De Rham Cohomology, Hodge Decomposition, and Vector Field Analysis

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1 Introduction

In differential geometry, the study of cohomology is essential for understanding the topology of smooth manifolds. De Rham cohomology provides a powerful tool for this purpose, and its connection to the Hodge decomposition is particularly insightful.

2 De Rham Cohomology

Let M be a smooth manifold. The De Rham cohomology groups $H^k_{\mathrm{dR}}(M)$ are defined as the cohomology groups of the de Rham complex.

Definition 2.1 (De Rham Complex)**.** *The De Rham complex on M is given by*

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots ,$$

where $\Omega^k(M)$ is the space of differential k-forms on M and d is the exterior derivative.

The cohomology groups $H^k_{\mathrm{dR}}(M)$ represent the equivalence classes of closed and cohomologous k-forms on M.

3 Hodge Decomposition

The Hodge decomposition on a compact, oriented Riemannian manifold M states that any differential form on M can be uniquely decomposed into a sum of three orthogonal components:

$$\Omega^k(M) = \mathcal{H}^k(M) \oplus d\Omega^{k-1}(M) \oplus \delta\Omega^{k+1}(M),$$

where $\mathcal{H}^k(M)$ is the space of harmonic forms, d is the exterior derivative, and δ is the codifferential.

Theorem 3.1 (Hodge Decomposition)**.** *For any differential form $\alpha \in \Omega^k(M)$, there exist unique forms $\omega \in \mathcal{H}^k(M)$, $\beta \in \Omega^{k-1}(M)$, and $\gamma \in \Omega^{k+1}(M)$ such that*

$$\alpha = \omega + d\beta + \delta\gamma.$$

4 Connection to Vector Field Analysis

Now, let's explore the connection between the Hodge decomposition and the analysis of vector fields.

Consider a vector field V on M. We can decompose V into its gradient, vorticity, and flux components.

Definition 4.1 (Vector Field Decomposition)**.** *Let V be a vector field on M. Then, there exist unique vector fields $W_{grad}, W_{vort}, W_{div}$ such that*

$$V = W_{grad} + W_{vort} + W_{div},$$

where

$$\begin{aligned} W_{grad} &= \nabla f \quad \textit{(gradient)}, \\ W_{vort} &= \nabla \times A \quad \textit{(vorticity)}, \\ W_{div} &= \nabla \cdot B \quad \textit{(flux)}, \end{aligned}$$

for suitable functions f and vector fields A, B.

This decomposition is analogous to the Hodge decomposition, where the gradient corresponds to the harmonic forms, the vorticity corresponds to the coexact forms, and the flux corresponds to the exact forms.

5 Approximation of Closed Forms

In this section, we consider the problem of finding the closest exact form to a given closed form on a manifold M. We will utilize the L^2 inner product on differential forms to quantify the distance between forms.

Definition 5.1 (L^2 Inner Product)**.** *For a compact Riemannian manifold M with volume form* Vol*, the L^2 inner product of two k-forms α, β is defined as*

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta,$$

where $*$ denotes the Hodge star operator.

For a closed form α that is not exact, we aim to find the closest exact form to α. Specifically, we want to find ϕ_0 such that $\|\alpha - d\phi_0\| = \inf_\phi \|\alpha - d\phi\|$, where $\|\cdot\|$ denotes the L^2 norm induced by the inner product.

Theorem 5.2 (Approximation of Closed Forms)**.** *Let M be a compact Riemannian manifold, and let α be a closed k-form on M that is not exact. Then, there exists a unique $(k-1)$-form ϕ_0 such that*

$$\|\alpha - d\phi_0\| \leq \|\alpha - d\phi\|$$

for all $(k-1)$-forms ϕ.

Proof. Consider the space $\Omega^{k-1}(M)$ of smooth $(k-1)$-forms on M. Define the set

$$\mathcal{A} = \{\phi \in \Omega^{k-1}(M) : d\phi \text{ is closed}\}.$$

For any $\phi \in \mathcal{A}$, the form $\alpha - d\phi$ is closed, and by Poincaré's lemma, there exists a $(k-2)$-form ψ such that $\alpha - d\phi = d\psi$. Now, let ϕ_0 be any element in $\mathcal{A}$ such that

$$\|\alpha - d\phi_0\|^2 = \langle \alpha - d\phi_0, \alpha - d\phi_0 \rangle$$

is minimized.

Since $\alpha - d\phi_0$ is exact, we have

$$\alpha - d\phi_0 = d\psi_0$$

for some $(k-2)$-form ψ_0. Then, for any $(k-1)$-form ϕ,

$$\begin{aligned}\|\alpha - d\phi_0\|^2 &\leq \langle \alpha - d\phi, \alpha - d\phi \rangle \\ &= \langle d\psi_0, \alpha - d\phi \rangle \\ &= \langle d\psi_0, \alpha \rangle - \langle d\psi_0, d\phi \rangle \\ &= \langle d\psi_0, \alpha \rangle - \langle d * d\psi_0, \phi \rangle,\end{aligned}$$

where we used the fact that $d\phi$ is exact, and thus, $d * d\phi = 0$. The equality holds when $\phi = \phi_0$, implying that ϕ_0 is the desired form.

