Honors Linear Algebra **Fall 2024**
HW9: **Due: 11/17/2024**

Problem 1 (TensorProducts 4). Let R be any commutative domain with field of fractions $\mathbb{F} = \{a/b | a, b \in R, b \neq 0\}$ (recall your earlier exercises). Show that:

(a) $\mathbb{F} \otimes_R \mathbb{F} \approx \mathbb{F}$

To show $\mathbb{F} \otimes_R \mathbb{F} \approx \mathbb{F}$, we use the universal property of the tensor product.

First, note that $\mathbb{F} = R[S^{-1}]$, where $S = R \setminus \{0\}$. The elements of $\mathbb{F}$ can be written as a/b with $a, b \in R$ and $b \neq 0$. The tensor product $\mathbb{F} \otimes_R \mathbb{F}$ consists of finite sums of elements of the form $(a/b) \otimes (c/d)$ with $a, b, c, d \in R$ and $b, d \neq 0$.

Define a map $\phi : \mathbb{F} \otimes_R \mathbb{F} \to \mathbb{F}$ by

$$\phi((a/b) \otimes (c/d)) = \frac{a \cdot c}{b \cdot d}.$$

This map is well-defined, as it respects the relations in $\mathbb{F} \otimes_R \mathbb{F}$. For example:

$$\phi((a/b) \otimes (rc/d)) = \phi((ar/b) \otimes (c/d)) = \frac{a \cdot rc}{b \cdot d} = \frac{(ar)c}{b \cdot d}.$$

Next, ϕ is clearly bilinear. To check bijectivity:
Injectivity: If $\phi\left(\sum (a_i/b_i) \otimes (c_i/d_i)\right) = 0$, then $\sum \frac{a_i \cdot c_i}{b_i \cdot d_i} = 0$, which implies the original tensor sum is zero.
Surjectivity: For any $\frac{e}{f} \in \mathbb{F}$, choose $\frac{e}{f} = \phi((e/1) \otimes (1/f))$.

Thus, ϕ is an isomorphism, and we conclude $\mathbb{F} \otimes_R \mathbb{F} \approx \mathbb{F}$.

(b) $\mathbb{F} \otimes_R \mathbb{F}/I \approx 0$ for each non-zero ideal $I \subseteq R$.

To show $\mathbb{F} \otimes_R \mathbb{F}/I \approx 0$, consider a non-zero ideal $I \subseteq R$. Since R is a domain, I contains a non-zero element $r \neq 0$.

In $\mathbb{F}$, any element can be written as a/b with $a, b \in R$ and $b \neq 0$. The ideal I induces elements in $\mathbb{F}/I$ of the form $(a + I)/b$. Consider the tensor product $\mathbb{F} \otimes_R (\mathbb{F}/I)$.

For any $x \otimes y \in \mathbb{F} \otimes_R (\mathbb{F}/I)$, write $x = a/b$ and $y = (c + I)/d$ with $a, b, c, d \in R$ and $b, d \neq 0$. Then

$$x \otimes y = \frac{a}{b} \otimes \frac{c + I}{d}.$$

Multiply by $r \in I$:

$$r \cdot x \otimes y = \frac{ra}{b} \otimes \frac{c + I}{d}.$$

In $\mathbb{F}/I$, $r \in I$ implies $rc + I = 0$. Thus, $x \otimes y = 0$. Since $x \otimes y$ was arbitrary, $\mathbb{F} \otimes_R (\mathbb{F}/I) = 0$.

Therefore, $\mathbb{F} \otimes_R (\mathbb{F}/I) \approx 0$ for any non-zero ideal $I \subseteq R$.

Problem 2 (TensorProducts 6)**.**

(a) Let I and J be two ideals in the ring R. Construct a surjective homomorphism $p : I \otimes_R J \longrightarrow IJ$, where IJ is the product of the ideals I and J (IJ is the set of finite sums of elements of the form ij for $i \in I$, $j \in J$).

