set of all non-invertible matrices is not a subspace.

(c) all A such that $AB = BA$, where B is some fixed matrix in V;

This set is closed under both addition and scalar multiplication. If A_1 and A_2 commute with B, then $(A_1 + A_2)B = A_1B + A_2B = BA_1 + BA_2 = B(A_1 + A_2)$, so $A_1 + A_2$ commutes with B. If A commutes with B, then for any scalar $\alpha \in \mathbb{F}$, $(\alpha A)B = \alpha(AB) = \alpha(BA) = B(\alpha A)$, so αA also commutes with B.

Thus, the set of all matrices that commute with a fixed matrix B is a subspace.

(d) all A such that $A^2 = A$

These matrices are called idempotent matrices. To check whether this set forms a subspace, we need to verify closure under addition and scalar multiplication. If $A_1^2 = A_1$ and $A_2^2 = A_2$, then in general, $(A_1 + A_2)^2 \neq A_1 + A_2$, so the set is not closed under addition. Similarly, for a scalar α, $(\alpha A)^2 = \alpha^2 A^2 = \alpha^2 A$, which is not necessarily equal to αA unless $\alpha = 0$ or 1.

Thus, the set of idempotent matrices is not a subspace.

Problem 2. Let V be the vector space of all functions from R into R; let V_e be the subset of even functions, $f(-x) = f(x)$; let V_o be the subset of odd functions, $f(-x) = -f(x)$.

(a) Prove that V_e and V_o are subspaces of V.

To prove that V_e and V_o are subspaces of V, we must check that each set satisfies the conditions for a subspace: closure under addition, closure under scalar multiplication, and that each contains the zero function.

First, consider V_e, the set of even functions. Let $f, g \in V_e$, i.e., $f(-x) = f(x)$ and $g(-x) = g(x)$ for all $x \in \mathbb{R}$. For closure under addition, we compute:

$$(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x),$$

which shows that $f + g \in V_e$. For scalar multiplication, let $c \in \mathbb{R}$. Then:

$$(cf)(-x) = cf(-x) = cf(x) = (cf)(x),$$

so $cf \in V_e$. Finally, the zero function $f(x) = 0$ is clearly even, as $0(-x) = 0(x)$ for all x. Hence, V_e is a subspace of V.

Now, consider V_o, the set of odd functions. Let $f, g \in V_o$, i.e., $f(-x) = -f(x)$ and $g(-x) = -g(x)$ for all $x \in \mathbb{R}$. For closure under addition, we compute:

$$(f+g)(-x) = f(-x) + g(-x) = -f(x) + -g(x) = -(f(x) + g(x)) = -(f+g)(x),$$

which shows that $f + g \in V_o$. For scalar multiplication, let $c \in \mathbb{R}$. Then:

$$(cf)(-x) = cf(-x) = c(-f(x)) = -(cf)(x),$$

so $cf \in V_o$. The zero function $f(x) = 0$ is also odd, as $0(-x) = -0(x)$ for all x. Hence, V_o is a subspace of V.

(b) Prove that $V_e + V_o = V$.

We need to show that any function $f \in V$ can be written as the sum of an even function and an odd function. Given $f \in V$, define two functions:

$$f_e(x) = \frac{f(x) + f(-x)}{2}, \quad f_o(x) = \frac{f(x) - f(-x)}{2}.$$

First, check that f_e is even:

$$f_e(-x) = \frac{f(-x) + f(x)}{2} = \frac{f(x) + f(-x)}{2} = f_e(x).$$

Thus, $f_e \in V_e$. Next, check that f_o is odd:

$$f_o(-x) = \frac{f(-x) - f(x)}{2} = -\frac{f(x) - f(-x)}{2} = -f_o(x).$$

Thus, $f_o \in V_o$.

Finally, observe that:

$$f(x) = f_e(x) + f_o(x),$$

which shows that any function $f \in V$ can be written as the sum of an even function and an odd function. Therefore, $V_e + V_o = V$.

(c) Prove that $V_e \cap V_o = \{0\}$.

Suppose $f \in V_e \cap V_o$. This means that f is both even and odd. Thus, for all $x \in \mathbb{R}$,

$$f(-x) = f(x) \quad \text{(since } f \text{ is even)},$$

and

$$f(-x) = -f(x) \quad \text{(since } f \text{ is odd)}.$$

Combining these, we get $f(x) = -f(x)$, which implies that $f(x) = 0$ for all $x \in \mathbb{R}$. Hence, $f = 0$, the zero function.

