Lecture 1 Introduction: What is real analysis?

Lecture 2 Elementary measure, Jordan measure

Lecture 3 Lebesgue outer measure

Lecture 4 Lebesgue measure — measure

Lecture 5 Lebesgue measure (continued)

Lecture 6 Measurable functions

Lecture 7 Convergences of measurable functions — fu...

Lecture 8 Approximation by simple and/or continuous functions

Lecture 9 Lebesgue Integrals of non-negative measurable functions

Lecture 10 Convergence of Lebesgue integrals: non-negative measurable funct...

Lecture 11 Lebesgue integrals of general measurable functions. — Int... g...

Lecture 12 Convergence of Lebesgue integrals: absolutely integrable functio...

Lecture 13 Lebesgue integrals V.S Riemann integrals, Fubini's theorem

Lecture 14 More on integrals (Lebesgue): Tonelli's theorem, Convolution

Lecture 15 Abstract measures.

Lecture 16 Integrations on abstract measure spaces — For

Lecture 17 From outer measures to measures — Abstract

Lecture 18 Metric V.S. measure — Measure

Lecture 19 The Riesz representation theorem for $C(X)$.

Lecture 20 L^p Spaces

Lecture 21 L^p Spaces as metric spaces — L^p-spaces

Lecture 22 L^2 Space as inner product space

Lecture 23 Riesz representation theorem for L^p Spaces — Riesz rep

Lecture 24 Signed measure, the Lebesgue-Radon-Nikodym theorem — theorem

Lecture 25 The Lebesgue differentiation theorem — differential

Lecture 26 The fundamental theorem of calculus

Lecture 27 The Rademacher differentiation theorem

答疑. 周日 15:55 ~ 17:30.

Real Analysis

- Lecturer: Zuoqin Wang 王作勤
- Office: 1601 @ 管理科研楼
- Email: wangzuoq @ ustc.edu.cn
- Monday 3-5
  Thursday 3-4.5 (答疑)
- TA: 张羽平. 于孟鑫

Course Information:

- website: http:// staff.ustc.edu.cn/~wangzuoq/course/18S-RealAnalysis/index.html

(course notes. psets → after each lecture)

- Reference Books:

① 周民强《实变函数论》. ② E. Stein Real Analysis - measure theory, integration & Hilbert Spaces

③ T. Tao An introduction to measure theory. Real Analysis

- Homeworks: Twice every week after class
  Collect on Mondays before class

- Score: 30% HW + 30% Midterm + 40% final
  A$^+$ essays

- Cover:
  - Measure theory
  - Integration theory
  - L^p - space

Lec 1

What is real analysis?

Why should we learn it?

· Wikipedia: Real Analysis deals with real numbers and real-valued functions of real-variables

In paticular, it deals with the analytic properties of real functions and sequences.

analytic
- limit and convergence
- continuity
- differentiation
- integration

History (Rough)

17^{th} centry $\longrightarrow$ 18^{th} centry $\longrightarrow$ 19^{th} centry

(Newton-Leibniz) invent Calculus 微积分

(Euler. Laplace ...) Applications of calculus

(Cauchy. Weierstrass ...) rigorous background ($\varepsilon-\delta$ language)

$\longrightarrow$ 20^{th} centry

(Lebesgue. Banach ...)

abstraction $\rightsquigarrow$ for more applications

Real Analysis (the 20^{th} centry's R-A)

Functional Analysis 泛函分析

What did we learn in Mathematical Analysis?

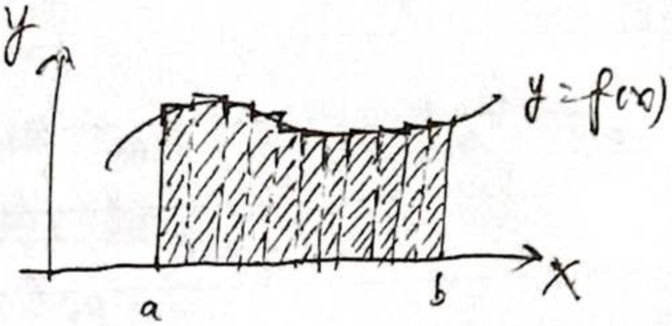
· Sequence and Series of
- real members
- real functions

complement of $\mathbb{R}$

- Talyor series
- Fourier series

· Differentiation $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ (slope. rate of change)

· Rimann Integral



$\int_a^b f(x)\,dx = \lim_{\Delta \to 0} \sum_i f(t_i)\Delta i$.

Sum up "length" of vertical intervals

Relationships between D and I

The fundamental therem of calculus

Suppose f is differentiatable on $[a,b]$ and f' is Rimann integable on $[a,b]$. then $\int_a^b f'(x)\,dx = f(b) - f(a)$.

Lebsgue Theorem || A bounded function f on $[a,b]$ is Rimann integrable if and only if the set of discontinuous points of f is of measure zero

|| A set $E \subset \mathbb{R}$ is of measure zero if $\forall \varepsilon > 0$. $\exists$ intervals (a_i, b_i) s.t. $(i = 1, 2, \ldots)$

(1) $E \subset \bigcup_i (a_i, b_i)$ (2) $\sum_i (b_i - a_i) < \varepsilon$

Shortcome of the Rimann Integration

(1) There exists simple and natural fuctions which are NOT Rimann integrable

e.g. $\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

-It's discontinuous everywhere → Not Riemann integrable

(2) The fundamental theorem of caculus fails if <u>f is not differentratable everywhere</u> or if <u>f' is not Riemann integrable.</u>

e.g. Volterra 1881. $\exists f$ which is differentratable every, But f' is not Riemann integrable.

(3) $f_n \to f$. f_n continuous $\not\Rightarrow$ f continuous, or $\int_a^b f_n(x)dx \to \int_a^b f$.

Need: uniformly convergence

e.g. $\exists$ $0 \le f_n \le 1$. continuous $f_n \downarrow f$ as $n \to \infty$. But: f is not Riemann integrable

(4) Besicovitch's example: $\exists f$

defined <u>on the plate</u>. Riemann integrable.

But, No matter how you choose <u>orthogonal</u> axis x and y.

正交的.

$\exists y$ s.t: $f(x,y)$ (as a func of x) is not Riemann integrable

In paticular. $\underbrace{\iint f \, dA}_{\text{double integral}} \neq \underbrace{\int \left(\int f(x,y) dx \right) dy}_{\text{iterated integral does Not exist}}$.

— closed related to <u>Kakeya neddle problem</u>

Find a set in R^n. as small as possible. s.t:

it contains unit length intevals of each direction.

(5) In mondern analysis, one regardness of functions is as "points" in some "space of functions".

one can define "distance" between two functions

e.g. $[a,b]$. $d(f,g) = \int_a^b |f(x) - g(x)| dx$.

For example R = all Riemann Integrable functions on $[a,b]$

$\rightsquigarrow$ "metric space" (R, d)

$\sim$ Cauchy sequence in (R, d) FACT. (R, d) is NOT complete
$\downarrow$
which in Functional Analysis is crucial

(6) Fourier Series. Suppose f is Riemann integrable on $[-\pi, \pi]$,
then $f \underset{\text{represented by}}{\sim} \sum_{-\infty}^{\infty} a_n e^{inx}$, $a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx$.

We know:

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx \quad \text{(Parseval Identity)}.$$

Question:

Given a sequence (a_n) can we find a function f s.t.: its Fourier coefficients are a_ns ?

In particular, if a_n only satisfies, $\sum |a_n|^2 < \infty$.
can we find such as f ?

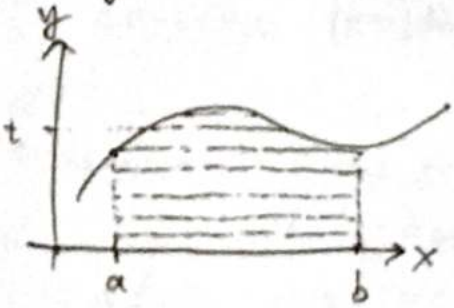
$\sum a_n e^{inx}$ $\underline{\text{if } \sum |a_n| < \infty} \rightarrow$ convergence.

$\downarrow$

add these "functions" into the space of consideration

Conclusion: Need new theory of integration

Rough idea & advantage of Lebesgue Theory



$$\int_a^b f(x)dx \xlongequal{f \text{ is nonnegative}} \int_0^\infty m(\{x : f(x) > t\})\, dt.$$

m: measure = size of set

(1) $\chi_{\mathbb{Q}}$ is Lebesgue integrable and $\int_a^b \chi_{\mathbb{Q}}(x)dx = 0$.

(2) The fundamental theorem of calculus holds for a very wide class of functions (Absolutely continuous).

(3) Simple criteria for convergence.
"Dominated convergence theorem"

(4) "Fubini theory": $\underset{\text{重积分}}{\underline{\text{Double integral}}} = \underset{\text{累次积分}}{\underline{\text{Iterated integral}}}$.

(5) $L^p([a,b]) = \{f : f^p \text{ is Lebesgue integrable}\}$ | complete
$d(f,g) = \left(\int_a^b |f-g|^p dx\right)^{1/p}$ $(p \geq 1)$

(6) Plancherel's theorem: $\sum |a_n|^2 < \infty \to \exists!\, f$ Lebesgue integrable whose Fourier coeff are a_ns

Moreover:
The process of establishing Lebesgue theory shed a light on how to define measure/integration theory in abstract sets

Key conceptions in Lebesgue's theory

- measure = "size of a set"
  interval → length rectangle → area Solid → volume.
- we mainly concentrate on subsets E of $\mathbb{R}$.

A measure m "should" be a function

$$m: P(\mathbb{R}) \longrightarrow [0,+\infty) \cup \{+\infty\}.$$

$P(\mathbb{R})$: all subsets of $\mathbb{R}$.

such that

(1) $m([a,b]) = b-a$.

(2) $m(\underbrace{E+x}_{\text{平移不变性}}) = m(E)$

(3) Suppose $E_i \in P(\mathbb{R})$ $E_i \cap E_j = \phi$ $(i \neq j)$.
then $m(\bigcup_i^{\infty} E_i) = \sum_i^{\infty} m(E_i)$

Unfortunately, such a "measure" does not exist if we allow axiom of choice

Reason: On $[0,1]$ we define $x \sim y \iff x-y \in \mathbb{Q}$.

- let $E = \{$ one number from each equivalence class$\} \subset [0,1]$
- For any $r \in \mathbb{Q}$. let

$$E_r = \{x+r \pmod{\mathbb{Z}} \mid x \in E\} \subset [0,1].$$

Then $[0,1] = \bigcup_{r \in \mathbb{Q}} E_r$ $\quad m(E_r) = m(E)$.

$\Rightarrow 1 = m([0,1]) = m(\bigcup E_r) = \sum m(E_r) = \sum m(E) = \infty \cdot m(E)$

$\Rightarrow m(E) = 0$.

$\Rightarrow m([0,1]) = \sum m(E_r) = 0$. (矛盾).

Even if we replace (3) by (3)': $m(\bigcup_{i=1}^{N} E_i) = \sum_{i=1}^{N} m(E_i)$,
the measure still doesn't exist. (Banach-Tarski paradox)

$\rightsquigarrow$ One can't define measure on all subsets of $\mathbb{R}$

$\rightsquigarrow$ measurable sets

$\rightsquigarrow$ measurable functions $\rightarrow$ for $\boxed{\overset{\text{a.e.}}{\text{almost every}}}$ t. the set.

$\{x \mid f(x) > t\}$ is measurable set

$\rightsquigarrow$ Lebesgue integrable function

A warning: There exists many wiered examples in Real Analysis which tell us the boundary of the theory.

REAL ANALYSIS IS NOT A SUBJECT OF WIERED EXAMPLES, INSTEAD, IT IS A SUBJECT THAT TELLS YOU ALL NICE PROPERTIES STILL HOLDS.

☆

Littlewood Three Principles:

(1) Every measurable set (in $\mathbb{R}$) is nearly a finite union of intervals

(2) Every measurable fuction is nearly continuous

(3) Every convergence sequence of measurable function is nearly uniformly convergence

Lec.2

Last time.

- Need a new theory of integration
- Lebesgue's idea: need to measure "complicated sets"
- No ways to define a measure for all sets

Today Elementary, Jordan } measure

1 Elementary measure:

- $I = [a,b]$. $[a,b)$, $(a,b]$, (a,b) intervals

$\rightsquigarrow m(I) = b-a. = |I|$

(we allow $a=b$ $<$ $[a,a] = \{a\}$, $(a,a) = \phi$)

- $B = I_1 \times I_2 \times \cdots \times I_d \longleftarrow$ box of dimension d.

$\rightsquigarrow m(B) = |B| = |I_1| \times \cdots \times |I_d|$

Def: A set $E \subset \mathbb{R}^d$ is called an Elementary set:
if it is the union of finitely many boxes
i.e. $E = B_1 \cup \cdots \cup B_n$. (Note: These boxes may intersect)

Notation: $\mathcal{E}_e$ = the set of all elementary sets in $\mathbb{R}^d$.

Observation: If $E, F \in \mathcal{E}_e$ then

- $E \cup F \in \mathcal{E}_e$
- $E \cap F \in \mathcal{E}_e$
- $E \setminus F \in \mathcal{E}_e$
- $E \Delta F = (E\setminus F) \cup (F \setminus E) = (E \cup F) - (E \cap F) \in \mathcal{E}_e$.

(Boolean Structure)

布尔型结构

Prop: Any elementary set can be represented as the union of finitely many non-intersecting boxes.

i.e. $E = B_1 \amalg B_2 \cdots \amalg B_n$. $(B_i \cap B_j = \phi \ i \neq j)$

Proof: First suppose $d=1$: $E = I_1 \cup I_2 \cdots \cup I_n$.

Denote the endpoints by a_j, b_j of I_j.

Rearrange these endpoints in increasing order as $c_1 \cdots c_{2N}$.

Let $I_1' = [c_1, c_1]$, $I_2' = (c_1, c_2)$, $I_3' = [c_2, c_2]$, $\cdots$, $I_{2N+1}' = [c_{2N}, c_{2N}]$

Let $K = \{k : I_k' \subset E\}$. Then $E = \bigcup_{k \in K} I_k'$ is a non-intersecting union.

· For $d>1$. Suppose $E = \tilde{B_1} \cup \cdots \cup \tilde{B_N}$ (may overlap).

where $B_j = I_{j,1} \times \cdots \times I_{j,d}$.

For each l apply the 1-dim arguement to $I_{1,l} \cdots I_{N,l}$.

Then consider all possible Cartesian products 笛卡尔积

of these new non-intersecting intervals

$\rightsquigarrow$ non-overlapping boxes $\rightsquigarrow E = \coprod_j B_j$

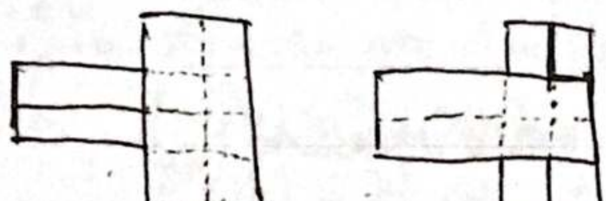
Def: The measure of an elementary set E is

$$m(E) = \sum_{j=1}^{n} m(B_j) \text{ where } E = \coprod_{j=1}^{n} B_j$$

$E = \bigcup_j \tilde{B_j}$ (maybe overlapped)

One need to check: For two different partitions of E into non-overlapping boxes, the formula give the same results

Method 1:



seek for a common division

Method 2: $m(E) = \lim_{N \to \infty} \frac{1}{N^d} \# (E \cap \frac{1}{N} \mathbb{Z}^d)$

"discretization formula": No partitions here. only related to E.

number of elements in finitely many sets

The following properties are easy to check via the definition

Prop: The measure of $m: \mathcal{E} \to [0,+\infty)$ satisfies:

(1) $m(B) = |B|$

(2) If $E_1, E_2 \in \mathcal{E}$, $E_1 \cap E_2 = \phi$ then $m(E_1 \cup E_2) = m(E_1) + m(E_2)$ ← finite additivity

(3) If $E \in \mathcal{E}$, $x \in \mathbb{R}^d$ Then $m(E+x) = m(E)$ ← translation invariance

non-negative

(4) If $E_1, E_2 \in \mathcal{E}$ then $m(E_1 \cup E_2) \le m(E_1) + m(E_2)$ ← subadditivity

(5) If $E_1, E_2 \in \mathcal{E}$, $E_1 \subset E_2$ then $m(E_1) \le m(E_2)$ ← monotonicity

Remark One can prove

If a function $\tilde{m}: \mathcal{E} \to [0,+\infty)$ satisfied (2), (3), with $\tilde{m}([0,1]^d) = 1$.

