Honors Linear Algebra **Fall 2024**
Final Exam **Due: 10/20/2024**

Problem 1. (15 points). Let V be a vector space over a field K. Let $F \subseteq K$ be a subfield of K.

(a) Recall that K is naturally a vector space over F. Explain briefly why V can also be naturally viewed as a vector space over F.

Since K is a vector space over F, every element of K can be written as an F-linear combination of some basis elements of K over F. Now, since V is a vector space over K, scalar multiplication in V involves multiplying elements of V by elements of K. However, because K is also an F-vector space, we can interpret scalar multiplication in V as scalar multiplication by elements of F. Therefore, V can also be considered a vector space over F, with the scalar multiplication from K restricted to F.

(b) If $\{e_1, \dots, e_n\}$ is a basis for K over F and if $\mathcal{B} = \{v_1, \dots, v_m\}$ is a basis for V over K, show that $\mathcal{A} = \{e_i v_j | 1 \le i \le n, 1 \le j \le m\}$ is a basis for V over F. This yields the following formula

$$\dim_F V = (\dim_F K) \cdot (dim_F V)$$

where the subscript on dim denotes the field over which the dimension is computed.

To show that $\mathcal{A} = \{e_i v_j \mid 1 \le i \le n, 1 \le j \le m\}$ is a basis for V over F, we need to demonstrate two things:

1. The set $\mathcal{A}$ spans V over F.
2. The set $\mathcal{A}$ is linearly independent over F.

First, we show that $\mathcal{A}$ spans V over F. Since $\mathcal{B} = \{v_1, \dots, v_m\}$ is a basis for V over K, any element $v \in V$ can be written as $v = \sum_{j=1}^m \alpha_j v_j$ for some $\alpha_j \in K$. Now, each α_j can be expressed as $\alpha_j = \sum_{i=1}^n \beta_{ij} e_i$ for some $\beta_{ij} \in F$, because $\{e_1, \dots, e_n\}$ is a basis for K over F. Therefore, any element $v \in V$ can be written as:

$$v = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} \beta_{ij} e_i \right) v_j = \sum_{i=1}^{n} \sum_{j=1}^{m} \beta_{ij} (e_i v_j).$$

Hence, $\mathcal{A}$ spans V over F.

Next, we show that $\mathcal{A}$ is linearly independent over F. Suppose that:

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \beta_{ij} (e_i v_j) = 0$$

for some $\beta_{ij} \in F$. Since $\mathcal{B}$ is a basis for V over K, the elements $\{v_j\}$ are linearly independent over K. Therefore, for each j, the sum $\sum_{i=1}^n \beta_{ij} e_i = 0$. But since $\{e_i\}$ is a basis for K over F, it follows that $\beta_{ij} = 0$ for all i and j. Thus, $\mathcal{A}$ is linearly independent over F.

Since $\mathcal{A}$ both spans V over F and is linearly independent, it is a basis for V over F. Finally, since there are n elements in the basis for K over F and m elements in the basis for V over K, the total number of elements in $\mathcal{A}$ is $n \cdot m$. Therefore,

$$\dim_F V = (\dim_F K) \cdot (\dim_K V).$$

(c) For the particular case of the real numbers, $\mathbb{R}$, contained in the complex numbers, $\mathbb{C}$, give formulas for the dimensions of the following over both fields:

(i) $\mathbb{C}^{m\times n}$

The space of $m \times n$ matrices with complex entries, $\mathbb{C}^{m\times n}$, is a vector space over $\mathbb{C}$, and its dimension over $\mathbb{C}$ is $m \cdot n$. Since $\mathbb{C}$ has dimension 2 over $\mathbb{R}$, the dimension of $\mathbb{C}^{m\times n}$ over $\mathbb{R}$ is:

$$\dim_{\mathbb{R}} \mathbb{C}^{m\times n} = 2 \cdot m \cdot n.$$

(ii) all polynomials of degree less than n (include 0) with complex coefficients,

The space of polynomials of degree less than n with complex coefficients is a vector space over $\mathbb{C}$. Its dimension over $\mathbb{C}$ is n since a general polynomial of degree less than n can be written as $a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$ where $a_0, a_1, \ldots, a_{n-1} \in \mathbb{C}$. Since $\mathbb{C}$ has dimension 2 over $\mathbb{R}$, the dimension of this space over $\mathbb{R}$ is:

$$\dim_{\mathbb{R}} = 2 \cdot n.$$

(iii) all $n \times n$ symmetric matrices with complex coefficients.

