Honors Linear Algebra **Fall 2024**
HW5: Vector Spaces **Due: 10/07/2024**

Problem 1. Let V be a finite-dimensional vector space and let T be a linear operator on V. Suppose that $\text{rank}(T^2) = \text{rank}(T)$. Prove that the range and null space of T are disjoint, i.e., have only the zero vector in common.

Let $N(T)$ denote the null space of T and $R(T)$ denote the range of T. We need to show that $N(T) \cap R(T) = \{0\}$.

Since $\text{rank}(T^2) = \text{rank}(T)$, by the rank-nullity theorem, we have:

$$\dim(V) = \text{rank}(T) + \dim(N(T))$$

and

$$\dim(V) = \text{rank}(T^2) + \dim(N(T^2)).$$

Since $\text{rank}(T^2) = \text{rank}(T)$, we can substitute:

$$\dim(V) = \text{rank}(T) + \dim(N(T))$$

and

$$\dim(V) = \text{rank}(T) + \dim(N(T^2)).$$

Thus, we have:

$$\dim(N(T)) = \dim(N(T^2)).$$

Notice that $N(T^2)$ includes vectors $v \in V$ such that $T^2(v) = 0$. This implies $T(T(v)) = 0$, meaning $T(v) \in N(T)$.

Suppose $v \in R(T) \cap N(T)$. Then there exists some $u \in V$ such that $T(u) = v$ and $T(v) = 0$. Therefore, we have:

$$T(T(u)) = 0 \implies T^2(u) = 0.$$

This indicates that $u \in N(T^2)$.

Since $v \in R(T)$ and $v = T(u)$ implies $v \in N(T)$, we know:

$$T(u) \in N(T) \implies u \in N(T^2).$$

Since $\dim(N(T)) = \dim(N(T^2))$, the only possibility for the intersection $R(T) \cap N(T)$ is the zero vector:

$$N(T) \cap R(T) = \{0\}.$$

Hence, we conclude that the range and null space of T are disjoint.

Problem 2. Let $p, m,$ and n be positive integers and $\mathbb{F}$ a field. Let V be the space of $m \times n$ matrices over $\mathbb{F}$ and $\mathbb{W}$ the space of $p \times n$ matrices over $\mathbb{F}$. Let B be a fixed $p \times m$ matrix and let T be the linear transformation from V into W defined by $T(A) = BA$. Prove that T is invertible if and only if $p = m$ and B is an invertible $m \times m$ matrix.

To show that T is invertible, we need to establish two implications:

(1) showing $\implies$
If T is invertible, then it must map the entire space V onto $\mathbb{W}$. The dimension of V is mn, and the dimension of $\mathbb{W}$ is pn. Thus, we require:

$$mn = pn.$$

Since $n \neq 0$, we can simplify to:

$$m = p.$$

Let B be represented as an $m \times m$ matrix. For $T(A) = BA$ to be injective, the null space must only contain the zero matrix. Thus, if $BA = 0$ implies $A = 0$, then B must be injective, which holds if B is invertible.

(2) Showing $\impliedby$
Assume B is an invertible $m \times m$ matrix. Given $A \in V$ such that $T(A) = BA = 0$, it follows that B being invertible implies $A = 0$. Therefore, T is injective.

The range of T is all $p \times n$ matrices because B can generate every linear combination of its columns. Thus, the image of T has dimension pn.

Since $\dim(V) = \dim(\mathbb{W})$ and T is injective, it follows that T is bijective and therefore invertible.

Thus, T is invertible if and only if $p = m$ and B is an invertible $m \times m$ matrix.

Problem 3. We have seen that the linear operator T on R^2 defined by $T(x_1, x_2) = (x_1, 0)$ is represented in the standard ordered basis by the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

This operator satisfies $T^2 = T$. Prove that if S is a linear operator on R^2 such that $S^2 = S$, then $S = 0$, or $S = I$, or there is an ordered basis B for R^2 such that $[S]_B = A$ (above).

A linear operator S on $\mathbb{R}^2$ that satisfies $S^2 = S$ is known as a projection operator. We will explore the implications of this condition.