Then $\tilde{m} = m$. more radical.

---

2. Jordan measure

Problem of E.M.

triangle, disc are not elementary sets

Archimede's Observation: one can use finitely many boxes "approximate" them both from inside and from outside.

Def. Let $A \subset \mathbb{R}^d$ be any bounded subset

(1) The Jordan inner measure $J_*(A) = \sup_{\mathcal{E} \ni E \subset A} m(E)$.

exists by (5) mono-

(2) The Jordan outer measure

$J^*(A) = \inf_{\mathcal{E} \ni F \supset A} m(F)$ → exists by non-negative

using finitely many boxes to approximate

(3) If $J^*(A) = J_*(A)$ then we say A is Jordan measurable, with Jordan measure $J(A) = J^*(A) = J_*(A)$.

Notation: $\mathcal{J}$ = the set of all Jordan measurable sets in R^d.

Remark:
(1) J_*. J^* are defined for any bounded set in R^d.
(2) J is only defined for "some sets".
(3) $J_* \le J^*$
(4) If $A \in \mathcal{E}_\Delta$ then $A \in \mathcal{J}$ and $J(A) = m(A)$.

Example: $A = \mathbb{Q} \cap [0,1]$
$J_*(A) = 0 \neq J^*(A) = 1$. $\mathcal{E}$ are formed by finite B (boxes)
Leave out finitely many of points in $R \backslash \mathbb{Q}$.

In fact. $A \in \mathcal{J} \Rightarrow A$ is "nearly" elementary

Thm: Let $A \subset R^d$ be a bounded subset, Then T.F.A.E. (The followings are equivalent)
(1) $A \in \mathcal{J}$
(2) $\forall \varepsilon > 0$ $\exists E, F \in \mathcal{E}_e$ s.t: $E \subset A \subset F$ and $m(F \backslash E) < \varepsilon$.
(3) $\forall \varepsilon > 0$. $\exists E \in \mathcal{E}_e$ s.t: $E \subset A$ and $J^*(A \backslash E) < \varepsilon$.
$\Longleftrightarrow \exists F \in \mathcal{E}_e$ s.t: $F \supset A$ and $J^*(F \backslash A) < \varepsilon$

a description of "near" using J^*, well defined

Lemma 1: Suppose $A_1 \subset A_2$ are bounded then $J_*(A_1) \le J_*(A_2)$.
$J^*(A_1) \le J^*(A_2)$.
monotonicity in outer and inner measure

Proof: Take a sequences of elementary sets. $E_n \subset A_1 \subset A_2 \subset F_n$.

s.t. $m(E_n) \to J_*(A_1)$

$m(F_n) \to J^*(A_2)$.

Then $E_n \subset A_2 \Rightarrow J_*(A_2) = \sup_{E' \subset A_2} m(E') \geq \lim_{n\to\infty} m(E_n) = J_*(A_1)$.

$F_n \supset A_1 \Rightarrow J^*(A_1) = \inf_{F' \supset A_1} m(F') \leq \lim_{n\to\infty} m(F_n) = J^*(A_2)$.

Lemma 2 | Suppose A_1, A_2 are bounded
Then $J^*(A_1 \cup A_2) \leq J^*(A_1) + J^*(A_2)$. finite subadditivity for outer Jordan measure

* (This property is not available in J_*).

Proof. $A_1 \subset E_n$, $A_2 \subset F_n$ $\quad A_1 \cup A_2 \subset \underline{E_n \cup F_n}$ ↑ sub-additivity available

Lemma 3 | Suppose $A \subset B$ ← box
Then $A \in \mathcal{J} \iff B \setminus A \in \mathcal{J}$

Proof: $\begin{cases} J^*(B\setminus A) = m(B) - J_*(A) \\ J_*(B\setminus A) = m(B) - J^*(A) \end{cases}$

Proof of Thm.

(1) ⇒ (2) Suppose $A \in \mathcal{J}$

Then $\forall \varepsilon > 0$. $\exists E, F \in \mathcal{E}_n$ s.t.

$m(F) - \varepsilon/2 \leq J^*(A) = J(A) = J_*(A) \leq m(E) + \frac{\varepsilon}{2}$.

$\Rightarrow m(F) - m(E) \leq \varepsilon$.

notice that: $F \supset E$. $F = E \amalg (F \setminus E)$.

$\Rightarrow m(F \setminus E) \leq \varepsilon$.

(2) ⇒ (3) Suppose (2) holds. Then $A \setminus E \subset F \setminus E$

$\Rightarrow J^*(A \setminus E) \le J^*(\underbrace{F \setminus E}_{\text{Elementary set}}) = m(F \setminus E) < \varepsilon.$

(2) ⇒ (4) the same as (2) ⇒ (3)

(3) ⇒ (1) Suppose (3) holds

$\forall \varepsilon > 0 \ \exists E_\varepsilon \subset A$ s.t. $J^*(A \setminus E_\varepsilon) < \varepsilon.$

Note $E_\varepsilon \in \mathcal{E}$ $\Rightarrow m(E_\varepsilon) \le J_*(A).$

By Lemma 2.

$J_*(A) \le J^*(A) \le J^*(A \setminus E_\varepsilon) + J^*(E_\varepsilon) < \varepsilon + m(E_\varepsilon) \le \varepsilon + J_*(A).$

let $\varepsilon \to 0$. $J_*(A) = J^*(A) \Rightarrow A \in \mathcal{J}$

(4) ⇒ (1) $\forall \varepsilon > 0 \ \exists F_\varepsilon \supset A$ s.t. $J^*(F_\varepsilon \setminus A) < \varepsilon.$

W.L.O.G. we can take a box $B \supset A$ s.t. $B \supset F_\varepsilon$.

let $E_\varepsilon = B \setminus F_\varepsilon \subset B \setminus A$. Then

$J^*((B \setminus A) \setminus E_\varepsilon) = J^*(F_\varepsilon \setminus A) < \varepsilon.$

By (3) ⇒ (1), see $B \setminus A \in \mathcal{J}$

By Lemma 3 $A \in \mathcal{J}$.

Remark:

(1) $J: \mathcal{J} \to [0, +\infty)$. satisfies

* finite addtivity

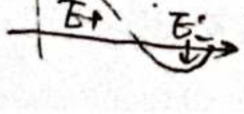
* translation-invariance } with Standardization $J([0,1]^d) = 1$

* Subaddtivity ⇓ radical properties,

* monotonicity

(2) Jordan measure ⟷ Riemann integral

· $A \in \mathcal{J} \Leftrightarrow \chi_A$ is Rieman integrable.

:  $\int f dx = J(E^+) - J(E^-)$.

Lec. 3

· Shortcome of Jordan measure:

- $\mathbb{Q}\cap[0,1]$ etc. are NOT Jordan measurable
- Not defined for unbounded sets

  → One way: define the Jordan (outer) measure of any unbounded set to be $+\infty$.

  But not reasonable to set $J^*(\mathbb{Z}) = +\infty$.

- There exists open sets / compact sets which are NOT measurable in Jordan

  (紧集).

  e.g. $\mathbb{Q}\cap[0,1] = \{q_1, q_2, q_3, \cdots\}$.

  For an $\varepsilon>0$ (small). We define $Q_\varepsilon := (q_1-\frac{\varepsilon}{2}, q_1+\frac{\varepsilon}{2}) \cup (q_2-\frac{\varepsilon}{4}, q_2+\frac{\varepsilon}{4}) \cup \cdots \cup (q_n - \frac{\varepsilon}{2^n}, q_n+\frac{\varepsilon}{2^n}) \cup \cdots$

  Then Q_ε is an open set, but not Jordan measurable

  $J^*(Q_\varepsilon) \geq 1$. $J_*(Q_\varepsilon) \leq 2\varepsilon$.

For compact set:

$[-1,2]\setminus Q_\varepsilon \longleftarrow$ bounded and closed

but not Jordan measurable.

- The "limit" of Jordan measurable sets may be Jordan-unmeasuable e.g. $Q_N = \{q_1, \cdots q_N\} \xrightarrow[N\to+\infty]{} \mathbb{Q}\cap[0,1]$.

Today: Lebesgue outer measure

$$J^*(A) = \inf_{\mathcal{E}\ni F\supset A} m(F) = \inf\{|B_1|+\cdots+|B_n| \mid A \subset \bigcup_{i=1}^n B_i\}.$$

$A\subset\bigcup_{i=1}^n B_i$ ([illegible])

Lebesgue's idea:

- Allow the outer measure to take value $+\infty$ (But not for all unbounded sets).
- Allow the number of boxes covering A to be infinitely m

} to make the definition make sense

Def: The Lebesgue outer measure of ANY subset $A \subset \mathbb{R}^d$ is:

$$m^*(A) = \inf \left\{ \sum_{i=1}^{\infty} |B_i| \ B_i\text{'s are boxes}, A \subset \bigcup_{i=1}^{\infty} B_i \right\}.$$

using countably many boxes. $\boxed{\text{L-cover}} \quad \boxed{I_i\text{'s are open boxes}}$

Example: $A = \mathbb{Q} \cap [0,1] = \{q_1, q_2, q_3, \cdots\}$

Then $A \subset \bigcup_{i=1}^{\infty} [q_i, q_i]$

$m^*(A) \leq \sum_{i=1}^{\infty} \left| [q_i, q_i] \right| = 0. \Rightarrow m^*(A) = 0.$

(or $A \subset Q_\varepsilon$. $m^*(A) \leq \sum_{i=1}^{\infty} 2^{1-i}\varepsilon = 2\varepsilon$. $\xrightarrow{\varepsilon \to 0} m^*(A) = 0$).

Example, Same Arguement

$\Downarrow$

Any countable set has Lesbegue outer measure Zero.

e.g. $\mathbb{Z}, \mathbb{Q}$.

Also by definition, one easily see

$$m^*(A) \leq J^*(A).$$

So we get

$$m^* : \underset{任意子集}{P(\mathbb{R}^d)} \to [0, +\infty]$$

It satisfies the following properties.

we will call them the axions of <u>abstract outer measures</u>

Prop: The Lebesgue outer measure

$$m^* \ P(\mathbb{R}^d) \longmapsto [0, +\infty]$$

Satisfies:

additivity is only for a measure, not for an outmeasure

(1) (Empty set) $m^*(\phi) = 0$.

(2) (monotonicity) If $A_1 \subset A_2$ then $m^*(A_1) \leq m^*(A_2)$

(3) (Countable subadditivity)

For any countable many sets $A_1, A_2, \cdots$

we have $m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^*(A_n)$.

Proof:

(1) $\phi \subset$ any box, $\forall \varepsilon.\ m^*(\phi) \le \varepsilon$

(2) Any coverings of A_2 is also a covering of A_1.

(3) (given yourself an ε of room).
try to prove $m^*(\bigcup_{i=1}^{\infty} A_i) \le \sum_{i=1}^{\infty} m^*(A_i) + \varepsilon$ for any ε

$\frac{\varepsilon}{2}\ \frac{\varepsilon}{4} \cdots \frac{\varepsilon}{2^n} \cdots$

For each A_i, we cover it by $B_{i1}, B_{i2}, \cdots$

$$A_i \subset \bigcup_{j=1}^{\infty} B_{ij} \quad \text{and} \quad \sum_{j=1}^{\infty} |B_{ij}| \le m^*(A_i) + 2^{-i}\varepsilon$$

Consider all boxes $\{B_{ij},\ i=1,2\cdots\ j=1,2\cdots\}$.

Then $\bigcup_{i=1}^{\infty} A_i \subset \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{\infty} B_{ij}$

$$\Rightarrow m^*(\bigcup_{i=1}^{\infty} A_i) \le \sum_{i,j} |B_{ij}| \le \sum_{i=1}^{\infty} m^*(A_i) + 2^{-i}\cdot\varepsilon.$$
$$= \sum_{i=1}^{\infty} m^*(A_i) + \varepsilon.$$

Remark: (1) Countable Subadditivity $\Rightarrow$ finite Subadditivity
$\not\Leftarrow$

In fact, Jordan outer measure satisfies finite subadditivity but not countable subadditivity

(2) Both Jordan outer measure and Lebesgue outer measure do NOT satisfy finite additivity

Prop. (Transition - invariance)

$$m^*(A+\{x\}) = m^*(A).$$

$B_1, B_2 \cdots\cdots$ of A

$\Leftrightarrow B_1+\{x\} \cdots\cdots$ of $A+\{x\}$.

Prop. For any elementray set $E \subset \mathbb{R}^d$ we have

$$m^*(E) = m(E) \leftarrow \left(m(E) = \sum_{i=1}^{n} |B_i| \text{, where } E = \bigcup_{i=1}^{n} B_i \text{, non-overlapping} \right)$$

($m^*(E) \le m(E)$:)

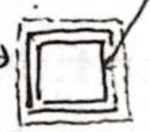
$$m^*(E) \le J^*(E) = m(E).$$

($m^*(E) \ge m(E)$:).

prove $\underbrace{m(E) \le m^*(E) + \varepsilon}$. for any ε.

we will use $\underbrace{\text{H-B}}$ property, (which $\mathbb{R}$ satisfies, but $\mathbb{Q}$ not)

open cover $\longleftrightarrow$ compact set.

for a box 

Step 1

Suppose $E = \bigcup_{i=1}^{n} B_i$, where B_i are non-overlapping.

Shrink B_i to $\widetilde{B_i}$ s.t. $\widetilde{B_i}$ is compact

. Such that $|\widetilde{B_i}| \ge |B_i| - \frac{\varepsilon}{n}$.

$\Rightarrow \widetilde{E} = \bigcup_{i=1}^{n} \widetilde{B_i}$ is a compact set, $\widetilde{E} \subset E$.

and $m(\widetilde{E}) \ge m(E) - \varepsilon$.

Step 2

For any covering of E by boxes $C_1, C_2, C_3 \cdots \leftarrow E \subset \bigcup_{i=1}^{\infty} C_i$ s.t. $\sum_{i=1}^{\infty} |C_i| \le m^*(E) + \varepsilon$.

we enlarge C_i to a larger $\underline{\text{open box}}$ $\widetilde{C_i}$ s.t. $C_i \subset \widetilde{C_i}$

and $|\widetilde{C_i}| \le |C_i| + 2^{-i} \cdot \varepsilon$.

Then $\underbrace{\bigcup_{i=1}^{\infty} \widetilde{C_i}}_{\text{open cover}} \supset \bigcup_{i=1}^{\infty} C_i \supset E \supset \underbrace{\widetilde{E}}_{\text{compact set}} \Rightarrow$ By H-B.

$\exists\ \widetilde{C}_{K(1)}, \dots \widetilde{C}_{K(N)}$ s.t. $\widetilde{E} \subset \bigcup_{i=1}^{N} \widetilde{C}_{K(i)}$.

$$\therefore \quad m(E) \le m(\widetilde{E}) + \varepsilon \le m\left(\bigcup_{i=1}^{N} \widetilde{C}_{K(i)}\right) + \varepsilon \le \sum_{i=1}^{N} |\widetilde{C}_{K(i)}| + \varepsilon$$

$$\le \sum_{i=1}^{\infty} |\widetilde{C_i}| + \varepsilon \le \sum_{i=1}^{\infty} |C_i| + 2\varepsilon \le m^*(E) + 3\varepsilon.$$

Def || We say two boxes B_1 and B_2 are <u>almost disjoint</u> if their interior do not intersect $\mathring{B}_1 \cap \mathring{B}_2 = \phi$.

Similarly define "almost disjoint" for more boxes

Obviously $m(\bigcup_{i=1}^{n} B_i) = \sum_{i=1}^{n} |B_i|$.

if B_i's are "almost disjoint".

Prop. || Let $A = \bigcup_{i=1}^{\infty} B_i$ be the union of almost disjoint boxes.

Then $m^*(A) = \sum_{i=1}^{\infty} |B_i|$.

Proof. $m^*(A) \leq \sum_{i=1}^{\infty} |B_i|$ <u>by definition</u> (or infinite subadditivity).

Conversely for any n. $\bigcup_{i=1}^{n} B_i \subset A$.

So by monotonicity $\sum_{i=1}^{n} |B_i| = m(\bigcup_{i=1}^{n} B_i) = m^*(\bigcup_{i=1}^{n} B_i) \leq m^*(A)$.

Let $n \to \infty$.

Thm. (Struction of open sets in $\mathbb{R}^d$).

