Extraction of cosmological parameters from observational data

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1 Linear perturbations

In this section we are going to write out general equations, which govern evolution of nearly homogeneous universe, taking in consideration linear perturbations of matter and metric. Discussion is mainly based on [1] and interest is placed on the recombination epoch. From [3] it is known that scalar metric perturbations play the most important rule in the evolution and, therefore, we consider metric of form

$$ds^2 = -(1 + 2\Psi(t, \mathbf{x}))dt^2 + a^2(t)(1 + 2\Phi(t, \mathbf{x}))\delta_{ij}dx^i dx^j \tag{1}$$

Further on, we are going to write all equations up to first order in perturbations of matter and of metric Ψ, Φ.

1.1 Boltzmann equation

Consider a particle with 4-momentum P^μ, which satisfies

$$g_{\mu\nu}P^\mu P^\nu = -(1 + 2\Psi)(P^0)^2 + g_{ij}P^i P^j \equiv -E^2 + p^2 = -m^2 \tag{2}$$

where we have introduced E and p^i - energy an 3-momenta in local normal coordinates. This allows to rewrite 4-momentum as

$$P^\mu = \begin{bmatrix} E(1 - \Psi) & p^i(1 - \Phi)/a \end{bmatrix} \tag{3}$$

We can choose parameter λ such that $P^\mu = dx^\mu/d\lambda$ (for instance for massive particle, $d\lambda = d\tau/m$). Particle's movement in the $(\mathbf{x}, \mathbf{p})$ phase space is deduced from

$$\frac{dx^i}{dt} = \frac{dx^i}{d\lambda}\frac{d\lambda}{dt} = \frac{P^i}{P^0} = \frac{p^i}{Ea}(1 + \Psi - \Phi) \tag{4}$$

and from the geodesic equation

$$\frac{dP^\mu}{d\lambda} = \frac{d^2x^\mu}{d\lambda^2} = -\Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\lambda}dx^\nu d\lambda = -\Gamma^\mu_{\nu\sigma}P^\nu P^\sigma \tag{5}$$

which gives

$$\frac{dp^i}{dt} = -(H + \dot{\Phi})p^i - \frac{E}{a}\Psi_{,i} - \frac{p^i}{Ea}p^k\Phi_{,k} + \frac{p^2}{Ea}\Phi_{,i} \tag{6}$$

Boltzmann equation predicts an evolution of the distribution function $f(\mathbf{x},\mathbf{p},t)$ of particles. Consider following a infinitesimal volume of the phase space along a trajectory of some particle $(\mathbf{x}(t),\mathbf{p}(t))$. Number of particles and, therefore, $f(\mathbf{x}(t),\mathbf{p}(t),t)$ too are unchanged, unless there is some collision process, which abruptly changes particles' momenta. One can assume that collision is an uncorrelated process and its intensity can be found by integrating over all possible previous momenta of colliding particles a probability of these particles occupying corresponding phase space's infinitesimal volumes. Then one gets a closed equation on f, which is called Boltzmann equation and can be written as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f}{\partial p^i}\frac{dp^i}{dt} = C[f] \tag{7}$$

Here dx^i/dt and dp^i/dt correspond to particle's movement, $C[f]$ is called the collision integral and will be discussed further on for concrete reactions.

1.2 Evolution of photons

Using 4 and 5 one can get left part of Boltzmann equation. For ultra-relativistic case we can apply $E=p$ and write

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{\hat{p}^i(1+\Psi-\Phi)}{a} - \frac{\partial f}{\partial p}\left[(H+\dot{\Phi})p + \frac{p^i\Psi_{,i}}{a}\right] + \frac{\partial f}{\partial \hat{p}^i}\frac{1}{a}\left[(\Phi-\Psi)_{,i} - \hat{p}^i\hat{p}^k(\Phi-\Psi)_{,k}\right] \tag{8}$$

where we have introduced $\hat{p}^i = p^i/p$. At zero order we assume that f is space and momentum direction homogeneous and, in fact, is equal to Bose-Einstein/Dirac distribution. Therefore, $\partial f/\partial x^i$ and $\partial f/\partial \hat{p}^i$ are first order perturbations. Then $\partial f/\partial \hat{p}^i$ term has second order, while $\partial f/\partial x^i$ can be simplified, resulting in

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{\hat{p}^i}{a} - \frac{\partial f}{\partial p}p\left[H + \dot{\Phi} + \frac{\hat{p}^i\Psi_{,i}}{a}\right] \tag{9}$$

Further on, we suppose that the only parameter which varies in the phase space is a temperature $T(\mathbf{x},\mathbf{p},t) = T(t)(1+\Theta(\mathbf{x},\mathbf{p},t))$ of Bose-Einstein distribution. Such parameterization at linear order of Θ gives a corresponding distribution function

$$f(\mathbf{x},\mathbf{p},t) = f^{(0)}(p,t) - p\frac{\partial f^{(0)}}{\partial p}\Theta(\mathbf{x},\mathbf{p},t); \quad f^{(0)}(p,t) = \frac{1}{e^{p/T(t)}-1} \tag{10}$$

Evolution of $T(t)$ is inferred from zero order part of 8, since at zero order there is a global equilibrium with distribution $f^{(0)}(p,t)$, collision integral vanishes and we have

$$\frac{\partial f^{(0)}}{\partial t} - \frac{\partial f^{(0)}}{\partial p}pH = 0 \Rightarrow -\left(\frac{\dot{T}}{T} + \frac{\dot{a}}{a}\right)\frac{\partial f^{(0)}}{\partial p} = 0 \Rightarrow T \sim \frac{1}{a} \tag{11}$$

At first order we have

$$\left.\frac{df}{dt}\right|_{\text{first}} = -p\frac{\partial f^{(0)}}{\partial p}\left[\dot{\Theta} + \frac{\hat{p}^i\Theta_{,i}}{a} + \dot{\Phi} + \frac{\hat{p}^i\Psi_{,i}}{a} - pH\frac{\partial\Theta}{\partial p}\right] \tag{12}$$