Therefore, $V_e \cap V_o = \{0\}$.

Problem 3. Let V be the vector space of all $n \times n$ matrices over the field $\mathbb{F}$, and let B be a fixed $n \times n$ matrix. if

$$T(A) = AB - BA$$

verify that T is a linear transformation from V into V.

Additivity:
Let $A_1, A_2 \in V$. We compute $T(A_1 + A_2)$ as follows:

$$T(A_1 + A_2) = (A_1 + A_2)B - B(A_1 + A_2).$$

$$T(A_1 + A_2) = A_1B + A_2B - (BA_1 + BA_2).$$

$$T(A_1 + A_2) = (A_1B - BA_1) + (A_2B - BA_2).$$

$$T(A_1 + A_2) = T(A_1) + T(A_2).$$

Thus, T is additive.

Homogeneity:
Let $c \in \mathbb{F}$ and $A \in V$. We compute $T(cA)$ as follows:

$$T(cA) = (cA)B - B(cA).$$

$$T(cA) = c(AB) - c(BA).$$

$$T(cA) = c(AB - BA) = cT(A).$$

Thus, T is homogeneous.

Since T satisfies both additivity and homogeneity, T is a linear transformation from V into V.

Problem 4. Let V be a vector space and T a linear transformation from V in V. Prove that the following two statements about T are equivalent.

(a) The intersection of the range of T and the null space of T is the zero subspace of V.

We want to show that the intersection of the range of T (aka the $\text{Im}(T)$) and the null space of T (aka the $\ker(T)$) is the zero subspace:

$$\text{Im}(T) \cap \ker(T) = \{0\}.$$

Let $v \in \mathrm{Im}(T) \cap \ker(T)$. By definition of intersection, $v \in \mathrm{Im}(T)$ and $v \in \ker(T)$.

Since $v \in \ker(T)$, we have $T(v) = 0$.
Since $v \in \mathrm{Im}(T)$, there exists some $u \in V$ such that $T(u) = v$.

Applying T to both sides of the equation $T(u) = v$, we get:

$$T(T(u)) = T(v) = 0.$$

Therefore, $T(T(u)) = 0$, which means $u \in \ker(T^2)$.

Now, since $v = T(u) \in \ker(T)$, and T is linear, this implies that u must also be in $\ker(T)$ (this will follow from the second subproblem). Hence, $v = T(u) = 0$.

Therefore, the only element in $\mathrm{Im}(T) \cap \ker(T)$ is the zero vector, so:

$$\mathrm{Im}(T) \cap \ker(T) = \{0\}.$$

(b) If $T(T\alpha) = 0$, then $T\alpha = 0$.

We need to show that if $T(T(\alpha)) = 0$, then $T(\alpha) = 0$.

Assume $T(T(\alpha)) = 0$ for some $\alpha \in V$. This means $T(\alpha) \in \ker(T)$, i.e., the vector $T(\alpha)$ is in the null space of T.

Now, consider the fact that the intersection of $\mathrm{Im}(T)$ and $\ker(T)$ is the zero subspace (from the first subproblem). Since $T(\alpha) \in \mathrm{Im}(T)$ and $T(\alpha) \in \ker(T)$, it follows that:

$$T(\alpha) = 0.$$

Therefore, if $T(T(\alpha)) = 0$, then $T(\alpha) = 0$.

This proves the implication.

Problem 5. Find two linear operations T and U on $\mathbb{R}^{\nvDash}$ such that $TU = 0$ but $UT \neq 0$.

Consider the following two linear maps T and U on $\mathbb{R}^2$ represented by matrices:

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

First, let's compute the product TU:

$$TU = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

This gives $TU = 0$, since every entry in the resulting matrix is zero.

Now, let's compute the product UT:

$$UT = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

This gives $UT = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, which is not the zero matrix.

Therefore, we have found that $TU = 0$ but $UT \neq 0$.

Problem 6. Let V be a vector space over the field $\mathbb{F}$ and T a linear operator on V. If $T^2 = 0$, what can you say about the relation of the range of T to the null space of T? Give an example of a linear operator T on $\mathbb{R}^2$ such that $T^2 = 0$ but $T \neq 0$.

The range of T is contained in the null space of T. This is because for any vector $v \in V$, if $T(v) \in \text{Range}(T)$, then applying T to $T(v)$ gives $T(T(v)) = 0$, so $T(v) \in \text{Null}(T)$.