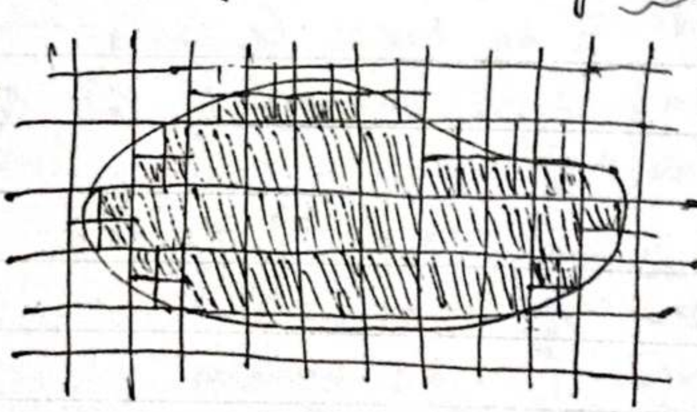
or disjoint semi-open and semi-closed boxes

$B = \times_{i=1}^{n} (a_i, b_i]$ $B_s \cap B_t = \phi$

Any open set $A \subset \mathbb{R}^d$ can be expressed as the union of countable many almost disjoint (closed) boxes.

($\Leftrightarrow$ $m^*(A) = \sum_{i=1}^{\infty} |B_i|$) for any open set A

Proof



Consider dyadic closed boxes of the form

$Q = [\frac{i_1}{2^n}, \frac{i_1+1}{2^n}] \times [\frac{i_2}{2^n}, \frac{i_2+1}{2^n}] \times \cdots \times [\frac{i_d}{2^n}, \frac{i_d+1}{2^n}]$

$i_j \in \mathbb{Z}$. $n \in \mathbb{N}$.

let $\mathcal{A} = \{Q \mid Q$ as above. $Q \subset A\}$.

Claim|| $\bigcup_{Q \in \mathcal{A}} Q = A$.

- $\bigcup_{Q \in \mathcal{A}} Q \subset A$ obvious.
- For any $x \in A$. A open thus exists a small ball $B_r(x) \subset A$. take n large s.t. $\frac{1}{2^n} < \frac{r}{10}$. Then we can find a $Q \ni x$ s.t. $Q \subset B_r(x) \subset A$. $\therefore A \subset \bigcup_{Q \in \mathcal{A}} Q$.

$\therefore \bigcup_{Q \in \mathcal{A}} Q = A$.

Observation: For any two such closed boxes Q_1, Q_2.
either they are almost disjoint
or one is contained in the other

Let $\mathcal{A}^* = \{Q \mid Q \subset A, Q$ is not contained in a large Q' in $\mathcal{A}^*\} \subset \mathcal{A}$ countable

Then $A = \bigcup_{Q \in \mathcal{A}^*} Q \leftarrow$ countable union of almost disjoint closed boxes.

Prop. (outer regularity) For any $A \subset \mathbb{R}^d$ $m^*(A) = \inf_{\substack{U \supset A \\ U \text{ open}}} m^*(U)$.

Proof. By monotonicity $A \subset U \Rightarrow m^*(A) \leq m^*(U)$
$\therefore m^*(A) \leq \inf_{U \supset A} m^*(U)$.

To prove the converse $\left(\inf_{U \supset A, U \text{ open}} m^*(U) \leq m^*(A)\right)$. W.L.O.G. assume $m^*(A) < +\infty$.

By definition of Lebesgue outer measure, $\forall \varepsilon > 0$ one can find boxes $B_1, B_2, \ldots$

s.t: $A \subset \bigcup_{i=1}^{\infty} B_i$ and $\sum_{i=1}^{\infty} |B_i| < m^*(A) + \varepsilon$.

Enlarge each B_i to $\widetilde{B_i}$ $\widetilde{B_i}$ is open box

and $|\widetilde{B_i}| \le |B_i| + 2^{-i}\varepsilon$. let $U_1 = \bigcup_{i=1}^{\infty} \widetilde{B_i}$

thus, $A \subset \bigcup_{i=1}^{\infty} B_i \subset \bigcup_{i=1}^{\infty} \widetilde{B_i} = U_1$

$$\Rightarrow \inf_{\substack{U \supset A \\ U \text{ open}}} m^*(U) \le \underbrace{m^*(U_1) \le \sum_{i=1}^{\infty} |\widetilde{B_i}|}_{\text{Sub-additivity}} \le \sum_{i=1}^{\infty} |B_i| + 2^{-i}\varepsilon \le m^*(A) + 2\varepsilon.$$

let $\varepsilon \to 0$. we get $\inf_{\substack{U \supset A \\ U \text{ open}}} m^*(U) \le m^*(A)$

Lec 4.

Additivity: (In condition of not overlapping)

→ If $\underset{\sim}{d(A_1, A_2) > 0}$. then $m^*(A_1 \cup A_2) = m^*(A_1) + m^*(A_2)$

(Stronger than $A_1 \cap A_2 = \phi$)

→ $m^*(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} |B_i|$ for almost disjoint boxes

→ Elementary set

In Jordan measurable, $J(A)$ exists $\Leftrightarrow$ $\forall \varepsilon > 0\ \exists F \in \mathcal{E}$ s.t. $J^*(F \setminus A) < \varepsilon$.

Def: || A set $A \subset \mathbb{R}^d$. is Lebesgue measurable if $\forall \varepsilon > 0$. $\exists$ open set $U \supset A$ s.t. $m^*(U \setminus A) < \varepsilon$.

- If A is Lebesgue measurable, we write $m(A) = m^*(A)$. and call it Lebesgue measure of A.
- We donate the set of all Lebesgue measuable sets by $\mathcal{L}$.

"Lebesgue measurable" ≈ nearly open

If we want to compare A_1, A_2.

we can consider $A_1 \Delta A_2 = (A_1 \setminus A_2) \cup (A_2 \setminus A_1)$.

Pset today

☆ difference between Lebesgue outer measure and Lebesgue measurable

$\forall \varepsilon > 0$. $\exists$ $U \supset A$ open s.t. $m^*(U) - m^*(A) < \varepsilon$.

$\forall \varepsilon > 0$ $\exists$ $U \supset A$. open s.t: $m^*(U \setminus A) < \varepsilon$.

Example: Any countable set is Lebesgue measuable with Lebesgue measurable $=0$.

- If $\underbrace{m^*(A)=0}_{\forall \varepsilon>0\ \exists\ U\supset A\ s.t:\ m^*(U)<\varepsilon\quad m^*(U\setminus A)<m^*(U)}$, then A is Lebesgue measuable, with $m(A)=0$.

Example: · Any open set is Lebesgue measuable.

· Any box is Lebesgue measuable.

Prop. If $A_i\in\mathcal{L}$. then $\bigcup_{i=1}^{\infty}A_i\in\mathcal{L}$.

Proof: $[\varepsilon=\frac{\varepsilon}{2}+\frac{\varepsilon}{4}+\cdots+\frac{\varepsilon}{2^n}+\cdots]$.

For any i. find an open set $U_i\supset A_i$ s.t: $m^*(U_i\setminus A_i)<\frac{\varepsilon}{2^i}$.

Let $U=\bigcup_{i=1}^{\infty}U_i$ Then U is open $U\supset\bigcup_{i=1}^{\infty}A_i$

$m^*(U\setminus A)\leq m^*(\bigcup_{i=1}^{\infty}(U_i\setminus A_i))\leq\sum_{i=1}^{\infty}m^*(U_i\setminus A_i)<\varepsilon$.

$\downarrow$

$\star\ \underline{U\setminus A\subset\bigcup_{i=1}^{\infty}(U_i\setminus A_i)}$

$A=\bigcup_{i=1}^{\infty}A_i$

· Any elementary is Lebesgue measuable.

Prop: Any $\underline{\text{compact}}$ set is Lebesgue measuable. $\to$ approach $m_*(A)$

Proof. Let $A\subset\mathbb{R}^d$ be compact

By def, ε we can find open set U. s.t: $U\supset A$.

and $\underline{m^*(U)\leq m^*(A)+\varepsilon}$ + additivity $\Rightarrow$ $m^*(A)=m(A)$.

Since $U\setminus A$ is open, we can write $U\setminus A=\bigcup_{i=1}^{\infty}B_i$

(B_is are almost disjoint closed boxes).

Set $\widetilde{B_n}=\bigcup_{i=1}^{n}B_i$. $\underline{\text{Then } \widetilde{B_n} \text{ is closed, and } \widetilde{B_n}\cap A=\phi}$.

$\Rightarrow \underline{d(\widetilde{B_n},A)>0}$. $\widetilde{B_n}$ A

So, $m^*(A)+m^*(\widetilde{B_n})=m^*(A\cup\widetilde{B_n})\leq m^*(A\cup(U-A))=m^*(U)\leq m^*(A)$

Note. A compact $\Rightarrow$ A is contained in a large box

$\Rightarrow m^*(A) < \infty$. So $m^*(\widehat{B_n}) < \varepsilon$.

$\Rightarrow m^*(\widehat{B_n}) = \sum_{i=1}^{n} m^*(B_i) = \sum_{i=1}^{n} |B_i| < \varepsilon \xrightarrow[n\to\infty]{} \sum_{i=1}^{\infty} |B_i| < \varepsilon$, i.e.

(for $\forall n$).

$m^*(U \setminus A) < \varepsilon$.

Cor. Any closed set is Lebesgue measurable.

proof A closed $\Rightarrow A_n := A \cap \overline{B(0,n)}$ is compact (for $\forall n \in N$)

and $\underbrace{A = \bigcup_{n=1}^{\infty} A_n}_{\text{prop 1.}}$. (develop compact sets to closed sets)

Prop: If $A \in \mathcal{L}$. then $A^c = \mathbb{R}^d \setminus A \in \mathcal{L}$.

Proof: Take open sets $U_n \supset A$. s.t: $m^*(U_n \setminus A) < \frac{1}{n}$.

Let $F_n = U_n^c$. and $F = \bigcup_{n=1}^{\infty} F_n$ Then $F \in \mathcal{L}$.

Since $A^c \setminus F \subset A^c \setminus F_n$. and $\underline{m^*(A^c \setminus F_n) = m^*(U_n \setminus A) < \frac{1}{n}}$.

$\Rightarrow m^*(A^c \setminus F) \le m^*(A^c \setminus F_n) < \frac{1}{n} \xrightarrow[n\to\infty]{} m^*(A^c \setminus F) = 0 \Rightarrow A^c \setminus F \in \mathcal{L}$

$\Rightarrow \begin{cases} F \in \mathcal{L} \\ A^c \setminus F \in \mathcal{L} \end{cases} \Rightarrow A^c \in \mathcal{L}$.

$A_1 \setminus A_2 = A_1 \cap A_2^c$
$= A_2^c \cap (A_1^c)^c = A_2^c \setminus A_1^c$.

Cor || $A_i \in \mathcal{L} \Rightarrow \bigcap_{i=1}^{\infty} A_i \in \mathcal{L}$.

Proof $(\bigcap_{i=1}^{\infty} A_i)^c = (\bigcup_{i=1}^{\infty} A_i^c) \in \mathcal{L}$.

$\bigcap_{i=1}^{\infty} A_i \in \mathcal{L}$

- F_σ sets (countable many closed sets' union) $\in \mathcal{L}$
- G_δ sets $\in \mathcal{L}$. $G_\delta = \bigcap_{i=1}^{\infty} A_i$. A_i's are open sets.

Cor || $A_1. A_2 \in \mathcal{L}. \Rightarrow A_1 \setminus A_2 \in \mathcal{L}$.

proof. $A_1 \setminus A_2 = A_1 \cap A_2^c$.

Prop: $A \in \mathcal{L} \Leftrightarrow \forall \varepsilon > 0\ \exists$ closed set $F \subset A$ s.t. $m^*(A \setminus F) < \varepsilon$.

Proof: $A \in \mathcal{L} \Leftrightarrow A^c \in \mathcal{L} \Leftrightarrow \exists\ U \supset A^c$ s.t. $m^*(U \setminus A^c) < \varepsilon$.

$\Leftrightarrow m^*(A \setminus U^c) < \varepsilon$

closed and $A \supset U^c$.

finite additivity

Thm (Countable additivity of Lebesgue measure)

$A_i \in \mathcal{L}$. disjoint $\Rightarrow m(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} m(A_i)$.

Proof: • Easy condition:

Assume all A_is are compact. Then (finite additivity for compact sets)

$\sum_{i=1}^{n} m(A_i) = m(\bigcup_{i=1}^{n} A_i) \le m(\bigcup_{i=1}^{\infty} A_i) \le \sum_{i=1}^{\infty} m(A_i)$. sub-additivity.

Let $n \to \infty$ we get $m(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} m(A_i)$.

• Middle case. Assume all A_i's are bounded.

$A_i \in \mathcal{L} \Rightarrow \exists$ closed $F_i \subset A_i$ s.t. $m(A_i \setminus F_i) < \varepsilon \cdot 2^{-i}$.

(compact)

Note: A_i's are disjoint. $\Rightarrow F_i$'s are disjoint.

$\sum_{i=1}^{\infty} m(A_i) \le \sum_{i=1}^{\infty} (m(F_i) + m(A_i \setminus F_i)) \le \sum_{i=1}^{\infty} m(F_i) + \varepsilon$.

$= m(\bigcup_{i=1}^{\infty} F_i) + \varepsilon \le m(\bigcup_{i=1}^{\infty} A_i) + \varepsilon. \le \sum_{i=1}^{\infty} m(A_i) + \varepsilon$.

let $\varepsilon \to 0 \Rightarrow m(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} m(A_i)$

General Case Only assume $A_i \in \mathcal{L}$.

Let $C_n = B(0,n) \setminus B(0,n-1)$.

Let $D_{i,n} = A_i \cap C_n$ They are bounded.

Then $\sum_{i=1}^{\infty} m(A_i) = \underbrace{\sum_{i=1}^{\infty} \sum_{n=1}^{\infty}}_{\text{Countable}} m(D_{i,n}) = m(\bigcup_{i,n} D_{i,n}) = m(\bigcup_{i=1}^{\infty} A_i)$.

Cor [Monotone convergence of measure]

Suppose $A_1 \subset A_2 \subset \cdots$ $A_i \in \mathcal{L}$ Then

$m(\bigcup_{i=1}^{\infty} A_i) = m(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} m(A_n)$.

Proof: Let $A_1' = A_1, A_2' = A_2 - A_1, \cdots A_k' = A_k - A_{k-1}, \cdots$

$\Rightarrow$ A_i's are disjoint and $\in \mathcal{L}$.

$\Rightarrow$ $m(\bigcup_{i=1}^{\infty} A_i) = m(\bigcup_{i=1}^{\infty} A_i') = \sum_{i=1}^{\infty} m(A_i') = \sum_{i=1}^{\infty} (m(A_i) - m(A_{i-1}))$.

$= \lim_{n\to\infty} m(A_n)$.

$\underline{A_i = A_i' \amalg A_{i-1}}$.

Lec 5

Last time

· A is Lebesgue measurable $\Leftrightarrow$ $\forall \varepsilon > 0$ $\exists$ open set $U \supset A$ s.t. $m^*(U \setminus A) < \varepsilon$

$\Leftrightarrow$ $\forall \varepsilon > 0$ $\exists$ closed set $F \subset A$ s.t. $m^*(A \setminus F) < \varepsilon$.

① $\emptyset \in \mathcal{L}$

② $A \in \mathcal{L} \Rightarrow A^c \in \mathcal{L}$

③ $A_n \in \mathcal{L} \Rightarrow \bigcup_n A_n \in \mathcal{L}$

$\bigcap_n A_n \in \mathcal{L}$

$\longrightarrow$ $\underline{\mathcal{L} \text{ is a } \sigma\text{-algebra}}$

$\mathcal{B} = \bigcap_{\substack{\Sigma \supset \{O \subset \mathbb{R}^d : O \text{ is open}\} \\ \Sigma \text{ is a } \sigma\text{-algebra}}} \Sigma$ $\forall B \in \mathcal{B}$. B is a Borel set.

Examples · countable sets

· $m^*(A) = 0$ (we call it null sets).

· open, closed, compact $\in \mathcal{L}$.

· $\underline{F_\sigma}$. $G_\delta \in \mathcal{L}$.

F_σ: Countable $\bigcup$ for closed sets

G_δ: Countable $\bigcap$ for open sets

· $\mathcal{E} \subset \mathcal{L}$

· $\mathcal{J} \subset \mathcal{L}$

Additivity (most significant property of m compared with m^*.)

Suppose $A_n \in \mathcal{L}$ are disjoint

then $m(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m(A_n)$. $\forall$ $m(A_n)$ can be $+\infty$.

Prop T.F.A.E

(1) $A \in \mathcal{L}$.