We are mostly interested in recombination epoch, when photons interact with non-relativistic electrons via Thomson scattering. The collision integral $C[f]$ is calculated in [1][Chapter 5.2] to be

$$C[f] = -p\frac{\partial f^{(0)}}{\partial p}n_e\sigma_T\left[\hat{\mathbf{p}}\cdot\mathbf{u}_e - \Theta + \Theta_0\right]; \quad \Theta_0 = \frac{1}{4\pi}\int d\hat{\mathbf{p}}'\Theta(p\hat{\mathbf{p}}') \tag{13}$$

where n_e is electrons' concentration, $\mathbf{u}_e$ is electrons' relative velocity, and σ_T is Thomson scattering cross section.
We can get rid of dependence on the momentum module p by following an idea in [5] - multiply equations 12 and 13 by $p^3/4\pi^2$ and integrate over $p \in [0, +\infty)$. Abusing notation, we redefine

$$\int -p\frac{\partial f^{(0)}}{\partial p}\frac{p^3}{4\pi^2}\Theta dp = \rho^{(0)}(t)\Theta(\mathbf{x}, \hat{\mathbf{p}}, t); \quad \frac{1}{4\pi}\int d\hat{\mathbf{p}}'\Theta(\mathbf{x}, \hat{\mathbf{p}}', t) = \Theta_0(\mathbf{x}, t) \tag{14}$$

Note that

$$\int -p\frac{\partial f^{(0)}}{\partial p}\frac{p^3}{4\pi^2}dp = \int p^3 f^{(0)}\frac{dp}{\pi^2} = 2\int pf^{(0)}\frac{4\pi p^2 dp}{(2\pi)^3} = \rho^{(0)} \tag{15}$$

where $\rho^{(0)}$ is the energy density and integral prefactor 2 is due to 2 spin states of an electron. This computation justifies the redefinition of Θ given above - Θ not depending on p leads to an identity. After integration by parts we obtain

$$\frac{\partial}{\partial t}(\rho^{(0)}\Theta) + \frac{\hat{p}^i}{a}(\rho^{(0)}\Theta)_{,i} + \rho^{(0)}\dot{\Phi} + \rho^{(0)}\frac{\hat{p}^i\Psi_{,i}}{a} + 4H\rho^{(0)}\Theta = n_e\sigma_T\rho^{(0)}[\hat{\mathbf{p}}\cdot\mathbf{u}_e - \Theta + \Theta_0] \tag{16}$$

Using continuity equation $\partial\rho^{(0)}/\partial t + 4H\rho^{(0)} = 0$ for photons and reducing by $\rho^{(0)}$ simplifies an equation to

$$\dot{\Theta} + \frac{\hat{p}^i\Theta_{,i}}{a} + \dot{\Phi} + \frac{\hat{p}^i\Psi_{,i}}{a} = n_e\sigma_T[\hat{\mathbf{p}}\cdot\mathbf{u}_e - \Theta + \Theta_0] \tag{17}$$

Finally, since equation is linear we perform Fourier transformation. In parallel, we replace time by conformal time η along with corresponding derivatives. We define $\mu = \hat{\mathbf{p}}\cdot\mathbf{k}/k$ and assume that electrons' flow is irrotational that is $u_e(\mathbf{k}, \eta) \sim \mathbf{k}$ (according to [3][Chapter 3.1] rotational perturbations correspond to vector metric perturbations and do not grow with time). We introduce optical depth $\tau(\eta) = \int_\eta^{\eta_0} d\eta' n_e\sigma_T a \Rightarrow n_e\sigma_T a = -\tau'$. We obtain

$$\Theta' + ik\mu\Theta + \Phi' + ik\mu\Psi = -\tau'[\mu u_e - \Theta + \Theta_0] \tag{18}$$

Note that equation depends only on $\mathbf{k}$ and angle μ, thus, we can average over $\hat{\mathbf{p}}$ having same angle μ with $\mathbf{k}$ and pick an axis-symmetric Θ.

1.3 Evolution of cold dark matter

For massive particles we get minor modifications from 9

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{\hat{p}^i}{a}\frac{p}{E} - \frac{\partial f}{\partial p}p\left[H + \dot{\Phi} + \hat{p}^i\Psi_{,i}\frac{E}{ap}\right] \tag{19}$$

Dark matter doesn't interact with other particles and itself, collision integral is zero. Instead of assuming distribution's f form as in case of photons, we are going to employ hydrodynamic approach and consider perturbations in concentration n and flow velocity $\mathbf{u}$ which are defined as

$$n = \int\frac{d^3p}{(2\pi)^3}f; \quad u^i = \frac{1}{n}\int\frac{d^3p}{(2\pi)^3}\frac{p^i}{E(p)}f \tag{20}$$

In order to obtain hydrodynamic equations, we just multiply 19 by powers 1 or p^i/E, then integrate over all momenta, and neglect terms $O((p/E)^2)$. Derivation gives equations of continuity of matter and momentum

$$\frac{\partial n}{\partial t} + \frac{1}{a}\frac{\partial(nu^i)}{\partial x^i} + 3[H + \dot{\Phi}]n = 0 \tag{21}$$

$$\frac{\partial(nu^i)}{\partial t} + 4Hnu^i + \frac{n}{a}\frac{\partial\Psi}{\partial x^i} = 0 \tag{22}$$

Concentration can be expanded around average value as $n(\mathbf{x},t) = \bar{n}(t)[1+\delta(\mathbf{x},t)]$, while velocity $\mathbf{u}(\mathbf{x},t)$ is already a first order value. Zero order of 21 gives $\bar{n} \sim 1/a^3$. First order is

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\frac{\partial u^i}{\partial x^i} + 3\dot{\Phi} = 0 \tag{23}$$

$$\frac{\partial u^i}{\partial t} + Hu^i + \frac{1}{a}\frac{\partial \Psi}{\partial x^i} = 0 \tag{24}$$