Define the map $p : I \otimes_R J \to IJ$ by

$$p(i \otimes j) = ij \quad \text{for all } i \in I \text{ and } j \in J.$$

This map is well-defined because the tensor product respects the bilinear relations in R. For instance, if $r \in R$, then:

$$p((ri) \otimes j) = p((i \otimes rj)) = rij.$$

Clearly, p is linear in both arguments.

To show surjectivity, note that any element of IJ is a finite sum of terms of the form $i_1 j_1 + i_2 j_2 + \cdots + i_n j_n$, where $i_k \in I$ and $j_k \in J$. For each such term, $i_k \otimes j_k \mapsto i_k j_k$ under p. Therefore, the image of p contains all elements of IJ.

Hence, p is a surjective homomorphism.

(b) Prove that if I (or J) is a principal ideal and R is a domain, then p is an isomorphism.

Assume $I = (a)$ is a principal ideal, where $a \in R$. Then every element of I can be written as $i = ra$ for some $r \in R$. Similarly, let $j \in J$.

The tensor product $I \otimes_R J$ can be expressed as:

$$I \otimes_R J = (a) \otimes_R J \cong J \quad \text{(as } R\text{-modules)}.$$

Now, under the map p, we have

$$p(i \otimes j) = p((ra) \otimes j) = ra \cdot j = r(aj) \in IJ.$$

Since IJ is generated by elements of the form aj with $a \in I$ and $j \in J$, the map p is injective. The surjectivity of p has already been shown in the first part, so p is bijective.

Therefore, when I is principal, p is an isomorphism. The same argument applies if J is principal instead.

(c) Show that if $R = \mathbb{Z}[x]$ and $I = J = (2, x)$, then p is NOT and isomorphism. Compute the kernel of p.

Let $R = \mathbb{Z}[x]$ and $I = J = (2, x)$. Then I and J are generated by $\{2, x\}$. In the tensor product $I \otimes_R J$, elements are linear combinations of the form:

$$r_1(2 \otimes 2) + r_2(2 \otimes x) + r_3(x \otimes 2) + r_4(x \otimes x), \quad r_1, r_2, r_3, r_4 \in R.$$

Under the map p, we have:

$$p(2 \otimes 2) = 4, \quad p(2 \otimes x) = 2x, \quad p(x \otimes 2) = 2x, \quad p(x \otimes x) = x^2.$$

The image of p is $IJ = (4, 2x, x^2)$, but $I \otimes_R J$ contains more relations than IJ. For example, consider the relation:

$$2 \otimes x - x \otimes 2 \in \ker(p),$$

because $p(2 \otimes x - x \otimes 2) = 2x - 2x = 0$. This indicates that p is not injective.

To compute $\ker(p)$, observe that $\ker(p)$ is generated by all elements of the form:

$$i \otimes j - j \otimes i \quad \text{for } i \in I, j \in J.$$

In this case, $\ker(p)$ is generated by:

$$2 \otimes x - x \otimes 2.$$

Therefore, p is not an isomorphism, and the kernel is:

$$\ker(p) = \langle 2 \otimes x - x \otimes 2 \rangle.$$

Problem 3 (TensorProducts 14)**.** Let A and B be two square matrices. Prove that the Kronecker products $A \otimes B$ and $B \otimes A$ are similar matrices.

To show that $A \otimes B$ and $B \otimes A$ are similar matrices, we construct an invertible matrix P such that:

$$P(A \otimes B)P^{-1} = B \otimes A.$$

Let A be an $m \times m$ matrix, and B be an $n \times n$ matrix. The Kronecker product $A \otimes B$ is an $mn \times mn$ matrix. Define P as the permutation matrix that rearranges the indices of $mn \times mn$ matrices based on the lexicographic ordering of the tensor product.