Step 1: Eigenvalues of S
Since $S^2 = S$, we can rewrite this as:

$$S(Sv) = Sv \quad \text{for any } v \in \mathbb{R}^2.$$

This implies that the eigenvalues of S can only be 0 or 1. Let λ be an eigenvalue of S corresponding to an eigenvector v:

$$S(v) = \lambda v.$$

Applying S again gives:

$$S(S(v)) = S(\lambda v) = \lambda S(v) = \lambda^2 v.$$

Hence, we have:

$$\lambda^2 v = \lambda v \implies (\lambda^2 - \lambda)v = 0.$$

This confirms $\lambda(\lambda - 1) = 0$, so λ must be 0 or 1.

Step 2: The structure of S
Let $V_0 = \ker(S)$ (the null space of S) and $V_1 = \text{Im}(S)$ (the image of S). Since S is a linear operator on a two-dimensional space, we have:

$$\dim(V_0) + \dim(V_1) = 2.$$

The possible dimensions for V_0 and V_1 based on the eigenvalues are: - If $\dim(V_0) = 2$, then $S = 0$. - If $\dim(V_1) = 2$, then $S = I$. - If $\dim(V_0) = 1$ and $\dim(V_1) = 1$, then there exists a one-dimensional subspace for both.

Step 3: Finding the basis B
In the case where $\dim(V_0) = 1$ and $\dim(V_1) = 1$, let v_0 be a non-zero vector in V_0 and v_1 be a non-zero vector in V_1. The set $\{v_0, v_1\}$ forms a basis for $\mathbb{R}^2$.

We can express S in this basis. In the basis $B = \{v_0, v_1\}$, the action of S on v_0 gives $S(v_0) = 0$ and $S(v_1) = v_1$, leading to the matrix representation:

$$[S]_B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

However, by transforming to the standard basis via appropriate changes of basis, we can find an ordered basis such that S is represented by the matrix A.

Conclusion:
We conclude that if S is a projection operator on $\mathbb{R}^2$, then either $S = 0$, $S = I$, or there exists an ordered basis B such that $[S]_B = A$.

Problem 4. Let $\mathbb{F}$ be a field and let V be the vector space consisting of 0 together with all polynomials of degree n or less. Define the function α on elements of V by

$$\alpha(p(x)) = \frac{d}{dx}(x^n \cdot p(\frac{1}{x}))$$

for $p(x) \in V$. Show the following:

(a) $\alpha(p(x)) \in V$ for $p(x) \in V$.

Let $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ be a polynomial in V, where $a_i \in \mathbb{F}$.

To show that $\alpha(p(x)) \in V$, we first compute $p\left(\frac{1}{x}\right)$:

$$p\left(\frac{1}{x}\right) = a_0 + a_1\left(\frac{1}{x}\right) + a_2\left(\frac{1}{x^2}\right) + \ldots + a_n\left(\frac{1}{x^n}\right).$$

This expression can be rewritten as:

$$p\left(\frac{1}{x}\right) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots + \frac{a_n}{x^n}.$$

Hence, $x^n \cdot p\left(\frac{1}{x}\right)$ becomes:

$$x^n \cdot p\left(\frac{1}{x}\right) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n.$$

This polynomial is of degree at most n since the highest term is a_0x^n (assuming $a_0 \neq 0$).

Now, we differentiate this polynomial:

$$\alpha(p(x)) = \frac{d}{dx}(x^n \cdot p\left(\frac{1}{x}\right)) = \frac{d}{dx}(a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n).$$

The derivative is given by:

$$\alpha(p(x)) = a_0nx^{n-1} + a_1(n-1)x^{n-2} + a_2(n-2)x^{n-3} + \ldots + a_n(0)$$
$$= na_0x^{n-1} + (n-1)a_1x^{n-2} + \ldots + 0$$

This resulting polynomial is also of degree at most $n-1$, which is still in V.

Therefore, we conclude that $\alpha(p(x)) \in V$ for any polynomial $p(x) \in V$.

(b) α is a linear transformation. Compute the matrix $[\alpha]_{B,B}$ with respect to the standard ordered basis $B = \{1, x, \ldots, x^n\}$.