To see this more concretely, let $T^2 = 0$ and let $W = \text{Range}(T)$. For any $w \in W$, there exists some $v \in V$ such that $w = T(v)$. Then applying T again, $T(w) = T(T(v)) = 0$, so $w \in \text{Null}(T)$. Thus, $W \subseteq \text{Null}(T)$.

Example of a linear operator T on $\mathbb{R}^2$ such that $T^2 = 0$ but $T \neq 0$.

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let's check that $T^2 = 0$:

$$T^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

So $T^2 = 0$. Also, $T \neq 0$ because T is not the zero matrix.

The range of T is:

$$\text{Range}(T) = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

which is a one-dimensional subspace of $\mathbb{R}^2$.

The null space of T is:

$$\text{Null}(T) = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \mid x \in \mathbb{R} \right\},$$

which is a one-dimensional subspace of $\mathbb{R}^2$.

$\text{Range}(T) \subseteq \text{Null}(T)$, satisfying the condition that the range of T is contained in the null space of T.

Problem 7. Let V be a vector space over the field $\mathbb{F}$. Assume W is a subspace of V and $S, S_i, i \in I$ are arbitrary subsets. Verify the following:

(a) $\text{Span}_{\mathbb{F}}(W) = W$

By definition, the span of a set W is the set of all linear combinations of elements of W. Since W is a subspace of V, it is closed under linear combinations and contains all such combinations of its elements. Thus, $\text{Span}_{\mathbb{F}}(W)$ includes every element of W, and every element of $\text{Span}_{\mathbb{F}}(W)$ is in W. Therefore:

$$\text{Span}_{\mathbb{F}}(W) = W.$$

(b) $\text{Span}_{\mathbb{F}}(\text{Span}_{\mathbb{F}}(S)) = \text{Span}_{\mathbb{F}}(S)$

Let $T = \text{Span}_{\mathbb{F}}(S)$. By definition, $\text{Span}_{\mathbb{F}}(T)$ is the set of all linear combinations of elements in T. Since T is already the span of S, it means T consists of all linear combinations of elements in S. Therefore:

$$\text{Span}_{\mathbb{F}}(\text{Span}_{\mathbb{F}}(S)) = \text{Span}_{\mathbb{F}}(S).$$

(c) $\text{Span}_{\mathbb{F}}(\bigcup\limits_{i\in I} S_i) = \sum\limits_{i\in I} \text{Span}_{\mathbb{F}}(S_i)$.

Let $\mathcal{S} = \bigcup_{i\in I} S_i$. The span of $\mathcal{S}$ is the set of all linear combinations of elements in $\mathcal{S}$. Each element in $\mathcal{S}$ belongs to some S_i, so every linear combination of elements in $\mathcal{S}$ can be expressed as a linear combination of elements in S_i for some $i \in I$. Thus:

$$\text{Span}_{\mathbb{F}}\left(\bigcup_{i\in I} S_i\right) = \sum_{i\in I} \text{Span}_{\mathbb{F}}(S_i),$$

where $\sum_{i\in I} \text{Span}_{\mathbb{F}}(S_i)$ denotes the set of all finite sums of elements where each element is from some $\text{Span}_{\mathbb{F}}(S_i)$.

(d) $\text{Span}_{\mathbb{F}}(\bigcap\limits_{i\in I} S_i) \subseteq \bigcap\limits_{i\in I} \text{Span}_{\mathbb{F}}(S_i)$. Equality may not hold; give an explicit example of this.

Let us show the inclusion first. If $x \in \text{Span}_{\mathbb{F}}\left(\bigcap_{i\in I} S_i\right)$, then x is a linear combination of elements in $\bigcap_{i\in I} S_i$. Each such element belongs to every S_i, so x is in every $\text{Span}_{\mathbb{F}}(S_i)$. Therefore:

$$\text{Span}_{\mathbb{F}}\left(\bigcap_{i\in I} S_i\right) \subseteq \bigcap_{i\in I} \text{Span}_{\mathbb{F}}(S_i).$$

Example where equality does not hold:

Consider $V = \mathbb{R}^2$, and let:

$$S_1 = \{(x, 0) \mid x \in \mathbb{R}\}, \quad S_2 = \{(0, y) \mid y \in \mathbb{R}\}.$$