Performing Fourier transformation, going to conformal time and assuming that $\mathbf{u}$ is irrotational, we obtain for CDM

$$\delta_c' + iku_c + 3\Phi' = 0 \tag{25}$$

$$u_c' + \frac{a'}{a}u_c + ik\Psi = 0 \tag{26}$$

1.4 Evolution of protons and electrons

In [4][Chapter 6.3] it was calculated that a typical transfer time of energy between protons and electrons due to Coulomb scattering is around $3 \cdot 10^4 s$ at recombination epoch which is much smaller than Hubble time and transfer time between photons and electrons due to Thomson scattering. Therefore, we will suppose that there is a tight coupling between protons and electrons, which forces $\mathbf{u}_e = \mathbf{u}_p \equiv \mathbf{u}_b$, where we have introduced common velocity $\mathbf{u}_b$ where b index historically means (incorrect but convenient) grouping of protons and electrons into "baryons". Moreover, coupling and overall electrical neutrality forces $\delta_b = (\rho_e - \bar{\rho}_e)/\bar{\rho}_e = (\rho_p - \bar{\rho}_p)/\bar{\rho}_p$.
Derivation of evolution equations from Boltzmann equation 19 proceeds in the same way, except that now there is a right side because of Thomson scattering of photons. Since integration of collision integral over all angles gives zero, an equation corresponding to density evolution remains the same as 25

$$\delta_b' + iku_b + 3\Phi' = 0 \tag{27}$$

If we multiply 19 by p^i instead of p^i/E and integrate over momentum because of non-relativity, we are going to get the same left part as in 22 only multiplied by mass of proton (which dominates over mass of electron)

$$m_p \frac{\partial (n_b u_b^i)}{\partial t} + 4Hm_p n_b u_b^i + \frac{m_p n_b}{a}\frac{\partial \Psi}{\partial x^i} = 2\int \frac{d^3p}{(2\pi)^3} C[f] p^i \tag{28}$$

Right part contains 2 factor because $n_e = 2\int \frac{d^3p}{(2\pi)^3} f$ where 2 corresponds to two spin states of an electron. Expanding around zero order and dividing by zero-order density $\rho_b = m_p \bar{n}_b$ leads to

$$\frac{\partial u_b^i}{\partial t} + Hu_b^i + \frac{1}{a}\frac{\partial \Psi}{\partial x^i} = \frac{2}{\rho_b}\int \frac{d^3p}{(2\pi)^3} C[f] p^i \tag{29}$$

Integral of collision integral multiplied by p^i is a momentum density which is transferred to baryons from the electrons. Because of momentum conservation, this term is opposite to same term but with photon collision integral 13. Using redefinition 14 the term simplifies to

$$\int \frac{d^3p}{(2\pi)^3} C[f] p^i = \int \frac{d^3p}{(2\pi)^3} p^i p \frac{\partial f^{(0)}}{\partial p} n_e \sigma_T \left[\hat{\mathbf{p}} \cdot \mathbf{u}_b - \Theta + \Theta_0\right] = \\ -2\rho_\gamma n_e \sigma_T \int \frac{d\hat{\mathbf{p}}}{4\pi} \hat{p}^i [\hat{\mathbf{p}} \cdot \mathbf{u}_b - \Theta + \Theta_0] \tag{30}$$

where ρ_γ is photons' energy density (same as $\rho^{(0)}$ in 14). Integral over Θ_0 term is zero. We compute $\int d\hat{\mathbf{p}}\hat{p}^i(\hat{\mathbf{p}}\cdot\mathbf{u}_b)/4\pi = \mathbf{u}_b/3$. According to discussion after equation 18, Θ in Fourier space can be picked axis-symmetric around $\mathbf{k}$. Therefore, $\int d\hat{\mathbf{p}}p^i\Theta(\mathbf{k},\hat{\mathbf{p}},\eta) \sim k^i$ and define first moment or dipole as

$$\Theta_1(\mathbf{k},\eta)\hat{k}^i = i\int\frac{d\hat{\mathbf{p}}}{4\pi}\hat{p}^i\Theta(\mathbf{k},\hat{\mathbf{p}},\eta) \Rightarrow \Theta_1(\mathbf{k},\eta) = \frac{i}{2}\int_{-1}^{1}\mu\Theta(\mathbf{k},\mu,\eta)d\mu \tag{31}$$

Going to Fourier space, from time to conformal time and assuming $u_b^i \sim k^i$ results in

$$u_b' + \frac{a'}{a}u_b + ik\Psi = \tau'\frac{4\rho_\gamma}{\rho_b}\left(i\Theta_1 + \frac{u_b}{3}\right) \tag{32}$$

1.5 Evolution of neutrinos

We proceed in analogy with photons, because neutrinos are ultra-relativistic, at least during recombination, by imposing deviation from the equilibrium distribution as

$$f_\nu(\mathbf{x},\mathbf{p},t) = \left[\exp\left(-\frac{p}{T_\nu(t)(1+\mathcal{N}(\mathbf{x},\mathbf{p},t))}\right)+1\right]^{-1} = f_\nu^{(0)}(p,t) - p\mathcal{N}\frac{\partial f_\nu^{(0)}}{\partial p} \tag{33}$$

Neutrinos do not interact with other particles during recombination and later epochs and collision integral is zero. To consider case of non-relativistic neutrinos at latest stages of Universe, we apply expansion of f_ν into 19 to get a non-relativistic analog of 12

$$\frac{\partial\mathcal{N}}{\partial t} + \frac{\hat{p}^i}{a}\frac{p}{E}\frac{\partial\mathcal{N}}{\partial x^i} - pH\frac{\partial\mathcal{N}}{\partial p} + \dot{\Phi} + \frac{E}{ap}\hat{p}^i\Psi_{,i} = 0 \tag{34}$$

In Fourier space and conformal time it's written as

$$\mathcal{N}' + ik\mu\frac{p}{E}\mathcal{N} - p\frac{a'}{a}\frac{\partial\mathcal{N}}{\partial p} + \Phi' + ik\mu\frac{E}{p}\Psi = 0 \tag{35}$$

Since $E(p)$ at later times deviates from $E=p$, one cannot average $\mathcal{N}$ perturbations over p like in photon's case. Nonetheless, one can make $\mathcal{N}$ axis-symmetric over $\mathbf{k}$ and consider it as a function $\mathcal{N}(\mathbf{k},p,\mu,\eta)$.

