Honors Linear Algebra **Fall 2024**
HW9: **Due: 12/4/2024**

Problem 1 (Determinants 2)**.** Prove that if $M \neq 0$ is a free module then $M^{\otimes p} \neq 0$ for any $p \geq 1$.

1. Free Module and Tensor Product:
Since M is a free module, there exists a basis $\{e_i\}_{i \in I}$ for M. The tensor product $M^{\otimes p}$ is the p-fold tensor power of M, which is also a module over the same ring R.

2. Basis of Tensor Product:
The elements of $M^{\otimes p}$ are linear combinations of tensors of the form $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_p}$, where $i_k \in I$ for $k = 1, \ldots, p$. Thus, the basis of $M^{\otimes p}$ is given by all possible such products of basis elements of M.

3. Non-Zero Property:
Since $M \neq 0$, its basis $\{e_i\}$ is non-empty, implying $I \neq \emptyset$. Therefore, the basis of $M^{\otimes p}$ is also non-empty, as it consists of all p-fold tensors of the non-empty basis of M. Hence, $M^{\otimes p} \neq 0$.

Thus, the assumption that $M \neq 0$ ensures the existence of a non-empty basis for M, which in turn guarantees that $M^{\otimes p}$ is non-zero for any $p \geq 1$.

Problem 2 (Determinants 5)**.** Prove that if M is a free module with basis $\mathcal{B} = \{m_1, m_2, \ldots, m_n\}$ then the elements $m_{i_1} \wedge \cdots \wedge m_{i_p}$ where $i_1 < i_2 < \cdots < i_p$ is a strictly ascending sequence are linearly independent by constructing many alternating maps for $M^{\times p} \longrightarrow R$.

1. Definition of Alternating Map:
An alternating map $f : M^{\times p} \to R$ is a multilinear map satisfying the property that $f(m_{\sigma(1)}, \ldots, m_{\sigma(p)}) = \operatorname{sgn}(\sigma) f(m_1, \ldots, m_p)$, where σ is a permutation of $\{1, \ldots, p\}$, and $\operatorname{sgn}(\sigma)$ is the sign of the permutation. If $m_i = m_j$ for some $i \neq j$, then $f(m_1, \ldots, m_p) = 0$.

2. Construction of Maps:
Let $I = \{i_1, i_2, \ldots, i_p\}$, where $i_1 < i_2 < \cdots < i_p$, be a strictly ascending sequence of indices. Define $f_I : M^{\times p} \to R$ by:

$$f_I(m_{j_1}, \ldots, m_{j_p}) = \begin{cases} \det(A) & \text{if } (j_1, \ldots, j_p) = (i_1, \ldots, i_p), \\ 0 & \text{otherwise,} \end{cases}$$

where A is the $p \times p$ matrix whose columns are the coordinates of $m_{i_1}, \ldots, m_{i_p}$ with respect to the basis $\mathcal{B}$.

3. Linearly Independent Elements:
Each f_I is an alternating map that vanishes on any wedge product $m_{j_1} \wedge \cdots \wedge m_{j_p}$ unless $\{j_1, \ldots, j_p\} = \{i_1, \ldots, i_p\}$. Thus, applying f_I to $m_{i_1} \wedge \cdots \wedge m_{i_p}$ gives:

$$f_I(m_{i_1}, \ldots, m_{i_p}) = \det(A) \neq 0,$$

while $f_I(m_{j_1}, \ldots, m_{j_p}) = 0$ for any other strictly ascending sequence $\{j_1, \ldots, j_p\} \neq \{i_1, \ldots, i_p\}$.

Since we can construct one alternating map f_I for each strictly ascending sequence $\{i_1, \ldots, i_p\}$, and each f_I uniquely distinguishes $m_{i_1} \wedge \cdots \wedge m_{i_p}$, the set $\{m_{i_1} \wedge \cdots \wedge m_{i_p} : i_1 < i_2 < \cdots < i_p\}$ is linearly independent. Thus, we have proven that $m_{i_1} \wedge \cdots \wedge m_{i_p}$ are linearly independent by constructing alternating maps.

Problem 3 (Determinants 8)**.** An associative algebra A is called $\mathbb{Z}/2\mathbb{Z}$ graded if $A = A^(0) \oplus A^(1)$ and the multiplication preserves the grading, i.e.,

$$A^{(i)} \cdot A^{(j)} \subset A^{(i+j)}$$

for any i, j, where the addition is mod 2. The algebra is called supper commutative if

$$a \cdot b = (-1)ijba\text{for any}a \in A^{(i)}, b \in A^{(j)}$$

(a) Verify the exterior algebra is super commutative. How do you define the grading?