Then:

$$\bigcap_{i\in\{1,2\}} S_i = \{(0, 0)\},$$

and:

$$\operatorname{Span}_{\mathbb{F}}\left(\bigcap_{i\in\{1,2\}} S_i\right) = \operatorname{Span}_{\mathbb{F}}\{(0,0)\} = \{(0,0)\}.$$

However:

$$\operatorname{Span}_{\mathbb{F}}(S_1) = \operatorname{Span}_{\mathbb{F}}\{(1,0)\} = \mathbb{R}^2,$$

and:

$$\operatorname{Span}_{\mathbb{F}}(S_2) = \operatorname{Span}_{\mathbb{F}}\{(0,1)\} = \mathbb{R}^2.$$

Therefore:

$$\bigcap_{i\in\{1,2\}} \operatorname{Span}_{\mathbb{F}}(S_i) = \mathbb{R}^2,$$

which is strictly larger than:

$$\operatorname{Span}_{\mathbb{F}}\left(\bigcap_{i\in\{1,2\}} S_i\right) = \{(0,0)\}.$$

Thus, in this example:

$$\operatorname{Span}_{\mathbb{F}}\left(\bigcap_{i\in I} S_i\right) \neq \bigcap_{i\in I} \operatorname{Span}_{\mathbb{F}}(S_i).$$

Problem 8. (Direct Sums)

(a) Show that the operation of direct sums is "commutative": that is, there is a natural isophorism

$$V_1 \oplus V_2 \approx V_2 \oplus V_1$$

Consider the vector space $V_1 \oplus V_2$. An element of $V_1 \oplus V_2$ can be written as a pair (v_1, v_2), where $v_1 \in V_1$ and $v_2 \in V_2$.

Define a map $\phi : V_1 \oplus V_2 \to V_2 \oplus V_1$ by

$$\phi((v_1, v_2)) = (v_2, v_1).$$

We will show that ϕ is a linear isomorphism.

Linearity:
For $(v_1, v_2), (w_1, w_2) \in V_1 \oplus V_2$ and $c \in \mathbb{F}$:

$$\phi((v_1, v_2) + (w_1, w_2)) = \phi((v_1 + w_1, v_2 + w_2)) = (v_2 + w_2, v_1 + w_1).$$

$$\phi((v_1, v_2)) + \phi((w_1, w_2)) = (v_2, v_1) + (w_2, w_1) = (v_2 + w_2, v_1 + w_1).$$

Thus, ϕ preserves addition.

For scalar multiplication:

$$\phi(c(v_1, v_2)) = \phi((cv_1, cv_2)) = (cv_2, cv_1),$$

$$c\phi((v_1, v_2)) = c(v_2, v_1) = (cv_2, cv_1).$$

Thus, ϕ preserves scalar multiplication.

Bijectivity:
To show that ϕ is bijective, we need to find its inverse. Define $\psi : V_2 \oplus V_1 \to V_1 \oplus V_2$ by

$$\psi((v_2, v_1)) = (v_1, v_2).$$

It is straightforward to verify that ψ is the inverse of ϕ since:

$$\phi \circ \psi((v_2, v_1)) = \phi((v_1, v_2)) = (v_2, v_1),$$

$$\psi \circ \phi((v_1, v_2)) = \psi((v_2, v_1)) = (v_1, v_2).$$

Thus, ϕ is an isomorphism, proving that $V_1 \oplus V_2 \cong V_2 \oplus V_1$.

(b) Explain the difference between the vector spaces $(V_1 \oplus V_2) \oplus V_3$ and $V_1 \oplus (V_2 \oplus V_3)$.

Note that an element of $(V_1 \oplus V_2) \oplus V_3$ can be represented as $((v_1, v_2), v_3)$, where $v_1 \in V_1$, $v_2 \in V_2$, and $v_3 \in V_3$. Similarly, an element of $V_1 \oplus (V_2 \oplus V_3)$ can be represented as $(v_1, (v_2, v_3))$, where $v_1 \in V_1$, $v_2 \in V_2$, and $v_3 \in V_3$.