(2) $A = G \setminus N$, where G is G_δ set, N is a null set

(3) $A = F \cup N$. Where F is a F_σ set, N is a null set

weaker than E. F open closed. Stronger than ε.

Proof: (1) ⇒ (2). For any n. Let U_n be an open set s.t: $U_n \supset A$, $m^*(U_n \setminus A) < \frac{1}{n}$.

Let $G = \bigcap_{n=1}^{\infty} U_n \in G_\delta$ and $G \supset A$.

Note. Let $N = G \setminus A$. Then $N \subset U_n \setminus A$ (for each n)

By monotonicity of m^*, $m^*(N) \le m^*(U_n \setminus A) < \frac{1}{n}$.

let $n \to \infty$. $m^*(N) = 0$. i.e. N is a null set

(2) ⇒ (1) $G \in \mathcal{L}$, $N \in \mathcal{L}$. $A = \underbrace{G \setminus N}_{= G \cap N^c} \in \mathcal{L}$.

(1) ⇒ (3) For any n let F_n be an closed set. s.t: $F_n \subset A$ $m^*(A \setminus F_n) < \frac{1}{n}$

Let $F = \bigcup_{n=1}^{\infty} F_n \in F_\sigma$. and $F \subset A$.

Note. Let $N = A \setminus F$. Then $N \subset A \setminus F_n$

$m^*(A \setminus F) \le m^*(A \setminus F_n) < \frac{1}{n}$. let $n \to \infty$ $m^*(A \setminus F) = 0$.

i.e. $N = A \setminus F$ is a null set.

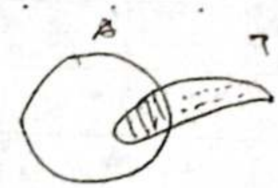
(3) ⇒ (1) $A = F \cup N$, $F \in \mathcal{L}$. $N \in \mathcal{L}$ $\therefore A \in \mathcal{L}$.

Other way to prove (1) ⇔ (3).

$A = G \setminus N = G \cap N^c$ $A^c = G^c \cup N$. $A \in \mathcal{L}$. $A^c \in \mathcal{L}$

a F_σ set

Prop. Suppose $A \subset \mathbb{R}^d$ is Lebesgue measurable
Then for any $T \subset \mathbb{R}^d$.
$m^*(T) = m^*(T \cap A) + m^*(T \setminus A)$



Proof. First assume T is open, then by additivity of m.
$m(T) = m(T \cap A) + m(T \setminus A)$ ①
Now for general $T \subset \mathbb{R}^d$. we take an open set $U \supset T$.
s.t: $\underline{m^*(U) \le m^*(T) + \varepsilon}$ definition of m^*.

$\boxed{\text{Cannot write } m^*(U \setminus T) < \varepsilon \text{ here}}$

Then $m^*(T) \le m^*(T \cap A) + m^*(T \setminus A)$
$\le m^*(U \cap A) + m^*(U \setminus A)$ ← using ①
$= m^*(U) \le m^*(T) + \varepsilon$ let $\varepsilon \to 0$.
$\Rightarrow m^*(T) = m^*(T \cap A) + m^*(T \setminus A)$.

Thm. T.F.A.E. (Carathedory Criteria)
(1) $A \in \mathcal{L}$
(2) For any box B. $|B| = m^*(B \cap A) + m^*(B \setminus A)$
(3) For any $E \in \mathcal{E}$ $m(E) = m^*(E \cap A) + m^*(E \setminus A)$
(4) For any set T. $m^*(T) = m^*(T \cap A) + m^*(T \setminus A)$
definition of measurable in textbook

This is used to define measurable sets on abstract outer measure space (without open sets).

(1) ⇒ (4) ✓ ⇒ (3) ⇒ (2) ✓
(2) ⇒ (1) ((2) ⇒ (3). (2)+(3) ⇒ (1)).

(2) ⇒ (3) Suppose $E \in \mathcal{E}$ Then $E = \bigcup_{i=1}^{n} B_i$
So $m(E) = \sum_{i=1}^{n} |B_i| = \sum_{i=1}^{n} m^*(B_i \cap A) + \sum_{i=1}^{n} m^*(B_i \setminus A)$
$m(E) \le m^*(E \cap A) + m^*(E \setminus A) \le \sum m^*(B_i \cap A) + \sum m^*(B_i \setminus A)$

$\boxed{\text{First assume } m^*(A) < \infty}$

(2)
(3) $\Rightarrow$ (1) Take disjoint boxes B_i's s.t. $A \subset \bigcup_{i=1}^{\infty} B_i$

$\sum_{i=1}^{\infty} |B_i| \le m^*(A) + \varepsilon$.

Enlarge each B_i to an open $\overset{\text{box}}{\widetilde{B_i}} \supset B_i$.

s.t. $|\widetilde{B_i}| < |B_i| + \frac{\varepsilon}{2^i}$

Then $\bigcup_{i=1}^{\infty} B_i = U$ is a open set and $U \supset A$. and

$m^*(A) = m^*(A \cap U) = m^*(\bigcup_i (A \cap \widetilde{B_i}))$

$\Leftarrow m^*(A) + m^*(U \setminus A) \le \sum_{i=1}^{\infty} m^*(\widetilde{B_i} \cap A) + \sum_{i=1}^{\infty} m^*(\widetilde{B_i} \setminus A)$.

$= \sum_{i=1}^{\infty} (m^*(\widetilde{B_i} \cap A) + m^*(\widetilde{B_i} \setminus A)) = \sum_{i=1}^{\infty} m^*(\widetilde{B_i}) \le \sum_{i=1}^{\infty} |B_i| + \varepsilon$.

$\le m^*(A) + 2\varepsilon$.

$\Rightarrow m^*(U \setminus A) \le 2\varepsilon$.

$\boxed{\text{Second. assume } m^*(A) = +\infty}$

Let D_n = box in $\mathbb{R}^d$. centered at 0. Side length is n.

Then $A \cap D_n = A_n \rightsquigarrow m^*(A_n) \le m^*(D_n) < \infty$

and $A = A \cap \mathbb{R}^d = A \cap (\bigcup_{i=1}^{\infty} D_i) = \bigcup_{i=1}^{\infty} (A \cap D_i) = \bigcup_{i=1}^{\infty} A_n$

If we can prove each $A_n \in \mathcal{L}$ the $A = \bigcup_{i=1}^{\infty} A_n \in \mathcal{L}$

By 1st part, we only need to prove

$|B| = m^*(B \cap A_n) + m^*(B \setminus A)$ (for any B)

We need $B \setminus A_n = B \cap A_n^c = B \cap (A \cap D_n)^c = B \cap (A^c \cup D_n^c)$.

$= (B \cap A^c) \cup (B \cup D_n^c) = (B \cap A^c \cap D_n) \cup \underbrace{(B \cap A^c \cap D_n^c) \cup (B \cup D_n^c)}_{\text{subset}}$

$= (B \cap A^c \cap D_n) \cup (B \cup D_n^c)$.

Note $|B| \le m^*(B \cap A_n) + m^*(B \setminus A_n)$

$\le \underbrace{m^*((B \cap D_n) \cap A) + m^*(B \cap A^c \cap D_n)}_{m^*(B \cap D_n)} + \underbrace{m^*(B \cap D_n^c)}_{= m^*(B \setminus D_n)}$

$m^*(B \cap A^c \cap D_n) = m^*((B \cap D_n) \setminus A)$, $\quad B \cap D_n \cap A^c$, $\quad \in \mathcal{L}$.

$= m^*((B \cap D_n) \cap A) + m^*((B \cap D_n) \setminus A) + m^*(B \cap D_n^c)$

$= m^*(B \cap D_n) + m^*(B \setminus D_n) = m(B) = |B|$.

$\Rightarrow |B| = m^*(B \cap A_n) + m^*(B \setminus A_n)$.

Prop. Suppose $A \subset \mathbb{R}^d$ is Lebesgue measurable $\underline{m(A) > 0}$. ($m^*(A) > 0$)
The $\forall\ 0 < t < 1$ $\exists$ box B s.t: $m(B \cap A) > t|B|$. ($m^*(B \cap A) > t|B|$)
(Some $\underline{\text{convergence property}}$ for measurable sets).

$\frac{m(B \cap A)}{|B|}$ kind of density

Proof: First assume $m(A) < +\infty$.
$\xrightarrow{\frac{(1-t)m(A)}{t}} \forall\ \varepsilon > 0$. One can take open set $U \supset A$. s.t: $m^*(U \setminus A) < \varepsilon$.
By structure theorem, we can write $U = \bigcup_{n=1}^{\infty} B_n$. almost disjoint boxes.
$\underline{\text{WANT}}$ $\exists$ k s.t: $t|B_k| < m(B_k \cap A)$.
By contradiction. Assume $m(B_n \cap A) \leq t|B_n|$ ($\forall n$)

$$m(A) \leq \sum_{n=1}^{\infty} m(A \cap B_n) \leq t \sum_{n=1}^{\infty} |B_n| = t\, m(U) \leq t(m(U \setminus A) + m(A))$$

$$\leq t\, m(A) + \underbrace{t\varepsilon < m(A)}_{\text{to get contradiction.}}$$

need $\varepsilon < \frac{(1-t)\, m(A)}{t}$.

For $m(A) = +\infty$.
Then $\exists$ n s.t: $A_n = \underline{A \cap B(0, n)}$ s.t: $m(A_n) > 0$ (monotone convergence)
Apply 1st part to A_n.

Prop. (Steinhaus) Suppose $A \in \mathcal{L}$ $m(A) > 0$. Then
$A - A := \{x - y \mid x, y \in A\}$.
Contains a small box B centered at 0.

Thm (Uniqueness of Lebesgue measure)
Suppose $\tilde{m} : \mathcal{L} \longrightarrow [0, +\infty]$ Satisfies
(1) $\tilde{m}(\emptyset) = 0$
(2) For disjoint $A_n \in \mathcal{L}$ one has $\tilde{m}(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \tilde{m}(A_n)$.
(3) $\tilde{m}(x + A) = \tilde{m}(A)$
(4) $\tilde{m}([0, 1]^d) = 1$.

Then $\tilde{m} = m$ $\boxed{\text{i.e. } \tilde{m}(A) = m(A)}$

Proof · By lecture 2. (P Set 1-2. Problem 2) we see that for $E \in \mathcal{E}_e$. then $\tilde{m}(E) = m(E)$

· By additivity and non-negativity, we have $\xrightarrow{\text{stronger than}}$ monotonicity, sub-additivity.

Step 1: $m(A) = 0 \Rightarrow \tilde{m}(A) = 0$.

Proof: Cover A by boxes B_n's s.t. $A \subset \bigcup_{n=1}^{\infty} B_n$. and $\sum_{n=1}^{\infty} |B_n| < \varepsilon$.

Then $\tilde{m}(A) \le \tilde{m}(\bigcup_{n=1}^{\infty} B_n) \le \sum_{n=1}^{\infty} \tilde{m}(B_n) = \sum_{n=1}^{\infty} m(B_n) < \varepsilon$.

Let $\varepsilon \to 0$. We get $\tilde{m}(A) = 0$.

Step 2: If U is open $m(U) < \infty$. then $\tilde{m}(U) = m(U)$.

By structure theorem. $U = \bigcup_{n=1}^{\infty} B_n$. B_n's are almost disjoint closed boxe

$\Rightarrow$ $\mathring{B}_n$ are disjoint open boxes.

Let $F = U \setminus \bigcup_{n=1}^{\infty} \mathring{B}_n \in \mathcal{L}$

Then $m(F) = m(U) - m(\bigcup_{n=1}^{\infty} \mathring{B}_n) = \sum_{n=1}^{\infty} |B_n| - \sum_{n=1}^{\infty} |\mathring{B}_n| = 0$.

$\Rightarrow \tilde{m}(F) = 0$.

So $\tilde{m}(U) = \tilde{m}(F) + \tilde{m}(\bigcup_{n=1}^{\infty} \mathring{B}_n) = \tilde{m}(\bigcup_{n=1}^{\infty} \mathring{B}_n) = \sum_{n=1}^{\infty} \tilde{m}(\mathring{B}_n)$
$= \sum_{n=1}^{\infty} m(\mathring{B}_n) = m(U) - m(F) = m(U)$

Step 3. $\tilde{m}(A) \le m(A)$ (for $\forall A \in \mathcal{L}$).

· It is trivially true for $m(A) = +\infty$

· Now assume $m(A) < +\infty$.

Choose open set $U \supset A$ s.t. $m(U \setminus A) = m(U) - m(A) < \varepsilon$.

So. $\tilde{m}(A) \le \tilde{m}(U) = m(U) < m(A) + \varepsilon$. $\varepsilon \to 0$

$\Rightarrow \tilde{m}(A) \le m(A)$.

thought of A^c

Step 4: If $A \in \mathcal{L}$ is bounded then $\tilde{m}(A) = m(A)$.

Take a box B s.t. $A \subset B$.

By additivity. $\tilde{m}(B \setminus A) = |B| - m(A)$.

Then. $\tilde{m}(A) \le m(A) = m(B) - m(B \setminus A) \le \tilde{m}(B) - \tilde{m}(B \setminus A) = \tilde{m}(A$

Step 5. For general $A \in \mathcal{L}$.

Let $C_n = B(0,n) \setminus B(0,n-1)$ $A_n = A \cap C_n$

Then $A = \bigcup_{n=1}^{\infty} A_n$. disjoint union. each A_n is bounded. and $A_n \in \mathcal{L}$.

So $\tilde{m}(A) = \sum_{n=1}^{\infty} \tilde{m}(A_n) = \sum_{n=1}^{\infty} m(A_n) = m(A)$.

Today Lec 6.

Recall (Lec 1).

Lebesgue's idea of integration

$$\int_a^b f(x)\,dx = \int_0^\infty m(\{x \mid f(x) > t\})\,dt$$

$(f(x) \geq 0)$.

Def: We say a d variable function f is (Lebesgue) measurable if for any (a.e.) $t \in \mathbb{R}$. the set

$\{x \mid f(x) > t\}$ is Lebesgue measurable.

Remarks:

1) We allow the domain of f to be a set A. instead of $\mathbb{R}^n$.

In this case. we require $A \in \mathcal{L}$.

Example: For any set $A \subset \mathbb{R}^d$ consider the characteristic function:

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Then χ_A is measurable $\Leftrightarrow$ $A \in \mathcal{L}$.

2) we allow the value of f to be $\pm\infty$.
in which case we say f is $\mathbb{R}^*$-valued.
$\overset{\shortparallel}{\mathbb{R}\cup\{\pm\infty\}}$

WARNING: $(+\infty)+(-\infty)$ makes no sense.

$f(x)=x^2$

In the case all values of f are in $\mathbb{R}$. we say f is $\underline{\text{finite-value}}$
or $\mathbb{R}$-valued. $\not\Leftrightarrow$ bounded functi

3) We say two functions f and g are equal almost everywhere.
(or a.e. equal).
if the set $\{x : f(x) \neq g(x)\}$. is a null set
Example: $\chi_{\mathbb{Q}} = 0$. a.e.

More generally we say a proposition P holds a.e.
if those points where P fails form a null set

Prop: Let $f: A \subset \mathbb{R}^d \to \mathbb{R}$. where $A \in \mathcal{L}$.
T.F.A.E. definition of measurable function
(1) $\underline{\forall\, t \in \mathbb{R} \text{ the set } \{x \mid f(x) > t\} \in \mathcal{L}}$.
(2) ---------- $\geq$ ---
(3) ---------- $<$ ---
(4) ---------- $\leq$ ---

Proof: (1)$\Leftrightarrow$(4). $\{x \mid f(x) > t\}^c = \{x \mid f(x) \leq t\}$
(2)$\Leftrightarrow$(3) $(<, \geq)$
(2) $\Rightarrow$ (1). $\{x \mid f(x) > t\} = \bigcup_{n=1}^{\infty} \{x \mid f(x) \geq t + \frac{1}{n}\}$. ☆ $\underline{\text{Some set is}}$
a way to prove measurable.

(4) $\Rightarrow$ (3) SAME method

$$\begin{matrix} (1) & \Longleftrightarrow & (4) \\ \Uparrow & & \Downarrow \\ (2) & \Longleftrightarrow & (3) \end{matrix}$$

Remark: ① We are not saying for a given t_0
$\{x \mid f(x) > t_0\} \in \mathcal{L} \Leftrightarrow \{x \mid f(x) \geq t_0\} \in \mathcal{L}$.

② (PSet Today)
(5) $\forall$ open set $U \subset \mathbb{R}$. $f^{-1}(U) \in \mathcal{L}$.
(6) $\forall$ closed set $F \subset \mathbb{R}$. $f^{-1}(F) \in \mathcal{L}$.