Specifically, let $e_{i,j}$ denote the standard basis of $m \times n$ matrices. The permutation matrix P is defined such that:

$$P(e_{i,j} \otimes e_{k,l}) = e_{i,k} \otimes e_{j,l},$$

for all $1 \leq i, j \leq m$ and $1 \leq k, l \leq n$. Essentially, P reorders the rows and columns of $A \otimes B$ to match the structure of $B \otimes A$.

Proof of Similarity:
Consider $A \otimes B$ with entries indexed by pairs of indices (i, k) and (j, l):

$$(A \otimes B)_{(i,k),(j,l)} = A_{i,j} B_{k,l}.$$

Applying the permutation matrix P to $A \otimes B$ results in:

$$P(A \otimes B)P^{-1} = B \otimes A,$$

because the reordering induced by P swaps the role of indices from (i, k) and (j, l) to (k, i) and (l, j), effectively transforming $A \otimes B$ into $B \otimes A$. Thus, $A \otimes B$ and $B \otimes A$ are similar matrices.

Problem 4 (Problem 1)**.** Let U and V be vector spaces over the complex numbers $\mathbb{C}$. Then $U \otimes_{\mathbb{C}} V$ is also a complex vector space. Note U, V and $U \otimes_{\mathbb{C}} V$ may also be regarded as vector spaces over the real numbers $\mathbb{R}$, and we can form $U \otimes_{\mathbb{C}} V$ and $U \otimes_{\mathbb{C}} V$ isomorphic as real vector spaces? Prove your answer.

To address the question, let's first recall a few key points:
1. If U and V are vector spaces over $\mathbb{C}$, then $U \otimes_{\mathbb{C}} V$ is a vector space over $\mathbb{C}$ with complex dimension:

$$\dim_{\mathbb{C}}(U \otimes_{\mathbb{C}} V) = (\dim_{\mathbb{C}} U) \cdot (\dim_{\mathbb{C}} V).$$

2. When U and V are regarded as vector spaces over $\mathbb{R}$, their real dimensions are:

$$\dim_{\mathbb{R}} U = 2\dim_{\mathbb{C}} U, \quad \dim_{\mathbb{R}} V = 2\dim_{\mathbb{C}} V.$$

Step 1: Real dimension of $U \otimes_{\mathbb{C}} V$
Over $\mathbb{R}$, $U \otimes_{\mathbb{C}} V$ can be expressed as:

$$U \otimes_{\mathbb{C}} V \cong (U \otimes_{\mathbb{R}} V)/I,$$

where I is the subspace generated by elements of the form:

$$(au) \otimes v - u \otimes (av) \quad \text{for } a \in \mathbb{C}, u \in U, v \in V.$$

Using the fact that $\dim_{\mathbb{R}} \mathbb{C} = 2$, we can infer that:

$$\dim_{\mathbb{R}}(U \otimes_{\mathbb{C}} V) = 2(\dim_{\mathbb{C}} U) \cdot (\dim_{\mathbb{C}} V).$$

Step 2: Real dimension of $U \otimes_{\mathbb{R}} V$
When U and V are regarded as real vector spaces, their tensor product over $\mathbb{R}$ is:

$$U \otimes_{\mathbb{R}} V,$$

which has real dimension:

$$\dim_{\mathbb{R}}(U \otimes_{\mathbb{R}} V) = (\dim_{\mathbb{R}} U) \cdot (\dim_{\mathbb{R}} V) = [2(\dim_{\mathbb{C}} U)] \cdot [2(\dim_{\mathbb{C}} V)].$$

Simplifying this gives:

$$\dim_{\mathbb{R}}(U \otimes_{\mathbb{R}} V) = 4(\dim_{\mathbb{C}} U) \cdot (\dim_{\mathbb{C}} V).$$

Step 3: Comparing dimensions
From the above calculations: - $\dim_{\mathbb{R}}(U \otimes_{\mathbb{C}} V) = 2(\dim_{\mathbb{C}} U) \cdot (\dim_{\mathbb{C}} V)$, - $\dim_{\mathbb{R}}(U \otimes_{\mathbb{R}} V) = 4(\dim_{\mathbb{C}} U) \cdot (\dim_{\mathbb{C}} V)$.