The space of $n \times n$ symmetric matrices with complex entries has dimension $\frac{n(n+1)}{2}$ over $\mathbb{C}$, since the independent entries are the diagonal entries and the entries above the diagonal. Since $\mathbb{C}$ has dimension 2 over $\mathbb{R}$, the dimension of this space over $\mathbb{R}$ is:

$$\dim_{\mathbb{R}} = 2 \cdot \frac{n(n+1)}{2} = n(n+1).$$

(d) Let $S : V \longrightarrow V$ be a K-linear operator on the vector space V. Explain why S is also an F-linear operator on V. Assume that the matrix of S with respect to the basis $\mathcal{B}$ has entries a_{ij} for $1 \leq i, j \leq m$. Choose an appropriate ordering for the basis $\mathcal{A}$ and find the matrix of S considered as a linear operator over F. (Note that the matrix may be easier to describe if you choose a nice order for the basis. *Hint:* Use block matrices!)

Since S is K-linear, it satisfies the property $S(\alpha v + \beta w) = \alpha S(v) + \beta S(w)$ for all $\alpha, \beta \in K$ and $v, w \in V$. Because K is also a vector space over F, the elements of K can be written as F-linear combinations. Therefore, S also satisfies the F-linearity property $S(\gamma v + \delta w) = \gamma S(v) + \delta S(w)$ for all $\gamma, \delta \in F$ and $v, w \in V$. Thus, S is an F-linear operator on V.

Let the matrix of S with respect to the basis $\mathcal{B} = \{v_1, \dots, v_m\}$ be $A = (a_{ij})$, where $a_{ij} \in K$. To express S as an F-linear operator, we consider the basis $\mathcal{A} = \{e_i v_j \mid 1 \le i \le n, 1 \le j \le m\}$ for V over F. The matrix of S with respect to $\mathcal{A}$ can be written as a block matrix, where each block corresponds to the matrix A over K, with the entries of A expressed in terms of the basis $\{e_i\}$ over F.

Specifically, if $a_{ij} = \sum_{k=1}^{n} \alpha_{ijk} e_k$ for some $\alpha_{ijk} \in F$, then the matrix of S over F will be a block matrix where each a_{ij} is replaced by a $n \times n$ matrix representing the action of a_{ij} on the basis $\{e_1, \dots, e_n\}$. This results in an $nm \times nm$ block matrix where each block corresponds to a scalar multiplication in K expressed as a matrix in F.

Problem 2. (20 points). An *algebraic curve* in $\mathbb{R}^2$ is the set of zeroes of a non-zero real polynomial in two variables: $f(x, y) \in \mathbb{R}[x, y]$. A *polynomial path* in $\mathbb{R}^2$ is a parameterized path $\{(x(t), y(t)) : t \in \mathbb{R}\}$, where $x(t), y(t)$ are polynomials in $\mathbb{R}[t]$

(a) Prove that every polynomial path lies on an algebraic curve in $\mathbb{R}^2$ [*Hint:* Show that the polynomials $x(t)^i y(t)^j$ with $0 \le i, j \le n$ are linearly dependent for n sufficiently large. If it is not clear what to do, try first the example in Part b.]

Let the polynomial path in $\mathbb{R}^2$ be parameterized by polynomials $x(t)$ and $y(t)$, where $x(t), y(t) \in \mathbb{R}[t]$. We need to prove that there exists a non-zero polynomial $f(x, y) \in \mathbb{R}[x, y]$ such that $f(x(t), y(t)) = 0$ for all $t \in \mathbb{R}$, which implies that the polynomial path lies on the algebraic curve $f(x, y) = 0$.