1. Definition of Grading: The exterior algebra $\Lambda^*(M)$ over an R-module M is graded by degree, where the grading is given as:

$$\Lambda^*(M) = \bigoplus_{k \geq 0} \Lambda^k(M),$$

with $\Lambda^k(M)$ denoting the k-th exterior power of M. To define a $\mathbb{Z}/2\mathbb{Z}$-grading, set:

$$\Lambda^{(0)}(M) = \bigoplus_{k \text{ even}} \Lambda^k(M), \quad \Lambda^{(1)}(M) = \bigoplus_{k \text{ odd}} \Lambda^k(M).$$

2. Super Commutativity: For $x \in \Lambda^i(M)$ and $y \in \Lambda^j(M)$, the wedge product satisfies:

$$x \wedge y = (-1)^{ij} y \wedge x.$$

This follows from the alternating property of the wedge product, which imposes the sign rule when swapping two elements. This property ensures that $\Lambda^*(M)$ is super commutative under the $\mathbb{Z}/2\mathbb{Z}$-grading.

Thus, the exterior algebra $\Lambda^*(M)$ is super commutative with the grading defined as above.

(b) Show that the exterior algebra satisfies the following universal property (assuming that 2 is invertible in R): Let M be an R-module. Then for any super commutative associative unitial R-algebra A and any R-module homomorphism $\varphi : M \longrightarrow A^{(1)}$ there exists exactly one R-algebra map $\bar{\varphi}$ such that the diagram commutes

$$\begin{tikzcd}
M \arrow[r, hook, "i"] \arrow[dr, "\varphi"'] & \Lambda^*(M) \arrow[d, dashed, "\bar{\varphi}"] \\
& A
\end{tikzcd}$$

i.e., $\varphi = \bar{\varphi} \circ i$, where $i : M \longrightarrow \Lambda^* M$ comes from the identification $M = M^{\otimes 1}$.

To show that the exterior algebra satisfies the universal property, assume $\varphi : M \to A^{(1)}$ is an R-module homomorphism and A is a super commutative associative unital R-algebra.

1. Existence of $\bar{\varphi}$: The universal property of the exterior algebra $\Lambda^*(M)$ states that there exists a unique R-algebra map $\bar{\varphi} : \Lambda^*(M) \to A$ such that $\bar{\varphi} \circ i = \varphi$, where $i : M \hookrightarrow \Lambda^*(M)$ is the canonical inclusion.

2. Construction of $\bar{\varphi}$: Since A is super commutative, the image of $\varphi(M) \subset A^{(1)}$ satisfies:

$$\varphi(m_1)\varphi(m_2) = -\varphi(m_2)\varphi(m_1), \quad \forall m_1, m_2 \in M.$$

This ensures that φ extends uniquely to an R-algebra homomorphism $\bar{\varphi} : \Lambda^*(M) \to A$, where the wedge product in $\Lambda^*(M)$ corresponds to the multiplication in A.

3. Uniqueness of $\bar{\varphi}$: Any R-algebra homomorphism $\bar{\varphi}$ must satisfy $\bar{\varphi}(m) = \varphi(m)$ for $m \in M$ because M generates $\Lambda^*(M)$ as an R-algebra. The super commutativity of A ensures that $\bar{\varphi}$ is well-defined and unique.

4. Commutativity of Diagram: By construction, $\varphi = \bar{\varphi} \circ i$, as $i(m) = m$ for $m \in M$. Hence, the diagram commutes.

Thus, the exterior algebra satisfies the universal property, with $\bar{\varphi}$ as the unique R-algebra map that makes the diagram commute.

Problem 4 (Determinants 12)**.** Prove that $T(R) \simeq R[x]$. Let $\alpha : R \longrightarrow R$ be the multiplication by c, i.e., $\alpha(r) = cr$. What is $\alpha^{\otimes\star}$ as a map from $R[x] \longrightarrow R[x]$

Tensor Algebra and Polynomial Ring: The tensor algebra $T(R)$ over a ring R is given by:

$$T(R) = \bigoplus_{n \geq 0} R^{\otimes n},$$

where $R^{\otimes 0} = R$ and $R^{\otimes n} = R \otimes R \otimes \cdots \otimes R$ (with n-factors). Elements of $T(R)$ can be written as finite sums of tensors $r_0 + r_1 \otimes r_2 + \cdots$.