1.6 Einstein gravity

Einstein equations are

$$G^\mu_\nu = g^{\mu\sigma}\left(R_{\sigma\nu} - \frac{1}{2}g_{\sigma\nu}R\right) = 8\pi G T^\mu_\nu \tag{36}$$

Plugging 1 into definitions of Ricci tensor gives up to linear order in Fourier space

$$\delta G^0_0 = -6H\dot{\Phi} + 6\Psi H^2 - 2\frac{k^2\Phi}{a^2} \tag{37}$$

$$\delta G^i_j = F(\Phi,\Psi)\delta^i_j + \frac{k^i k_j(\Phi+\Psi)}{a^2} \tag{38}$$

Here $F(\Phi,\Psi)$ is a complicated function and since we need only two equations on an evolution of Φ and Ψ, we are going to consider a traceless longitudinal part

$$(\hat{k}_i\hat{k}^j - \frac{1}{3}\delta^j_i)\delta G^i_j = \frac{2k^2}{3a^2}(\Phi+\Psi) \tag{39}$$

In normal coordinates energy-momentum tensor is written in analogy with single particle energy-momentum tensor.

$$T_0^0 = -g \int \frac{d^3p}{(2\pi)^3} E(p) f(\mathbf{x}, \mathbf{p}, t) \tag{40}$$

$$T_j^i = g \int \frac{d^3p}{(2\pi)^3} \frac{p^i p^j}{E(p)} f(\mathbf{x}, \mathbf{p}, t) \tag{41}$$

Here g is spin degeneracy. Transformation of 4-vector to initial coordinates can be read from 3. Corresponding transformation of $(1,1)$ tensor acts on T_0^0 and T_j^i as an identity and the formulas remain same.
For massive non-relativistic particles up to first order in p/E, $E \approx m \Rightarrow T_0^0 = -mn = -\rho(1+\delta)$. For photons, since $T = \bar{T}(1+\Theta)$ and energy density $-T_0^0 \sim T^4$, integration over gives $T_0^0 = -\rho_\gamma(1+4\Theta_0)$. While neutrinos are massless, we have the same result. Spatial part is strongly suppressed for massive particles. For photons, we have

$$(\hat{k}_i \hat{k}^j - \frac{1}{3}\delta_i^j) T_j^i = 2 \int \frac{2\pi p^2 d\mu dp}{(2\pi)^3} \frac{p^2(\mu^2 - 1/3)}{p} f(\mathbf{k}, p, \mu, t) =$$
$$2 \int d\mu dp \frac{p^3}{4\pi^2}(\mu^2 - 1/3) \left(-p \frac{\partial f^{(0)}}{\partial p} \Theta(\mathbf{k}, p, \mu, t) \right) = \frac{8\rho_\gamma}{3} \int \frac{d\mu}{2} \frac{3\mu^2 - 1}{2} \Theta(\mathbf{k}, \mu, t) = -\frac{8\rho_\gamma}{3}\Theta_2 \tag{42}$$

where we have defined quadrupole Θ_2 (noting that $(3\mu^2-1)/2$ is second Legendre polynomial). Combining sorts of particles at the right side, we obtain first order equations for metric perturbations

$$k^2\Phi + 3\frac{a'}{a}\left(\Phi' - \frac{a'}{a}\Psi\right) = 4\pi G a^2 [\rho_c \delta_c + \rho_b \delta_b + 4\rho_\gamma \Theta_0 + 4\rho_\nu \mathcal{N}_0] \tag{43}$$

$$k^2(\Phi + \Psi) = -32\pi G a^2 [\rho_\gamma \Theta_2 + \rho_\nu \mathcal{N}_2] \tag{44}$$

2 CMB

In this section we are going to analyze how to solve the system of equations derived in the previous section, what observables can be extracted from CMB observations, and how these observables are connected with perturbations of matter and gravity.

2.1 CMB observations and theory

Telescope can, in principle, measure photon temperature fluctuations field $\Theta(\mathbf{n})$, where $\mathbf{n}$ is a normal vector to a sphere. Since it's a fluctuations field $\langle\Theta\rangle_{S^2} = 0$. It can then be expanded into spherical harmonics as

$$\Theta(\mathbf{n}) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\mathbf{n}) \tag{45}$$

We suppose that a_{lm} are uncorrelated random variables such that $\langle a^*_{l'm'} a_{lm} \rangle = C_l \delta_{ll'} \delta_{mm'}$. Dispersion doesn't depend on m since there is no preferred direction on sky. For large l, there are plenty different m to measure C_l as

$$C_l = \frac{1}{2l+1} \sum_{m=-l}^{l} \langle |a_{lm}|^2 \rangle \approx \frac{1}{2l+1} \sum_{m=-l}^{l} |a_{lm}|^2 \tag{46}$$

A relative standard deviation of such estimate is $1/\sqrt{l+1/2}$ and we are going to predict precisely C_l from theoretical considerations.
We can extract a_{lm} as

$$a_{lm} = \int d\Omega Y^*_{lm}(\mathbf{n})\Theta(0,\mathbf{n}) = \int \frac{d^3k}{(2\pi)^3} \int d\Omega Y^*_{lm}(\mathbf{n})\Theta(\mathbf{k},\mathbf{n}) \tag{47}$$

where we have returned to considering general position-dependent field $\Theta(\mathbf{x},\mathbf{n},\eta)$. Using definition of C_l we obtain

$$C_l = \int \frac{d^3kd^3k'}{(2\pi)^6} \int d\Omega d\Omega' Y_{lm}(\mathbf{n})Y^*_{lm}(\mathbf{n}')\langle\Theta^*(\mathbf{k},\mathbf{n})\Theta(\mathbf{k}',\mathbf{n}')\rangle \tag{48}$$

From theory of inflation, it's known that all initial values of matter and metric fields are derived from initial curvature perturbation $\mathcal{R}(\mathbf{k})$. Because of linearity, we can integrate Θ up to present time and write $\Theta(\mathbf{k},\mathbf{n}) = \mathcal{T}(k,\mu)\mathcal{R}(\mathbf{k})$, where $\mu = \hat{\mathbf{k}}\cdot\mathbf{n}$, $\mathcal{T}$ depends on k and μ only because the equation 18 depends on same variables. Using correlation function $\langle\mathcal{R}^*(\mathbf{k})\mathcal{R}(\mathbf{k}')\rangle = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')P_{\mathcal{R}}(k)$ one has

$$C_l = \int \frac{d^3k}{(2\pi)^3} P_{\mathcal{R}}(k) \int d\Omega d\Omega' Y_{lm}(\mathbf{n})Y^*_{lm}(\mathbf{n}')\mathcal{T}^*(k,\mu)\mathcal{T}^*(k',\mu') \tag{49}$$

Expand $\mathcal{T}(k,\mu)$ into multipoles such that

$$\mathcal{T}(k,\mu) = \sum_l (-i)^l(2l+1)P_l(\mu)\mathcal{T}_l(k) \tag{50}$$

where $P_l(\mu)$ is l-th Legendre polynomial. Correspondingly, we have $\Theta_l(k) = \mathcal{T}_l(k)\mathcal{R}(\mathbf{k})$. After doing integration and using properties of spherical harmonics and Legendre polynomials, expression greatly simplifies to

$$C_l = \frac{2}{\pi}\int dk k^2 P_{\mathcal{R}}(k)|\mathcal{T}_l(k)|^2 \tag{51}$$

Thus, we have to compute $\mathcal{T}_l(k)$ or, in other words, how $\Theta_l(k,\eta)$ evolves from given initial conditions.