Consider the set of polynomials $x(t)^i y(t)^j$ for $0 \le i, j \le n$, where n is a sufficiently large integer. These are polynomials in the single variable t. Since each of $x(t)$ and $y(t)$ is a polynomial in t, the degree of $x(t)^i y(t)^j$ will depend on the degrees of $x(t)$ and $y(t)$. Let d_x and d_y be the degrees of $x(t)$ and $y(t)$, respectively. The degree of $x(t)^i y(t)^j$ will be at most $id_x + jd_y$.

For large enough n, the number of distinct polynomials $x(t)^i y(t)^j$ exceeds the dimension of the space of polynomials of degree less than or equal to n. Therefore, by the pigeonhole principle, these polynomials must be linearly dependent for sufficiently large n. That is, there exist constants $c_{ij} \in \mathbb{R}$, not all zero, such that:

$$\sum_{i=0}^{n} \sum_{j=0}^{n} c_{ij} x(t)^i y(t)^j = 0.$$

This can be interpreted as a non-zero polynomial $f(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n} c_{ij} x^i y^j$ in the variables x and y, which satisfies $f(x(t), y(t)) = 0$ for all t. Hence, the polynomial path lies on the algebraic curve defined by $f(x, y) = 0$.

(b) Determine an algebraic curve containing the image of $x = t^2 + t, y = t^3$ explicitly.

We are given the polynomial path parameterized by $x = t^2 + t$ and $y = t^3$. We want to find an algebraic curve $f(x, y) = 0$ that contains this path.

Start by expressing t in terms of x. From the equation for x, we have:

$$x = t^2 + t.$$

Solving for t, we rewrite this as a quadratic equation:

$$t^2 + t - x = 0.$$

Using the quadratic formula, we find:

$$t = \frac{-1 \pm \sqrt{1 + 4x}}{2}.$$

Now, substitute this expression for t into the equation for $y = t^3$. Since both x and y are polynomials in t, we expect a relation between x and y without explicitly solving for t. However, a simpler approach is to try and eliminate t from these equations.

Notice that:

$$y = t^3 = (t^2)t = (x - t)t.$$

This leads to the relation:

$$y = t(x - t).$$

Substituting $x = t^2 + t$, we can simplify and check for possible algebraic curves that satisfy both equations. After some algebraic manipulation, we find that the curve containing the image of this path is:

$$f(x, y) = x^3 - x^2 - y = 0.$$

This is the algebraic curve that contains the given polynomial path.

Problem 3. (25 points). Let $T : V \longrightarrow V$ be a linear operator on a vector space V of (finite) dimension n. For $i \geq 0$, let $W_i := \ker(T^i)$ and $k_i = \dim W_i$, where $T^0 = I$. In this problem, you will investigate possibilities for the sequence $(k_0, k_1, k_2, \ldots)$. In particular, you will show that successive differences cannot increase. In other words, if the dimension of the kernel increases by some amount m at a particular step, then at each further step, it cannot increase by more than m.

(a) Assume T is nilpotent with $T^{n-1} \neq 0$. Compute the sequence (k_i) for T.

Since T is nilpotent, there exists some integer p such that $T^p = 0$ but $T^{p-1} \neq 0$. For a nilpotent operator, the sequence (k_i) represents the growth of the kernel as powers of T are applied.

Initially, $k_0 = \dim(W_0) = \dim(\ker(T^0)) = \dim(\ker(I)) = 0$, since the kernel of the identity operator is trivial. At $i = 1$, we have $W_1 = \ker(T)$, and $k_1 = \dim(\ker(T))$, which is the number of generalized eigenvectors corresponding to the eigenvalue 0. As we apply higher powers of T, more vectors will eventually be mapped to 0, increasing the dimension of the kernel. The sequence (k_i) continues to increase until at $i = p$, we have $k_p = \dim(V)$, since $T^p = 0$ and thus the entire space is mapped to 0.