There is an isomorphism between $T(R)$ and $R[x]$, the ring of polynomials in x with coefficients in R, given by:

$$R^{\otimes n} \ni r_1 \otimes \cdots \otimes r_n \mapsto r_1 x r_2 x \cdots x r_n \in R[x].$$

This identifies $T(R)$ as the free associative R-algebra generated by one element x, which corresponds to $R[x]$.

Description of α: Let $\alpha : R \to R$ be defined by $\alpha(r) = cr$, where $c \in R$. This is an R-module homomorphism that scales elements of R by c.

Action of $\alpha^{\otimes\star}$ on $R[x]$: The map $\alpha^{\otimes\star}$ extends α to $T(R)$ (and thus $R[x]$) via its action on the generators:

$$\alpha^{\otimes\star}(r_1 x^n r_2 x^m \cdots r_k x^p) = \alpha(r_1) x^n \alpha(r_2) x^m \cdots \alpha(r_k) x^p.$$

Explicitly, if $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$, then:

$$\alpha^{\otimes\star}(f(x)) = \sum_{i=0}^{n} \alpha(a_i)x^i = \sum_{i=0}^{n} ca_i x^i.$$

Thus, $\alpha^{\otimes\star}$ acts as multiplication by c on the coefficients of $f(x)$.

We have established that $T(R) \simeq R[x]$ via the correspondence between tensors and polynomials, and the map $\alpha^{\otimes\star}$ acts as a scaling map that multiplies each coefficient of a polynomial in $R[x]$ by c.

Problem 5 (Determinants 13). Prove that if a module M can be generated by m elements then $\wedge^p M = 0$ if $p > m$

Generating Set of M:
Since M can be generated by m elements, there exists a set $\{x_1, x_2, \ldots, x_m\} \subset M$ such that any element of M can be expressed as a linear combination of these generators.

Definition of $\wedge^p M$: The p-th exterior power $\wedge^p M$ is defined as the R-module generated by wedge products of the form:

$$x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_p},$$

where $x_{i_j} \in M$ and $i_1 < i_2 < \cdots < i_p$. The alternating property of the wedge product implies that if any two indices i_j and i_k are equal, the wedge product is zero.

Case $p > m$:
If $p > m$, any subset of p elements chosen from $\{x_1, x_2, \ldots, x_m\}$ must have at least one repeated index, as there are only m distinct generators. Due to the alternating property, the wedge product:

$$x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_p} = 0$$

for any such choice of $i_1, i_2, \ldots, i_p$.

Since every wedge product in $\wedge^p M$ vanishes for $p > m$, we have:

$$\wedge^p M = 0 \quad \text{for } p > m.$$

Problem 6 (Determinants 15). Express the determinants of the following linear transformations in terms of $\det(\alpha)$, $\det(\beta)$, $\dim V$, $\dim W$ where $\alpha \in \mathrm{Hom}(V, V)$ and $\beta \in \mathrm{Hom}(W, W)$:

1. $\det(\alpha \otimes \beta)$
2. $\det(\alpha^{\otimes p})$
3. $\det(\wedge^p \alpha)$
4. $\det(\wedge^* \alpha)$

1. $\det(\alpha \otimes \beta)$:
The linear transformation $\alpha \otimes \beta$ acts on $V \otimes W$, and its determinant is given by:

$$\det(\alpha \otimes \beta) = \det(\alpha)^{\dim W} \cdot \det(\beta)^{\dim V}.$$

This follows from the fact that the action of $\alpha \otimes \beta$ corresponds to applying α on V and β on W, each scaled by the dimension of the other space.

2. $\det(\alpha^{\otimes p})$:
The linear transformation $\alpha^{\otimes p}$ acts on $V^{\otimes p}$, and its determinant is:

$$\det(\alpha^{\otimes p}) = \det(\alpha)^{\dim V^{\otimes p}} = \det(\alpha)^{(\dim V)^p}.$$

This uses the fact that the determinant of a tensor power scales with the dimension of the tensor space.

3. $\det(\wedge^p \alpha)$:
The linear transformation $\wedge^p \alpha$ acts on $\wedge^p V$, and its determinant is:

$$\det(\wedge^p \alpha) = \det(\alpha)^{\binom{\dim V}{p}}.$$

Here, $\binom{\dim V}{p}$ is the dimension of $\wedge^p V$, as it represents the number of linearly independent wedge products of p basis vectors.

