The Rubik's Cube and Group Theory

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Introduction

In this project, we explore the structure of the Rubik's Cube group from a rigorous algebraic perspective. We will begin by defining the cube group and its standard generators, and then investigate how this group can be decomposed into two important subgroups: one acting on the edges and another on the corners. We will construct group homomorphisms that capture the orientation data of these subgroups and analyze the conditions under which edge and corner configurations combine to yield legal cube states.

The Basic Rubik's Cube Group

The Rubik's Cube is a classic example of a mathematical object that can be modeled using the principles of group theory. Each configuration of the cube can be reached by a finite sequence of moves from the solved state, and these moves together form a mathematical group under the operation of *move sequence* composition.

To better understand the group-theoretic behavior of the Rubik's Cube, it is helpful to examine its physical structure. Although the cube appears to be composed of $3 \times 3 \times 3 = 27$ smaller cubes, only 26 of these are visible and out of those 26, only 20 of them are movable; the 27th cube lies at the very center and serves as a stationary mechanical core with no visible color. It is not counted among the manipulable pieces because it does not participate in the puzzle's configuration space.

The 26 visible pieces can be classified into three types:

- **Corner pieces:** There are 8 corner pieces, each displaying three colors. They occupy the corners of the cube and can be permuted and oriented.

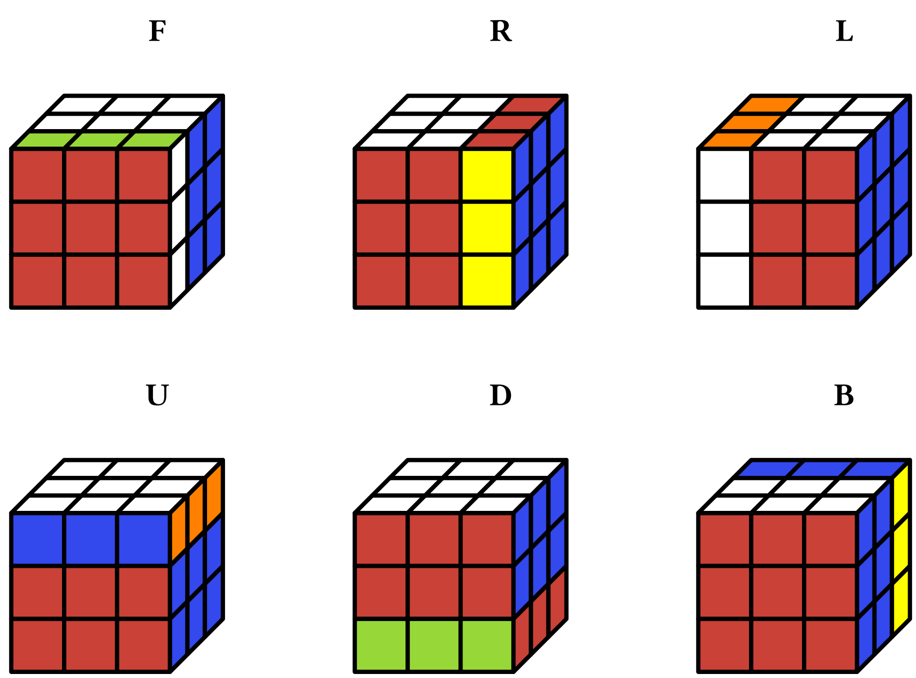
- **Edge pieces:** There are 12 edge pieces, each displaying two colors. These lie between corners on the edges of each face and can also be permuted and oriented.

- **Center pieces:** There are 6 center pieces, one on each face, each displaying a single color. These are fixed in position relative to each other and do not move under any legal cube operation. They define the orientation of the cube. And moreover, define the color of the face.

The standard $3 \times 3 \times 3$ Rubik's Cube has six faces: up (U), down (D), left (L), right (R), front (F), and back (B). Each of these faces can be rotated $90°$ clockwise, counterclockwise, or $180°$. Typically, we denote a $90°$ clockwise rotation of face X as X, a $90°$ counterclockwise rotation as X', and a $180°$ rotation as X^2. However, in most Rubik's Cube notation algorithms, X^2 is often denoted $2X$. But for this project we will stick with the more algebraic notation, X^2. Additionally, we may treat X' as the inverse of X.

We define the generating set $\mathcal{R}$ as the set of all face turns on the Rubik's Cube. Each legal configuration of the cube corresponds to a group element, with the group operation given by composition of move sequences. Since each face turn has order 4, the inverse of a move, denoted X', can be expressed as X^3. Therefore, the generating set need not explicitly include inverse face turns.

$$\mathcal{R} = \{F, R, L, U, D, B\}$$



With inverses and combinations, this set generates the entire group of Rubik's Cube permutations. We'll call this Rubik's Cube group, G such that $G = \langle \mathcal{R} \rangle$—the group G is generated by the set of moves $\mathcal{R}$.