Cor: If f is Lebesgue measurable, then for any t, the level set
$f^{-1}(t) = \{x \mid f(x) = t\}$ is lebesgue measuable.
$= \{x \mid f(x) \geq t\} \cap \{x \mid f(x) \leq t\}$.

Examples of measurable functions.
(1) χ_A, where $A \in \mathcal{L}$.
(2) Any monotone function is measurable. (R)
(3) Any continuous function is measurable
$\Leftrightarrow$ for any open set U. $f^{-1}(U)$ is open $\in \mathcal{L}$.
Remark ② (5).
(4) $\Phi \circ f$ where $\Phi: \mathbb{R} \to \mathbb{R}$ is continuous
and f is Lebesgue measurable.
$$(\Phi \circ f)^{-1}(\underset{\uparrow \text{open}}{U}) = f^{-1} \circ \underset{\text{open}}{\underbrace{\Phi^{-1}(U)}} \in \mathcal{L}.$$

WARNING The composition of two measuable functions need not to be measuable function.

(5) Prop: If f, g are $\mathbb{R}$-valued measurable function.
then (1) cf, (2) $f \pm g$, (3) fg are measurable.
$\underbrace{\qquad\qquad}_{\text{Algebraic Structure}}$ (except for $\mathbb{R}^*$-valued functions. large)

Proof (1) If $c>0$. $\{x \mid c f(x) > t\} = \{x \mid f(x) > \frac{t}{c}\}$.

$c<0$

$c=0$ $cf = 0$.

(2) [Trick: decomposition a set to a countable union]

Suppose $f(x)+g(x)>t$.

$\iff f(x) > t - g(x)$

$\iff \exists\, r \in \mathbb{Q}$ s.t. $f(x) > r > t - g(x)$.

$\iff \exists\, r \in \mathbb{Q}$ s.t. $f(x) > r$. AND $g(x) > t - r$.

So:

$$\underbrace{\{x \mid f(x)+g(x) > t\}}_{\in \mathcal{L}} = \bigcup_{r\in\mathbb{Q}} \Big(\underbrace{\{x \mid f(x) > r\}}_{\in \mathcal{L}} \cap \underbrace{\{x \mid g(x) > t-r\}}_{\in \mathcal{L}} \Big)$$

(3) First note: f is measurable $\Rightarrow f^2$ is measurable.

$$\{x \mid f^2(x) > t\} \underset{(t\geq 0)}{=} \{x \mid |f(x)| > \sqrt{t}\} = \underbrace{\{x \mid f(x) > \sqrt{t}\}}_{\in \mathcal{L}} \cup \underbrace{\{x \mid f(x) < -\sqrt{t}\}}_{\in \mathcal{L}}$$

Now suppose f, g measurable. Then

$$fg = \frac{1}{2}\left((f+g)^2 - f^2 - g^2\right) \text{ is measurable.}$$

(6)

Thm || If f_n are measurable functions and $\lim_{n\to\infty} f_n = f$ then f is measurable.

Remark: we know that limit of continuous functions need not to be continuous.

So measurable functions are much "nicer" than continuous functions under taking limit

Proof: Suppose

$f(x) = \lim_{n\to\infty} f_n(x) > t. \Rightarrow \exists\, k$ s.t. $\lim_{n\to\infty} f_n(x) > t + \frac{1}{k}$.

OURSTORYBEGINS

$\Rightarrow \exists k, \exists N$ s.t: $\forall n > N$ $f_n(x) > t + \frac{1}{k}$.

So $\{x \mid f(x) > t\} = \bigcup_k \{x \mid \lim_{n\to\infty} f(x) > t + \frac{1}{k}\}$.

$= \bigcup_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n>N} \{x \mid f_n(x) > t + \frac{1}{k}\} \in \mathcal{L}$

$\exists \rightsquigarrow \cup$

$\forall \rightsquigarrow \cap$

In fact, we don't need to assume f_n to be convergent.

Prop: Let f_n be any sequence of measurable functions

then $\lim_{n\to\infty} \sup f_n$ $\quad \lim_{n\to\infty} \inf f_n$

$\sup_n f_n$ $\quad \inf_n f_n$

are all measurable functions

for a fixed x,

$\sup_n f_n(x) > t \iff \exists n$ s.t: $f_n(x) > t$.

$\{x \mid \sup_n f_n(x) > t\} = \bigcup_n \{x \mid f_n(x) > t\}$.

$\lim_{n\to\infty} \sup f_n(x) > t \Rightarrow \exists k.$ $\lim_{n\to\infty}\sup f_n(x) > t + \frac{1}{k}$.

$\Rightarrow \exists k. \forall N. \exists n > N.$ s.t: $f_n(x) > t + \frac{1}{k}$.

$\{x \mid \lim_{n\to\infty}\sup f_n(x) > t\} = \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{x \mid f_n(x) > t + \frac{1}{k}\}$.

Cor: f is measurable $\Rightarrow |f|$ is measurable.

$|f| = \sup\{f, -f\}$

Any measurable function is the limit of a sequence of simple function

Lec 7

Today: convergent sequence of measurable function

1 Convergence in measure

· Let's take a closer look at a.e. convergence

$\lim_{n\to\infty} f_n(x) \neq f(x)$ $\exists \varepsilon > 0$ s.t: $\forall N \in \mathbb{N}$, $\exists k \geq N$ s.t: $|f_k(x) - f(x)| > \varepsilon$.

($\lim_{n\to\infty} f_n(x) = f(x)$) $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t: $\forall k > N$ $|f_k(x) - f(x)| < \varepsilon$.)

Let $A_k = \{x \in A : |f_k(x) - f(x)| > \varepsilon\}$.

$\lim_{n\to\infty} f_n(x) \neq f(x) \Leftrightarrow \exists \varepsilon > 0.$ $x \in \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} A_k(\varepsilon)$.

$f_n \to f$ a.e $\Leftrightarrow m(\{x : \lim_{n\to\infty} f_n(x) \neq f(x)\}) = 0 \Leftrightarrow m(\{x : \exists \varepsilon > 0$ s.t: $x \in \bigcap_{N=1}^{\infty}\bigcup_{k=N}^{\infty} A_k(\varepsilon)\})$

$\Leftrightarrow m(\underbrace{\bigcup_{\varepsilon}}_{\text{infinite union}} (\bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} A_k(\varepsilon))) = 0. \Leftrightarrow \boxed{\forall \varepsilon > 0, m(\limsup_{n\to\infty} A_n(\varepsilon)) = 0}$

Def: For any sequence A_k of sets we define its upper limit

$$\limsup_{k\to\infty} A_k = \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} A_k$$

Prop: || $\limsup_{n\to\infty} A_k = \{x : x \in A_k$ for infinitely many A_k's$\}$

proof. $x = \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} A_k \Leftrightarrow \forall N.\ x \in \bigcup_{k=N}^{\infty} A_k$

$\Leftrightarrow \forall N\ \exists k \geq n$ s.t: $x \in A_k$

$\Leftrightarrow x \in A_k$ for infinitely many A_k's.

Borel-Cantelli lemma || If $A_k \in \mathcal{L}$, $\sum_{k=1}^{\infty} m(A_k) < \infty$, then

$A_\infty := \limsup_{k\to\infty} A_k \in \mathcal{L}$ and $m(A_\infty) = 0$.

$= \lim_{N\to\infty} \bigcup_{k=N}^{\infty} A_k$

Def. Let f_n be a sequence of a.e. finite and measurable functions defined on $A \in \mathcal{L}$.

We say f_n converges to f in measure if $\forall \varepsilon > 0$.

$$\boxed{\lim_{k\to\infty} m(A_k(\varepsilon)) = 0}$$

OURSTORYBEGINS

Prop 11 If $m(A)<\infty$. then $f_n \to f$ a.e $\Rightarrow$ $f_n \to f$ in measure.

Proof. Recall "downward monotone convergence"

$\forall \varepsilon>0$ fixed, we have

$A \supset \bigcup_{k=1}^{\infty} A_k(\varepsilon) \supset \bigcup_{k=2}^{\infty} A_k(\varepsilon) \cdots$

$\Rightarrow$ By downward monotone convergence

$\lim_{N\to\infty} m\left(\bigcup_{k=N}^{\infty} A_k(\varepsilon)\right) = m\left(\lim_{N\to\infty}\bigcup_{k=N}^{\infty} A_k(\varepsilon)\right) = \underbrace{m\left(\bigcap_{N=1}^{\infty}\bigcup_{k=N}^{\infty} A_k(\varepsilon)\right) = 0}_{f_n\to f \text{ a.e}}$

$\Rightarrow m(A_N(\varepsilon)) \le m\left(\bigcup_{k=N}^{\infty} A_k(\varepsilon)\right) \xrightarrow[N\to\infty]{} 0$

$\Rightarrow$ $f_n \to f$ in measure.

Example Let $E_n = \left[\frac{k-1}{2^l}, \frac{k}{2^l}\right] \subset [0,1]$. where $n = 2^l + k < 2^{l+1}$

e.g. $E_1=[0,1]$. $E_2=[0,1]$. $E_3=[0,\frac{1}{2}]$. $E_4=[\frac{1}{2},1]$. $E_5=[0,\frac{1}{4}]$....

Let $f_n = \chi_{E_n}$.

Then $f_n \to 0$ in measure $\left[m(A_n(\varepsilon)) = m(E_n) = \frac{1}{2^l} \to 0\right]$.

But $f_n \not\to 0$, $f_n \not\to 0$ a.e ($\forall x \in [0,1]$. $f_n(x) \not\to 0$)

$\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty} A_n(\varepsilon) = \bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty} E_n = \bigcap_{N=1}^{\infty}[0,1] = [0,1]$

Observe $f_1, f_2, f_3, f_5, f_9, f_{17} \cdots \to 0$ a.e.

Thm (Riesz) If $f_n \to f$ in measure $\to \exists$ Subsequence $\{f_{n_k}\}$ s.t. $\underset{a.e}{f_{n_k} \to f}$

Proof. For any $k \in \mathbb{N}$, choose n_k st:

$m\left(A_n\left(\frac{1}{2^k}\right)\right) < \frac{1}{2^k}$ $\forall n \ge n_k$.

Note we can always arrange s.t. $n_1 < n_2 < n_3 \cdots$

Let $A_\infty = \limsup_{k\to\infty} A_{n_k}\left(\frac{1}{2^k}\right)$.

Then $\sum_{k=1}^{\infty} m\left(A_{n_k}\left(\frac{1}{2^k}\right)\right) < 1 < \infty$

$\xRightarrow[\text{B-C Lemma}]{} m(A_\infty) = 0$.

Note $x \notin A_\infty \Leftrightarrow x \in A_{n_k}\left(\frac{1}{2^k}\right)$ for only finitely many k

$\Leftrightarrow \exists \underset{(\ge n_k)}{K}$ s.t: $x \notin A_k\left(\frac{1}{2^k}\right) \Rightarrow \forall n \ge K$ $|f_n(x) - f(x)| < \frac{1}{2^k}$

$\therefore f_n(x) \to f(x)$.

Uniform convergence $\Rightarrow$ Locally uniformly convergence

(pointwise) Convergence $\Leftarrow$ Locally uniformly convergence

Convergence $\Rightarrow$ a.e Convergence, a.e Convergence $\not\Rightarrow$ Convergence

a.e Convergence $\not\Rightarrow$ Locally uniformly convergence; a.e Convergence $\Rightarrow$ Locally uniformly convergence outside a set of measure $<\varepsilon$

Convergence $\not\Rightarrow$ Convergence in measure

Convergence in measure $\xRightarrow{\text{sub-sequence}}$ a.e Convergence; Convergence in measure $\not\Rightarrow$ a.e Convergence; a.e Convergence $\not\Rightarrow$ Convergence in measure; a.e Convergence $\xRightarrow{m(A)<+\infty}$ Convergence in measure

$\Leftrightarrow \forall \varepsilon>0.\ \exists N$ s.t. $\forall n\geq N\ A_n(\varepsilon)=\emptyset$

Recall: $f_n\to f$ uniformly $\Leftrightarrow \forall\varepsilon>0.\ \exists N$ s.t. $\forall n\geq N.\ \forall x\ |f_n(x)-f(x)|<\varepsilon$

Def. We say $f_n\to f$ locally uniformly if for any bounded set E, we have $f_n\to f$ uniformly

Example $f_n(x)=\frac{x}{n}$ $x\in\mathbb{R}$.

Then $f_n(x)\to 0$ locally, pointwise, a.e.

$f_n(x)\not\to 0$ uniformly

$f_n(x)\not\to 0$ in measure.

$f_n\to f$ a.e $\Leftrightarrow \forall\varepsilon>0$ $m(\limsup\limits_{k\to\infty} A_k(\varepsilon))=0$.

$f_n\to f$ in measure $\Leftrightarrow \forall\varepsilon>0$ $\lim\limits_{k\to\infty} m(A_k(\varepsilon))=0$.

Example $f_n(x)=\frac{1}{nx}\chi_{\mathbb{R}^+}=\begin{cases}\frac{1}{nx} & x>0\\ 0 & x\leq 0\end{cases}$

Then $f_n\to 0$ pointwise, a.e., in measure.

$f_n\not\to 0$ locally, uniformly.

Thm (Егоров.) f_n finite a.e. and measurable.

If $f_n\to f$ a.e. then $\forall\varepsilon>0.\ \exists A_\varepsilon\in\mathcal{L}$ with $m(A_\varepsilon)<\varepsilon$

s.t. $f_n\to f$ locally uniformly on $A\setminus A_\varepsilon$.

$\Leftrightarrow \forall x\ \exists$ open set $U\ni x$ s.t. $f_n\to f$ uniformly in U

Remark: (1) This explains Littewood's 3^{rd} principle.

= Every covergent seq is "nearly" uniformly convergent

(2) We don't require $m(A)<\infty$.

We say a proposition holds locally if $\forall x, \exists$ open set $U\ni x$

s.t. the proposition holds in U

Proof: Take A_0 s.t.: $m(A_0)=0$ and
$f_n \to f$ in $A \setminus A_0$

Denote $A_{N,m} = \bigcup_{n \geq N} (A_n(\frac{1}{m}) \cap (A \setminus A_0)) \in \mathcal{L}$.
$= \{x \mid x \in A \setminus A_0 : \exists\, n \geq N \text{ s.t.: } |f_n(x) - f(x)| > \frac{1}{m}\}$.

Then $\underbrace{\bigcap_{N=1}^{\infty} A_{N,m}}$ $= \phi$. (for a fixed m).
($x \in \Leftrightarrow \forall N \geq 1.\ \exists\, n \geq N$ s.t. $|f_n(x) - f(x)| > \frac{1}{m} \Leftrightarrow f_n(x) \not\to f(x)$.)

Note. $A_{1,m} \supset A_{2,m} \cdots \supset A_{n,m} \supset \cdots$
$\Rightarrow$ For any R. $(B(0,R) \cap A_{1,m}) \supset (B(0,R) \cap A_{2,m}) \supset \cdots$

By monotone convergence thm we get $\lim_{N \to \infty} m(B(0,R) \cap A_{N,m}) = 0$.
(using fixed m. $\lim_{N \to \infty} m(B(0;m) \cap A_{N,m}) = 0$).

In particular, we can find $N_m > 0$ s.t.:
$m(B(0,m) \cap A_{N,m}) < \frac{\varepsilon}{2^m}$, $\forall N \geq N_m$.

Not let $A_\varepsilon = \bigcup_{m=1}^{\infty} (B(0,m) \cap A_{N_m, m}) \cup A_0 \in \mathcal{L}$
$m(A_\varepsilon) < \varepsilon$ (By sub-additivity)

Finally. by construction, we have
$|f_n(x) - f(x)| < \frac{1}{m}$. $\forall m \geq 1$ and $\forall x \in (A \setminus A_\varepsilon) \cap B(0,m)$ and $\forall n \geq N_m$.

Now for any bounded set E, with $E \subset A \setminus A_\varepsilon$.
we take m large s.t.: $E \subset B(0,m)$.
Then $|f_n(x) - f(x)| < \frac{1}{m}$ $\forall x \in E$. $\forall n \geq N_m \Rightarrow f_n \to f$ uniformly in E.

2. Approximation by simple functions

Def: A function is simple function if it has the form
$f(x) = c_1 \chi_{A_1}(x) + c_2 \chi_{A_2}(x) + \cdots + c_n \chi_{A_n}(x)$.
where $c_1, c_2, \ldots, c_n$ are numbers. and $A_1, A_2, \ldots A_n \in \mathcal{L}$ finite
and $A_i \cap A_j = \phi$ ($i \neq j$.) $f(x) = 0$ ($x \notin \bigcup A_i$).