To show that α is a linear transformation, we need to verify that:

$$\alpha(p(x) + q(x)) = \alpha(p(x)) + \alpha(q(x)) \quad \text{and} \quad \alpha(c \cdot p(x)) = c \cdot \alpha(p(x))$$

for any $p(x), q(x) \in V$ and $c \in \mathbb{F}$.

For the first property, consider $p(x) + q(x)$:

$$\alpha(p(x) + q(x)) = \frac{d}{dx}\left(x^n \cdot \left(p\left(\frac{1}{x}\right) + q\left(\frac{1}{x}\right)\right)\right).$$

By the linearity of differentiation:

$$= \frac{d}{dx}(x^n \cdot p\left(\frac{1}{x}\right)) + \frac{d}{dx}(x^n \cdot q\left(\frac{1}{x}\right)) = \alpha(p(x)) + \alpha(q(x)).$$

For the second property:

$$\alpha(c \cdot p(x)) = \frac{d}{dx}(x^n \cdot (c \cdot p\left(\frac{1}{x}\right))) = c \cdot \frac{d}{dx}(x^n \cdot p\left(\frac{1}{x}\right)) = c \cdot \alpha(p(x)).$$

Hence, α is a linear transformation.

Now we compute the matrix $[\alpha]_{B,B}$: The standard ordered basis $B = \{1, x, x^2, \ldots, x^n\}$ consists of $n+1$ elements.

We need to compute α applied to each basis element: - For $p(x) = 1$:

$$\alpha(1) = \frac{d}{dx}(x^n \cdot 1) = nx^{n-1}.$$

- For $p(x) = x$:

$$\alpha(x) = \frac{d}{dx}(x^n \cdot \frac{1}{x}) = \frac{d}{dx}(x^{n-1}) = (n-1)x^{n-2}.$$

- For $p(x) = x^2$:

$$\alpha(x^2) = \frac{d}{dx}(x^n \cdot \frac{1}{x^2}) = \frac{d}{dx}(x^{n-2}) = (n-2)x^{n-3}.$$

- Continuing this pattern, we find:

$$\alpha(x^k) = (n-k)x^{n-k-1} \quad \text{for } k = 0, 1, 2, \ldots, n.$$

Now, we can construct the matrix $[\alpha]_{B,B}$:

$$[\alpha]_{B,B} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & n \\ 0 & 0 & 0 & \cdots & 0 & n-1 \\ 0 & 0 & 0 & \cdots & 0 & n-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

where the last row corresponds to $\alpha(x^n) = 0$.

Thus, the matrix $[\alpha]_{B,B}$ is an upper triangular matrix with entries decreasing from n to 1 in the last column.

Problem 5. (Nilpotentcy) A linear transformation $T : V \to V$ is called *nilpotent* if $T^i = 0$ for some $i > 0$. If $\dim(V) = n < \infty$ and T is nilpotent, prove that $T^n = 0$.

Given that T is a nilpotent linear transformation on a finite-dimensional vector space V with $\dim(V) = n$, we need to show that $T^n = 0$.

Step 1: Consider the sequence of subspaces
Let $\{V_k\}$ be the sequence of subspaces defined as follows:

$$V_k = \text{Im}(T^k) \quad \text{for } k \geq 0,$$

where $V_0 = V$.

Since T is a linear transformation, it follows that $V_{k+1} = T(V_k)$.

Step 2: Analyze the dimensions of the subspaces
By the properties of linear transformations, we have:

$$\dim(V_{k+1}) \leq \dim(V_k).$$

This implies that the dimensions of the subspaces $\{V_k\}$ form a non-increasing sequence:

$$\dim(V_0) \geq \dim(V_1) \geq \dim(V_2) \geq \dots .$$

Since the dimension of V is finite (specifically n), the sequence must eventually stabilize. Therefore, there exists some integer $m \leq n$ such that:

$$\dim(V_k) = 0 \quad \text{for all } k \geq m.$$

Step 3: Show that $T^m = 0$
If $\dim(V_m) = 0$, then $V_m = \{0\}$, which means:

$$\text{Im}(T^m) = 0.$$

This implies that $T^m(v) = 0$ for all $v \in V$.

Since T is nilpotent, this means $T^i = 0$ for all $i \geq m$.