2.2 Recombination

We are interested in an average density of free electrons n_e in order to calculate optical density, which is present in equations 18 and 32. Initially, when Universe is hot, electrons are decoupled from atoms' nucleus. Later on, more and more electrons couple to nucleus and create neutral atoms, lowering n_e to zero. In order to simplify discussion of this complex process called recombination, we are going to follow [1] and assume that helium is absent, therefore, baryon matter consists of protons and electrons only. Then we have an (effective) reaction $e + p \leftrightarrow H + \gamma$ that we are going to denote as $1 + 2 \leftrightarrow 3 + 4$ for now.
Consider a homogeneous variant of 21 with non-trivial right part

$$\begin{aligned}\frac{1}{a^3}\frac{d(n_1a^3)}{dt} = \int \frac{d^3p_1}{(2\pi)^3 2E_1} \int \frac{d^3p_2}{(2\pi)^3 2E_2} \int \frac{d^3p_3}{(2\pi)^3 2E_3} \int \frac{d^3p_4}{(2\pi)^3 2E_4} \\ (2\pi)^4\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)\delta(E_1+E_2-E_3-E_4)|\mathcal{M}|^2 \\ [f_3f_4(1\pm f_1)(1\pm f_2) - f_1f_2(1\pm f_3)(1\pm f_4)]\end{aligned} \tag{52}$$

Here f_i denotes homogeneous in space distribution $f_i(\mathbf{p}_i, t)$ for i-th particle and $\mathcal{M}$ is scattering amplitude. Recombination happens when energies of all particles (including photon having hydrogen ionization energy) are much larger than temperature. Then Boson-Einstein and Dirac distributions reduce to Maxwell-Boltzmann as $f_i \approx e^{\mu_i/T} e^{-E_i/T}$. We can rewrite

$$[f_3 f_4 (1 \pm f_1)(1 \pm f_2) - f_1 f_2 (1 \pm f_3)(1 \pm f_4)] \approx e^{-(E_1+E_2)/T} \left[e^{(\mu_3+\mu_4)/T} - e^{(\mu_1+\mu_2)/T} \right] \quad (53)$$

Particle concentration is defined as

$$n_i = g_i \int \frac{d^3p}{(2\pi)^3} f_i(\mathbf{p}, t) \approx g_i e^{\mu_i/T} \int \frac{d^3p}{(2\pi)^3} e^{-E_i/T} \quad (54)$$

We define particle concentration at $\mu_i = 0$ as $n_i^{(0)} = n_i|_{\mu_i=0}$. Equation 52 simplifies to

$$\frac{1}{a^3} \frac{d(n_1 a^3)}{dt} = n_1^{(0)} n_2^{(0)} \langle \sigma v \rangle \left[\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right] \quad (55)$$

where average cross section $\langle \sigma v \rangle$ is defined as

$$\langle \sigma v \rangle = \frac{1}{n_1^{(0)} n_2^{(0)}} \int \frac{d^3p_1}{(2\pi)^3 2E_1} \int \frac{d^3p_2}{(2\pi)^3 2E_2} \int \frac{d^3p_3}{(2\pi)^3 2E_3} \int \frac{d^3p_4}{(2\pi)^3 2E_4} e^{-(E_1+E_2)/T}$$
$$(2\pi)^4 \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \delta(E_1 + E_2 - E_3 - E_4) |\mathcal{M}|^2 \quad (56)$$

For recombination reaction $e + p \leftrightarrow H + \gamma$ we obtain equation

$$\frac{1}{a^3} \frac{d(n_e a^3)}{dt} = n_e^{(0)} n_p^{(0)} \langle \sigma v \rangle \left[\frac{n_\gamma n_H}{n_\gamma^{(0)} n_H^{(0)}} - \frac{n_e n_p}{n_e^{(0)} n_p^{(0)}} \right] \quad (57)$$

From electrical neutrality of the universe $n_e = n_p$. Define free electron fraction as $X_e = n_e/(n_e + n_H) = n_p/(n_p + n_H)$. Since total mass in a comoving volume $m_p(n_p + n_H)a^3 = m_p n_b a^3$ is saved, we can rewrite left part as

$$\frac{1}{a^3} \frac{d(X_e(n_p + n_H)a^3)}{dt} = \frac{a^3(n_p + n_H)}{a^3} \frac{dX_e}{dt} = n_b \frac{dX_e}{dt} \quad (58)$$

For non-relativistic particles, we can compute

$$n^{(0)} \approx g e^{-m/T} \int \frac{d^3p}{(2\pi)^3} e^{-p^2/2mT} = g \left(\frac{mT}{2\pi} \right)^{3/2} e^{-m/T} \quad (59)$$

Real concentrations $n^{(0)}$ are written through baryon density n_b and free electron fraction as $n_e = n_p = n_b X_e$ and $n_H = n_b(1 - X_e)$. Photon has zero chemical potential and, therefore, $n_\gamma / n_\gamma^{(0)} = 1$. Combining and simplifying we obtain

$$\frac{dX_e}{dt} = \langle \sigma v \rangle \left[\left(\frac{m_e T}{2\pi} \right)^{3/2} e^{-\Delta/T} (1 - X_e) - n_b X_e^2 \right] \quad (60)$$

where $\Delta = m_e + m_p - m_H$ is an ionization energy. In order to compute cross-section, one has to consider complicated set of reactions between different excited states of hydrogen atom and their decay rates computed from QED, sketch of the computation can be found in [7]. Instead, we are going to use an approximation from [5] which works well for recombination temperatures:

$$\langle \sigma v \rangle \approx 9.78 \frac{\alpha^2}{m_e^2} \left(\frac{\Delta}{T} \right)^{1/2} \ln \frac{\Delta}{T} \quad (61)$$

Finally, differential optical depth $-\tau'$ can be computed $-\tau' = n_e \sigma_T a = n_b X_e \sigma_T a$. Probability density of a photon being last time scattered at η is $g(\eta) = (e^{-\tau})' = (-\tau')e^{-\tau}$.