Define a map $\phi : (V_1 \oplus V_2) \oplus V_3 \to V_1 \oplus (V_2 \oplus V_3)$ by

$$\phi(((v_1, v_2), v_3)) = (v_1, (v_2, v_3)).$$

To show that ϕ is an isomorphism, note that:

$$\phi(((v_1, v_2), v_3)+((w_1, w_2), w_3)) = \phi(((v_1+w_1, v_2+w_2), v_3+w_3)) = (v_1+w_1, (v_2+w_2, v_3+w_3)),$$

$$\phi(((v_1, v_2), v_3))+\phi(((w_1, w_2), w_3)) = (v_1, (v_2, v_3))+(w_1, (w_2, w_3)) = (v_1+w_1, (v_2+w_2, v_3+w_3)).$$

For scalar multiplication:

$$\phi(c((v_1, v_2), v_3)) = \phi((cv_1, cv_2), cv_3) = (cv_1, (cv_2, cv_3)),$$

$$c\phi(((v_1, v_2), v_3)) = c(v_1, (v_2, v_3)) = (cv_1, (cv_2, cv_3)).$$

Thus, ϕ is a linear isomorphism, showing that $(V_1 \oplus V_2) \oplus V_3 \cong V_1 \oplus (V_2 \oplus V_3)$. The isomorphism preserves the structure of the vector spaces, so the direct sum operation is associative.

(c) Show that the operation of direct sum is "associative": that is, there is a natural isomorphism

$$(V_1 \oplus V_2) \oplus V_3 \approx V_1 \oplus (V_2 \oplus V_3)$$

Consider the vector space $(V_1 \oplus V_2) \oplus V_3$. An element of $(V_1 \oplus V_2) \oplus V_3$ is of the form $((v_1, v_2), v_3)$, where $v_1 \in V_1$, $v_2 \in V_2$, and $v_3 \in V_3$.

Define a map $\phi : (V_1 \oplus V_2) \oplus V_3 \to V_1 \oplus (V_2 \oplus V_3)$ by

$$\phi(((v_1, v_2), v_3)) = (v_1, (v_2, v_3)).$$

We need to show that ϕ is a linear isomorphism.

Linearity:
For $((v_1, v_2), v_3), ((w_1, w_2), w_3) \in (V_1 \oplus V_2) \oplus V_3$ and $c \in \mathbb{F}$:

$$\phi((((v_1, v_2), v_3)+((w_1, w_2), w_3))) = \phi(((v_1+w_1, v_2+w_2), v_3+w_3)) = (v_1+w_1, (v_2+w_2, v_3+w_3)).$$

$$\phi((v_1, (v_2, v_3)))+\phi((w_1, (w_2, w_3))) = (v_1, (v_2, v_3))+(w_1, (w_2, w_3)) = (v_1+w_1, (v_2+w_2, v_3+w_3)).$$

For scalar multiplication:

$$\phi(c((v_1, v_2), v_3)) = \phi((cv_1, cv_2), cv_3) = (cv_1, (cv_2, cv_3)),$$

$$c\phi(((v_1, v_2), v_3)) = c(v_1, (v_2, v_3)) = (cv_1, (cv_2, cv_3)).$$

Thus, ϕ preserves both vector addition and scalar multiplication, making it a linear map.

Bijectivity:
To find the inverse, define $\psi : V_1 \oplus (V_2 \oplus V_3) \to (V_1 \oplus V_2) \oplus V_3$ by

$$\psi((v_1, (v_2, v_3))) = ((v_1, v_2), v_3).$$

Verify that ψ is the inverse of ϕ:

$$\phi \circ \psi((v_1, (v_2, v_3))) = \phi(((v_1, v_2), v_3)) = (v_1, (v_2, v_3)),$$

$$\psi \circ \phi(((v_1, v_2), v_3)) = \psi((v_1, (v_2, v_3))) = ((v_1, v_2), v_3).$$

Hence, ϕ is an isomorphism, proving that $(V_1 \oplus V_2) \oplus V_3 \cong V_1 \oplus (V_2 \oplus V_3)$. The direct sum operation is associative.

(d) Give a definition of the direct sum of $k > 3$ vector spaces ove $\mathbb{F}$ using k-tuples. Give an inductive definition assuming the case $k = 2$ is given. Verify that the two definitions give isomorphic vector spaces.

The direct sum of k vector spaces $V_1, V_2, \ldots, V_k$ over a field $\mathbb{F}$ can be defined using k-tuples. Specifically, the direct sum is:

$$V_1 \oplus V_2 \oplus \cdots \oplus V_k = \{(v_1, v_2, \ldots, v_k) \mid v_i \in V_i \text{ for } i = 1, 2, \ldots, k\}.$$

This space is the set of all k-tuples where each component is an element from the corresponding vector space.