Group Structure and Axioms

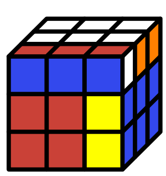
We verify that G satisfies the group axioms under the operation "$\circ$", move sequence composition:

- **Closure:** The composition of any two moves on the Rubik's Cube results in another legal move sequence, hence another element of G.
- **Associativity:** The operation of move composition is associative: for any move sequences $a, b, c \in G$, we have $(a \circ b) \circ c = a \circ (b \circ c)$.
- **Identity:** The identity element is the "empty" state of the cube, which corresponds to doing nothing, denoted by e. For any sequence $g \in G$, we have $g \circ e = e \circ g = g$.
- **Inverses:** Every sequence of moves has an inverse (i.e. if $g = RUR'U'$, then the inverse algorithm $g' = URU'R'$. Applying $g \circ g'$ would result in the identity state e), and thus every element $g \in G$ has an inverse g^{-1} such that $g \circ g^{-1} = g^{-1} \circ g = e$.

Therefore, G satisfies the group axioms and is indeed a group. From this point onward, we will omit the composition symbol "$\circ$" when writing sequences of moves, and interpret juxtaposition as move composition. It is important to note that G is not entirely commutative; that is, in general, the order of moves in a sequence matters. For example,

$$RU \neq UR$$

RU



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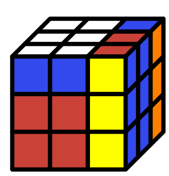


The Structure of Pieces on the Rubik's Cube

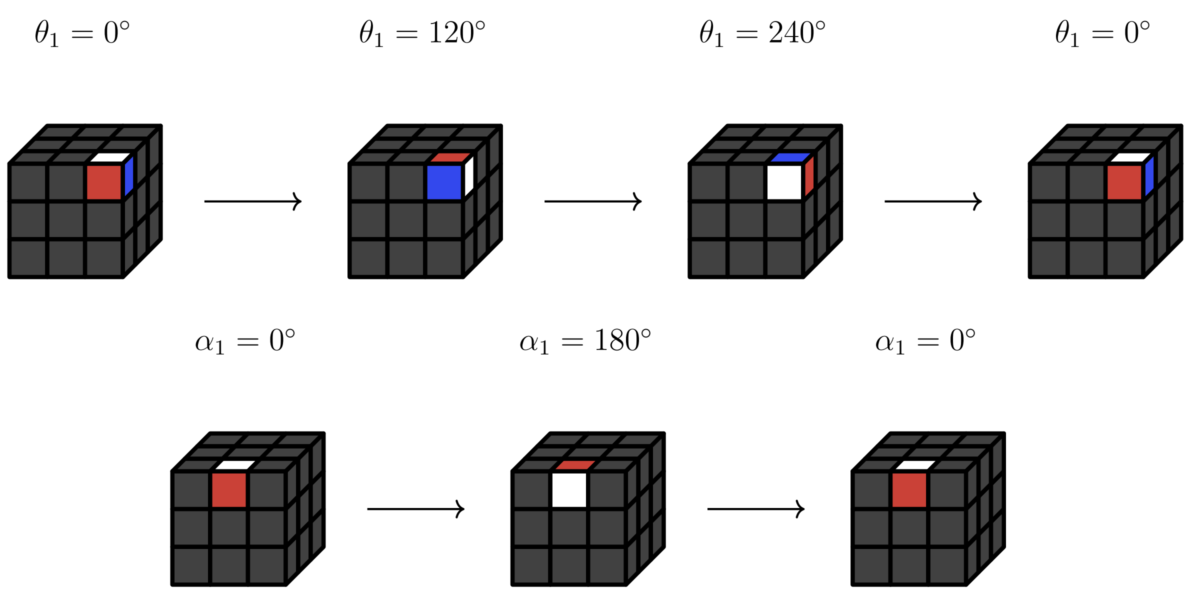
Now that we have developed some intuition for moves, let's go back to the three types of pieces: corner pieces, edge pieces, and center pieces. The centers are fixed relative to each other and serve as reference points for the cube's orientation. The structure and constraints of the movable pieces, corners and edges, are governed by the internal mechanism of the cube.

- The **corner orientation subgroup**, generated by sequences that twist corners without changing their positions, is isomorphic to $\mathbb{Z}_3^7$, with the constraint that the total corner twist must be zero modulo 3.

