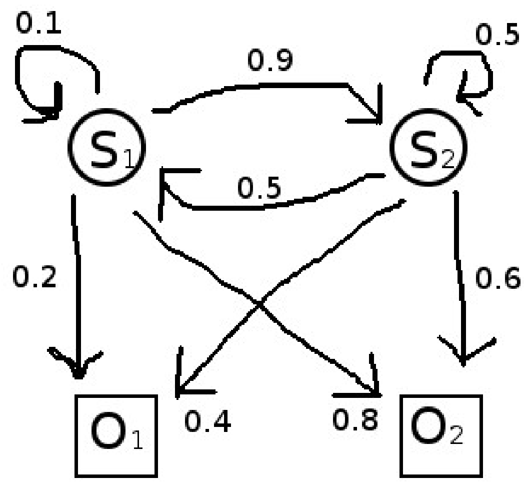

Figure 1: Graph of the model for exercise 1.a

Problem 1.a.

Problem 1.b. The two hidden coins determine the states and what visible coin will be thrown. The outcome of the visible coin thrown is the observation O. The two hidden coins are represented by the rows of the transition matrix A. The first column corresponds to throwing 'Heads', the second column corresponds to throwing 'Tails'. Thus for example hidden coin 1 has probability 0.1 that 'Heads' will appear and 0.9 that 'Tails' will appear. We have fixed here that 'Heads' of a hidden coin means we transition to state S_1 and 'Tails' means we transition to state S_2 and that when we are in state S_1 we throw hidden coin 1 and when in state S_2 we throw hidden coin 2.
The rows and columns of the emission matrix have the same meaning, only they correspond to the two visible coins. When in state S_1 we throw visible coin 1 and when in state S_2 we throw visible coin 2. For example visible coin 1 has probability 0.2 that the observation 'Heads' appears and 0.8 that 'Tails' appears.

Problem 1.c. We want to calculate

$$P((q_1, q_2) \mid (O_1, O_2) = (\text{tails}, \text{tails})).$$

Using Bayes' theorem we can rewrite this as

$$= \frac{P((O_1, O_2) = (\text{tails}, \text{tails}) \mid (q_1, q_2)) \cdot P((q_1, q_2))}{P((O_1, O_2) = (\text{tails}, \text{tails}))}.$$

Since we start in state S_1 we can assume that q_1. We need to calculate the probabilities for $q_2 = S_1$ and $q_2 = S_2$. The conditional probability can be calculated using the values from the emission matrix

$$P((O_1, O_2) = (\text{tails}, \text{tails}) \mid (q_1, q_2)) = \begin{cases} 0.8^2 \text{ if } q_2 = S_1 \\ 0.8 \cdot 0.6 \text{ if } q_2 = S_2 \,. \end{cases}$$

Further we have

$$\begin{aligned} P((S_1, q_2)) &= \pi_1 \cdot P(q_2 \mid S_1) \\ &= 1 \cdot P(q_2 \mid S_1) \\ &= \begin{cases} 0.1 \text{ if } q_2 = S_1 \\ 0.9 \text{ if } q_2 = S_2 \,. \end{cases} \end{aligned}$$

Lastly the probability that we observe tails twice is equal to

$$\begin{aligned} P((O_1, O_2) = (\text{tails}, \text{tails})) &= \pi_1 \cdot B_{12} \cdot (A_{11} B_{12} + A_{12} B_{22}) \\ &= 0.8 \cdot (0.1 \cdot 0.8 + 0.9 \cdot 0.6) \\ &= 0.496 \,. \end{aligned}$$

Putting the values together gives the following distribution

$$P((q_1, q_2) \mid (O_1, O_2) = (\text{tails}, \text{tails})) = \begin{cases} 0.129 \text{ if } q_2 = S_1 \\ 0.871 \text{ if } q_2 = S_2 \,. \end{cases}$$

The two probabilities sum up to 1 as expected.