Inductive Definition:
Assume that the direct sum is defined for k vector spaces, i.e.,

$$V_1 \oplus V_2 \oplus \cdots \oplus V_k = \{(v_1, v_2, \ldots, v_k) \mid v_i \in V_i \text{ for } i = 1, 2, \ldots, k\}.$$

For $k+1$ vector spaces, we can define:

$$V_1 \oplus V_2 \oplus \cdots \oplus V_k \oplus V_{k+1} = \{((v_1, v_2, \ldots, v_k), v_{k+1}) \mid v_i \in V_i \text{ for } i = 1, 2, \ldots, k+1\}.$$

This can be seen as taking the direct sum of the previously defined direct sum with V_{k+1}.

Verification of Isomorphism:
To verify that these definitions are isomorphic, we need to show that:

$$(V_1 \oplus V_2 \oplus \cdots \oplus V_k) \oplus V_{k+1} \cong V_1 \oplus (V_2 \oplus \cdots \oplus (V_k \oplus V_{k+1}) \cdots)$$

Define a map $\phi : (V_1 \oplus V_2 \oplus \cdots \oplus V_k) \oplus V_{k+1} \to V_1 \oplus (V_2 \oplus \cdots \oplus (V_k \oplus V_{k+1}) \cdots)$ by

$$\phi(((v_1, v_2, \ldots, v_k), v_{k+1})) = (v_1, (v_2, \ldots, (v_k, v_{k+1}) \cdots)).$$

To verify that ϕ is an isomorphism, check that it preserves addition and scalar multiplication, and is bijective with a clear inverse.

Thus, by induction, the direct sum operation is associative and can be defined using k-tuples.

Problem 9. Show that if the index set I is infinite and all vector spaces V_i are non-trivial (i.e., the have elements different from 0), then the direct sum $\bigoplus_{i \in I} V_i$ is a proper subspace of the direct product $\prod_{i \in I} V_i$. (This uses the axiom of choice).

Definitions:
The direct sum $\bigoplus_{i \in I} V_i$ consists of all tuples $(v_i)_{i \in I}$ where $v_i \in V_i$ and $v_i = 0$ for all but finitely many $i \in I$. The direct product $\prod_{i \in I} V_i$ consists of all tuples $(v_i)_{i \in I}$ where $v_i \in V_i$ for all $i \in I$.

Claim: $\bigoplus_{i \in I} V_i \subsetneq \prod_{i \in I} V_i$

Proof.

1. $\bigoplus_{i \in I} V_i \subseteq \prod_{i \in I} V_i$:

By definition, every element of $\bigoplus_{i \in I} V_i$ is an element of $\prod_{i \in I} V_i$, because the direct sum is a subset of the direct product. Thus, we have:

$$\bigoplus_{i \in I} V_i \subseteq \prod_{i \in I} V_i.$$

2. $\bigoplus_{i \in I} V_i$ is a proper subspace of $\prod_{i \in I} V_i$:

To prove this, we need to show that there exists at least one element in $\prod_{i \in I} V_i$ that is not in $\bigoplus_{i \in I} V_i$.

Since I is infinite, there exists an infinite subset $J \subset I$. For each $i \in J$, choose a non-zero vector $v_i \in V_i$ (possible because each V_i is non-trivial).

Define a tuple $(w_i)_{i \in I} \in \prod_{i \in I} V_i$ by:

$$w_i = \begin{cases} v_i & \text{if } i \in J, \\ 0 & \text{if } i \notin J. \end{cases}$$

This tuple $(w_i)_{i\in I}$ is an element of $\prod_{i\in I} V_i$ because each $w_i \in V_i$.

However, $(w_i)_{i\in I} \notin \bigoplus_{i\in I} V_i$, because J is infinite and thus there are infinitely many non-zero entries in $(w_i)_{i\in I}$. By definition of $\bigoplus_{i\in I} V_i$, it can only contain tuples with finitely many non-zero entries.

Therefore, the tuple $(w_i)_{i\in I}$ is not in $\bigoplus_{i\in I} V_i$, showing that $\bigoplus_{i\in I} V_i$ is a proper subset of $\prod_{i\in I} V_i$.

Thus, the direct sum $\bigoplus_{i\in I} V_i$ is indeed a proper subspace of the direct product $\prod_{i\in I} V_i$.

Problem 10. Let W_1 and W_2 be subspaces of a vector space V such that their set-theoretic union is also a subspace. Prove that one of the spaces W_i is contained in the other.

