Entanglement Area Law for Shallow and Deep Quantum Neural Network States

Zhih-Ahn Jia[1,2,3,4], Lu Wei[3], Yu-Chun Wu[1,2] and Guang-Can Guo[1,2]
Guo-Ping Guo[1,2]
[1]CAS Key Laboratory of Quantum Information, Chinese Academy of Sciences, School of Physics, University of Science and Technology of China, Hefei, Anhui, 230026, P. R. China
[2]Synergetic Innovation Center of Quantum Information and Quantum Physics, University of Science and Technology of China, Hefei, Anhui, 230026, P. R. China
[3]School of Physics, University of Science and Technology of China, Hefei, Anhui, 230026, P. R. China
[4]Microsoft Station Q and Department of Mathematics, University of California, Santa Barbara, California 93106-6105, USA

Neural network as an ansatz state

Boltzmann machine (BM):

$$E(\mathbf{s}) = -\frac{1}{2}\sum_{i,j;i\neq j} W_{i,j}s_i s_j - \sum_i b_i s_i$$

Restricted Boltzmann machine (RBM):

$$E(\mathbf{v},\mathbf{h}) = -\sum_i a_i v_i - \sum_j b_j h_j - \sum_{i,j} h_j W_{ji} v_i$$

Deep Boltzmann machine (DBM):

$$E(\mathbf{v},\mathbf{h},\mathbf{g}) = -\sum_i v_i a_i - \sum_k c_k g_k - \sum_j h_j b_j - \sum_{i,j;\langle ij\rangle} W_{ij} v_i h_j - \sum_{jk;\langle kj\rangle} W_{kj} h_j g_k$$

Ansatz state:

$$\Psi(\mathbf{v}) = \sum_{h_1,\cdots,h_m=0,1}\ \sum_{g_1,\cdots,g_l=0,1} \exp^{-E(\mathbf{v},\mathbf{h},\mathbf{g})}$$

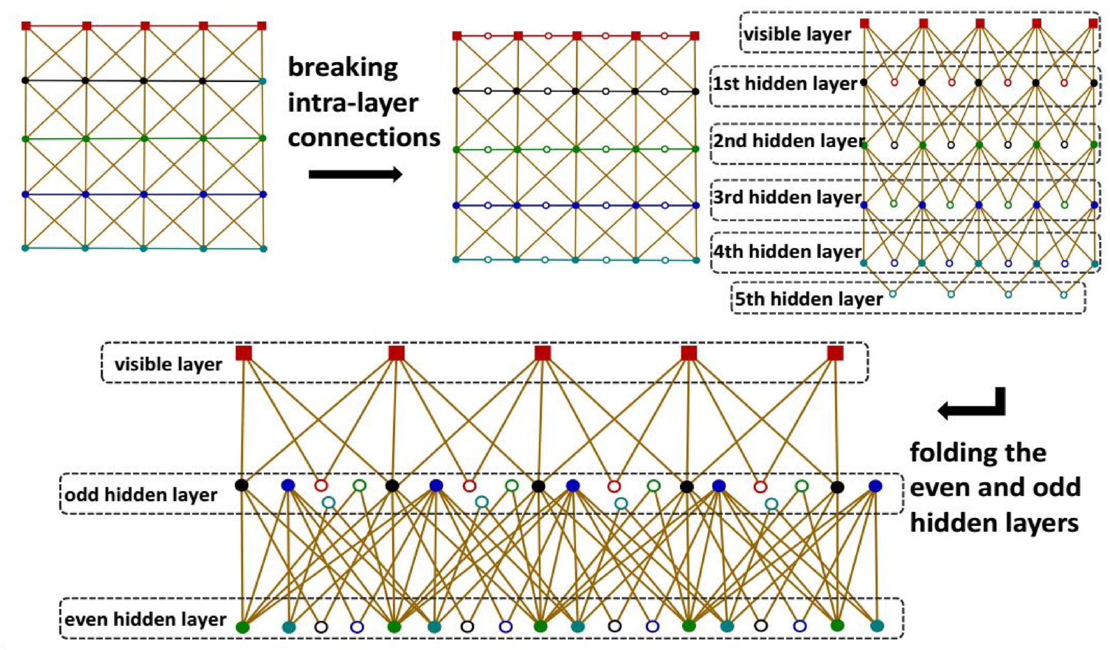


FIG. 1: The illustration of the procedure for reducing any BM neural network into DBM with only two hidden layers: the first step is to reduce a BM into DBM by adding hidden neurons in intra-layer connections; and the second step is to reduce multi-hidden-layer DBM into two-hidden-layer DBM using folding trick, i.e., folding the odd (resp. even) layers together.