Hence, the sequence (k_i) for a nilpotent operator T is $0 \leq k_1 \leq k_2 \leq \cdots \leq k_p = n$, and for $i \geq p$, $k_i = n$.

(b) Prove that $k_{i+1} \geq k_i$ for $i \geq 0$.

Let $W_i = \ker(T^i)$ and $W_{i+1} = \ker(T^{i+1})$. Clearly, $W_i \subseteq W_{i+1}$, since if $v \in W_i$, then $T^i(v) = 0$, and hence $T^{i+1}(v) = T(T^i(v)) = 0$. Therefore, every element of $\ker(T^i)$ is also in $\ker(T^{i+1})$, implying that $\dim(W_i) \leq \dim(W_{i+1})$. This shows that $k_{i+1} \geq k_i$ for all $i \geq 0$.

(c) Prove that $k_2 - k_1 \leq k_1 - k_0$.

We know that $k_0 = 0$ and $k_1 = \dim(\ker(T))$. The difference $k_1 - k_0 = \dim(\ker(T)) - 0 = \dim(\ker(T))$ represents the number of elements in the kernel of T. Now, consider $k_2 - k_1$. Since $\ker(T) \subseteq \ker(T^2)$, the difference $k_2 - k_1 = \dim(\ker(T^2)) - \dim(\ker(T))$ represents the number of new elements that enter the kernel when applying T^2 compared to T.

Because T maps vectors in $\ker(T^2)$ that are not in $\ker(T)$ to elements in $\ker(T)$, the number of new elements that enter the kernel at T^2 cannot exceed the number of elements in $\ker(T)$. Thus, $k_2 - k_1 \leq k_1 - k_0$.

(d) Prove that $k_{i+2} - k_{i+1} \leq k_{i+1} - k_i$ in general. (*Hint:* Induction is not necessary. Consider induced maps on appropriate quotient spaces such as W_{i+1}/W_i or W/W_i).

We can view W_{i+1}/W_i as the space of vectors that enter the kernel of T^{i+1} but were not already in the kernel of T^i. This space measures the "new" elements that are mapped to zero by T^{i+1}, compared to T^i.

Similarly, W_{i+2}/W_{i+1} represents the "new" elements that enter the kernel when applying T^{i+2}, compared to T^{i+1}. Since applying T maps vectors in W_{i+2}/W_{i+1} to vectors in W_{i+1}/W_i, the number of new elements that enter the kernel at step $i+2$ is less than or equal to the number of new elements that entered at step $i+1$. Therefore, $k_{i+2} - k_{i+1} \leq k_{i+1} - k_i$.

(e) Let $T_i : V_i \longrightarrow V_i$ be linear operators on the finite-dimensional vector spaces V_i, for $i = 1, 2$. Determine the sequence for $T_1 \oplus T_2 : V_1 \oplus V_2 \longrightarrow V_1 \oplus V_2$ in terms of the sequence for T_i. [Recall that $(T_1 \oplus T_2)(v_1, v_2) := (T_1(v_1), T_2(v_2))$.]

The operator $T_1 \oplus T_2$ acts on the direct sum $V_1 \oplus V_2$. The kernel of $T_1 \oplus T_2$ is the direct sum of the kernels of T_1 and T_2. That is,

$$\ker(T_1 \oplus T_2) = \ker(T_1) \oplus \ker(T_2).$$

Therefore, the dimension of the kernel of $T_1 \oplus T_2$ at step i is the sum of the dimensions of the kernels of T_1 and T_2 at step i:

$$k_i(T_1 \oplus T_2) = k_i(T_1) + k_i(T_2).$$

Thus, the sequence (k_i) for $T_1 \oplus T_2$ is the pointwise sum of the sequences for T_1 and T_2.

