Honors Algebra **Spring 2025**
HW 6: Groups, Modules, and Vector Spaces **3/20/2025**

Abstract Algebra: An Integrated Approach by J.H. Silverman.
Page 320-325: 10.13, 10.16, 10.22, 10.23, 10.27
Page 357–370: 11.24, 11.32, 11.37
Page 393–396: 12.16
Do problem 11.32 for modules over non-commutative rings, i.e., show that if M is left R module then the annihiliator Ann(M) is two sided ideal in R.

Problem 1 (10.13). For each of the following linear operators, find eigenvectors and eigenvalues as we did in Example 10.27.

(a) $L(e_1) = -e_1 + 4e_2$ and $L(e_2) = 4e_1 - e_2$

(b) $L(e_1) = e_2$ and $L(e_2) = -e_1$. (You may take $F = \mathbb{C}$ for this part.)

(c) $L(e_1) = e_1 - e_3$, $L(e_2) = 4e_3$, and $L(e_3) = 2e_1 + e_2 + 2e_3$

Problem 2 (10.16). Let V be a finite-dimensional vector space. Let $L1, L2 \in \mathrm{End}_F(V)$ be linear operators on V such that the following three statements are true:

(1) There is a basis for V consisting of eigenvectors of L_1

(2) There is a basis for V consisting of eigenvectors of L_2

(3) $L_1 L_2 = L_2 L_1$

Prove that there is a basis for V consisting of vectors that are simultaneously eigenvectors of $L1$ and eigenvectors of $L2$. This result is often stated as follows: "commuting diagonalizable matrices are simultaneously diagonalizable." (Hint. If you're not sure how to get started, first try the case that $L1$ has $\dim(V)$ distinct eigenvalues.)

Problem 3 (10.22). Let $L \in \mathrm{End}_F(V)$ be an invertible linear operator.

(a) Prove that

$$P_{L^{-1}}(T) = \det(L)^{-1} \cdot (-T)^{\dim V} \cdot P_L(T^{-1})$$

(b) Let $n = \dim V$, and let the eigenvalues of L be $\lambda_1, \ldots \lambda_n$ (repeated with appropriate multiplicity). Prove that the eigenvalues of L^{-1} are $\lambda_1^{-1}, \ldots, \lambda_n^{-1}$.

(c) If $L^d = I$, prove that the eigenvalues of L are dth-roots of unity.

Problem 4 (10.23). Let V be an n-dimensional F vector space, let $L \in \mathrm{End}_F(V)$, and let

$$P_L(T) = T^n - c_1(L)T^{n-1} + c_2(L)T^{n-2} - \cdots + (-1)^c c_n(L)$$

The *trace of* L is defined to be the quantity

$$\operatorname{tr}(L) = c_1(L)$$

(a) Let $J, L \in \operatorname{End}_F(V)$. Prove that

$$\operatorname{tr}(JL) = \operatorname{tr}(LJ)$$

In particular, if J is invertible, prove that

$$\operatorname{tr}(J^{-1}LJ) = \operatorname{tr}(L)$$

(b) Prove that

$$\operatorname{tr}(aL_1 + bL_2) = a\operatorname{tr}(L_1) + b\operatorname{tr}(L_2);$$

i.e., prove that the map

$$\operatorname{tr} : \operatorname{End}_F(V) \longrightarrow F$$

is an F-linear transformation.

(c) Suppose that F is algebraically closed and that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of L, repeated with appropriate multiplicities so that $P_L(T) = \prod(T - \lambda_i)$. Prove that

$$\operatorname{tr}(L) = \lambda_1 + \cdots + \lambda_n$$

(d) Let $\mathcal{B}$ be a basis for V, and let

$$\mathcal{M}_{\mathcal{L},\mathcal{B},\mathcal{B}} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \in \operatorname{Mat}_{n\times n}(F) \tag{1}$$

be the matrix associated to $\mathcal{L}$ for the basis $\mathcal{B}$. Prove that

$$\operatorname{tr}(L) = a_{11} + a_{22} + \cdots + a_{nn}$$

is the sum of the diagonal elements of the martix L.

Problem 5 (10.27)**.** We defined the product of an infinite list of vector spaces $V1, V2, V3, \ldots$, but suppose that we want to take the product of an uncountable number of vector spaces. In general, we take an arbitrary index set I, and we suppose that for each $iinI$ we are given an F-vector space V_i. We can no longer talk about ordered lists of infinite-tuples, since the index set I is arbitrary, so we define the direct product of the V_i over all $i \in I$ to be a vector space of functions,

$$\prod_{i\in I} V_i = \left\{ \text{functions } v : I \longrightarrow \bigcup_{i\in I} V_i \quad \text{satisfying } v(i) \in V_i \quad \text{for all } i \in I \right\}$$

Addition and scalar multiplication in $\prod_{i\in I} V_i$ are defined by

$$(v + w)(i) = v(i) + w(i) \quad \text{and} \quad (cv)(i) = cv(i)$$

(a) Prove that $\prod_{i\in I} v_i$ is an F-vector space.

(b) If $I = \mathbb{N}$, explain why the definition of $\prod_{i\in\mathbb{N}} V_i$ in this exercise is the same as the one given in Definition 10.48

(c) Explain how you would define the direct sum of the V_i for an arbitrary index set I.

Problem 6 (11.24)**.** An *Artinian ring* is a ring in which every descending list of ideals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

eventually stabilizes; i.e., there is a $k \geq 1$ so that $I_k = I_{k+i}$ for all $i \geq 0$. Although this resembles the definition of Noetherian ring, the Artinian condition is actually fare more restrictive.

(a) Let R be an Artinian ring, and let I be an ideal in R. Prove that R/I is an Artinian ring.

(b) Let R be an Artinian ring that is an integral domain. Prove that R is a field. (*Hint.* Let $a \in R$ and consider the ideals $aR \supseteq a^2R \supseteq a^3R \supseteq \cdots$)

(c) Let R be an Artinian ring. Prove that every prime ideal in R is a maximal ideal. (*Hint.* Use (a) and (b)).

(d) Let R be an Artinian ring. Prove that R has only finitely many maximal ideals.

Problem 7 (11.32)**.** Do problem 11.32 for modules over non-commutative rings, i.e., show that if M is left R module then the annihiliator Ann(M) is two sided ideal in R. Let M be an R-module. Prove that the annihilator

$$\mathrm{Ann}(M) = \{a \in R : am = 0 \quad \text{for all} \quad m \in M\}$$

is an ideal of R.

Problem 8 (11.37)**.** Let R be a commutative ring.

(a) Suppose that $a, b \in R$ have the property that $aR + bR = R$. Prove that for all $m, n \geq 1$ we have

$$a^m R + b^n R = R$$

(b) More generally, let $a_1, \ldots, a_t \in R$, and let $e_1, \ldots, e_t \geq 1$ be positive integers. Prove that

$$a_1R + a_2R + \cdots + a_tR = R \quad\iff\quad a_1^{e_1}R + a_2^{e_2}R + \cdots + a_t^{e_t}R = R.$$

Problem 9 (12.16)**.** This exercise give examples showing that the list of composition quotients of a finite group G are not enough to determine G.

(a) Prove that composition series for the cyclic group $\mathcal{C}_4$ and the product group $\mathcal{C}_2 \times \mathcal{C}_2$ have the same length and the same composition quotients.

(b) Prove that composition series for the cyclic group $\mathcal{C}_6$ and the symmetric group $\mathcal{S}_3$ have the same length and the same composition quotients.