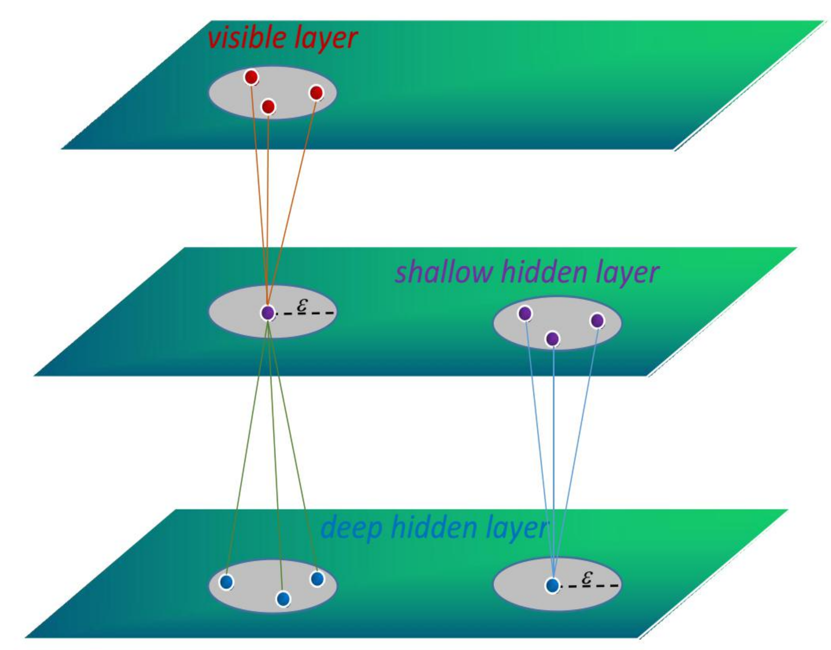


FIG. 2: The depiction of the K-local DBM states, where the geometries of each layer are the same and fixed at in advance. For each hidden neuron h we can define the ε-neighborhood $B(h;\varepsilon)$, h can only connect neurons lie in $B(h;\varepsilon)$. Under this local constraint, the maximum number K in the given DBM structure is defined as local connection numbers. Here is an example of 3-local DBM structure.

Entanglement area law of quasi-product states

Theorem 1. For an N-particle system $\mathcal{S}$, suppose that

$$|\Psi\rangle = \sum_{s_1,\cdots,s_N} \Psi(s_1,\cdots,s_N)|s_1\rangle\otimes\cdots\otimes|s_N\rangle$$

is a K-local quasi-product quantum states, then the Rényi entropies of the reduced density matrix $\rho_{\mathcal{A}} = \mathrm{Tr}_{\mathcal{A}^c}|\Psi\rangle\langle\Psi|$ with respect to the bipartition $\mathcal{S} = \mathcal{A}\sqcup\mathcal{A}^c$ satisfies the following area law

$$S_\alpha(\mathcal{A}) \leq \zeta(K)\,\mathrm{Area}(\mathcal{A}),\ \forall\alpha,$$

where $\mathrm{Area}(\mathcal{A})$ denotes the number of particles on the boundary of $\mathcal{A}$ and $\zeta(K)$ is a scaling factor only depends on the size of local cluster K.

Many ground states of 1d and 2d local gapped quantum systems can be represented as local quasi-product states