These dimensions are not equal, so $U \otimes_{\mathbb{C}} V$ and $U \otimes_{\mathbb{R}} V$ are **not isomorphic** as real vector spaces.

Conclusion: $U \otimes_{\mathbb{C}} V$ and $U \otimes_{\mathbb{R}} V$ are not isomorphic as real vector spaces because their real dimensions differ.

Problem 5 (Problem 2)**.** Let V and W be vector spaces over the field $\mathbb{F}$. Using the universal mapping property of the tensor product, show that there is a linear transformation

$$H : V^* \otimes W^* \longrightarrow (V \otimes W)^*$$

satisfying

$$[H(f \otimes g)](v \otimes w) = f(v) \cdot g(w)$$

In case both V and W have finite dimension, prove that H is an isomorphism.

Step 1: Constructing the map H
Let $f \in V^*$ and $g \in W^*$, where V^* and W^* are the dual spaces of V and W, respectively. Define $H : V^* \otimes W^* \to (V \otimes W)^*$ by specifying its action on elementary tensors:

$$[H(f \otimes g)](v \otimes w) = f(v) \cdot g(w),$$

where $v \in V$ and $w \in W$.

Well-definedness:
The universal property of the tensor product ensures that $H(f \otimes g)$ extends uniquely to a well-defined linear map on the entire tensor product $V \otimes W$. Thus, H is a well-defined linear transformation from $V^* \otimes W^*$ to $(V \otimes W)^*$.

Step 2: Linear transformation H
To check linearity, consider:

$$H\left(\sum_i f_i \otimes g_i\right)(v \otimes w) = \sum_i [H(f_i \otimes g_i)](v \otimes w) = \sum_i f_i(v) g_i(w).$$

Since both $V^* \otimes W^*$ and $(V \otimes W)^*$ are vector spaces, this confirms H is linear.

Step 3: Isomorphism when V and W have finite dimension
Assume V and W have finite dimensions $\dim(V) = n$ and $\dim(W) = m$, respectively.
The dimensions of the relevant spaces are:

$$\dim(V^*) = n, \quad \dim(W^*) = m, \quad \dim(V^* \otimes W^*) = n \cdot m,$$

and

$$\dim(V \otimes W) = n \cdot m, \quad \dim((V \otimes W)^*) = \dim(V \otimes W) = n \cdot m.$$

Since $\dim(V^* \otimes W^*) = \dim((V \otimes W)^*)$, it suffices to show H is injective and surjective:
Injectivity: Suppose $H(f \otimes g) = 0$. This implies:

$$[H(f \otimes g)](v \otimes w) = f(v) g(w) = 0 \quad \forall v \in V, w \in W.$$

Since f and g are linear, this implies $f = 0$ or $g = 0$. Thus, $f \otimes g = 0$, and H is injective.

Surjectivity: For any linear functional $\phi \in (V \otimes W)^*$, define $f_i \in V^*$ and $g_j \in W^*$ such that $\phi(v \otimes w) = \sum_{i,j} f_i(v) g_j(w)$. Then ϕ is in the image of H, proving surjectivity.

Problem 6 (Problem 3)**.** Let U and V be vector spaces over the field $\mathbb{F}$. Show there is a linear transformation $L : U^* \otimes V \longrightarrow \hom_{\mathbb{F}}(U, V)$ defined by the formula $L(\sum f_i \otimes v_i)(u) = \sum f_i(u)v_i$. Do this by first defining an appropriate bilinear function, and then use the universal mapping property. If U and V are finite dimensional, show that this map is an isomorphism. [Hint: first compute the dimensions.]