- The **edge orientation subgroup**, which flips edges without changing their locations, is isomorphic to $\mathbb{Z}_2^{11}$, under the constraint that the number of edge flips is even.

You can think of corner twisting and edge flipping as invisible bookkeeping. When solving the cube or analyzing move sequences, these twists and flips must be corrected or kept in check for the puzzle to return to a valid and solvable configuration. Now, what is considered legal and illegal will be discussed in the 'Actions on Corners and Edges' section on page 5, but for now we will consider the space of illegal moves.

Each corner piece can be oriented in 3 possible ways, corresponding to an element of $\mathbb{Z}_3$. This would be the set of corner orientations $\{0°, 120°, 240°, 0°, 120°, \ldots\}$. And each edge piece can be oriented in 2 possible ways, corresponding to an element of $\mathbb{Z}_2$, $\{0°, 180°, 0°, \ldots\}$ (see the model below). Therefore, the total set of all possible corner orientations should be $\mathbb{Z}_3^8$, where each component $\theta_i \in \mathbb{Z}_3^8$ represents the twist orientation of the i-th corner piece. And for the edges, the total set of all possible orientations should be $\mathbb{Z}_2^{12}$, where each component $\alpha_i \in \mathbb{Z}_2^{12}$ represents the i-th edge flip orientation.



Each move in $\mathcal{R}$ acts as a permutation on the set of pieces. Let Ω be the set of oriented piece types (e.g., a corner with a specific twist), then:

$$G \subseteq \mathrm{Sym}(\Omega)$$

and each generator $m \in \{R, L, U, D, F, B\}$ corresponds to a permutation $\rho_m \in \mathrm{Sym}(\Omega)$. These permutations not only rearrange the pieces but also alter their orientation states in a predictable, algebraically constrained manner.

The group G is therefore realized as the subgroup of $\mathrm{Sym}(\Omega)$ generated by the permutations $\rho_R, \rho_L, \rho_U, \rho_D, \rho_F, \rho_B$. Each ρ_m operates on Ω by mapping each oriented piece to a new location and orientation, in accordance with the mechanical movement of the cube.

More precisely, let a piece $\omega \in \Omega$ be represented as a tuple (p, o), where p denotes the piece's position and o represents the piece orientation (i.e., a twist for corners or a flip for edges). Then each ρ_m acts by:

$$\rho_m(p, o) = (p', o')$$

where p' is the new position and o' is the new orientation, determined by the structure of the move m. The orientation update follows consistent rules depending on the axis of rotation: for example, quarter-turns of adjacent faces may increment or decrement the orientation modulo 3 for corners, and modulo 2 for edges. Thus, G encodes not just the spatial symmetries of the cube but also the group-theoretic constraints on the orientation. This interpretation enables us to look at the configuration space and cosets corresponding to equivalence classes under specific constraints. We will use this structure when constructing the homomorphisms in the next section.

Actions on Corners and Edges

Each element of G, generated by the basic face moves, induces a specific permutation and orientation transformation on the edges and corners. This action naturally decomposes into two intertwined but distinct components: the edge action and the corner action. These define homomorphisms from G into the respective semidirect products of position and orientation groups.

We define two subgroups corresponding to the cube's two types of movable components. Let G_1 be the group of edge pieces and let G_2 be the group of corner pieces:

$$
\begin{aligned}
G_1 &\subseteq S_8 \ltimes C_3^8 \quad \text{(corner permutations and twists)} \\
G_2 &\subseteq S_{12} \ltimes C_2^{12} \quad \text{(edge permutations and flips)}
\end{aligned}
$$

Not every element of $S_8 \ltimes C_3^8$ or $S_{12} \ltimes C_2^{12}$ corresponds to a physically realizable state of the Rubik's Cube. The cube's mechanical design and the structure of its allowable moves impose global constraints on orientation: the total twist of the corners must be zero modulo 3, and the total flip of the edges must be zero modulo 2. These constraints arise because every legal move is a permutation from a fixed generating set (face turns), and no combination of these moves can produce an isolated twist or flip.

For the corners, this means that the sum of the corner twists,

$$\sum_{i=0}^{7} \delta_i^{(c)} \equiv 0 \pmod 3,$$

must hold in any legal state. This restriction limits C_3^8 to the subset of twist configurations that are actually realizable through permutations in the cube group G. Likewise, for the edges, the condition

$$\sum_{i=0}^{11} \delta_i^{(e)} \equiv 0 \pmod 2$$

must hold for legal edge flips. Thus, the legal cube states form proper subgroups of $S_8 \ltimes C_3^8$ and $S_{12} \ltimes C_2^{12}$, and these subgroups are precisely the images of G under the homomorphisms into G_1 and G_2.

