Honors Algebra **Spring 2025**
HW 7: Groups, Structures of Abelian Groups, Automorphisms **3/25/2025**

Abstract Algebra: An Integrated Approach by J.H. Silverman.
Page 393-396: 12.10, 12.11, 12.12, 12.14, 12.15, 12.20, 12.26

Problem 1 (12.10)**.** For each $pi \in \mathcal{S}_n$, we define a linear transformation $\rho(\pi) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by

$$\rho(\pi)(e_i) = e_{\pi(i)}$$

By abuse of notation, we write $\rho(\pi)$ for the n-by-n matrix of $\rho(\pi)$ relative to the basis $\{e_1, \ldots, e_n\}$. The matrix $\rho(\pi)$, whose entries are all equal to 0 or 1, is called the permutation matrix associated to π.

(a) Prove that the matrix $\rho(\pi)$ can also be described by the following formula:

$$(i,j)\text{-entry of } \rho(\pi) = \begin{cases} 1 & \text{if } \pi(j) = i, \\ 0 & \text{if } \pi(j) \neq i \end{cases}$$

(b) Write down the six 3-by-3 matrices corresponding to the six elements of $\mathcal{S}_3$.

(c) Prove that every row of $\rho(\pi)$ has exactly one entry equal to 1 and similarly that every column of $\rho(\pi)$ has exactly one entry equal to 1.

(d) Prove that the map

$$\rho : \mathcal{S}_n \longrightarrow \mathrm{GL}_n \mathbb{Z}$$

is an injective group homomorphism; i.e., prove that ρ is injective and satisfies

$$\rho(\pi_1\pi_2) = \rho(\pi_i)\rho(\pi_2) \text{ for all } \pi_1, \pi_2 \in \mathcal{S}_n$$

(e) Let $\pi \in \mathcal{S}_n$. Prove that

$$\text{sign}(\pi) = \det \rho(\pi)$$

(f) Prove that the eigenvalues of $\rho(\pi)$ are roots of unity.

Problem 2 (12.11)**.** Let G be a group, let $H \subset G$ be a subgroup of G, and let G/H be the collection of cosets of H.

(a) Prove that there is a well-defined group homomorphism

$$\pi_H : G \longrightarrow \mathcal{S}_{G/H}, \;\; \pi_H(g)(aH) = gaH$$

(*Hint.* Generalize the proof of Cayley's Theorem, which was the case $H = \{e\}$.)

(b) Prove that the kernel of π_H is contained in H.

(c) Let K be a normal subgroup of G, and suppose that K is contained in H. Prove that $K \subseteq \ker(\pi_H)$. Thus we may describe $\ker(\pi_H)$ as the largest normal subgroup of G that is contained in H.

Problem 3 (12.12)**.** Let G be a group of order n, and let $H \nsubseteq G$ be a subgroup of order m, and suppose that n does nto divide $(n/m)!$. Prove that H contains a non-trivial normal subgroup of G. (*Hint.* Use Exercise 12.11.)

Problem 4 (12.14)**.** Let G be a group. We say that a normal subgroup $N \nsubseteq G$ is a maximal normal subgroup of G if it has the following property:

$$N \subseteq N' \subseteq G \text{ and } N' \text{ normal in } G \quad\Longrightarrow\quad N' = N \text{ or } N' = G$$

Prove that

$$N \text{ is a maximal normal subgroup} \quad\Longleftrightarrow\quad G/N \text{is a simple group}$$

Problem 5 (12.15)**.** For this exercise, you may use the first part of Sylow's Theorem that says that p-Sylow subgroups exist but do not use the second and third parts.

(a) Let p and q be distinct primes, and let G be a group of order pq. Prove that G is not a simple group.

(b) Prove that a group of order 36 is not a simple group.

(c) Let p and q be distinct primes, let $i, j \geq 1$, and suppose that $p^i < q$. Prove that a group of order $p^i < q^j$ is not a simple group.

(*Hint.* Use Exercise 12.12. We mention that we proved (a) using all three parts of Sylow's Theorem in Example 6.37.)

Problem 6 (12.20)**.**

(a) Prove that $\mathrm{Aut}(\mathbb{Z})$ is a cyclic group of order 2.

(b) More generally, prove that $\mathrm{Aut}(\mathbb{Z}^{\ltimes}) \cong \mathrm{GL}_n(\mathbb{Z})$.

Problem 7 (12.26)**.** Let G be the set of affine linear maps on $\mathbb{R}$, which by definition is the group

$$G = \{\text{maps } \phi : \mathbb{R} \longrightarrow \mathbb{R} \text{ of the form } \phi(x) = ax + b \text{ for some } a \in \mathbb{R}^* \text{ and } b \in \mathbb{R}\}$$

(a) Prove that G is group, where the group law is composition of maps.

(b) Prove that G is isomorphic to the semidirect product $\mathbb{R} \rtimes \mathbb{R}^*$. Be sure to describe the homomorphism from $\mathbb{R}^*$ to $\mathrm{Aut}(\mathbb{R})$ used to define the semidirect product.