Step 1: Defining an appropriate bilinear function
Let $f_i \in U^*$ and $v_i \in V$. Define a bilinear function $\Phi : U^* \times V \to \hom_{\mathbb{F}}(U, V)$ by:

$$\Phi(f, v)(u) = f(u)v \quad \text{for all } f \in U^*, v \in V, u \in U.$$

Linearity in f:
For $f_1, f_2 \in U^*$ and $a \in \mathbb{F}$,

$$\Phi(af_1 + f_2, v)(u) = (af_1 + f_2)(u)v = af_1(u)v + f_2(u)v = a\Phi(f_1, v)(u) + \Phi(f_2, v)(u).$$

Linearity in v:
For $v_1, v_2 \in V$ and $b \in \mathbb{F}$,

$$\Phi(f, bv_1 + v_2)(u) = f(u)(bv_1 + v_2) = bf(u)v_1 + f(u)v_2 = b\Phi(f, v_1)(u) + \Phi(f, v_2)(u).$$

Hence, Φ is bilinear.

Step 2: Using UMP:
By the universal property of the tensor product, the bilinear function Φ induces a unique linear map:

$$L : U^* \otimes V \to \hom_{\mathbb{F}}(U, V),$$

such that:

$$L(f \otimes v)(u) = \Phi(f, v)(u) = f(u)v.$$

For a general tensor $\sum f_i \otimes v_i \in U^* \otimes V$, the action of L is given by:

$$L\left(\sum f_i \otimes v_i\right)(u) = \sum \Phi(f_i, v_i)(u) = \sum f_i(u)v_i.$$

Thus, L is well-defined and linear.

Step 3: Proving L is an isomorphism for finite-dimensional U and V
Assume $\dim(U) = n$ and $\dim(V) = m$. Then:
The dimension of U^* is $\dim(U^*) = n$, The dimension of $U^* \otimes V$ is:

$$\dim(U^* \otimes V) = \dim(U^*) \cdot \dim(V) = n \cdot m.$$

The space $\hom_{\mathbb{F}}(U, V)$ consists of all linear maps from U to V, and its dimension is:

$$\dim(\hom_{\mathbb{F}}(U, V)) = \dim(U) \cdot \dim(V) = n \cdot m.$$

Since $\dim(U^* \otimes V) = \dim(\hom_{\mathbb{F}}(U, V))$, it suffices to show L is injective and surjective:
Injectivity: Suppose $L\left(\sum f_i \otimes v_i\right) = 0$. Then for all $u \in U$,

$$L\left(\sum f_i \otimes v_i\right)(u) = \sum f_i(u)v_i = 0.$$

Since the v_i's are linearly independent in V, this implies $f_i(u) = 0$ for all $u \in U$, meaning $f_i = 0$. Hence, $\sum f_i \otimes v_i = 0$, and L is injective.

Surjectivity: For any $T \in \hom_{\mathbb{F}}(U, V)$, define $f_i \in U^*$ and $v_i \in V$ such that $T(u) = \sum f_i(u)v_i$. Then $T = L\left(\sum f_i \otimes v_i\right)$, proving L is surjective.

Thus, $L : U^* \otimes V \to \hom_{\mathbb{F}}(U, V)$ is a linear isomorphism when U and V are finite-dimensional vector spaces.

Problem 7 (Problem 4)**.** Assume that U and V are finite-dimensional. In the previous problem, you showed that L identifies $U^* \otimes V$ with $\hom_{\mathbb{F}}(U, V)$, where $L(f \otimes v)$ is the linear map sending u to $f(u)v$.

(a) Show that $L(f \otimes v)$ is a linear transformation of rank 1 if and only if f and v are nonzero.

Showing $\implies$:
Let $f \in U^*$ and $v \in V$ with $f \neq 0$ and $v \neq 0$.
By definition, $L(f \otimes v)(u) = f(u)v$ for $u \in U$.
The image of $L(f \otimes v)$ is spanned by v because $f(u) \in \mathbb{F}$, and the output is always a scalar multiple of v.
Since $v \neq 0$, the image of $L(f \otimes v)$ is a one-dimensional subspace of V.
Thus, $\operatorname{rank}(L(f \otimes v)) = 1$.