Starting with the corner pieces, we define the first homomorphism. Let $\phi_1 : G \to G_1$ where for each $g \in G$, we set

$$\phi_1(g) = (\pi_g^{(c)}, \delta_g^{(c)})$$

where $\pi_g^{(c)} \in S_8$ is the induced permutation of the 8 corner pieces, and $\delta_g^{(c)} \in C_3^8 \cong (\mathbb{Z}/3\mathbb{Z})^8$ represents the corner orientation. The image of ϕ_1 lies within the subgroup,

$$S_8 \ltimes \left\{ \delta_i \in C_3^8 \;\middle|\; \sum_{i=0}^{8} \delta_i \equiv 0 \pmod 3 \right\} \cong S_8 \ltimes C_3^7$$

Next, we define the edge homomorphism. Let $\phi_2 : G \to G_2$ where for each $g \in G$, we define

$$\phi_2(g) = (\pi_g^{(e)}, \delta_g^{(e)})$$

where $\pi_g^{(e)} \in S_{12}$ is the induced permutation of the 12 edge pieces, and $\delta_g^{(e)} \in C_2^{12} \cong (\mathbb{Z}/2\mathbb{Z})^{12}$ is the edge orientation. The image of ϕ_2 lies within the subgroup,

$$S_{12} \ltimes \left\{ \delta_i \in C_2^{12} \;\middle|\; \sum_{i=0}^{12} \delta_i \equiv 0 \pmod 2 \right\} \cong S_{12} \ltimes C_2^{11}$$

We see that indeed, G is a proper subgroup of $G_1 \times G_2$, that is $\phi : G \hookrightarrow G_1 \times G_2$ is injective, but not surjective as the many theoretically possible configurations in $G_1 \times G_2$ are physically impossible to reach with just moves in $\mathcal{R}$.

These homomorphisms separate the action of the full cube group G into its edge and corner components, revealing the constrained structure of orientation states within the full configuration space. The reason why these spaces are semidirect products is because a piece's orientation is not invariant under all positional changes. This means that certain moves simultaneously permute and rotate other pieces. This structure mirrors the cube's physical constraints and explains why some configurations are unreachable by legal moves, even if they appear superficially scrambled.

Special Symmetries and Commutativity

Although the full cube group G is intricate and highly non-abelian, the structure revealed by the homomorphisms ϕ_1 and ϕ_2 helps us understand how individual edges and corners respond to moves. This decomposition not only shows which configurations are legal, but also helps identify special subsets of moves that exhibit additional structure. In particular, certain subsets of G behave like *abelian groups*: their elements commute, despite the complexity of the full group.

For a group H to be an abelian group it must be the case that for all $h_1, h_2 \in H$, h_1 and h_2 commute, that is $h_1 h_2 = h_2 h_1$. For example, two face rotations on opposite and non-interacting faces, such as U and D, will satisfy commutativity:

$$UD = DU$$

UD $\qquad\qquad$ DU

Commutativity in the Rubik's Cube group often emerges when generators act on disjoint sets of pieces. The moves U and D affect entirely separate sets of pieces and have no overlap in their action. As a result, the order in which they are applied does not matter: $UD = DU$.

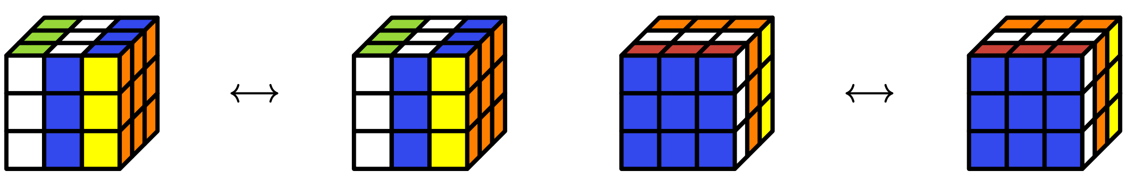
This means the subgroup they generate is abelian:

$$\langle U, D \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_4$$

Here, each of the individual cyclic subgroups $\langle U \rangle$ and $\langle D \rangle$ is of order 4, since applying the same face turn four times returns the cube to its original state. But more importantly, because U and D act independently, their compositions commute. The same structure holds for other pairs of moves with disjoint action:

$$\langle L, R \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_4 \qquad \langle F, B \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_4$$

$LR = RL$ $\qquad\qquad$ $FB = BF$



In general, for any pair of moves $X, Y \in \mathcal{R}$ with disjoint support, the moves are commutative. Hence $XY = YX$, as seen in the homomorphic model above. More formally, a necessary and sufficient condition for commutative moves is:

$$\operatorname{supp}(X) \cap \operatorname{supp}(Y) = \emptyset$$

where $\operatorname{supp}(X)$ denotes the set of Rubik's Cube pieces (edges and/or corners) affected by the generator $\langle X \rangle$. Not only does this disjoint-action criterion provides a constructive way to identify abelian subgroups within G, but it allows us to observe the kinds of cycles on the Rubik's cube.