Showing that $W_1 \cap W_2 \neq \emptyset$:

Since $W_1 \cup W_2$ is a subspace, it must contain the zero vector. Therefore, $0 \in W_1 \cup W_2$. This implies that 0 is in at least one of W_1 or W_2, but more importantly, $0 \in W_1 \cap W_2$. Hence, $W_1 \cap W_2$ is non-empty.

Proving that $W_1 \cup W_2$ being a subspace implies W_1 and W_2 are contained in each other.

Since $W_1 \cup W_2$ is a subspace, it is closed under addition and scalar multiplication. Consider any vectors $u \in W_1$ and $v \in W_2$. Since $W_1 \cup W_2$ is a subspace, their sum $u + v$ must also be in $W_1 \cup W_2$.

There are two cases to consider:

Case 1 $u + v \in W_1$:
If $u + v \in W_1$, since $u \in W_1$ and $u + v \in W_1$, it follows that v must be in W_1 (because W_1 is a subspace and closed under subtraction). Therefore, $W_2 \subseteq W_1$ because v was an arbitrary element of W_2.

Case 2 $u + v \in W_2$:
Similarly, if $u + v \in W_2$, then since $v \in W_2$ and $u + v \in W_2$, it follows that u must be in W_2 (because W_2 is a subspace and closed under subtraction). Thus, $W_1 \subseteq W_2$ because u was an arbitrary element of W_1.

In either case, we find that one of the subspaces is contained in the other.

Problem 11. (Subspaces)

(a) Let R be an commutative ring (we assume that R is also associative and has a unit). Show that the set $R*$ of invertible elements in R is an abelian group, i.e., there is an operation $\boxplus$ which is commutative, associative, there exist a zero element and "negatives"

To show that R^*, the set of invertible elements in a commutative ring R, forms an abelian group under multiplication, we need to verify the following group properties:

Closure under Multiplication:
Let $a, b \in R^*$. Since a and b are invertible, there exist a^{-1} and b^{-1} in R such that:

$$a \cdot a^{-1} = 1 \quad \text{and} \quad b \cdot b^{-1} = 1.$$

We need to show that $a \cdot b$ is also invertible. Consider:

$$(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}.$$

$$(a \cdot b) \cdot (b^{-1} \cdot a^{-1}) = a \cdot (b \cdot b^{-1}) \cdot a^{-1} = a \cdot 1 \cdot a^{-1} = a \cdot a^{-1} = 1.$$
$$(b^{-1} \cdot a^{-1}) \cdot (a \cdot b) = b^{-1} \cdot (a^{-1} \cdot a) \cdot b = b^{-1} \cdot 1 \cdot b = b^{-1} \cdot b = 1.$$

Therefore, $a \cdot b$ is invertible, and $(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$, confirming that R^* is closed under multiplication.

Associativity:
Multiplication in R is associative, and since R^* is a subset of R, the operation of multiplication is associative in R^*. Specifically, for any $a, b, c \in R^*$:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$

Existence of Identity Element:
The identity element in R under multiplication is 1, and $1 \in R^*$ because 1 is invertible with $1^{-1} = 1$. Therefore, 1 serves as the identity element for R^*.

Existence of Inverses:
By definition, each element $a \in R^*$ has an inverse $a^{-1} \in R$. Thus, for every $a \in R^*$, the element a^{-1} is also in R^*, satisfying the requirement for inverses in the group.

Commutativity:
Since R is commutative, for any $a, b \in R^*$:

$$a \cdot b = b \cdot a.$$

Hence, R^* inherits this commutativity from R, and so R^* is abelian.

Therefore, $(R^*, \cdot)$ is an abelian group where the group operation is multiplication. It is closed, associative, has an identity element, and every element has an inverse, and the operation is commutative.

(b) Let $\mathbb{F}_p$ be the field with p elements, p a prime. Let A be an abelian group. Find a "natural" condition in order that A will be a vector space over $\mathbb{F}_p$ (in a unique way). [Hint: 1 does what?]

The element 1 in the field $\mathbb{F}_p$ must act as the multiplicative identity on the abelian group A. This requirement ensures that A can be given a vector space structure over $\mathbb{F}_p$ where the scalar multiplication by 1 leaves each vector unchanged.

The "natural" condition for A to be a vector space over $\mathbb{F}_p$ is that A must be a finite abelian group of order p^n for some positive integer n. This condition ensures that A has a structure that allows it to be a vector space over $\mathbb{F}_p$ in a unique way.