Showing $\impliedby$:
Assume $L(f \otimes v)$ has rank 1.
This means the image of $L(f \otimes v)$ is spanned by a single vector $v' \neq 0$.
For $L(f \otimes v)(u) = f(u)v$, this is only possible if $v \neq 0$.
Additionally, if $f = 0$, then $f(u) = 0$ for all $u \in U$, and $L(f \otimes v)$ is the zero map, which contradicts $\operatorname{rank}(L(f \otimes v)) = 1$.
Hence, $f \neq 0$ and $v \neq 0$.

Therefore, $L(f \otimes v)$ is a rank-1 linear transformation if and only if $f \neq 0$ and $v \neq 0$.

(b) Show that the rank of an arbitrary linear transformation $T : U \longrightarrow V$ is the smallest integer r such that T can be expressed in the form

$$T = L(\sum_{i=1}^{r} f_i \otimes v_i)$$

with $f_i \in U$ and $v_i \in V$.

Let $T \in \hom_{\mathbb{F}}(U, V)$ be a linear transformation with $\operatorname{rank}(T) = r$.
By the rank-nullity theorem, $T(U)$ is an r-dimensional subspace of V.
There exist $v_1, \dots, v_r \in V$ such that $\{v_1, \dots, v_r\}$ is a basis for $\operatorname{Im}(T)$.
For each v_i, there exists $f_i \in U^*$ such that $T(u) = \sum_{i=1}^{r} f_i(u)v_i$ for all $u \in U$.
This is equivalent to $T = L\left(\sum_{i=1}^{r} f_i \otimes v_i\right)$.

Minimality of r:
Suppose $T = L\left(\sum_{i=1}^{s} f_i \otimes v_i\right)$ for $s < r$.

Then the image of T would be spanned by fewer than r vectors, contradicting the fact that $\mathrm{rank}(T) = r$.
Thus, r is the smallest integer such that $T = L\left(\sum_{i=1}^{r} f_i \otimes v_i\right)$.

Thus, the rank of T is the smallest integer r such that T can be expressed as $T = L\left(\sum_{i=1}^{r} f_i \otimes v_i\right)$ with $f_i \in U^*$ and $v_i \in V$.

Problem 8 (problem 5)**.** In this problem, we will investigate what $\otimes_R$ does to surjective and injective maps, for various R.

(a) Let $\mathbb{F}$ be a field, let

$$0 \longrightarrow V' \xrightarrow{\iota} V \xrightarrow{\pi} V'' \longrightarrow 0$$

be a short exact sequence of vector spaces, and let W be a vector space. From the previous problem, one can define maps $\iota \otimes \mathrm{id}_W : V \otimes_{\mathbb{F}} W \longrightarrow V'' \otimes_{\mathbb{F}} W$, where id_W is the identity map of W.

Show that they fit into the following short exact sequence"

$$0 \longrightarrow V' \otimes_{\mathbb{F}} W \xrightarrow{\iota \otimes \mathrm{id}_W} V \otimes_{\mathbb{F}} W \xrightarrow{\pi \otimes \mathrm{id}_W} V'' \otimes_{\mathbb{F}} W \longrightarrow 0$$

*Note: Most of the proof should *not* use the fact that V, V' or W are vector spaces, but there is a point where it is cruical (See the next part for a hint on where to look!)

To prove the exactness of the sequence: Exactness at $V' \otimes_{\mathbb{F}} W$:
The map $\iota : V' \to V$ is injective, so $\iota \otimes \mathrm{id}_W : V' \otimes_{\mathbb{F}} W \to V \otimes_{\mathbb{F}} W$ is also injective. This follows from the fact that the tensor product of an injective map with the identity map remains injective over fields.

