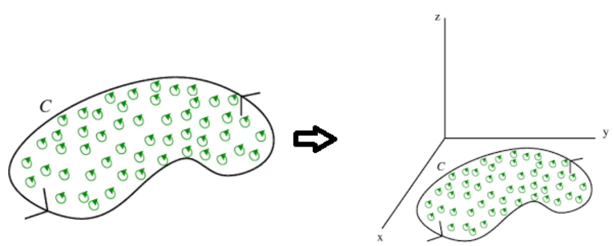
**Subject** : Calculus

**Question** : How to do stoke's theorem? Explain using a problem.

**Answer** :

Stoke's theorem was named after Irish mathematician Sir George Gabriel Stokes.

Stoke's theorem is a generalization of Green's theorem which determines the circulation in a planar region to circulation along a surface in three dimensions.




**Statement of Stoke's theorem:**

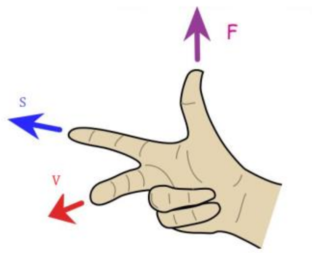
Stoke's theorem states that the surface integral of curl F over a surface S is the vector field F around the boundary of the surface.

Let *S* be an oriented smooth surface that is bounded by a simple, closed, smooth boundary curve *C* with positive orientation with F as a vector field, then

$$\oint_C F \cdot ds = \iint_S curl F \cdot \hat{n} dS \quad \text{--->1}$$

Where n is a unit normal vector to the surface. Applying right hand rule when we point the thumb of right hand perpendicular to a surface, fingers will curl in a direction parallel to the surface.




The relationship between the curve C and the surface S is identical to the relationship between the curve C and the region D as in Green's theorem.

$$\oint_C F \cdot ds = \iint_D (CurlF) \cdot \hat{n} dA$$

$$\oint_C F \cdot ds = \iint_S curlF \cdot \hat{n} dS$$

Hence Stoke's theorem is written as,

$$\oint_C F \cdot ds = \iint_S curlF \cdot \hat{n} dS = \iint_S curlF \cdot dS \text{ ---->2}$$

The integral on the right is the surface integral of the vector field curl F.

**Problem:**

Now let's see an illustrative problem to verify the Stokes theorem for the vector **F** taken round the rectangle bounded by the lines x given as below.

$$\vec{F} = (x^2 + y^2)\hat{i} - 2xy\,j$$

$$x = \pm a, y = 0, y = b$$

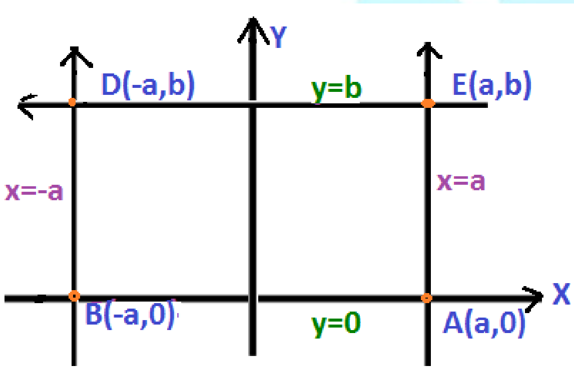
**Solution:**

To prove, $\oint_C \vec{F}.d\vec{r} = \iint_S curl\vec{F}.ndS$

Let C denote the boundary of the rectangle.

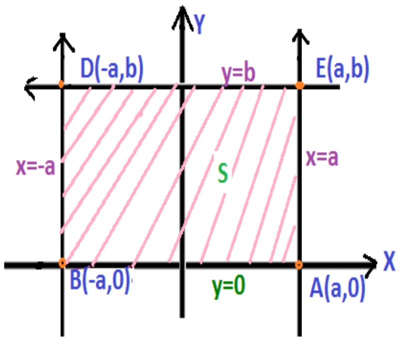
Given, $x = \pm a$, which implies $\Rightarrow x = a, x = -a$ and also $y = 0, y = b$

ABDE be the rectangle. The curve consists of four lines $AB$, $BD$, $DE$ and $EA$.




$S$ be the surface enclosed by the rectangular lines.




Now let us find the closed integral over **F** vector dot d r. vector

$$\oint_C \vec{F}.d\vec{r} = \oint_C \left[(x^2+y^2)\hat{i} - 2xyj\right].\left[\hat{i}dx + jdy\right]$$

We get the dot product value by multiplying the corresponding i, j and k terms.

$$\oint_C \vec{F}.d\vec{r} = \oint_C (x^2+y^2)dx - 2xydy$$

Along $AE$,

$x = a$

This implies that $\Rightarrow dx = 0$

$y$ varies from $0$ to $b$.