"Step function". special case of simple functions with each A_i = box

Note: Any simple function is measurable.

Thm. A function f is measurable if and only if f is the limit of a sequence of simple function $f_n(x)$.

Moreover

(1) If $f \geq 0$, then we can take $0 \leq f_1 \leq f_2 \cdots \nearrow$.

(2) In general, we can take $|f_1| \leq |f_2| \leq \cdots \leq |f|$.

(3) If f is bounded function, we can take $f_n \to f$ uniformly.

Proof. (a) If $f \geq 0$.

Fix n. define $f_n(x) = \begin{cases} \frac{k-1}{2^n} & \text{if } \left(\frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n}\right) \quad (k = 1, 2, \cdots, n2^n). \\ n & \text{if } (f(x) \geq n) \end{cases}$

$\to$ simple function.

Check
- $f_n \leq f_{n+1}$.
- $f_{n+1}(x) - f_n(x) \leq \frac{1}{2^{n+1}}$ if $f < n$
- $f_{n+1}(x) - f_n(x) \leq 1$. if $f \geq n$.

(b) General.

Define f^+. f^-. $f^+ = \max(f, 0)$. $f^- = \max(-f, 0)$

$f = f^+ - f^-$

$$\begin{tikzcd}
U.C. \arrow[r, Rightarrow] \arrow[drr, Rightarrow] & L.U.C \arrow[r, Rightarrow] & P.C \arrow[r, Rightarrow] & a.e.C \arrow[lll, bend right, "\text{Egorov}"'] \arrow[dl, bend left, "m(A)<\infty"] \\
 & & C.\ \text{in measure} \arrow[ur, "\text{Riesz}"] &
\end{tikzcd}$$

In all these modes. the limit f must be measurable

Lec 8

1 Approximation by simple functions

Thm 1: Let $f \geq 0$ be a non-negative measurable function defined on $A \in \mathcal{L}$. Then there exists an increasing sequence of simple functions $0 \leq f_1(x) \cdots \leq f_n(x) \leq \cdots \leq f(x)$.
Such that $\lim_{n\to\infty} f_n(x) = f(x)$. Moreover, if f is bounded, the $f_n \to f$ uniformly

Proof. define $f_n(x)$ like last lec.

$$f_n(x) = \begin{cases} \frac{k-1}{2^n} & \text{if } \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n} \quad k = 1, 2, \cdots, n\cdot 2^n \\ n & \text{if } f(x) \geq n. \end{cases}$$

Then, since f is measurable, the sets

$$\{x: \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n}\} \quad \{x: f(x) \geq n\}$$

are measurable. So $f_n(x)$ is a simple function

By def, $f_n(x) \leq f(x)$

We need to compare f_n and f_{n+1}.

(1) If $f(x) < n$, then $\exists k \leq n\cdot 2^n$ s.t. $\frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n}$

$$\frac{k-1}{2^n} = \frac{2k-2}{2^{n+1}} \quad \frac{2k-1}{2^{n+1}} \quad \frac{k}{2^n} = \frac{2k}{2^{n+1}}$$

$\rightarrow f_n(x) = \frac{k-1}{2^n}$

$\rightarrow f_{n+1}(x) = \frac{2k-2}{2^{n+1}}$ or $\frac{2k-1}{2^{n+1}}$

Thus $f_n(x) \leq f_{n+1}(x)$ and $0 \leq f_{n+1}(x) - f_n(x) \leq \frac{1}{2^{n+1}}$ ☆

(2) If $f(x) \geq n$, then $f_n(x) = n$.

Since $n = n\frac{2^{n+1}}{2^{n+1}} \leq f(x) \Rightarrow f_{n+1}(x) = \frac{n\cdot 2^{n+1}}{2^{n+1}}, \frac{n2^{n+1}+1}{2^{n+1}}, \cdots, \frac{n\cdot 2^{n+1}+2^{n+1}-1}{2^{n+1}}, n+1$

$\Rightarrow 0 \leq f_{n+1}(x) - f_n(x) \leq 1$.

So we have

$$0 \leq f_1 \leq \cdots \leq f_n \leq \cdots \leq f.$$

and $\lim_{n\to\infty} f_n(x) = f(x)$ — $f(x) = \infty$ $\forall n$. $f_n(x) > n$.
$f(x) < \infty$. $\{f_n(x)\}$ cauchy 列收敛.

Moreover, if $f(x) < M$, then for $n > M$ only (1) occurs.

$\Rightarrow \forall x.\ |f_n(x) - f(x)| \leq \frac{1}{2^n} \Rightarrow f_n \to f$ uniformly.

Thm2 Let f be a measurable function defined on $A \in \mathcal{L}$.
There exists a sequence of simple function
$0 \le |f_1| \le |f_2| \le \cdots \le |f_n| \le \cdots \le |f|$
s.t. $\lim_{n\to\infty} f_n(x) = f(x)$.
Moreover, if $|f(x)| < M$, then $f_n \to f$ uniformly.

Proof. Let $f^+ = \max(f, 0)$ $f^- = \max(-f, 0)$.
then f^+, f^- are measurable, non-negetive and $\underline{f = f^+ - f^-}$.
Apply 1 to f^+, f^- respectively. we get
$0 \le \underbrace{g_1 \le g_2 \le \cdots}_{\text{simple}} \le f^+$. $0 \le \underbrace{h_1 \le h_2 \le \cdots}_{\text{simple}} \le f^-$.
$g_n \to f^+$ $h_n \to f^-$ let $\underline{f_n = g_n - h_n}$.

$\to$ If $f(x) > 0$ $|f(x)| = f^+(x)$. $f^-(x) = 0$. $\to h_n(x) = 0 \to |f_n(x)| = |g_n(x)| = g_n(x)$
$\le g_{n+1}(x) = |f_{n+1}(x)|$
$\to$ If $f(x) < 0$ Similarly.

Finally. if $|f(x)| < M$, then $g_n \to f^+$ uniformly, $h_n \to f^-$ uniformly
$\Rightarrow f_n \to f$ uniformly.

Cor: f is measurable $\Leftrightarrow$ f is the limit of a sequence of simple functions.

Cor: If f is measurable, then $\exists$ a sequence of step functions.
$f_1, f_2 \cdots$ s.t: $f_n \to f$ a.e.
Exercise

2 Approximation by continuous functions
Thm (Lusin) Let f be any a.e. finite and measurable function.
defined on $A \in \mathcal{L}$. Then $\forall \varepsilon > 0$ $\exists$ $\boxed{\text{closed set}}$ $F \subset A$ with
$m(A \setminus F) < \varepsilon$ s.t: f is continuous on F. for extension
Littlewood's 2nd principle:

"Every measurable function is nearly continuous"

(For each n. take closed set F_n with $m(A\setminus F_n) < \frac{\varepsilon}{2^n}\cdot\frac{1}{3}$ s.t:

Proof 1: Take a sequence of step functions $f_n \to f$ a.e (Cor).

f_n is continuous on F_n.

By Egorov, $\exists F_\varepsilon$ with $m(F_\varepsilon) < \frac{\varepsilon}{3}$ s.t: $f_n \to f$ uniformly on $F\setminus F_\varepsilon$

Then $A\setminus\underbrace{\left(\left(\bigcup_n A\setminus F_n\right)\cup F_\varepsilon\right)} \in \mathcal{L}$.

Take $F \subset A\setminus\underbrace{\left(\left(\bigcup_n A\setminus F_n\right)\cup F_\varepsilon\right)}$ closed s.t: $m\left(\left(A\setminus\left(\left(\bigcup A\setminus F\right)\cup F_\varepsilon\right)\right)\setminus F\right) < \frac{\varepsilon}{3}$

Then $m(A\setminus F) < \varepsilon$ and $f_n \to f$ uniformly on F. f_n is continuous on F.

$\Rightarrow$ f is continuous on F.

Proof 2: · First assume f is simple function. i.e

$f = \sum_{i=1}^{n} c_i \chi_{A_i}$ $A_i \in \mathcal{L}$ $A_i \cap A_j = \phi$ and $A = \bigcup_{i=1}^{n} A_i$

For each A_i find $\underset{\text{closed}}{F_i \subset A_i}$. $m(A_i\setminus F_i) < \frac{\varepsilon}{n}$.

Then $F = \bigcup_{i=1}^{n} F_i$ closed, $m(A\setminus F) \le \sum_{i=1}^{n} m(A_i\setminus F_i) < \varepsilon$.

f is continuous on each F_i. since it is a constant function

f is continuous on F, since each F_i is closed. and $F_i \cap F_j = \phi$.

· Second assume f is bounded

Then simple functions $f_1, f_2, \ldots$ s.t: $f_n \to f$ uniformly.

By step 1. $\exists F_n \subset A$ with $m(A\setminus F_n) < \frac{\varepsilon}{2^n}$ s.t: f_n is continuous on F_n. Let $F := \bigcap_{n=1}^{\infty} F_n$. then F is closed and $m(A\setminus F) < \varepsilon$.

Now on F. f_n are continuous $f_n \to f$ uniformly on F

$\Rightarrow$ f is continuous on F.

· Finally, for general case $\mathbb{R} \to [0,1]$ one to one

☆ let $g(x) = \frac{f(x)}{1+|f(x)|}$ By step 2. $\exists F$ closed with $m(A\setminus F) < \varepsilon$ s.t. g is continuous on F. $f(x) = \frac{g(x)}{1-g(x)}$ is continuous

Cor: f a.e. finite, measurable on $A \in \mathcal{L}$.
$\exists$ a continuous function g on R^n s.t: $m(\{x : f(x) \neq g(x)\}) < \varepsilon$.

$\rightarrow$ Lusin + Tietze extension thm: $\forall F \subset R^n$ closed
f continuous on $F \Rightarrow \exists g$ continuous on R^n
s.t: $g = f$ on F
Moreover, if $|f| \leq M$, then can take $|g| \leq M$.

ε.

Cor: f a.e finite measurable
Then $\exists$ a sequence of continuous function $f_1, f_2, \cdots$ on R^n
s.t: $f_n \to f$ $\boxed{\text{a.e.}}$

By the previous corollary, one can find a sequence of continuous function f_n s.t: $f_n \to f$ $\boxed{\text{in measure}}$

$\Rightarrow \exists$ subseq $f_{n_k} \to f$ a.e.

$\frac{\varepsilon}{3}$

ε.

ε.

Lec 9

Idea: To define $\int_A f(x)dx$, we first define $\int_A f_n(x)dx$, where f_n is simple. It's simpler if we first let $f_n \to f$ to be non-negative

· Consider infinite sum $\sum_{n=1}^{\infty} a_n$

We have two different theories

(1) "non-negative theory"

If each $a_n \geq 0$, then $\sum_{n=1}^{\infty} a_n$ always converges (may be $+\infty$)

and we change order $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_{\sigma(n)}$ $\quad \sigma: \mathbb{N}\to\mathbb{N}$ bijective

$= \sup\{\sum_{n\in S} a_n,\ S\subset\mathbb{N},\ S \text{ is finite}\}$

(2) "absolute convergence theory"

If $\sum_{n=1}^{\infty} |a_n| < +\infty$, then $\sum_{n=1}^{\infty} a_n$ converges, and we can change the order, In particular we have $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^-$

Similarly, for Lebesgue's integrals, one also have

(1) "Nonnegative theory"

→ Only need $f \geq 0$.

→ Generally $\int f dx = \int f^+ - \int f^-$

→ Convergence theorems

(Lev.) monotone convergence

Fatou lemma

(2) "absolute convergence theory"

need $\int_{\mathbb{R}^d} |f| dx < +\infty$.

⟷ (Lebesgue) dominated convergence theorem

1. Integrals of non-negative simple functions.

- Let $A \in \mathcal{L}$.
  for $f(x) = \chi_A(x)$ is non-negative simple function.
  $\to$ define $\int_{\mathbb{R}^d} \chi_A(x)\,dx = m(A)$.

- For any non-negative simple function $f(x) = \sum_{n=1}^{N} \underbrace{a_n}_{\text{can be } +\infty} \chi_{A_n}(x)$.
  we must define $\int_{\mathbb{R}^d} f(x)\,dx = \sum_{n=1}^{N} a_n m(A_n)$
  $\to$ to ensure linear character.

Remark:

(1) One can check that the definition is independent of representation of f
if $\sum_{n=1}^{N} a_n \chi_{A_n} = \sum_{m=1}^{M} \tilde{a}_m \chi_{\tilde{A}_m}$ (A_i's, $\tilde{A}_i$'s can be overlapped).
then $\sum_{n=1}^{N} a_n m(A_n) = \sum_{m=1}^{M} \tilde{a}_m m(\tilde{A}_m)$

So we always let $A_i \cap A_j = \phi$, and a_i is non-zero

(2) If f is defined on $A \in \mathcal{L}$ we can define $\int_A f\,dx = \int_{\mathbb{R}^d} f(x)\chi_A(x)\,dx$. $\leftarrow$ also simple $\Uparrow$ $\chi_{A_1}(x)\cdot\chi_{A_2}(x) = \chi_{A_1\cap A_2}(x)$.

So we can always extend f to be a simple function on $\mathbb{R}^d$
s.t. $f = 0$ on A^c. $\Rightarrow$ It's enough to handle $\int_{\mathbb{R}^d} f(x)\,dx$,
and we can assume $\bigcup_{i=1}^{n} A_i = \mathbb{R}^d$.

Prop: Let $f(x) = \sum_{i=1}^{n} a_i \chi_{A_i}$ $g(x) = \sum_{j=1}^{m} b_j \chi_{B_j}$ be non-negative simple functions (assume $A_i \cap A_j = \phi$. $\bigcup A_i = \mathbb{R}^d$. $B_i \cap B_j = \phi$ $\bigcup B_i = \mathbb{R}^d$).

① For any $c_1, c_2 \geq 0$. $\int_{\mathbb{R}^d} (c_1 f + c_2 g)\,dx = c_1\int_{\mathbb{R}^d} f + c_2 \int_{\mathbb{R}^d} g$. Linearity

② If $f \leq g$ $\int_{\mathbb{R}^d} f \leq \int_{\mathbb{R}^d} g$ Monotonicity

③ $\int_{\mathbb{R}^d} f(x)\,dx = 0 \iff f = 0$ a.e.

④ $\int_{\mathbb{R}^d} f(x)\,dx < +\infty \iff f$ is a.e. finite. and $m(\underbrace{\mathrm{Supp}\, f}_{\{x:\, f(x)\neq 0\}}) < +\infty$

different from $cl(\{x: f(x) \neq 0\})$

⑥ $|\int \varphi| \le \int |\varphi|$. Triangle inequality

⑦ $\int_{E\cup F} \varphi = \int_E \varphi + \int_F \varphi$. E, F are disjoint. Additivity

OURSTORYBEGINS

⑤ For any $y \in \mathbb{R}^d$. $\int_{\mathbb{R}^d} f(x+y)dx = \int_{\mathbb{R}^d} f(x)dx$.

Proof. (1) $c_1 f + c_2 g = \sum_{i=1}^n c_1 a_i \chi_{A_i} + \sum_{j=1}^m c_2 b_j \chi_{B_j} = (\sum_{i=1}^n c_1 a_i \chi_{A_i}) \cdot \underbrace{(\sum_{j=1}^m \chi_{B_j})}_{=1}$

$+ (\sum_{j=1}^m c_2 b_j \chi_{B_j}) \cdot (\sum_{i=1}^n \chi_{A_i}) = \sum_{i=1}^n \sum_{j=1}^m (c_1 a_i + c_2 b_j) \chi_{A_i} \chi_{B_j}$

$= \sum_{i=1}^n \sum_{j=1}^m (c_1 a_i + c_2 b_j) \chi_{A_i \cap B_j}$

$(A_i \cap B_j) \cap (A_{i'} \cap B_{j'}) = \phi$ $(i,j) \neq (i',j')$

$\Rightarrow c_1 f + c_2 g \ge 0$. and

$\int_{\mathbb{R}^d} (c_1 f + c_2 g) dx = \sum_i \sum_j (c_1 a_i + c_2 b_j) m(A_i \cap B_j)$ $\quad \bigcup_{i,j} (A_i \cap B_j) = (\bigcup A_i) \cap (\bigcup B_j) = \mathbb{R}^d$.

$= \sum_{i=1}^n c_1 a_i \underbrace{(\sum_{j=1}^m m(A_i \cap B_j))}_{= m(A_i)}$

$+ \sum_{j=1}^m c_2 b_j \underbrace{(\sum_{i=1}^n m(A_i \cap B_j))}_{= m(B_j)}$

$= \sum_{i=1}^n c_1 a_i m(A_i) + \sum_{j=1}^m c_2 b_j m(B_j) = c_1 \int_{\mathbb{R}^d} f + c_2 \int_{\mathbb{R}^d} g$

(2) Similarly. Let $c_1 = -1$. $c_2 = 1$.

