Honors Algebra **Spring 2025**
HW 9: Galois Theory, Automorphisms of Fields, Splitting Fields **4/15/2025**

Abstract Algebra: An Integrated Approach by J.H. Silverman.
Page 285-294: 9.3, 9.4, 9.5, 9.9, 9.13, 9.14, 9.15, 9.16, 9.20

Problem 1 (9.3)**.** Let L/F be an extension of fields, and let $\alpha_1, \ldots, \alpha_r \in L$ be algebraic over F.

(a) Prove that
$$F[\alpha_1 \ldots, \alpha_r] = F(\alpha, \ldots, \alpha_r)$$

(b) Prove that
$$[F(\alpha_1 \ldots, \alpha_r) : F] \leq \prod_{i=1}^{r} [F(\alpha_i) : F]$$

(c) Suppsoe that the degrees $[F(\alpha_i) : F]$ are pairwise relatively prime. Prove that the inequality in (b) is an equality

(d) Suppose that
$$F(\alpha_i) \cap F(\alpha_j) = F \quad \text{for all } i \neq j$$
Does this imply that the inequality in (b) is an equality? Either prove that (b) is an equality or give a counterexample.

Problem 2 (9.4)**.** Let $L/K/F$ be a tower of fields. Prove that
$$(L \text{ is algebraic over } K) \text{ and } (K \text{ is algebraic over } F) \implies (L \text{ is algebraic over } F)$$

Problem 3 (9.5)**.** Compute the minimal polynomials of the indicated numbers over the indicated fields; cf. Example 9.9.

	α	F	$\Phi_{F,\alpha}(x)$
(a)	$\sqrt{3}$	$\mathbb{Q}$	answer goes here
(b)	$\sqrt{3}$	$\mathbb{Q}(\sqrt{2})$	answer goes here
(c)	$\sqrt{3}$	$\mathbb{Q}(\sqrt{(3)})$	answer goes here
(d)	i	$\mathbb{R}$	answer goes here
(e)	i	$\mathbb{C}$	answer goes here
(f)	$i + \sqrt{3}$	$\mathbb{Q}$	answer goes here
(g)	$i + \sqrt{3}$	$\mathbb{Q}(i)$	answer goes here
(h)	$i + \sqrt{3}$	$\mathbb{R}$	answer goes here

Problem 4 (9.9). Let F be a field, let K/F be an extension field, and assume that K is algebraically closed. Let

$$L = \{\alpha \in K : \alpha \text{ is algebraic over } F\}$$

Prove that L is an algebraically closed field. (Note that we do not assume that K/F is an algebraic extension.)

Problem 5 (9.13). Let K/F be a finite extension of fields, and let

$$\phi : K \longrightarrow K$$

be a field homomorphism that fixes the elements of F; i.e., $\phi(c) = c$ for every $c \in F$. Prove that ϕ is an isomorphism. (Hint. You'll need to use the fact that K/F is finite, since Exercise 9.14 shows that the assertion may be false for infinite extensions.)

Problem 6 (9.14). Let F be a field, and let $F(T)$ be the field of rational function as described in Example 7.31 and Definition 7.32. Define maps

$$\sigma, \tau : F(T) \longrightarrow F(T) \quad \text{by} \quad \sigma(p(T)) = p(T^{-1}) \quad \text{and} \quad \tau(p(T)) = p(T^2)$$

(a) Prove that σ and τ are field homomorphisms $F(T) \longrightarrow F(T)$ that fix F. Prove that σ is a field automophism of $F(T)$, but that τ is not.

(b) Prove that $\sigma^2 = e$ but that no iterate of τ is the identity element.

(c) Find an element $u \in F(T)$ so that

$$\{p(T) \in F(T) : \sigma(p(T)) = p(T)\} = F(u)$$

(d) What are the element of $F(T)$ that are fixed by τ?

Problem 7 (9.15). Show that Lemma 9.23 is false for

$$F_1 = F_2 = \mathbb{Q}, \quad f_1(x) = f_2(x) = x^4 - 5x^2 + 6, \quad \alpha_1 = \sqrt{2}, \quad \alpha_2 = \sqrt{3}$$

Why does this not provide a counterexample to Lemma 9.23?

Problem 8 (9.16). Let F be a field of characteristic 0, let $f(x) \in F[x]$, and let K/F be a splitting field for $f(x)$ over F. This exercise asks you to prove Proposition 9.34, which states the K is the splitting field of a seperable polynomial in $F[x]$.

(a) We know from Corollary 7.20 that we can factor $f(x)$ as a product of irreducible polynomials, say

$$f(x) = cg_1(x)^{e_1} g_2(x)^{e_2} \cdots g_r(x)^{e_r}$$

where $g_1(x), \ldots, g_r(x) \in F[x]$ are distinct monic irreducible polynomials. Prove that

$$g_i(x) \text{ and } g_j(x) \text{ have a common root} \iff i = j$$

(b) Let $g(x) = g_1(x)g_2(x) \cdots g_r(x)$. Prove that $g(x)$ is a seperable polynomial.

(c) Prove that K is the splitting field of $g(x)$ over F.

Problem 9 (9.20)**.** Let F be a separable field, and let K/F and L/F be field extensions. Suppose that K/F is a finite extension and that L is algebraically closed. Prove that there are exactly $[K : F]$ embeddings $\sigma : K \hookrightarrow L$ that are the identity map on F.