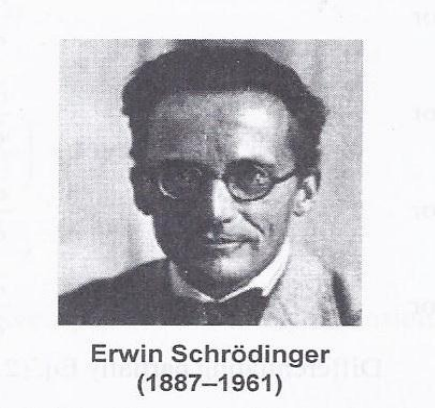
Schrodinger's Wave Equations

To describe the motion of a particle of atomic dimensions in space and time, Schrodinger derived differential equations, which control the space-time behaviour of the wave function ψ of de Broglie waves associated with that particle. These differential equations are known as Schrodinger's wave equations.



Erwin Schrodinger, the famous Austrian physicist, contributed to the creation of quantum mechanics and formulated the famous Schrodinger wave equation.

Derivation of time-independent and time-dependent one-dimensional Schrodinger's wave equations

Suppose that $\psi(r, t)$ is the wave displacement of de Broglie waves associated with a moving particle of rest mass m_0. Then the one-dimensional general wave-equation for these waves can be expressed as

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} \qquad (1)$$

where v is the particle velocity.

The general solution of Eq. (1) is given by

$$\psi = \exp\{i(kx - \omega t)\} \qquad (2)$$

where ω is the angular frequency and k is the propagation constant

If E is the energy and p the momentum of the particle, then we have

$$E = h\nu = \frac{h}{2\pi} 2\pi\nu = \hbar\omega \qquad \left(since\ \hbar = \frac{h}{2\pi}\ and\ \omega = 2\pi\upsilon \right)$$

and $$p = \frac{h}{\lambda} = \frac{h}{2\pi}\frac{2\pi}{\lambda} = \hbar k \qquad \left(since\ \hbar = \frac{h}{2\pi}\ and\ k = \frac{2\pi}{\lambda} \right)$$

Using these relations in Eq. (2), we have

$$\psi = \exp\left\{ \frac{i}{\hbar}\left(px - Et \right) \right\} \qquad (3)$$

Differentiating partially Eq. (3) w.r.t. x, we have

$$\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}\left[\exp\left\{ \frac{i}{\hbar}\left(px - Et \right) \right\} \right]$$

or $$\frac{\partial \psi}{\partial x} = \frac{ip}{\hbar}\exp\left\{ \frac{i}{\hbar}\left(px - Et \right) \right\}$$

or $$\frac{\partial \psi}{\partial x} = \frac{ip}{\hbar}\psi \qquad (4)$$

Differentiating partially Eq. (4) w.r.t. x, we have

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{ip}{\hbar}\psi \right)$$

or $$\frac{\partial^2 \psi}{\partial x^2} = \frac{ip}{\hbar}\frac{\partial \psi}{\partial x}$$

or $$\frac{\partial^2 \psi}{\partial x^2} = \frac{ip}{\hbar}\left(\frac{ip}{\hbar} \right)\psi$$ [from Eq. (4)]

or $$\frac{\partial^2\psi}{\partial x^2} = \left(\frac{ip}{\hbar}\right)^2 \psi$$

or $$\frac{\partial^2\psi}{\partial x^2} = -\frac{p^2}{\hbar^2}\psi \qquad (5)$$

Differentiating partially Eq. (5) w.r.t. time t, we gave

$$\frac{\partial\psi}{\partial t} = \frac{\partial}{\partial t}\left[\exp\left\{\frac{i}{\hbar}(px - Et)\right\}\right]$$

or $$\frac{\partial\psi}{\partial t} = \frac{-iE}{\hbar}\exp\left\{\frac{i}{\hbar}(px - Et)\right\}$$

or $$\frac{\partial\psi}{\partial t} = \frac{-iE}{\hbar}\psi \qquad (6)$$

The energy of a particle is expressed as

$$E = KE + PE$$

i.e. $$E = \frac{p^2}{2m_0} + V \qquad (7)$$

Operating wave function ψ on both sides of Eq. (7) , we have

$$E\psi = \frac{p^2\psi}{2m_0} + V\psi \qquad (8)$$

Using Eq. (5) in Eq. (8), we get

$$E\psi = \frac{1}{2m_0}\left(-\hbar^2\frac{\partial^2\psi}{\partial x^2}\right) + V\psi$$

$$E\psi - V\psi = \frac{-\hbar^2}{2m_0}\frac{\partial^2\psi}{\partial x^2}$$

Or $$\frac{-2m_0}{\hbar^2}(E-V)\psi = \frac{\partial^2\psi}{\partial x^2}$$

or $$\frac{\partial^2\psi}{\partial x^2} + \frac{2m_0}{\hbar^2}(E-V)\psi = 0$$

This is the one-dimensional time-independent Schrodinger's wave equation. The three-dimensional generalization of this equation is given by

$$\nabla^2\psi + \frac{2m_0}{\hbar^2}(E-V)\psi = 0$$

Now, using Eqs. (5) and (6) in Eq. (8), we get

$$-\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = \frac{1}{2m_0}\left(-\hbar^2\frac{\partial^2\psi}{\partial x^2}\right) + V\psi$$

i.e., $$i\hbar\frac{\partial\psi}{\partial t} = \frac{-\hbar^2}{2m_0}\frac{\partial^2\psi}{\partial x^2} + V\psi$$

This is the one-dimensional time-dependent Schrodinger's wave equation. The three dimensional generalization of this equation is given by

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{-\hbar^2}{2m_0}\nabla^2\psi + V\psi$$

Significance of Schrodinger's wave equation

Schrodinger presented his famous wave equation as a development of de Broglie ideas of the wave properties. Schrodinger's equation is the fundamental equation of wave mechanics in the same sense as the Newton's second law of motion of classical mechanics. It is the differential equation for the de Broglie waves associated with particles and it describes the motion of particles.