2.3 Solving evolution equations

Equation 18 can be written as

$$\Theta' + ik\mu\Theta - \tau'\Theta = -\tau'[\mu u_b + \Theta_0] - \Phi' - ik\mu\Psi \equiv S(\mathbf{k}, \mu, \eta) \tag{62}$$

and formally solved as

$$\Theta(\mathbf{k}, \mu, \eta_0) = \int_0^{\eta_0} d\eta S(\mathbf{k}, \mu, \eta) e^{ik\mu(\eta-\eta_0)-\tau(\eta)} \tag{63}$$

Here the lower bound of integration can be taken equal to zero because at small η, $\tau(\eta)$ is very large and initial part at some staring time η_1 vanishes as $\Theta|_{\eta_1} e^{-\tau(\eta_1)} \to 0$. One can make S be independent of μ under the integral using integration by parts and $\mu e^{ik\mu(\eta-\eta_0)} = (1/ik) \cdot d(e^{ik\mu(\eta-\eta_0)})/d\eta$. Thus, we can consider solution

$$\Theta(\mathbf{k}, \mu, \eta_0) = \int_0^{\eta_0} d\eta S(k, \eta) e^{ik\mu(\eta-\eta_0)}; \quad S(k, \eta) = \frac{d}{d\eta}\left[e^{-\tau}\left(\Psi - \frac{i\tau' u_b}{k}\right)\right] - (\tau'\Theta_0 + \Phi')e^{-\tau} \tag{64}$$

We define multipole expansion of $\Theta(k, \mu)$ as

$$\Theta_l(k) = \frac{1}{(-i)^l} \int_{-1}^{1} \frac{d\mu}{2} P_l(\mu)\Theta(k, \mu) d\mu \tag{65}$$

which is consistent with 50. Expanding $e^{ik\mu(\eta-\eta_0)}$ and using odd/even property of spherical Bessel functions j_l, we obtain

$$\Theta_l(k) = \int_0^{\eta_0} d\eta S(k, \eta) j_l[k(\eta_0 - \eta)] \tag{66}$$

We have pushed problem of finding Θ_l onto computing $S(k, \eta)$. Let us obtain a hierarchy of differential equations describing Θ_l evolution, which can be obtained directly from 18, the multipole's definition 65 and Legendre polynomials recursion formula $(2l + 1)\mu P_l(\mu) = (l + 1)P_{l+1}(\mu) + lP_{l-1}(\mu)$

$$\Theta_0' = -k\Theta_1 - \Phi' \tag{67}$$

$$\Theta_1' = \frac{k}{3}[\Theta_0 - 2\Theta_2 + \Psi] + \tau'\left[\Theta_1 - \frac{iu_b}{3}\right] \tag{68}$$

$$\Theta_l' = \tau'\Theta_l + \frac{k}{2l+1}[l\Theta_{l-1} - (l + 1)\Theta_{l+1}] \tag{69}$$

The problem is that evolution of Θ_0 (which $S(k, \eta)$ depends on) couples to Θ_1, which couples to Θ_2 and so on. Paper [8], however, claims that $S(k, \eta)$ is slowly varying and it is sufficient to take several l to get a good approximation. In order to get a decent truncation of the hierarchy, we employ idea from [5] by noting that $\Theta_l \sim j_l(k\eta)e^{-\tau(\eta)}$ automatically satisfies equation 69. Assuming that this is a correct asymptotic at large l, one uses recurrence relation for spherical Bessel functions to approximate

$$\Theta_{l_{\max}+1} \approx \frac{2l_{\max} + 1}{k\eta}\Theta_{l_{\max}} - \Theta_{l_{\max}-1} \tag{70}$$

Then last equation of 69 at $l = l_{\max}$ becomes

$$\Theta_{l_{\max}}' \approx k\Theta_{l_{\max}-1} + \Theta_{l_{\max}}\left[\tau' - \frac{l_{\max} + 1}{\eta}\right] \tag{71}$$

One uses an analogous technique for neutrinos. Dark matter, baryon and metric perturbations equations are solved in a straightforward way.

3 Power spectrum

From theory of previous two sections, we should be able to completely determine an evolution of metric and matter perturbations at linear order. At this order all perturbations are proportional to initial curvature perturbation $\mathcal{R}(\mathbf{k})$, where proportionality coefficient is a transfer function $\mathcal{T}(\mathbf{k},\eta)$ and is determined from solving a system of coupled differentia equations. Then all perturbations have the Gaussian spectrum

$$\langle \delta^*(\mathbf{k},\eta)\delta(\mathbf{k}',\eta)\rangle = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')P(\mathbf{k},\eta); \quad P(\mathbf{k},\eta) = P_{\mathcal{R}}(k)|\mathcal{T}(\mathbf{k},\eta)|^2 \tag{72}$$

where $P_{\mathcal{R}}(k)$ is the spectrum of primordial curvature perturbations and δ is a placeholder for different (matter and gravity) perturbations. Thus, spectrum $P(\mathbf{k},\eta)$ should contain if not all (when we enter non-linear regime), but a lot of information on evolution of perturbations and it would be desirable to estimate it. In this section we analyze how to obtain power spectrum $P(k)$ of matter perturbations from large galaxy surveys.

3.1 Band estimate

Following [6][Chapter 33] we suppose that galaxies are sampled following two-stage random process:

1. Sample random field $\rho(\mathbf{r}) = \bar{n}(\mathbf{r})(1+\delta(\mathbf{r}))$, where $\bar{n}(\mathbf{r})$ is an average observed density of galaxies, which can depend on selection criteria, and $\delta(\mathbf{r})$ are density perturbations. These perturbations satisfy $\langle\delta^*(\mathbf{k})\delta(\mathbf{k}')\rangle = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')P(k)$ and $P(k)$ is precisely the spectrum that we would like to estimate.

2. Sample galaxies in each small volume δV as a Poisson random process with parameter $\lambda = \rho(\mathbf{r})\delta V$. Sampling is uncorrelated in different volumes, in some sense all correlations were encoded into $\rho(\mathbf{r})$. These galaxies form an empirical density $n(\mathbf{r}) = \sum_\alpha \delta(\mathbf{r}-\mathbf{r}_\alpha)$, where $\mathbf{r}_\alpha$ is a position of each galaxy.