Step 4: Relate m and n
Now we know that if T is nilpotent, we can choose m such that $m \leq n$. In particular, we can take $m = n$ without loss of generality, thus:

$$T^n = 0.$$

Therefore, we conclude that if T is nilpotent and $\dim(V) = n < \infty$, then:

$$T^n = 0.$$

Problem 6. (Trace) Let n be a positive integer.

(a) Let $\text{tr} : \mathbb{F}^{n\times n} \to \mathbb{F}$ denote the trace function on $n \times n$ matrices over the field F. Show $\text{tr}(AB) = \text{tr}(BA)$ for any $A, B \in \mathbb{F}^{n\times n}$. Show that tr is never the zero linear functional.

To prove that $\text{tr}(AB) = \text{tr}(BA)$, we start by using the definition of the trace. The trace of a matrix is defined as the sum of its diagonal entries.

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be matrices in $\mathbb{F}^{n\times n}$.

The (i, j)-th entry of the matrix product AB is given by:

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Therefore, the trace of AB is:

$$\operatorname{tr}(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki} = \sum_{k=1}^{n} \sum_{i=1}^{n} a_{ik} b_{ki}.$$

Now consider BA: The (i, j)-th entry of BA is:

$$(BA)_{ij} = \sum_{k=1}^{n} b_{ik} a_{kj}.$$

Thus, the trace of BA is:

$$\operatorname{tr}(BA) = \sum_{i=1}^{n} (BA)_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ik} a_{ki} = \sum_{k=1}^{n} \sum_{i=1}^{n} b_{ik} a_{ki}.$$

By renaming indices in the summation, we see that:

$$\operatorname{tr}(AB) = \sum_{k=1}^{n} \sum_{i=1}^{n} a_{ik} b_{ki} = \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ik} a_{ki} = \operatorname{tr}(BA).$$

Thus, we have shown that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Show that tr is never the zero linear functional:
A linear functional $f : \mathbb{F}^{n \times n} \to \mathbb{F}$ is zero if $f(X) = 0$ for all $X \in \mathbb{F}^{n \times n}$. However, we can evaluate $\operatorname{tr}(I) = n$ where I is the identity matrix. Since $n \neq 0$ for positive integers, tr is never the zero functional.

(b) Show that there do not exist $A, B \in \mathbb{F}^{n \times n}$ for $\mathbb{F}$ the field of complex numbers, such that $AB - BA = I$. What happens for an arbitrary field?

Assume for the sake of contradiction that there exist $A, B \in \mathbb{F}^{n \times n}$ such that $AB - BA = I$.

Taking the trace of both sides yields:

$$\operatorname{tr}(AB - BA) = \operatorname{tr}(I).$$

Using the property of trace we showed earlier, we have:

$$\operatorname{tr}(AB) - \operatorname{tr}(BA) = 0.$$

Hence:

$$0 = \operatorname{tr}(I) = n.$$

This is a contradiction since $n \neq 0$.

Thus, no such matrices A and B exist in the field of complex numbers.

For an arbitrary field $\mathbb{F}$:
If $\mathbb{F}$ has characteristic p, the same argument holds unless $p = n$. If $n \equiv 0 \mod p$, it's possible to have a solution in that case. However, if $n \not\equiv 0 \mod p$, then the argument remains valid and $AB - BA = I$ cannot hold.

(c) Let $T : V \to V$ be a linear transformation where V is a finite dimenstional vector space over a field $\mathbb{F}$. Choose a basis B for V and define $\operatorname{tr}(T) = \operatorname{tr}([T]_{B,B})$. Show that the definition does not depend on the choice of basis B.

Let B_1 and B_2 be two different bases for V. The change of basis matrix from B_1 to B_2 is denoted by P, which is invertible.

The matrix representation of T with respect to B_1 is $[T]_{B_1}$ and with respect to B_2 is $[T]_{B_2}$. We have the relation:

$$[T]_{B_2} = P^{-1}[T]_{B_1}P.$$

The trace of a matrix has the property that:

$$\operatorname{tr}(P^{-1}AP) = \operatorname{tr}(A) \quad \text{for any square matrix } A.$$

Therefore, we get:

$$\operatorname{tr}([T]_{B_2}) = \operatorname{tr}(P^{-1}[T]_{B_1}P) = \operatorname{tr}([T]_{B_1}).$$

This shows that $\operatorname{tr}(T)$ is invariant under the choice of basis B.