~~$g - f = \sum_i \sum_j (b_j - a_i) \chi(A_i \cap B_j)$. is a simple function~~

Since $g \ge f$, $b_j \ge a_i$ if $A_i \cap B_j \neq \phi$.

$\Rightarrow \int_{\mathbb{R}^d} f - g = \sum_{i,j} (b_j - a_i) m(A_i \cap B_j) \ge 0$.

$\Rightarrow \int_{\mathbb{R}^d} f + \int_{\mathbb{R}^d} g - f = \int_{\mathbb{R}^d} g$. $\Rightarrow \int_{\mathbb{R}^d} g \ge \int_{\mathbb{R}^d} f$.

(3) • If $\underset{= \sum a_i \chi_{A_i}}{f} = 0$ a.e. then if $a_i \neq 0$. then $A_i \subset \{x : f(x) \neq 0\} \Rightarrow m(A_i) = 0$.

$\Rightarrow \int_{\mathbb{R}^d} f = 0$.

• If $\int_{\mathbb{R}^d} f\, dx = \sum_{i=1}^n a_i m(A_i) = 0$. then $\forall i$ $a_i m(A_i) = 0$. $\Rightarrow$ if $a_i \neq 0$. $m(A_i) = 0$.

$\Rightarrow \{x : f(x) \neq 0\} = \bigcup_{a_i \neq 0} \{x : f(x) = a_i\} = \bigcup_{a_i \neq 0} A_i$.

$\Rightarrow m(\{x : f(x) \neq 0\}) = 0$.

(4) • If f is finite a.e. and $m(\operatorname{supp} f) < \infty$

Then let $a = \underbrace{\sup\{a_i, a_i < +\infty\}}_{\text{Finite set}} < +\infty$

we have

$$\int_{\mathbb{R}^d} f(x)dx = \sum_{i=1}^n a_i m(A_i) = \sum_{0<a_i<\infty} a_i m(A_i) \le a \sum_{0<a_i<\infty} m(A_i)$$
$$= a\, m\Big(\bigcup_{0<a_i<\infty} A_i\Big) \le a \cdot m(\widetilde{\operatorname{supp}} f) < +\infty.$$

- If $m(\{x: f(x) = +\infty\}) > 0$, then $\int_{\mathbb{R}^d} f\, dx = \sum_{i=1}^n a_i m(A_i) \ge +\infty \cdot m(\{x: f(x) = +\infty\}) = +\infty$.

If $m(\widetilde{\operatorname{supp}} f) = +\infty$ let $\tilde{a} = \inf\{a_i : a_i > 0\} > 0$.

Then $\int_{\mathbb{R}^d} f\, dx = \sum_{i=1}^n a_i m(A_i) \ge \sum_{a_i>0} \tilde{a} m(A_i) = \tilde{a} \cdot m(\widetilde{\operatorname{supp}} f) = +\infty$.

(5) As a function of x

$f(x+y) = \sum_{i=1}^n a_i \chi_{A_i - y}(x)$. $\quad A_i - y = \{x - y : x \in A_i\}$.

By translation invariance of Lebesgue measure, we have $A_i - y \in \mathcal{L}$.

and $m(A_i - y) = m(A_i)$

$\Rightarrow f \ge 0$ and is a simple function and

$$\int_{\mathbb{R}^d} f(x+y)\, dx = \sum_{i=1}^n a_i m(A_i - y) = \sum_{i=1}^n a_i m(A_i) = \int_{\mathbb{R}^d} f(x)\, dx.$$

2. Integration of non-negative measurable function

Now let f be a non-negative measurable function

want: define $\int_{\mathbb{R}^d} f(x)\, dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)\, dx$ where $f_n \nearrow f$

f_n simple.

Then: need to check it's well-defined $\Rightarrow$ complicated.

Instead we define (for measurable function).

$$\int_{\mathbb{R}^d} f(x)\, dx = \sup\Big\{\int_{\mathbb{R}^d} h(x)dx.\ h \text{ is non-negative, simple and } h \le f\Big\}.$$

This is always well defined.

Remark. (1) If f is defined on $A \in \mathcal{L}$. then

$$\int_A f(x) = \int_{\mathbb{R}^d} f(x) \chi_A(x)\, dx.$$

So. it's enough to handle $\int_{\mathbb{R}^d} f(x) dx$.

(2) In the case f is non-measurable (still ≥ 0)
one can define lower Lebesgue integral $\int_{\mathbb{R}^d} f(x)\,dx :=$
$\sup\{\int_{\mathbb{R}^d} h(x)\,dx : h \geq 0.$ simple and $h \leq f\}$. ($\underline{\int}_{\mathbb{R}^d} f(x)\,dx$)
and upper Lebesgue integral $\int_{\mathbb{R}^d} f(x)\,dx :=$
$\inf\{\int_{\mathbb{R}^d} h(x)\,dx : h \geq 0.$ simple and $h \geq f\}$. ($\overline{\int}_{\mathbb{R}^d} f(x)\,dx$)

$\int_{\mathbb{R}^d} = \underline{\int}_{\mathbb{R}^d}$

One can prove: if f is measurable and bounded and $m(\operatorname{supp} f) < +\infty$.
then $\underline{\int}_{\mathbb{R}^d} f\,dx = \overline{\int}_{\mathbb{R}^d} f\,dx$.

Prop: Let $f(x).\ g(x)$ be non-negative measurable functions

① $c_1, c_2 \geq 0$. $\int_{\mathbb{R}^d} (c_1 f + c_2 g)\,dx = c_1 \int f + c_2 \int g$

② If $f \leq g$ then $\int_{\mathbb{R}^d} f\,dx \leq \int_{\mathbb{R}^d} g\,dx$.

③ $\int_{\mathbb{R}^d} f(x)\,dx = 0 \iff f = 0$ a.e.

④ [Markov's inequality] For any $0 < \lambda < +\infty$, one has $m(\{x \in \mathbb{R}^d : f(x) \geq \lambda\}) \leq \frac{1}{\lambda}\int_{\mathbb{R}^d} f(x)\,dx$

⑤ For any $y \in \mathbb{R}^d$ $\int_{\mathbb{R}^d} f(x+y)\,dx = \int_{\mathbb{R}^d} f(x)\,dx$.

Proof (1) We first prove $\forall c > 0$. $\int_{\mathbb{R}^d} cf\,dx = c\int_{\mathbb{R}^d} f\,dx$.
This is true because

$$\begin{aligned}\int_{\mathbb{R}^d} cf(x)\,dx &= \sup\{\int_{\mathbb{R}^d} h(x)\,dx,\ h \geq 0.\ \text{simple},\ h \leq cf\}\\ &= \sup\{c\int_{\mathbb{R}^d} \tilde{h}(x)\,dx,\ \tilde{h} \geq 0\ \text{simple},\ \tilde{h} \leq f\} \quad \tilde{h} = \frac{h}{c}.\\ &= c\sup\{\int_{\mathbb{R}^d} \tilde{h}(x)\,dx.\ \tilde{h} \geq 0.\ \text{simple},\ \tilde{h} \leq f\}.\\ &= c\int_{\mathbb{R}^d} f(x)\,dx.\end{aligned}$$

It remains to prove $\int_{\mathbb{R}^d} (f+g)\,dx = \int_{\mathbb{R}^d} f\,dx + \int_{\mathbb{R}^d} g\,dx$.
[We will postpone this to next time].

(2) If $0 \le h \le f$ then $0 \le h \le g$.

$$\int_{\mathbb{R}^d} f(x)dx = \sup\{\int_{\mathbb{R}^d} h\,dx.\ h \ge 0.\ \text{simple},\ h \le f\}.$$
$$\le \sup\{\int_{\mathbb{R}^d} h\,dx.\ h \ge 0.\ \text{simple}.\ h \le g\} = \int_{\mathbb{R}^d} g(x)dx.$$

(4) Let $g(x) = \lambda \chi_{\{x:\ f(x) \ge \lambda\}}$. Then $g \ge 0$. simple.

$g(x) \le f(x) \quad \forall x$.

~~By (2)~~ $\int_{\mathbb{R}^d} g(x) \le \int_{\mathbb{R}^d} f(x)dx$.

(definition of $\int_{\mathbb{R}^d} f\,dx$). $\lambda \cdot (m(\{x: f(x) \ge \lambda\}))$

(3) · If $f = 0$ a.e. and $0 \le h \le f$ then $h = 0$ a.e. $\int_{\mathbb{R}^d} h(x)dx = 0$

$\Rightarrow \int_{\mathbb{R}^d} f(x)dx = \sup\{\int_{\mathbb{R}^d} h(x)dx,\ 0 \le h \le f.\ h \text{ simple}\} = 0.$

· Conversely. Suppose $\int_{\mathbb{R}^d} f(x)dx = 0$.

Then $m(\{x: f(x) \ne 0\}) = m(\bigcup_{k=1}^{\infty} \{x: f(x) > \frac{1}{k}\}) \le \sum_{k=1}^{\infty} m(\{x: f(x) \ge \frac{1}{k}\})$.

$\le \sum_{k=1}^{\infty} k \int_{\mathbb{R}^d} f(x)dx = 0. \quad \Rightarrow f = 0$ a.e.

(5) If h is simple $0 \le h \le f$.

then as a function of x. $h(x+y)$ is simple and in fact

$0 \le h(x+y) \le f(x+y)$

$\Rightarrow \int_{\mathbb{R}^d} f(x+y)dx \ge \sup\{\int_{\mathbb{R}^d} h(x+y)dx.\ h \text{ simple},\ 0 \le h \le f\}.$

$= \sup\{\int_{\mathbb{R}^d} h(x)dx.\ h \text{ simple}.\ 0 \le h \le f\} = \int_{\mathbb{R}^d} f(x)dx.$

Conversely, if $0 \le h(x+y) \le f(x+y)$ the $h(x)$ Simple and $0 \le h(x) \le f(x)$.

Similarly.

Def: We say a non-negative measurable f is integrable

$$\int_{\mathbb{R}^d} f(x)\,dx < +\infty$$

We have seen

A non-negative simple function f is integrable

$\Leftrightarrow$ f is finite a.e. and $m(\widetilde{\text{supp}}\, f) < +\infty$.

This is no longer true for measurable functions $\not\Rightarrow$ $\not\Leftarrow$

e.g. $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \chi_{[n,n+1]}$ $\rightarrow$ limit of a series of simple functions measurable.

$\int_{\mathbb{R}^d} f(x)dx = \int_{\mathbb{R}^d} \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^2} \chi_{[n,n+1]} dx$

$= \lim_{N\to\infty} \int \sum_{n=1}^{N} \frac{1}{n^2} \chi_{[n,n+1]} dx = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty$. Integrable.

But. $\widetilde{\text{supp}}\, f = +\infty$.

Prop: Suppose f is integrable, $\Rightarrow$ then f is a.e. finite.

Proof: $m(\{x : f(x) = +\infty\}) = m(\bigcap_{k=1}^{\infty} \{x : f(x) > k\}) \le m(\{x : f(x) > K\})\ \forall K > 0$

$\le \frac{1}{K} \int_{\mathbb{R}^d} f(x)dx$

Let $K \to \infty$. we get $m(\{x : f(x) = +\infty\}) = 0$.

Lec 10

Today: Limits of integrals

Recall (Lec. 4) Monotonicity of measure

$A_1 \subset A_2 \subset \cdots$ $A_i \in \mathcal{L}$. $\Rightarrow \lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n \in \mathcal{L}$

and $m(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} m(A_n)$

Interpret via "integrals"

$\chi_{A_1} \le \chi_{A_2} \le \cdots \Rightarrow \lim_{n\to\infty} \chi_{A_n}$ is simple, and

$\underline{\int_{\mathbb{R}^d} \lim_{n\to\infty} \chi_{A_n} dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} A_n dx} \Leftrightarrow \int_{\lim_{n\to\infty} A_n} 1\, dx = \lim_{n\to\infty} \int_{A_n} 1\, dx$.

Levi's monotone convergence theorem

In $0 \le f_1 \le f_2 \le f_3 \le \cdots$ is a monotone increasing sequence of non-negative measurable functions

then ($\lim_{n\to\infty} f_n$ is a non-negative measurable and)

$\| \quad \int_{R^d} \lim_{n\to\infty} f_n dx = \lim_{n\to\infty} \int_{R^d} f_n(x)\,dx.$

Proof: Let's first prove a simple case

$\|$ Lemma: Suppose $A_1 \subset A_2 \subset \cdots$ and $h(x) = \sum_{i=1}^{m} a_i \chi_{B_i}$ is simple

$\|$ $h \geq 0$. Then $\lim_{n\to\infty} \int_{A_n} h\,dx = \int_{\lim_{n\to\infty} A_n} h\,dx$

(let $f_n = h \cdot \chi_{A_n}$. Then $f_n \geq 0$, $f_n \nearrow$).

Proof: $\int_{A_n} h\,dx = \sum_{i=1}^{m} a_i \int_{A_n} \chi_{B_i}\,dx = \sum_{i=1}^{m} a_i\, m(\underbrace{A_n \cap B_i}_{\downarrow n\to\infty \; (\lim_{n\to\infty} A_n)\cap B_i})$

$\rightarrow \sum_{i=1}^{m} a_i\, m((\lim_{n\to\infty} A_n) \cap B_i)$

$= \sum_{i=1}^{m} a_i \int_{\lim_{n\to\infty} A_n} \chi_{B_i}\,dx = \int_{\lim_{n\to\infty} A_n} h\,dx.$

Proof of Levi's

$\because f_n \leq \lim_{n\to\infty} f_n \Rightarrow \int_{R^d} f_n dx \leq \int_{R^d} \lim_{n\to\infty} f_n dx \Rightarrow \lim_{n\to\infty} (\int_{R^d} f_n dx) \leq \int_{R^d} (\lim_{n\to\infty} f_n) dx$

· Take h any non-negative simple function, $0 \leq h \leq \lim_{n\to\infty} f_n$, Fixed ε.

Let $A_n \triangleq \{x : f_n(x) \geq (1-\varepsilon) h(x)\}$.

Then · $A_n \subset A_{n+1}$

· $\bigcup_{n=1}^{\infty} = \lim_{n\to\infty} A_n = R^d$.

By lemma. $\int_{A_n} h(x)\,dx \rightarrow \int_{\lim_{n\to\infty} A_n} h(x)\,dx = \int_{R^d} h(x)\,dx.$

$\Rightarrow \int_{R^d} f_n(x)\,dx \geq \int_{A_n} f_n(x)\,dx \geq (1-\varepsilon) \int_{A_n} h(x)\,dx$

$\Rightarrow \lim_{n\to\infty} \int_{R^d} f_n(x)\,dx \geq (1-\varepsilon) \lim_{n\to\infty} \int_{A_n} h(x)\,dx = (1-\varepsilon) \int_{R^d} h(x)\,dx.$

$\Rightarrow \lim_{n\to\infty} \int_{R^d} f_n(x)\,dx \geq (1-\varepsilon) \sup \{ \int_{R^d} h(x)\,dx.\ 0 \leq h \leq \lim f_n.\ h \text{ simple} \}.$

$= (1-\varepsilon) \int_{R^d} \lim_{n\to\infty} f(x)\,dx$

let $\varepsilon \to 0$. $\lim_{n\to\infty} \int_{R^d} f_n(x)\,dx \geq \int_{R^d} \lim_{n\to\infty} f(x)\,dx$

OURSTORYBEGINS

another def of $\int_{\mathbb{R}^d} f(x)dx$.

Cor || If $0 \le f_1 \le f_2 \le \cdots \nearrow f$ $\underset{\sim}{f_n \text{ is simple}}$. $\int_{\mathbb{R}^d} f dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)dx$.

Cor || $\int_{\mathbb{R}^d} (f+g)dx = \int_{\mathbb{R}^d} f\,dx + \int_{\mathbb{R}^d} g\,dx$.

Proof: Take simple function $f_n \nearrow f$ $g_n \nearrow g$.

