Path continuity - addendum

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1 Definition of aligned motion mode

Aligned motion mode $\theta_{ta}(u)$ and its derivatives are defined as:

$$\zeta(u) = \text{arctan2}(C'_y(u), C'_x(u)) \tag{1}$$

$$\theta_{ta}(u) = \zeta(u) + \alpha \tag{2}$$

$$\theta'_{ta}(u) = \frac{\mathbf{C}'(u) \times \mathbf{C}''(u)}{\|\mathbf{C}'(u)\|_2^2} \tag{3}$$

$$\theta''_{ta}(u) = \frac{\mathbf{C}'(u) \times \mathbf{C}'''(u)}{\|\mathbf{C}'(u)\|_2^2} - 2\frac{(\mathbf{C}'(u) \cdot \mathbf{C}''(u))\,(\mathbf{C}'(u) \times \mathbf{C}''(u))}{\|\mathbf{C}'(u)\|_2^4} \tag{4}$$

2 G_0 continuity of motion mode

$$\theta_{ta2}(0) = \theta_{ta1}(1) \tag{5}$$

$$\zeta_2(0) + \alpha_2 = \zeta_1(1) + \alpha_1 \tag{6}$$

From this, it can be seen that for motion mode to have G_0 continuity, angle offsets must be equal $\alpha_2 = \alpha_1$ and corresponding curves must be G_1 continuous:

$$\mathbf{C}'_2(0) = \beta_1 \mathbf{C}'_1(1), \quad \beta_1 \in \mathbf{R}^+ \tag{7}$$

3 G_1 continuity of motion mode

Motion mode G_1 continuity equivalent for path static continuity:

$$\theta'_{ta2}(0) = \beta_1 \theta'_{ta1}(1) \tag{8}$$

$$\frac{\mathbf{C}'_2(0) \times \mathbf{C}''_2(0)}{\|\mathbf{C}'_2(0)\|_2^2} = \beta_1 \frac{\mathbf{C}'_1(1) \times \mathbf{C}''_1(1)}{\|\mathbf{C}'_1(1)\|_2^2} \Bigg/ \frac{1}{\beta_1 \|\mathbf{C}'_1(1)\|_2} \tag{9}$$

$$\frac{\mathbf{C}'_2(0) \times \mathbf{C}''_2(0)}{\|\mathbf{C}'_2(0)\|_2^3} = \frac{\mathbf{C}'_1(1) \times \mathbf{C}''_1(1)}{\|\mathbf{C}'_1(1)\|_2^3} \tag{10}$$

$$\kappa_2(0) = \kappa_1(1) \tag{11}$$

We can see in case of subsequent tangent aligned motion modes, that for path to be statically continuous, path has to be G_2 continuous:

$$\mathbf{C}''(0) = \beta_1^2 \mathbf{C}''(1) + \beta_2 \mathbf{C}'(1), \quad \beta_2 \in \mathbf{R} \tag{12}$$

4 G_2 continuity of motion mode

Similarly, it can be shown that for motion modes G_2 continuity equivalent, path has to have constant rate of change of curvature $\frac{d\kappa(u)}{ds}$:

$$\theta''_{ta2}(0) = \beta_1^2 \theta''_{ta1}(1) + \beta_2 \theta'_{ta1}(1) \tag{13}$$

$$\begin{aligned}
&\frac{\mathbf{C}'_2(0) \times \mathbf{C}'''_2(0)}{\|\mathbf{C}'_2(0)\|_2^2} - 2\frac{(\mathbf{C}'_2(0) \cdot \mathbf{C}''_2(0))\,(\mathbf{C}'_2(0) \times \mathbf{C}''_2(0))}{\|\mathbf{C}'_2(0)\|_2^4} = \\
&\quad \beta_1^2 \left(\frac{\mathbf{C}'_1(1) \times \mathbf{C}'''_1(1)}{\|\mathbf{C}'_1(1)\|_2^2} - 2\frac{(\mathbf{C}'_1(1) \cdot \mathbf{C}''_1(1))\,(\mathbf{C}'_1(1) \times \mathbf{C}''_1(1))}{\|\mathbf{C}'_1(1)\|_2^4} \right) + \\
&\qquad + \beta_2 \frac{\mathbf{C}'_1(1) \times \mathbf{C}''_1(1)}{\|\mathbf{C}'_1(1)\|_2^2} \Bigg/ \frac{1}{\beta_1^2 \|\mathbf{C}'_1(1)\|_2^2}
\end{aligned} \tag{14}$$

$$\begin{aligned}
&\frac{\mathbf{C}'_2(0) \times \mathbf{C}'''_2(0)}{\|\mathbf{C}'_2(0)\|_2^4} - 2\frac{(\mathbf{C}'_2(0) \cdot \mathbf{C}''_2(0))\,(\mathbf{C}'_2(0) \times \mathbf{C}''_2(0))}{\|\mathbf{C}'_2(0)\|_2^6} = \\
&\quad \frac{\mathbf{C}'_1(1) \times \mathbf{C}'''_1(1)}{\|\mathbf{C}'_1(1)\|_2^4} - 2\frac{(\mathbf{C}'_1(1) \cdot \mathbf{C}''_1(1))\,(\mathbf{C}'_1(1) \times \mathbf{C}''_1(1))}{\|\mathbf{C}'_1(1)\|_2^6} + \\
&\qquad + \frac{\beta_2}{\beta_1^2} \frac{\mathbf{C}'_1(1) \times \mathbf{C}''_1(1)}{\|\mathbf{C}'_1(1)\|_2^4}
\end{aligned} \tag{15}$$