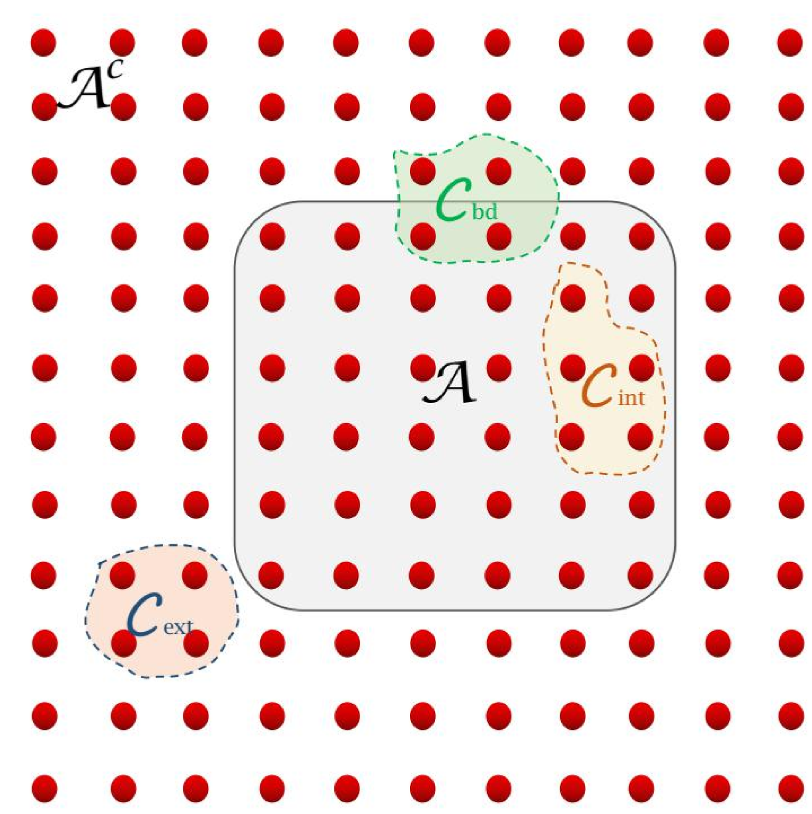
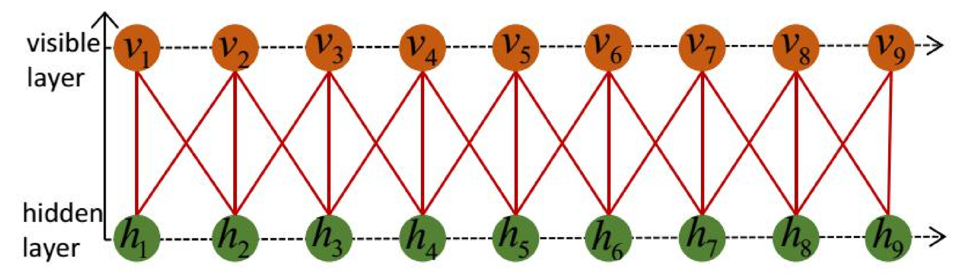
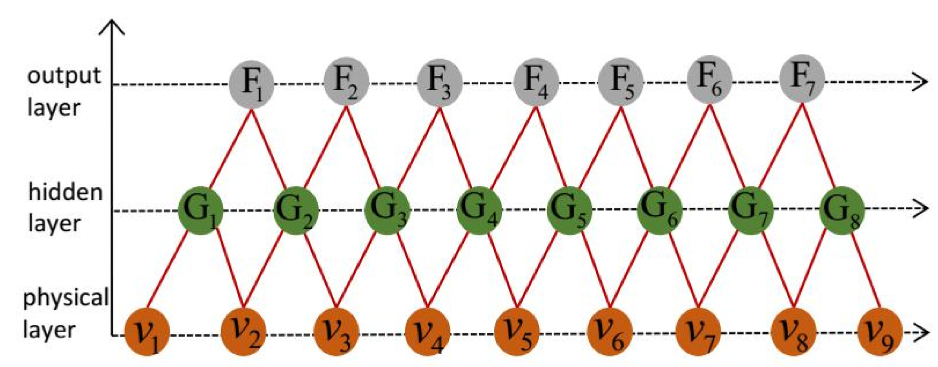


FIG. 3: The depiction of internal cluster $\mathcal{C}_{int}$, external cluster $\mathcal{C}_{ext}$ and boundary cluster $\mathcal{C}_{bd}$ for a bipartition $\mathcal{A}$ and $\mathcal{A}^c$ of the system.



(a) local RBM states.



(b) local feed-forward neural network states.

local RBM states and local feed-forward states are special cases of local quasi-product states. Thus they all obey the entanglement area law

Entanglement area-law of DBM states

Theorem 2. For any K-local DBM state $|\Psi\rangle$, Rényi entropy of the reduced density matrix $\rho_{\mathcal{A}} = \mathrm{Tr}_{\mathcal{A}^c}(|\Psi\rangle\langle\Psi|)$ satisfy the following area law

$$S_R^\alpha(\mathcal{A}) \leq C(K)\,\mathrm{Area}(\mathcal{A}),\ \forall\alpha,$$

where $\mathrm{Area}(\mathcal{A})$ denotes the number of particles on the boundary of $\mathcal{A}$ and $C(K)$ is a scaling factor only depends on local connection number K.

Entanglement entropy as a quantifier of efficiency of image classification problem

Theorem 3. For any target function $f_{\mathcal{T}}$ of locally smooth image classification problem, then Rényi entanglement entropy satisfy the area law

$$S_\alpha(\mathcal{A}) \leq \zeta\mathrm{Area}(\mathcal{A}),$$

where ζ is a scaling factor depending on B and not depending on the size of the images (number of pixels).

Reference:

1. Z.-A Jia, Y.-H Zhang, Yu-Chun Wu, Guang-Can Guo, and Guo-Ping Guo， Efficient Machine Learning Representations of Surface Code with Boundaries, Defects, Domain Walls and Twists, arxiv:1802.03738.
2. Z.-A Jia, Lu Wei, Yu-Chun Wu, Guang-Can Guo and Guo-Ping Guo, Entanglement Area Law for Shallow and Deep Quantum Neural Network States

中科院量子信息重点实验室，中国科学技术大学
CAS Key Laboratory of Quantum Information,University of Science and Technology of China
微软Q站研究院和数学系，加州大学圣芭芭拉分校
Microsoft Station Q and Department of Mathematics, University of California, Santa Barbara, California