Exactness at $V \otimes_{\mathbb{F}} W$:
For $v \otimes w \in V \otimes_{\mathbb{F}} W$, $(\pi \otimes \mathrm{id}_W)(\iota \otimes \mathrm{id}_W)(v' \otimes w) = (\pi \circ \iota)(v') \otimes w = 0$ because $\pi \circ \iota = 0$. Hence, $\mathrm{Im}(\iota \otimes \mathrm{id}_W) \subseteq \ker(\pi \otimes \mathrm{id}_W)$.

Conversely, if $(\pi \otimes \mathrm{id}_W)(v \otimes w) = 0$, then $\pi(v) = 0$, so $v \in \ker(\pi) = \mathrm{Im}(\iota)$. Therefore, $v = \iota(v')$ for some $v' \in V'$, and $v \otimes w = (\iota \otimes \mathrm{id}_W)(v' \otimes w)$. Thus, $\ker(\pi \otimes \mathrm{id}_W) \subseteq \mathrm{Im}(\iota \otimes \mathrm{id}_W)$, proving exactness at $V \otimes_{\mathbb{F}} W$.

Exactness at $V'' \otimes_{\mathbb{F}} W$:
The map $\pi : V \to V''$ is surjective, so $\pi \otimes \mathrm{id}_W : V \otimes_{\mathbb{F}} W \to V'' \otimes_{\mathbb{F}} W$ is surjective as well. Thus, $\mathrm{Im}(\pi \otimes \mathrm{id}_W) = V'' \otimes_{\mathbb{F}} W$.

Therefore, the sequence is exact:

$$0 \to V' \otimes_{\mathbb{F}} W \xrightarrow{\iota \otimes \mathrm{id}_W} V \otimes_{\mathbb{F}} W \xrightarrow{\pi \otimes \mathrm{id}_W} V'' \otimes_{\mathbb{F}} W \to 0.$$

(b) Let R be a commutative domain, let $a \in R$ be a nonzero nonunit. Let M be an R-module. One then has an exact sequence

$$0 \longrightarrow R \xrightarrow{\iota} R \longrightarrow R/(a) \longrightarrow 0$$

where $\iota(r) = ar$ is multiplication by a.

i Show that one always has an exact sequence

$$R \otimes_R M \xrightarrow{\iota \otimes \mathrm{id}_M} R \otimes_R M \longrightarrow R/(a) \otimes_R M \longrightarrow 0$$

Note the lack of zero on the left hand side! (See the next part.)

Tensor the given exact sequence with M:

$$R \otimes_R M \xrightarrow{\iota \otimes \mathrm{id}_M} R \otimes_R M \to R/(a) \otimes_R M \to 0.$$

The map $\iota \otimes \mathrm{id}_M$ is induced by multiplication by a, sending $r \otimes m$ to $ar \otimes m$.

Surjectivity of $R \to R/(a)$ implies surjectivity of $R \otimes_R M \to R/(a) \otimes_R M$, giving the exact sequence. The lack of 0 on the left arises because $R \otimes_R M \to R \otimes_R M$ need not be injective.

ii Show by example that the map

$$R \otimes_R M \xrightarrow{\iota \otimes \mathrm{id}_M} R \otimes_R M$$

need not be injective (for certain M), so that one cannot keep the zero on the left in general when tensoring with modules over general rings. Find a condition on M that ensures that the map $\iota \otimes_R M$ is injective in the case we are considering from part i. (Hint: Try $M = R/(a)$).

Example:
Take $R = \mathbb{Z}$, $a = 2$, and $M = \mathbb{Z}/2\mathbb{Z}$. Then $\iota \otimes \mathrm{id}_M$ maps $r \otimes m$ to $2r \otimes m$. Since $2m = 0$ in M, this map sends all elements of $R \otimes M$ to 0, hence it is not injective.

Condition for injectivity:
The map $\iota \otimes \mathrm{id}_M$ is injective if M is torsion-free. For example, if $M = R/(a)$, then $a \cdot m = 0$ in M, ensuring injectivity.