Before we continue, we expand the following expression:

$$\frac{\beta_2}{\beta_1^2} = \frac{\beta_2 \|\mathbf{C}'_1(1)\|_2^2}{\beta_1^2 \|\mathbf{C}'_1(1)\|_2^2} + \frac{\beta_1^2 (\mathbf{C}'_1(1) \cdot \mathbf{C}''_1(1))}{\beta_1^2 \|\mathbf{C}'_1(1)\|_2^2} - \frac{\mathbf{C}'_1(1) \cdot \mathbf{C}''_1(1)}{\|\mathbf{C}'_1(1)\|_2^2} \tag{16}$$

$$= \frac{1}{\beta_1^2} \frac{\mathbf{C}'_{1_x}(1) \left(\beta_1^2 \mathbf{C}''_{1_x}(1) + \beta_2 \mathbf{C}'_{1_x}(1) \right)}{\|\mathbf{C}'_1(1)\|_2^2} + \tag{17}$$

$$+ \frac{1}{\beta_1^2} \frac{\mathbf{C}'_{1_y}(1) \left(\beta_1^2 \mathbf{C}''_{1_y}(1) + \beta_2 \mathbf{C}'_{1_y}(1) \right)}{\|\mathbf{C}'_1(1)\|_2^2} - \frac{\mathbf{C}'_1(1) \cdot \mathbf{C}''_1(1)}{\|\mathbf{C}'_1(1)\|_2^2}$$

$$= \frac{1}{\beta_1^2} \frac{\mathbf{C}'_{1_x}(1)\mathbf{C}''_{2_x}(0) + \mathbf{C}'_{1_y}(1)\mathbf{C}''_{2_y}(0)}{\|\mathbf{C}'_1(1)\|_2^2} - \frac{\mathbf{C}'_1(1) \cdot \mathbf{C}''_1(1)}{\|\mathbf{C}'_1(1)\|_2^2} \tag{18}$$

$$= \frac{1}{\beta_1^3} \frac{\mathbf{C}'_2(0) \cdot \mathbf{C}''_2(0)}{\|\mathbf{C}'_1(1)\|_2^2} - \frac{\mathbf{C}'_1(1) \cdot \mathbf{C}''_1(1)}{\|\mathbf{C}'_1(1)\|_2^2} \tag{19}$$

For next step, we expand aligned motion mode G_1 continuity:

$$\beta_1 = \frac{\theta'_{ta2}(0)}{\theta'_{ta1}(1)} \tag{20}$$

$$\beta_1 = \frac{(\mathbf{C}'_2(0) \times \mathbf{C}''_2(0)) \, \|\mathbf{C}'_1(1)\|_2^2}{(\mathbf{C}'_1(1) \times \mathbf{C}''_1(1)) \, \|\mathbf{C}'_2(0)\|_2^2} \tag{21}$$

$$\beta_1^3 = \frac{\mathbf{C}'_2(0) \times \mathbf{C}''_2(0)}{\mathbf{C}'_1(1) \times \mathbf{C}''_1(1)} \tag{22}$$

Last preparatory step we need, is to (19) (22) expansions into the last term of (15):

$$\begin{aligned}\frac{\beta_2}{\beta_1^2} \frac{\mathbf{C}'_1(1) \times \mathbf{C}''_1(1)}{\|\mathbf{C}'_1(1)\|_2^4} &= \left(\frac{1}{\beta_1^3} \frac{\mathbf{C}'_2(0) \cdot \mathbf{C}''_2(0)}{\|\mathbf{C}'_1(1)\|_2^2} - \frac{\mathbf{C}'_1(1) \cdot \mathbf{C}''_1(1)}{\|\mathbf{C}'_1(1)\|_2^2} \right) \frac{\mathbf{C}'_1(1) \times \mathbf{C}''_1(1)}{\|\mathbf{C}'_1(1)\|_2^4} \\ &= \frac{(\mathbf{C}'_2(0) \cdot \mathbf{C}''_2(0)) \, (\mathbf{C}'_2(0) \times \mathbf{C}''_2(0))}{\|\mathbf{C}'_2(0)\|_2^6} - \frac{(\mathbf{C}'_1(1) \cdot \mathbf{C}''_1(1)) \, (\mathbf{C}'_1(1) \times \mathbf{C}''_1(1))}{\|\mathbf{C}'_1(1)\|_2^6}\end{aligned} \tag{23}$$

By inserting substituting (23) into (15), we get:

$$\begin{aligned}&\frac{\mathbf{C}'_2(0) \times \mathbf{C}'''_2(0)}{\|\mathbf{C}'_2(0)\|_2^4} - 3 \frac{(\mathbf{C}'_2(0) \cdot \mathbf{C}''_2(0)) \, (\mathbf{C}'_2(0) \times \mathbf{C}''_2(0))}{\|\mathbf{C}'_2(0)\|_2^6} = \\ &\qquad \frac{\mathbf{C}'_1(1) \times \mathbf{C}'''_1(1)}{\|\mathbf{C}'_1(1)\|_2^4} - 3 \frac{(\mathbf{C}'_1(1) \cdot \mathbf{C}''_1(1)) \, (\mathbf{C}'_1(1) \times \mathbf{C}''_1(1))}{\|\mathbf{C}'_1(1)\|_2^6}\end{aligned} \tag{24}$$

$$\left. \frac{d\kappa_2(u)}{ds} \right|_{u=0} = \left. \frac{d\kappa_1(u)}{ds} \right|_{u=1} \tag{25}$$

This means that for G_2 continuity of motion modes, path must have constant rate of change of curvature, i.e, it has to be G_3 continuous:

$$\mathbf{C}'''_2(0) = \beta_1^3 \mathbf{C}'''_1(1) + 3\beta_1\beta_2 \mathbf{C}''_1(1) + \beta_3 \mathbf{C}'_1(1), \quad \beta_3 \in \mathbf{R} \tag{26}$$