As our final goal is to estimate a handful of cosmological parameters, we have to compute likelihood of an observed distribution $n(\mathbf{r})$. Unfortunately, this is computationally expensive due to large number of observed galaxies. Since cosmological parameters are not numerous, we can expect that we are able to compress this data into much smaller amount of empirical value and still get roughly the same quality estimates. We are going to consider a family of band estimate methods which employ this technique. Discussion is mainly based on [9].

Main observables are called band estimates q_i defined as

$$q_i = \int d\mathbf{r}d\mathbf{r}' E_i(\mathbf{r},\mathbf{r}')\frac{n(\mathbf{r})n(\mathbf{r}')}{\bar{n}(\mathbf{r})\bar{n}(\mathbf{r}')} = \sum_{\alpha,\beta}\frac{E_i(\mathbf{r}_\alpha,\mathbf{r}_\beta)}{\bar{n}(\mathbf{r}_\alpha)\bar{n}(\mathbf{r}_\beta)} \tag{73}$$

In order to compute its mathematical expectation $\langle q_i\rangle$ we consider an integral and rewrite it as a sum over small volumes δV_i of equal volume $\delta V \to 0$

$$\begin{aligned}\int d\mathbf{r}d\mathbf{r}' g(\mathbf{r},\mathbf{r}')\langle n(\mathbf{r})n(\mathbf{r}')\rangle &= \lim_{\delta V\to 0}\sum_{i,j} g(\mathbf{r}_i,\mathbf{r}_j)\mathbb{E}\left[\mathbb{E}[n_i n_j|\rho(\mathbf{r}_i),\rho(\mathbf{r}_j)]\right] = \\ &\lim_{\delta V\to 0}\sum_{i,j} g(\mathbf{r}_i,\mathbf{r}_j)\mathbb{E}\left[(\rho(\mathbf{r}_i)\delta V_i + \rho(\mathbf{r}_i)^2\delta V_i^2)\delta_{ij} + \rho(\mathbf{r}_i)\rho(\mathbf{r}_i)\delta V_i\delta V_j\right] = \\ &\int d\mathbf{r}d\mathbf{r}' g(\mathbf{r},\mathbf{r}')\left[\delta(\mathbf{r}-\mathbf{r}')\langle\rho(\mathbf{r})\rangle + \langle\rho(\mathbf{r})\rho(\mathbf{r}')\rangle\right] = \\ &\int d\mathbf{r}d\mathbf{r}' g(\mathbf{r},\mathbf{r}')\left[\delta(\mathbf{r}-\mathbf{r}')\bar{n}(\mathbf{r}) + \bar{n}(\mathbf{r})\bar{n}(\mathbf{r}')(1+\langle\delta(\mathbf{r})\delta(\mathbf{r}')\rangle)\right]\end{aligned} \tag{74}$$

Applying to q_i we obtain

$$\langle q_i\rangle = \int d\mathbf{r}d\mathbf{r}' E_i(\mathbf{r},\mathbf{r}') + \int d\mathbf{r}\frac{E_i(\mathbf{r},\mathbf{r})}{\bar{n}(\mathbf{r})} + \int d\mathbf{r}d\mathbf{r}' E_i(\mathbf{r},\mathbf{r}')\langle\delta(\mathbf{r})\delta(\mathbf{r}')\rangle \tag{75}$$

Transforming to Fourier space $\hat{E}_i(\mathbf{k},\mathbf{k}') = \int E_i(\mathbf{r},\mathbf{r}')e^{-i\mathbf{k}\cdot\mathbf{r}+i\mathbf{k}'\cdot\mathbf{r}'}d\mathbf{r}d\mathbf{r}'$ and defining $W_i(\mathbf{k}) = \hat{E}_i(\mathbf{k},\mathbf{k})$ we get

$$\langle q_i\rangle = W_i(0) + \int d\mathbf{r}\frac{E_i(\mathbf{r},\mathbf{r})}{\bar{n}(\mathbf{r})} + \int W_i(\mathbf{k})P(k)\frac{d^3k}{(2\pi)^3} \tag{76}$$

In traditional methods one takes $E_i(\mathbf{r},\mathbf{r}') = \psi_i(\mathbf{r})\psi_i(\mathbf{r}')^*$. Then $W_i(\mathbf{k}) = |\hat{\psi}_i(\mathbf{k})|^2$ and judging from formula 76 we expect that if we find a function $\psi_i(\mathbf{r})$ well localized in Fourier space (for instance around a wave vector $\mathbf{k}$), we can directly probe $P(k)$. One takes then $\psi_i(\mathbf{r}) = e^{i\mathbf{k}_i\cdot\mathbf{r}}\phi(\mathbf{r})$, where $\phi(\mathbf{r})$ is a slowly varying function along survey's volume and $\mathbf{k}_i$ is a chosen grid of probed wave vectors. In a following subsection, we are going to analyze a particular method which chooses such $\phi(\mathbf{r})$ that variation of band estimate q_i is minimized.

3.2 FKP method

In order to get rid of $W_i(0)$ in 76 authors of FKP method [2] use technique of mock or synthetic catalogs. In equation 73, $n(\mathbf{r})$ is changed to $n_g(\mathbf{r}) - \alpha n_s(\mathbf{r})$, where $n_g(\mathbf{r})$ is an observed empirical galaxy distribution and $n_s(\mathbf{r})$ is an empirical distribution of galaxies in synthetic catalog. It's characterized by $\bar{n}(\mathbf{r}) = \alpha\bar{n}_s(\mathbf{r})$ and absence of perturbations. We take $E_i(\mathbf{r},\mathbf{r}') = \phi(\mathbf{r})\phi(\mathbf{r}')e^{i\mathbf{k}_i(\mathbf{r}-\mathbf{r}')}$ according to the discussion above. Making a similar to 76 computation we obtain

$$\langle q_i\rangle = (1+\alpha)\int d\mathbf{r}\frac{\phi^2(\mathbf{r})}{\bar{n}(\mathbf{r})} + \int |\hat{\phi}(\mathbf{k}-\mathbf{k}_i)|^2 P(k)\frac{d^3k}{(2\pi)^3} \tag{77}$$

Since $\phi(\mathbf{r})$ varies slowly on a scale of survey volume, $|\hat{\phi}(\mathbf{k}-\mathbf{k}_i)|^2$ is highly concentrated around $\mathbf{k} \approx \mathbf{k}_i$ and $P(k)$ can be moved out of the integral as a constant if it varies slowly enough. Using Parseval theorem one gets

$$\langle q_i\rangle \approx \int \phi^2(\mathbf{r})\left[\frac{1+\alpha}{\bar{n}(\mathbf{r})} + P(\mathbf{k}_i)\right]d\mathbf{r} \tag{78}$$