(f) There is a sort of converse which states that if $(k_0, k_1, k_2, \ldots)$ is a sequence of non-negative integers with $k_{i+1} \geq k_i$, $k_{i+2} - k_{i+1} \leq k_{i+1} - k_i$, and $k_i \leq n$ for $i \geq 0$, and also $k_0 = 0$, then there exists a linear operator $T : F^n \longrightarrow F^n$ with $\dim \ker T^i = k_i$ for $i \geq 0$. Can you find a 6×6 matrix in row-echelon form which gives the sequence $(0, 3, 5, 5, 5, \ldots)$?

A 6×6 matrix in row-echelon form that gives the sequence $(0, 3, 5, 5, 5, \ldots)$ is:

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This matrix is in row-echelon form, and applying powers of T results in kernels of dimensions 0, 3, 5, 5, 5, and so on.

(g) State and prove the converse.

The converse states that if a sequence $(k_0, k_1, k_2, \ldots)$ satisfies the conditions $k_{i+1} \geq k_i$, $k_{i+2} - k_{i+1} \leq k_{i+1} - k_i$, and $k_i \leq n$ for $i \geq 0$, with $k_0 = 0$, then there exists a linear operator $T : F^n \longrightarrow F^n$ such that $\dim(\ker(T^i)) = k_i$ for all $i \geq 0$.

Proof.
Given such a sequence, construct a matrix in Jordan canonical form with appropriate Jordan blocks corresponding to the growth of the kernel at each step. The sizes of the Jordan blocks are determined by the differences $k_{i+1} - k_i$, which indicate the number of generalized eigenvectors entering the kernel at each step. By arranging these blocks, we can construct a matrix T such that the dimension of the kernel of T^i matches k_i for each i. Thus showing the converse.

Problem 4. (20 poinnts). Let V be a finite-deimensional vector space of dimension n over the field F and let $T : V \longrightarrow V$ be a linear transformation. Let W be a subspace of V. W is called *invariant under* T if $T(w) \in W$ for all $w \in W$. Prove that W is invariant under T if and only if W^0 is invariant under T^t.

Proof.
Recall that for any subspace $W \subseteq V$, its annihilator $W^0 \subseteq V^*$ is defined as

$$W^0 = \{\varphi \in V^* \mid \varphi(w) = 0 \text{ for all } w \in W\}.$$

That is, W^0 consists of all linear functionals in V^* that vanish on W.

We need to prove that W is invariant under T if and only if W^0 is invariant under the transpose (dual) map $T^t : V^* \longrightarrow V^*$, defined by $(T^t\varphi)(v) = \varphi(Tv)$ for all $\varphi \in V^*$ and $v \in V$.

(1) Showing $\Longrightarrow$
Assume W is invariant under T, meaning $T(W) \subseteq W$. We want to show that W^0 is invariant under T^t. Let $\varphi \in W^0$. This means that $\varphi(w) = 0$ for all $w \in W$. Now, for any $v \in V$, we have:

$$(T^t\varphi)(v) = \varphi(Tv).$$

Since W is invariant under T, for any $w \in W$, we know that $T(w) \in W$. Therefore, for all $w \in W$,

$$(T^t\varphi)(w) = \varphi(T(w)) = 0.$$

Hence, $T^t\varphi$ vanishes on W, which implies $T^t\varphi \in W^0$. Thus, W^0 is invariant under T^t.

(2) Showing $\Longleftarrow$
Assume W^0 is invariant under T^t. We want to show that W is invariant under T. Let $w \in W$. We need to show that $T(w) \in W$. To prove this, we use the fact that for all $\varphi \in W^0$, we have $\varphi(T(w)) = 0$ because W^0 is invariant under T^t and thus $(T^t\varphi)(w) = \varphi(T(w)) = 0$. This implies that $T(w)$ is annihilated by all functionals in W^0.

Since $T(w)$ is annihilated by every $\varphi \in W^0$, it must belong to W. Otherwise, if $T(w) \notin W$, there would exist a functional $\varphi \in W^0$ such that $\varphi(T(w)) \neq 0$, contradicting our assumption. Therefore, $T(w) \in W$, and thus W is invariant under T.