The field $\mathbb{F}_p$ has exactly p elements. For A to be a vector space over $\mathbb{F}_p$, A must be isomorphic to $\mathbb{F}_p^n$ for some n. Therefore, A must have p^n elements.

The order of A must be p^n because this is the number of elements in $\mathbb{F}_p^n$. For A to have this order, it must be that A is a finite abelian group whose order is a power of p.

Given A is a finite abelian group of order p^n, there is a unique (up to isomorphism) vector space structure on A over $\mathbb{F}_p$. This is because every finite abelian group of order p^n is isomorphic to a direct sum of n copies of $\mathbb{F}_p$, thus it has a unique vector space structure over $\mathbb{F}_p$.

Therefore, the natural condition for A to be a vector space over $\mathbb{F}_p$ is that A must be a finite abelian group whose order is a power of p. This ensures that A can be given a unique vector space structure over $\mathbb{F}_p$.

(c) Show that if R is the ring of n-tuples of elements in $\mathbb{F}_4$ with componentwise addition and multiplication, then $R*$ is a vector space over $\mathbb{F}_3$.

Firstly, recall that R is defined as:

$$R = (\mathbb{F}_4)^n,$$

where $\mathbb{F}_4$ is the finite field with 4 elements. We can describe $\mathbb{F}_4$ as $\mathbb{F}_2[x]/(x^2+x+1)$ with elements $\{0, 1, \alpha, \alpha+1\}$, where α is a root of x^2+x+1 in $\mathbb{F}_4$.

Next, R^* denotes the additive group of R, so:

$$R^* = (\mathbb{F}_4)^n,$$

which is the same as R under addition. Hence, R^* consists of all n-tuples over $\mathbb{F}_4$ with componentwise addition.

To show that R^* is a vector space over $\mathbb{F}_3$, we need to define scalar multiplication by elements of $\mathbb{F}_3$. Given an element $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in R^*$ and a scalar $c \in \mathbb{F}_3$, we define scalar multiplication as:

$$c \cdot \mathbf{v} = (c \cdot v_1, c \cdot v_2, \ldots, c \cdot v_n),$$

where $c \cdot v_i$ denotes the multiplication of the scalar $c \in \mathbb{F}_3$ with the element $v_i \in \mathbb{F}_4$ (computed in $\mathbb{F}_4$).

To verify that R^* is a vector space, we check the vector space axioms over $\mathbb{F}_3$.

Firstly, R^* is closed under addition. Given two vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in R^*, their sum is:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n).$$

Since addition in $\mathbb{F}_4$ is closed, $\mathbf{u} + \mathbf{v}$ is also in R^*, so R^* is closed under addition.

Next, R^* is closed under scalar multiplication. Given a vector $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in R^*$ and a scalar $c \in \mathbb{F}_3$, scalar multiplication is defined as:

$$c \cdot \mathbf{v} = (c \cdot v_1, c \cdot v_2, \ldots, c \cdot v_n).$$

Since multiplication in $\mathbb{F}_4$ is closed and $\mathbb{F}_4$ is a vector space over $\mathbb{F}_3$, each component $c \cdot v_i$ remains in $\mathbb{F}_4$, so $c \cdot \mathbf{v} \in R^*$. Hence, R^* is closed under scalar multiplication.

The properties of associativity and commutativity of addition hold because $\mathbb{F}_4$ itself is an associative and commutative ring.

The zero vector $\mathbf{0} = (0, 0, \ldots, 0)$ is in R^*, and for any $\mathbf{v} \in R^*$, $\mathbf{v} + \mathbf{0} = \mathbf{v}$, ensuring the existence of an additive identity.

For any vector $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in R^*$, its additive inverse is $-\mathbf{v} = (-v_1, -v_2, \ldots, -v_n)$. Since $\mathbb{F}_4$ is closed under additive inverses, R^* also contains all additive inverses.

Scalar multiplication distributes over vector addition and field addition:

$$c \cdot (\mathbf{u} + \mathbf{v}) = c \cdot \mathbf{u} + c \cdot \mathbf{v},$$

$$(c + d) \cdot \mathbf{v} = c \cdot \mathbf{v} + d \cdot \mathbf{v},$$

and scalar multiplication is associative:

$$c \cdot (d \cdot \mathbf{v}) = (c \cdot d) \cdot \mathbf{v}.$$

Since R^* satisfies all these axioms, it follows that R^* is a vector space over $\mathbb{F}_3$.