Now let us find the value of **F** vector dot d r vector along this line.

$$\int_{AE} \vec{F}.d\vec{r} = \int_{AE} (x^2+y^2)dx - 2xydy$$

We substitute the values of x equal to **a**, dx equal to zero and then differentiate it with respect to x.

$$= \int_0^b (a^2+y^2)(0) - 2aydy$$

Let's apply the limit values and simplify it $= -2a\left[\dfrac{y^2}{2}\right]_0^b$

Along the line AE we get, $\int\limits_{AE} \vec{F}.d\vec{r} = -ab^2$

Along the line ED, $y = b$

This implies that, $\Rightarrow dy = 0$

Here x varies from $a$ to $-a$.

Now let us find the value of **F** vector dot d r vector along this line,

$$\int\limits_{ED} \vec{F}.d\vec{r} = \int\limits_{ED} (x^2 + y^2)dx - 2xydy$$

We substitute the values of y equal to **b**, dy equal to zero and then differentiate it with respect to y $= \int\limits_{a}^{-a} (x^2 + b^2)dx - 2xb(0)$

Let's apply the limit values and simplify it, $= \left[\dfrac{x^3}{3} + b^2 x\right]_a^{-a}$

Along the line ED, we get the value as, $\int\limits_{ED} \vec{F}.d\vec{r} = -\dfrac{2a^2}{3} - 2ab^2$

Along the line DB, $x = -a$

This implies that dx value $\Rightarrow dx = 0$

Y varies from $b$ to $0$

Now let us find the value of **F** vector dot d r vector along this line,

$$\int_{DB} \vec{F}.d\vec{r} = \int_{DB} (x^2 + y^2)dx - 2xydy$$

We substitute the values of x equal to **minus a**, dx equal to zero and then differentiate it with respect to x $= \int_b^0 ((-a)^2 + y^2)(0) - 2(-a)ydy$

Let's apply the limit values and simplify it, $= 2a\left[\frac{y^2}{2}\right]_b^0$

Along the line DB, we get the value as, $\int_{DB} \vec{F}.d\vec{r} = -ab^2$

Along the line BA, $y = 0$

This implies that $\Rightarrow dy = 0$,

x varies from $-a$ to $a$.

Now let us find the value of F vector dot d r vector along this line,

$$\int_{BA} \vec{F}.d\vec{r} = \int_{BA} (x^2 + y^2)dx - 2xydy$$

We substitute the values of y equal to zero, dy equal to zero and then differentiate it with respect to y.

$$= \int_a^{-a} (x^2 + 0)dx - 2x(0)$$

Along the line BA, we get the value as, $\int_{BA} \vec{F}.d\vec{r} = \frac{2a^3}{3}$,

Now we shall add all the four lines,

$$\oint_C \vec{F}.d\vec{r} = \int_{AE} \vec{F}.d\vec{r} + \int_{ED} \vec{F}.d\vec{r} + \int_{DB} \vec{F}.d\vec{r} + \int_{BA} \vec{F}.d\vec{r}$$

We can simplify by cancelling the similar terms with different sign and adding the remaining,

$$= -ab^2 - \frac{2a^2}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3},$$

On simplifying we get the answer as, $\oint_C \vec{F}.d\vec{r} = -4ab^2 \qquad (1)$

Next we must find the curl of **F** vector,

$$curl\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix}$$

The results of the two cross two matrices are,

$$= \frac{\partial}{\partial z}(-2xy)\hat{i} - \left(-\frac{\partial}{\partial z}(x^2 + y^2)\right)\hat{j} + \left(\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(x^2 + y^2)\right)\hat{k}\,\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(x^2 + y^2)$$

We simplify it by differentiating with respect to the corresponding x, y and z.

$$= 0 + 0 + (-2y - 2y)k$$

For the surface **S**, we have $n = k.$, The dot product of Curl **F** and unit normal is $curl\vec{F}.n = -4yk.k = -4y$

Now apply the integration to the dot product value, $\iint_S curl\vec{F}.ndS = \int_0^b \int_{-a}^{a} -4ydxdy$ ,

On simplifying,

$$= -8a\frac{b^2}{2}$$

The value of double integral over the surface is,

$$\iint_S curl\vec{F}.ndS = -4ab^2 \qquad (2)$$

We can see that both the integrals one and two are same.

$$(1) = (2)$$

Hence the **Stokes' theorem** is verified.