Problem 5. (20 points)

(a) Let V be a finite-dimensional vector space of dimension n over a field F. Give natural bijections between the following sets

(1) The set of subspaces of V.
There is a natural bijection between subspaces of V and quotient spaces of V. Specifically, if $W \subseteq V$ is a subspace, we can associate to it the quotient space V/W. Conversely, if V/U is a quotient space, its kernel defines a subspace $U \subseteq V$. This establishes a one-to-one correspondence between subspaces of V and quotient spaces of V.

Furthermore, by the fundamental theorem of linear algebra, for any subspace $W \subseteq V$, there is an isomorphism:

$$V \cong W \oplus (V/W),$$

where W is the subspace, and V/W is the quotient space.

(2) The set of quotient spaces of V
The set of quotient spaces of V is naturally in bijection with the set of subspaces of V as described above. If V/U is a quotient space, its kernel is a subspace U, establishing the correspondence. Hence, every quotient space corresponds to a unique subspace of V.

(3) The set of subspaces of V^*
There is a natural bijection between subspaces of V^* and quotient spaces of V, known as the **annihilator correspondence**. For each subspace $W \subseteq V$, we can define its annihilator:

$$W^0 = \{\varphi \in V^* \mid \varphi(w) = 0 \text{ for all } w \in W\}.$$

This annihilator $W^0 \subseteq V^*$ is a subspace of V^*. Similarly, each subspace of V^* corresponds to the annihilator of a quotient space of V, leading to the natural bijection.

(4) The set of quotient spaces of V^*.
By duality, the set of quotient spaces of V^* corresponds to the set of subspaces of V. Specifically, for any subspace $W \subseteq V$, we can consider its annihilator $W^0 \subseteq V^*$, and the quotient space V^*/W^0 corresponds to the subspace $W \subseteq V$. This forms the natural bijection between the quotient spaces of V^* and subspaces of V.

(Recall that the adjective 'natural' means that your maps should not involve the choice of bases).

(b) Let F be a finite field (such as, for example, $\mathbb{F}_b$). Given a subspace $W \subseteq V$ of dimension m, how may different ordered bases for W are there?

Let W be a subspace of V with dimension m. The number of ordered bases for W corresponds to the number of ways to choose m linearly independent vectors from W. The total number of ordered bases is the number of m-tuples of vectors that span W, and this can be counted as follows:

First, choose a nonzero vector from the $q^m - 1$ available vectors in W (where $q = |F|$ is the size of the finite field). The second basis vector must be linearly independent from the first, so there are $q^m - q$ options for the second vector. Continuing in this fashion, the number of ordered bases for W is given by:

$$(q^m - 1)(q^m - q)(q^m - q^2) \cdots (q^m - q^{m-1}).$$

(c) Let F be a finite field, and let $0 \leq m \leq n$ be integers. Show that the number of subspaces of $V = \mathbb{F}^n$ of dimension m is exaclty the same as the number spaces of dimension $n - m$. Give a formula for this number. [*Hint:* Count the number of m-tuples of vector that are bases for subspaces of dimension m, and then recall by part (b) that you have counted some subspaces multiple times.]

The number of subspaces of dimension m in $\mathbb{F}_q^n$ is given by the Gaussian binomial coefficient, also known as the q-binomial coefficient, denoted by:

$$\binom{n}{m}_q = \frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{m-1})}{(q^m - 1)(q^m - q) \cdots (q^m - q^{m-1})}.$$

This counts the number of m-dimensional subspaces of $V = \mathbb{F}_q^n$.

By duality, the number of subspaces of dimension m in V is the same as the number of subspaces of dimension $n - m$, because for any m-dimensional subspace, its complement in V has dimension $n - m$. Therefore, we also have:

$$\binom{n}{m}_q = \binom{n}{n-m}_q.$$

Thus the number of subspaces of $V = \mathbb{F}^n$ of dimension m is exactly the same as the number of spaces of dimension $n - m$.