(d) Let $f : \mathbb{F}^{n\times n} \to \mathbb{F}$ be a linear functional which satisfies $f(AB) = f(BA)$ for all $A, B \in \mathbb{F}^{n\times n}$. Prove that $f = a \cdot \operatorname{tr}$ for some scalar $a \in \mathbb{F}$. Further show that if the characterisitc of $\mathbb{F}$ is 0, then $f = \operatorname{tr}$ precisely when $f(1) = n$. What happens if the characteristic is not 0? Can you find another way decide if $f = \operatorname{tr}$ by computing a single value?

Since f is a linear functional that satisfies $f(AB) = f(BA)$, we can utilize the basis E_{ij} of matrix units, where $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ (the matrix with 1 in the (i, j) position and 0 elsewhere).

For any $A \in \mathbb{F}^{n\times n}$:

$$f(A) = f\left(\sum_{i,j} a_{ij}E_{ij}\right) = \sum_{i,j} a_{ij}f(E_{ij}).$$

By the property of f:

$$f(E_{ij}) = f(E_{ji}),$$

thus indicating that $f(E_{ij})$ depends only on the diagonal elements:

$$f(E_{ii}) = c_i \quad \text{and} \quad f(E_{ij}) = 0 \text{ for } i \neq j.$$

Therefore, we can express $f(A)$ as:

$$f(A) = \sum_{i=1}^{n} a_{ii}c_i,$$

indicating f can be written as:

$$f(A) = a \cdot \operatorname{tr}(A) \quad \text{where } a = c_1 + c_2 + \ldots + c_n.$$

To check if $f = \operatorname{tr}$, we calculate $f(1)$:

$$f(1) = c_1 + c_2 + \ldots + c_n = n \Rightarrow f = \operatorname{tr} \text{ if the characteristic of } \mathbb{F} \text{ is } 0.$$

If the characteristic of $\mathbb{F}$ is not 0, then the condition $f(1) = n$ may not uniquely determine f. In this case, computing $f(E_{ij})$ for $i = j$ gives us diagonal information but may allow for scalars to differ by multiples of the characteristic, hence leading to non-uniqueness.

One can also decide if $f = \operatorname{tr}$ by computing $f(E_{ii})$ for all i and checking whether all these values equal 1.

(e) Let $S = \operatorname{span}_{\mathbb{F}}(\{AB - BA : A, B \in \mathbb{F}^{n\times n}\})$. Prove that $S = \ker(\operatorname{tr})$. (Hint: Compute the dimension of $\ker(\operatorname{tr})$, and find a basis for S using some well-known matrices.)

First, we calculate $\ker(\operatorname{tr})$:

$$\ker(\operatorname{tr}) = \{X \in \mathbb{F}^{n\times n} : \operatorname{tr}(X) = 0\}.$$

The dimension of $\ker(\operatorname{tr})$ can be computed as follows. The space $\mathbb{F}^{n\times n}$ has dimension n^2, and the image of tr is $\mathbb{F}$, which has dimension 1. Therefore:

$$\dim(\ker(\operatorname{tr})) = n^2 - 1.$$

Now consider the space $S = \operatorname{span}_{\mathbb{F}}(\{AB - BA : A, B \in \mathbb{F}^{n\times n}\})$.

The matrices $AB - BA$ are trace-zero since:

$$\operatorname{tr}(AB - BA) = \operatorname{tr}(AB) - \operatorname{tr}(BA) = 0.$$

Thus, $S \subseteq \ker(\operatorname{tr})$.

To show $\ker(\operatorname{tr}) \subseteq S$:
Take any matrix $X \in \ker(\operatorname{tr})$ such that $\operatorname{tr}(X) = 0$. The matrices E_{ij} where $i \neq j$ yield $E_{ij}E_{ji} - E_{ji}E_{ij}$ for non-diagonal matrices. Since all such combinations can form a linear combination to construct any X in $\ker(\operatorname{tr})$, we conclude that every element of $\ker(\operatorname{tr})$ can be expressed as an element of S.

Hence, we conclude that $S = \ker(\operatorname{tr})$.