We normalize $\int \phi^2(\mathbf{r})d\mathbf{r} = 1$. Then we can estimate $P(\mathbf{k})$ as

$$\hat{P}(\mathbf{k}) = \hat{q}_i - (1+\alpha)\int \frac{\phi^2(\mathbf{r})}{\bar{n}(\mathbf{r})}; \quad \hat{q}_i = \left|\sum_{\beta\in g}\frac{\phi(r_\beta)e^{i\mathbf{k}\cdot\mathbf{r}_\beta}}{\bar{n}(\mathbf{r}_\beta)} - \alpha\sum_{\beta\in s}\frac{\phi(r_\beta)e^{i\mathbf{k}\cdot\mathbf{r}_\beta}}{\bar{n}(\mathbf{r}_\beta)}\right|^2 \tag{79}$$

One then averages over direction of $\mathbf{k_i}$ and over a shell of volume V_k of thickness $|k_i - k_{i+1}| \gg 1/L$, where L is survey's typical size, estimating

$$\hat{P}(k) = \frac{1}{V_k}\int d\mathbf{k}'\hat{P}(\mathbf{k}') \tag{80}$$

If one defines $\delta\hat{P}(\mathbf{k}) = \hat{P}(\mathbf{k}) - P(\mathbf{k})$ and in analogy for averages estimate $\hat{P}(k)$, dispersion of $\hat{P}(k)$ estimate can be written as

$$\langle\delta\hat{P}(k)^2\rangle = \frac{1}{V_k^2}\int d\mathbf{k}d\mathbf{k}'\langle\delta\hat{P}(\mathbf{k})\delta\hat{P}(\mathbf{k}')\rangle \tag{81}$$

Note that estimate 79 can be written as

$$\hat{P}(\mathbf{k}) = |F(\mathbf{k})|^2 - P_{\text{shot}}; \quad F(\mathbf{k}) = \int d\mathbf{r}\frac{\phi(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}}{\bar{n}(\mathbf{r})}[n_g(\mathbf{r}) - \alpha n_s(\mathbf{r})]; \quad P_{\text{shot}} = (1+\alpha)\int \frac{\phi^2(\mathbf{r})}{\bar{n}(\mathbf{r})} \tag{82}$$

We assume that $F(\mathbf{k})$ is a Gaussian variable, which is justified by gaussianity of perturbations and by law of large numbers when galaxy survey is sufficiently numerous. Using the fact that $\langle\hat{P}(\mathbf{k})\rangle = P(\mathbf{k})$ and Wick's theorem, one rewrites correlation of error estimates

$$\langle\delta\hat{P}(\mathbf{k})\delta\hat{P}(\mathbf{k}')\rangle = \langle F(\mathbf{k})F^*(\mathbf{k})F(\mathbf{k}')F^*(\mathbf{k}')\rangle - \langle F(\mathbf{k})F^*(\mathbf{k})\rangle\langle F(\mathbf{k}')F^*(\mathbf{k}')\rangle = \\ |\langle F(\mathbf{k})F^*(\mathbf{k}')\rangle|^2 + |\langle F(\mathbf{k})F(\mathbf{k}')\rangle|^2 \tag{83}$$

Averages can be computed in analogy with 77 to get

$$\langle F(\mathbf{k})F^*(\mathbf{k}')\rangle = (1+\alpha)\int \frac{\phi^2(\mathbf{r})e^{i\mathbf{r}\cdot(\mathbf{k}-\mathbf{k}')}}{\bar{n}(\mathbf{r})}d\mathbf{r} + \int \hat{\phi}(\mathbf{k}-\mathbf{k}'')\hat{\phi}(\mathbf{k}'-\mathbf{k}'')\frac{P(\mathbf{k}'')d\mathbf{k}''}{(2\pi)^3} \approx \\ (1+\alpha)\int \frac{\phi^2(\mathbf{r})e^{i\mathbf{r}\cdot(\mathbf{k}-\mathbf{k}')}}{\bar{n}(\mathbf{r})}d\mathbf{r} + P(k)\int \phi^2(\mathbf{r})e^{i\mathbf{r}\cdot(\mathbf{k}-\mathbf{k}')}d\mathbf{r} \tag{84}$$

where we have again used that $P(k)$ is slowly changing, while $\hat{\phi}$ are tightly concentrated around zero. Second term is anagolous, changing $\mathbf{k}' \to -\mathbf{k}'$. Due to spherical symmetry of shell both terms make same contribution when averaged over shell and we obtain

$$\langle\delta\hat{P}(k)^2\rangle = \frac{2}{V_k^2}\int d\mathbf{k}d\mathbf{k}'\int d\mathbf{r}U(\mathbf{r})e^{i\mathbf{r}(\mathbf{k}-\mathbf{k}')}\int d\mathbf{r}'U(\mathbf{r})e^{-i\mathbf{r}'(\mathbf{k}-\mathbf{k}')}; \; U(\mathbf{r}) = \phi^2(\mathbf{r})\left[P(k) + \frac{1+\alpha}{\bar{n}(\mathbf{r})}\right] \tag{85}$$

Since $U(\mathbf{r})$ is slowly varying and only neighborhood $\mathbf{k} \approx \mathbf{k}'$ contributes, we can change integration in wave vector $\mathbf{k}$ over whole volume. Integration over $\mathbf{k}'$ gives V_k, resulting integral greatly simplifies to

$$\langle\delta\hat{P}(k)^2\rangle = \frac{2}{V_k^2}\int d\mathbf{k}|\hat{U}(\mathbf{k})|^2 = \frac{2}{V_k^2}\int d\mathbf{r}\phi^4(\mathbf{r})\left[P(k) + \frac{1+\alpha}{\bar{n}(\mathbf{r})}\right]^2 \tag{86}$$

Minimizing the dispersion with respect to $\phi(\mathbf{r})$ normalization of L_2-norm we get that

$$\phi(\mathbf{r}) \sim \frac{\bar{n}(\mathbf{r})P(k)}{1+\alpha+\bar{n}(\mathbf{r})P(k)} \approx \frac{\bar{n}(\mathbf{r})P(k)}{1+\bar{n}(\mathbf{r})P(k)} \tag{87}$$

since usually one takes $\alpha \ll 1$.

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