This establishes the existence of ϕ_0. Uniqueness follows from the fact that if $\alpha - d\phi_0 = d\psi_0$, then ϕ_0 is unique up to the addition of an exact form. $\square$

This theorem provides a constructive way to find the closest exact form to a closed form, which is crucial in applications involving differential forms on manifolds.

6 Harmonic Forms on Manifolds with Boundary

Now, let's consider a manifold M^n embedded in $\mathbb{R}^n$ with boundary $\partial M = \varnothing$. We are interested in harmonic forms on M.

Definition 6.1 (Harmonic Forms)**.** *Let M be a compact Riemannian manifold with boundary $\partial M = \varnothing$. A differential k-form α on M is called harmonic if it satisfies the following conditions:*

$$\begin{aligned}&\textit{(1)}\quad d\alpha = 0,\\ &\textit{(2)}\quad \delta\alpha = 0,\end{aligned}$$

where d is the exterior derivative and δ is the codifferential.

The space of harmonic k-forms is denoted by $\mathcal{H}^k(M)$. We are interested in the following result:

Theorem 6.2 (Existence of Harmonic Representatives)**.** *Let M^n be a compact Riemannian manifold with boundary $\partial M = \varnothing$. For any k-form α on M, there exists a unique $(k-1)$-form ω_0 such that*

$$\int_M |\alpha - d\omega_0|^2 \leq \int_M |\alpha - d\omega'|^2$$

for all $(k-1)$-forms ω' if and only if $d * (\alpha - d\omega_0) = 0$.

Proof. Let $\mathcal{A} = \{\omega \in \Omega^{k-1}(M) : d\omega \text{ is closed}\}$, and consider the set

$$\mathcal{B} = \{\omega \in \mathcal{A} : \delta(\alpha - d\omega) = 0\}.$$

Similar to the previous proof, we can show that there exists a unique element ω_0 in $\mathcal{B}$ that minimizes $\|\alpha - d\omega_0\|^2$. This implies that ω_0 is the desired form.

The condition $d * (\alpha - d\omega_0) = 0$ ensures that $\delta(\alpha - d\omega_0) = 0$, and thus, ω_0 lies in $\mathcal{B}$. The proof of uniqueness is analogous to the previous theorem.

Therefore, the existence of ω_0 is guaranteed if and only if $d * (\alpha - d\omega_0) = 0$. $\square$

Here are some images of 3-dimensional manifolds I created in Java Processing.

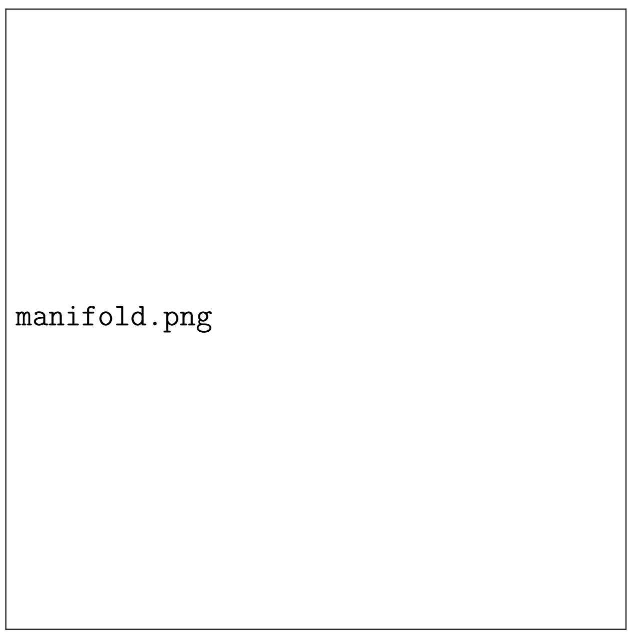

Figure 1: Here is a closed manifold in $\mathbb{R}^3$ in the shape of a morphed bean.

7 Conclusion

The interplay between De Rham cohomology, Hodge decomposition, and vector field analysis provides a rich framework for understanding the geometric

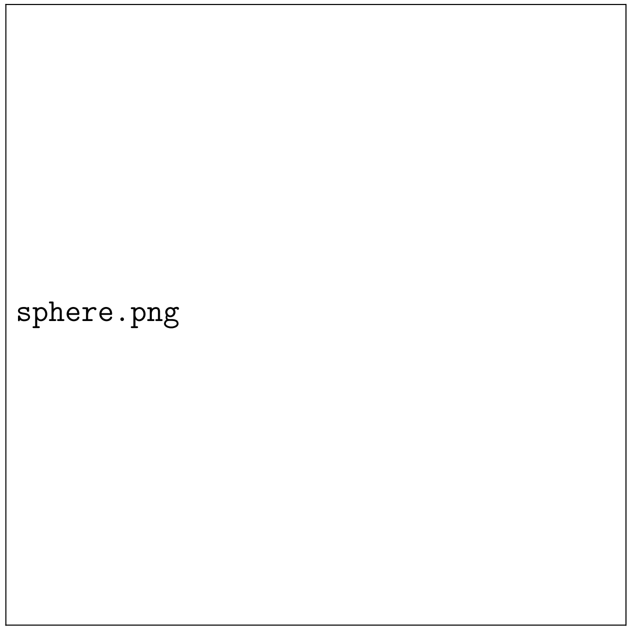

Figure 2: This is an image of a sphere.

and topological properties of smooth manifolds. The decomposition of vector fields into gradient, vorticity, and flux components reflects the underlying structure revealed by De Rham cohomology.