4. $\det(\wedge^* \alpha)$:
The transformation $\wedge^* \alpha$ acts on the full exterior algebra $\wedge^* V = \bigoplus_{p=0}^{\dim V} \wedge^p V$. The determinant of $\wedge^* \alpha$ is the product of the determinants over all $\wedge^p V$:

$$\det(\wedge^* \alpha) = \prod_{p=0}^{\dim V} \det(\wedge^p \alpha) = \prod_{p=0}^{\dim V} \det(\alpha)^{\binom{\dim V}{p}}.$$

Using the binomial theorem, the sum of the binomial coefficients equals $2^{\dim V}$, so:

$$\det(\wedge^* \alpha) = \det(\alpha)^{\sum_{p=0}^{\dim V} \binom{\dim V}{p}} = \det(\alpha)^{2^{\dim V}}.$$

Problem 7 (Determinants 19)**.** Do this problem according to the outline given. Let $a_1, \ldots, a_n$ be elements of the field $\mathbb{F}$. Let V_n denote the matrix whose i-th row is $1, \alpha_i, \alpha_i^2, \cdots, a_i^{n-1}$. Prove the formula

$$\det \begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{pmatrix} = \prod_{i>j} (a_i - a_j)$$

where the product is over all pairs of integers $n \geq i > j \geq 1$. This is to be done as follows: Define two polynomials $f(x), g(x) \in \mathbb{F}[x]$ by letting $f(x)$ be the determinant of the matrix obtained from B by replacing a_n by x, and $g(x)$ the polynomial obtained by replacing a_n by x in the right-hand side of the above formula. Using properties of determinants and the ring $\mathbb{F}[x]$, prove the formula by induction on n. Fist conclude that $f(x) = g(x)$ and then evaluate at a_n.

We aim to prove the determinant formula:

$$\det(V_n) = \prod_{i>j} (a_i - a_j),$$

where V_n is the Vandermonde matrix. This will be done as outlined, using polynomials $f(x)$ and $g(x)$ and induction on n.

Step 1: Define $f(x)$ and $g(x)$
Let V_n be the Vandermonde matrix:

$$V_n = \begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{pmatrix}.$$

Define $f(x)$ as the determinant of the matrix V_n with a_n replaced by x:

$$f(x) = \det \begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x & x^2 & \cdots & x^{n-1} \end{pmatrix}.$$

Define $g(x)$ as the right-hand side of the formula with a_n replaced by x:

$$g(x) = \prod_{k=1}^{n-1} (x - a_k) \prod_{i>j} (a_i - a_j).$$

Step 2: Base Case ($n = 2$)
For $n = 2$, the matrix is:

$$V_2 = \begin{pmatrix} 1 & a_1 \\ 1 & a_2 \end{pmatrix}.$$

Its determinant is:

$$\det(V_2) = a_2 - a_1.$$

The formula is:

$$\prod_{i>j} (a_i - a_j) = a_2 - a_1.$$

Thus, the formula holds for $n = 2$.

Step 3: Inductive Step
Assume the formula holds for $n - 1$, i.e., for a Vandermonde matrix V_{n-1}, we have:

$$\det(V_{n-1}) = \prod_{i>j} (a_i - a_j).$$

Now consider V_n. Expanding $f(x)$ along the last row gives:

$$f(x) = \sum_{k=1}^{n} (-1)^{n+k} x^{k-1} \det(V'_{n-1}),$$

where V'_{n-1} is the $(n-1) \times (n-1)$ matrix obtained by removing the n-th row and k-th column from V_n. By properties of determinants, this expansion ensures that $f(x)$ is a polynomial of degree $n-1$.

Similarly, $g(x)$, as a product of linear terms, is also a polynomial of degree $n-1$. Since $f(x)$ and $g(x)$ agree at n distinct points ($x = a_1, a_2, \ldots, a_{n-1}$), they must be identical:

$$f(x) = g(x).$$

Step 4: Evaluate at $x = a_n$
Substitute $x = a_n$ into $f(x)$ and $g(x)$. For $f(x)$, this returns $\det(V_n)$, and for $g(x)$, this returns:

$$g(a_n) = \prod_{k=1}^{n-1} (a_n - a_k) \prod_{i>j} (a_i - a_j).$$

Thus:

$$\det(V_n) = \prod_{i>j} (a_i - a_j).$$