We can define this idea of disjoint sequences more clearly. Let some permutation $\sigma \in S_{12}$ be a permutation of the 12 edge pieces. For example, we let $\sigma(i) = j$ to mean $i \longmapsto j$.

$$\begin{array}{llllll} \sigma(1) = 6, & \sigma(2) = 11, & \sigma(3) = 2 & \sigma(4) = 5, & \sigma(5) = 4, & \sigma(6) = 1 \\ \sigma(7) = 9, & \sigma(8) = 12, & \sigma(9) = 7 & \sigma(10) = 3, & \sigma(11) = 8, & \sigma(12) = 10 \end{array}$$

or

$$\begin{gathered} 1 \longmapsto 6, 2 \longmapsto 11, 3 \longmapsto 2, 4 \longmapsto 5, 5 \longmapsto 4, 6 \longmapsto 1, \\ 7 \longmapsto 9, 8 \longmapsto 12, 9 \longmapsto 7, 10 \longmapsto 3, 11 \longmapsto 8, 12 \longmapsto 10 \end{gathered}$$

In cycle notation, the permutation σ can be expressed as:

$$\sigma = (2, 11, 3, 7, 9, 8, 12, 10)(1, 6)(4, 5)$$

Permutations can be operated under sequence composition. That is, for some permutation τ, $\sigma \circ \tau \in S_{12}$. Similarly, these cycles are disjoint if they have no mappings in common; if $\operatorname{supp}(\tau)$ and (σ) are disjoint if they share no common numbers.

We can relate this geometric condition to our algebraic framework with the homomorphisms:

$$\phi_1 : G \to S_{12} \ltimes C_2^{12}, \quad \phi_2 : G \to S_8 \ltimes C_3^8$$

Each move $g \in G$ induces an action on the edge and corner pieces through $\phi_1(g) = (\pi_g^{(e)}, \delta_g^{(e)})$ and $\phi_2(g) = (\pi_g^{(c)}, \delta_g^{(c)})$. The support of g is reflected in the nontrivial coordinates of these image components.

Thus, if $\phi_1(X)$ and $\phi_1(Y)$ act on disjoint sets of edge indices, and likewise $\phi_2(X)$ and $\phi_2(Y)$ act on disjoint sets of corners, then:

$$[\phi_1(X), \phi_1(Y)] = e \quad \text{and} \quad [\phi_2(X), \phi_2(Y)] = e$$

which implies:

$$[X, Y] \in \ker(\phi_1) \cap \ker(\phi_2)$$

Since this kernel intersection is very small (often just the identity or a small central subgroup), the disjoint-support condition effectively guarantees that X and Y commute. Therefore, any collection of moves with pairwise disjoint support will generate an abelian subgroup of G.

Conclusion

In this project, we analyzed the Rubik's Cube group G by constructing explicit group homomorphisms that encode the action of cube moves on both edge and corner pieces. Through the maps ϕ_1 and ϕ_2, we saw how cube states are constrained by parity conditions and how these constraints arise naturally from the structure of semidirect products. These homomorphisms allowed us to isolate and interpret the legal configurations of the cube and provided a framework for identifying subgroup structures within G.

One significant outcome of this analysis was the identification of abelian subgroups generated by moves with disjoint support. By understanding when certain generators commute, we gained insight into the cube's internal symmetries and its algebraic richness. The connection between geometric intuition (disjoint piece movement) and algebraic formality (commutativity and kernel intersections) reveals the elegance of group theory as a language for modeling complex systems.

This framework offers natural extensions. For instance, similar techniques can be applied to other $n \times n \times n$ puzzles, or the Rubik's Cube can be studied through the lens of coset decomposition, normal subgroups, or even the theory of solvable groups. Beyond pure mathematics, these ideas also inform computational approaches for solving and optimizing move sequences.

In sum, the Rubik's Cube serves as a remarkably tangible and accessible model for exploring deep concepts in group theory. Its study continues to bridge the abstract and the concrete, offering both pedagogical value and mathematical intrigue.

References

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