$\Rightarrow f_n + g_n \nearrow f+g$

Then $\int_{\mathbb{R}^d} (f+g)\,dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} (f_n+g_n)\,dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} f_n dx + \lim_{n\to\infty} \int_{\mathbb{R}^d} g_n dx$

$= \int_{\mathbb{R}^d} f dx + \int_{\mathbb{R}^d} g\,dx$

Cor || $\int_{\mathbb{R}^d} (\sum_{i=1}^{n} f_i)\,dx = \sum_{i=1}^{n} \int_{\mathbb{R}^d} f_i\,dx$ where f_i are non-negetive

Cor (Integration by terms) || let $f_1 \cdots f_n \cdots$ be nonnegative measurable. Then $\int_{\mathbb{R}^d} (\sum_{i=1}^{\infty} f_i)\,dx = \sum_{i=1}^{\infty} \int_{\mathbb{R}^d} f_i\,dx$.

Proof. Let $g_N = \sum_{i=1}^{N} f_i$ then $g_1 \le g_2 \le g_3 \cdots$

$\Rightarrow \int_{\mathbb{R}^d} \sum_{i=1}^{\infty} f_i dx = \int_{\mathbb{R}^d} \lim_{N\to\infty} g_N\,dx = \lim_{N\to\infty} \int_{\mathbb{R}^d} g_N dx = \lim_{N\to\infty} \sum_{i=1}^{N} \int_{\mathbb{R}^d} f_i dx$

$= \sum_{i=1}^{\infty} \int_{\mathbb{R}^d} f_i(x)\,dx$

Cor: Suppose $A_i \in \mathcal{L}$. $A_i \cap A_j = \phi$. Then for any non-negative measurable function f $\int_{\bigcup_{i=1}^{\infty} A_i} f\,dx = \sum_{i=1}^{\infty} \int_{A_i} f\,dx$.

proof. let $g_n = f(x) \cdot \chi_{\bigcup_{i=1}^{n} A_i}$ Then $g_n \le g_{n+1}$

$\Rightarrow \int_{\bigcup_{i=1}^{\infty} A_n} f\,dx = \int_{\mathbb{R}^d} \lim_{n\to\infty} g_n\,dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} g_n dx =$

$\lim_{n\to\infty} \int_{\mathbb{R}^d} f(x) \chi_{\bigcup_{i=1}^{n} A_i} = \lim_{n\to\infty} \int_{\mathbb{R}^d} f(x) \overset{\nearrow A_i \cap A_j = \phi}{\sum_{i=1}^{n}} \chi_{A_i}(x)\,dx$.

$= \lim_{n\to\infty} \sum_{i=1}^{n} \int_{\mathbb{R}^d} f(x) \chi_{A_i}(x)\,dx = \sum_{n=1}^{\infty} \int_{A_i} f\,dx$.

Remark "Abstract measure"

$\mu: \mathcal{F} \to [0, +\infty]$ s.t: $\mu(\phi) = 0$

$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ $A_i \cap A_j = \phi$

countable additivity

Ex. We can define $M_f: \mathcal{L} \to [0, +\infty]$.
s.t: $M_f(A) = \int_A f\,dx$ ($\forall f$, $f \geq 0$ and measurable)
Then $M_f(\phi) = \int_\phi f\,dx = 0$.
$M_f(\bigcup_{i=1}^{\infty} A_i) = \int_{\bigcup_{i=1}^{\infty} A_i} f\,dx = \sum_{i=1}^{\infty} \int_{A_i} f\,dx = \sum_{i=1}^{\infty} M_f(A_i)$.

Cor Let $f^{\geq 0}$ be an a.e. finite and measurable on $A \in \mathcal{L}$ and $\boxed{m(A) < +\infty}$. Then $\int_A f(x)dx$ ~~$= \int_0^\infty m(\{x: f(x) > t\})\,dt$~~ ↑Riemann sense
$= \lim_{\delta \to 0} \sum_{k=0}^{\infty} (t_{k+1} - t_k)\, m(\underbrace{\{x: f(x) \geq t_k\}}_{A_k})$.
where $|t_{k+1} - t_k| \leq \delta$

Proof. By def $A_k \supset A_{k+1} \supset \cdots$
$t_k\, m(A_k \setminus A_{k+1}) \leq \int_{A_k \setminus A_{k+1}} f(x)dx \leq t_{k+1}\, m(A_k \setminus A_{k+1})$

$\Rightarrow \sum_{k=0}^{\infty} t_k(m(A_k) - m(A_{k+1})) \leq \int_A f(x)dx \leq \sum_{k=0}^{\infty} t_{k+1}(m(A_k) - m(A_{k+1}))$.

$t_0 m(A_0) = 0$

$\sum_{k=0}^{\infty} t_{k+1} m(A_k) - t_k m(A_k)$

$= \sum_{k=0}^{\infty} (t_{k+1} - t_k) m(A_k)$

$t_0 m(A_0) = 0$

$\sum_{k=0}^{\infty} t_{k+1} m(A_{k+1}) - t_k m(A_{k+1})$ $\curlyvee$

$= \sum_{k=0}^{\infty} (t_{k+1} - t_k)\, m(A_{k+1})$

$= \sum_{k=0}^{\infty} (t_{k+1} - t_k)(m(A_{k+1}) - m(A_k))$

$+ \sum_{k=0}^{\infty} (t_{k+1} - t_k)\, m(A_k)$.

$\leq \delta \sum_{k=0}^{\infty} m(A_{k+1}) + m(A_k)$.

$= \delta\, m(A) \to 0$ as $\delta \to 0$

$m(A) < +\infty$.

Fatou Lemma || Suppose f_n is a seq, $f_n \geq 0$ measurable
Then $\int_{\mathbb{R}^n} \liminf_{n\to\infty} f_n\,dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^n} f_n\,dx$.

$f_n = \begin{cases} n & 0 < x < \frac{1}{n} \\ 0 & \text{else} \end{cases}$

Proof: For $\forall j \geq k$. we have $\inf_{n \geq k} f_n \leq f_j$.
$\to \int_{\mathbb{R}^d} \inf_{n\geq k} f_n\,dx \leq \inf_{j \geq k} \int_{\mathbb{R}^d} f_j\,dx$. $\to \lim_{k\to\infty} \inf_{j\geq k} \int_{\mathbb{R}^d} f_j = \liminf_{n\to\infty} \int_{\mathbb{R}^d} f_n$.

$\int_{\mathbb{R}^d} g_k\,dx \xrightarrow{k\to\infty} \int_{\mathbb{R}^d} \liminf_{n\to\infty} f_n\,dx$

Let $g_k = \inf_{n \geq k} f_n \leq g_{k+1} \leq \cdots$

PSet 3 – Part 1

2. lemma 1 $\mathcal{L}: \mathbb{R}^d \to \mathbb{R}^d$ L: Linear transformation

Then $m^*(L(A)) = |\det L| \cdot m^*(A)$.
$\forall A \in \mathcal{L}$

Pf:

① box ② open set ③ $A \in \mathcal{L}$.
Open U C'ed

② $m^*(L(U)) = m^*(L(\bigcup_i B_i)) = m^*(\bigcup_i (L(B_i)) = \sum m^*(L(B_i)) = |\det L| \sum_i m^*(B_i)$
$= |\det L| \, m^*(U)$.

PSet 3 – Part 2

4. f measuriable find $C \in \mathcal{L}$ $f^{-1}(C) \notin \mathcal{L}$.

Pf: 若 $\underset{\text{可逆}}{f}$ 满足 $\exists B \subset A$ ($m(B) > 0$) $m(f(B)) = 0$. $\Rightarrow$ $\exists$ 不可测 $W \subset B$.

$0 \le m(f(W)) \le m(f(B)) = 0$. $\therefore$ Choose $f(W)$ as C.

$$\begin{tikzcd} B \arrow[r] \arrow[d, phantom, "\cup" rotate=-90] & f(B) \arrow[d, phantom, "\cup" rotate=-90] \\ W_{\notin \mathcal{L}} & f(W) \in \mathcal{L} \end{tikzcd}$$

Claim: $\exists W \subset \underbrace{f(C)}_{m(f(C)) > 0}$ W不可测，正测集有不可测子集

不妨设 $m(A) > 0$. $A \subset [0,1]$

令 $W = A/\mathbb{Q}$: (定义~关系) $x \sim y$: $x - y \in \mathbb{Q}$. 在每个等价类中取一个元素构成W. 令 $W_r = W + r$, $r \in \mathbb{Q} \cap [-1,1]$

则 $A \subset \bigcup_r W_r \subset [-1,2]$ 若 $W \in \mathcal{L}$. $0 < m(A) \le \sum m(W_r) \le 3$ 矛盾.

$\rightarrow$ W 不可测.

Lec 11

1 Definition of Lebesgue integral for general measurable functions (Absolute integrability)

- Let f be a measurable function on R^d

(It's enough to assume f is a.e measurable).

$f(x) = f^+(x) - f^-(x)$, $|f(x)| = f^+(x) + f^-(x)$

$\Rightarrow$ both f^- and f^+ are non-negative measurable functions

Define. $\int_{R^d} f\,dx = \int_{R^d} f^+\,dx - \int_{R^d} f^-\,dx$.

(1) If at least one of $\int_{R^d} f^+\,dx$, $\int_{R^d} f^-\,dx$ is finite, then we can define

$$\int_{R^d} f\,dx = \int_{R^d} f^+dx - \int_{R^d} f^-\,dx.$$

(2) we say f is integrable if

$\int_{R^d} f^+$ and $\int_{R^d} f^-$ are finite

($\Rightarrow$ $\int_{R^d} f(x)\,dx$ is finite)

Notation: $L^1(R^d)$ = the set of all integrable functions on R^d.

Rmk: • For any measurable set A and a measurable function f on A. we can define

$$\int_A f = \int_{R^d} f \cdot \chi_A \rightarrow \begin{cases} f(x) & x \in A \\ 0 & x \notin A \end{cases}$$ <u>zero extension</u> of f.

- If $f = g$ a.e. on A. then $\int_A f\,dx = \int_A g\,dx$.
- Notation $L^1(A)$. = integrable functions on A.

Prop: $f \in \mathcal{L}^1(\mathbb{R}^d) \iff |f| \in \mathcal{L}^1(\mathbb{R}^d)$ ($\int_{\mathbb{R}^d} |f| dx < +\infty$).

Proof: • $f \in \mathcal{L}^1(\mathbb{R}^d) \Rightarrow \int_{\mathbb{R}^d} f^+ < \infty$. $\int_{\mathbb{R}^d} f^- < \infty \Rightarrow \int_{\mathbb{R}^d} |f| dx = \int_{\mathbb{R}^d} f^+ + \int_{\mathbb{R}^d} f^- < \infty$.

($\int_{\mathbb{R}^d} f^-$)

• $\int_{\mathbb{R}^d} |f| dx < \infty \Rightarrow \int_{\mathbb{R}^d} f^+ \leq \int_{\mathbb{R}^d} |f| < +\infty \Rightarrow f \in \mathcal{L}^1(\mathbb{R}^d)$.

Cor: $f \in \mathcal{L}^1(\mathbb{R}^d) \Rightarrow f$ is a.e. finite.

Proof. $\int_{\mathbb{R}^d} |f| < +\infty \Rightarrow |f|$ is a.e finite $\Rightarrow f$ is a.e finite.

Cor: $\mathcal{L}^1(\mathbb{R}^d)$ is a linear space.

Proof: Suppose $f_1, f_2 \in \mathcal{L}^1(\mathbb{R}^d)$ $c_1, c_2 \in \mathbb{R}$ then

$$\int_{\mathbb{R}^d} |c_1 f_1 + c_2 f_2| dx \leq \int_{\mathbb{R}^d} (|c_1||f_1| + |c_2||f_2|) dx.$$

$$= |c_1| \int_{\mathbb{R}^d} |f_1| dx + |c_2| \int_{\mathbb{R}^d} |f_2| dx < \infty.$$

$$\Rightarrow c_1 f_1 + c_2 f_2 \in \mathcal{L}^1(\mathbb{R}^d).$$

Def: Let $f \in \mathcal{L}^1(\mathbb{R}^d)$ We call the quantity

$$\|f\|_{\mathcal{L}^1(\mathbb{R}^d)} := \int_{\mathbb{R}^d} |f| dx < \infty$$

the <u>$\mathcal{L}^1$-norm</u> of f.

It's obvious that

① $\|cf\|_{\mathcal{L}^1(\mathbb{R}^d)} = |c| \, \|f\|_{\mathcal{L}^1(\mathbb{R}^d)}$

② $\|f_1 + f_2\|_{\mathcal{L}^1(\mathbb{R}^d)} \leq \|f_1\|_{\mathcal{L}^1(\mathbb{R}^d)} + \|f_2\|_{\mathcal{L}^1(\mathbb{R}^d)}$

③ $\|f\|_{\mathcal{L}^1(\mathbb{R}^d)} = 0 \iff f = 0$ a.e.

Given $f, g \in \mathcal{L}^1(\mathbb{R}^d)$ we can define $\mathcal{L}^1$-distance.

$$d_{\mathcal{L}^1(\mathbb{R}^d)}(f, g) = \|f - g\|_{\mathcal{L}^1(\mathbb{R}^d)}.$$

One can prove

$(\overset{=}{\mathcal{L}^1}(\mathbb{R}^d), d_{\mathcal{L}^1})$ is complete metric space

For $p \geq 1$ $\|f\|_{L^p(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |f|^p \, dx\right)^{1/p}$

$L^p(\mathbb{R}^d) = \{f : \|f\|_{L^p} < \infty\}$.

2. Properties of Lebesgue integrals (Absolute integral).

Prop: Let $f_1, f_2 \in L^1(\mathbb{R}^d)$ Then

(1) (linearity) $\int_{\mathbb{R}^d} (c_1 f_1 + c_2 f_2) dx = c_1 \int_{\mathbb{R}^d} f_1 + c_2 \int_{\mathbb{R}^d} f_2$. $\forall c_1, c_2 \in \mathbb{R}$

(2) (monotonicity) If $f_1 \leq f_2$ a.e. then $\int_{\mathbb{R}^d} f_1 \leq \int_{\mathbb{R}^d} f_2$.

(3) (translation invariance) $\int_{\mathbb{R}^d} f(x+y) dx = \int_{\mathbb{R}^d} f(x) \, dx$.

Proof: (1) It is enough to prove.

(a) $\int_{\mathbb{R}^d} cf \, dx = c \int_{\mathbb{R}^d} f \, dx$ $(c \in \mathbb{R})$.

(b) $\int_{\mathbb{R}^d} f_1 + f_2 \, dx = \int_{\mathbb{R}^d} f_1 \, dx + \int_{\mathbb{R}^d} f_2 \, dx$

To prove (a)

If $c \geq 0$ then $(cf)^+ = cf^+$, $(cf)^- = cf^-$

$\Rightarrow \int_{\mathbb{R}^d} cf \, dx = \int_{\mathbb{R}^d} (cf)^+ dx - \int_{\mathbb{R}^d} (cf)^- dx$.

$= c\int_{\mathbb{R}^d} f^+ dx - c\int_{\mathbb{R}^d} f^- dx = c\left(\int_{\mathbb{R}^d} f^+ dx - \int_{\mathbb{R}^d} f^- dx\right) = c\int_{\mathbb{R}^d} f \, dx$

If $c < 0$, $(cf)^+ = (-c) \cdot f^-$, $(cf)^- = (-c) \cdot f^+$

$\Rightarrow \int_{\mathbb{R}^d} cf \, dx = \int_{\mathbb{R}^d} (-c) \cdot f^- dx - \int_{\mathbb{R}^d} (-c) \cdot f^+ dx$

$= (-c)\left(\int_{\mathbb{R}^d} f^- dx - \int_{\mathbb{R}^d} f^+ dx\right) = c\left(\int_{\mathbb{R}^d} f^+ dx - \int_{\mathbb{R}^d} f^- dx\right) = c\int_{\mathbb{R}^d} \cdot$

To prove (b), we denote $g = f_1 + f_2$ Then

$g^+ - g^- = g = f_1 + f_2 = (f_1^+ - f_1^-) + (f_2^+ - f_2^-)$

$\Rightarrow g^+ + f_1^- + f_2^- = g^- + f_1^+ + f_2^+$

$\Rightarrow \int_{\mathbb{R}^d} g^+ + \int_{\mathbb{R}^d} f_1^- + \int_{\mathbb{R}^d} f_2^- = \int_{\mathbb{R}^d} g^- + \int_{\mathbb{R}^d} f_1^+ + \int_{\mathbb{R}^d} f_2^+$

(each term is $< \infty$, thus can change order)

$\Rightarrow \int_{\mathbb{R}^d} (f_1 + f_2) dx = \int_{\mathbb{R}^d} g^+ dx - \int_{\mathbb{R}^d} g^- dx$

$= \left(\int_{\mathbb{R}^d} f_1^+ dx - \int_{\mathbb{R}^d} f_1^- dx\right) + \left(\int_{\mathbb{R}^d} f_2^+ dx - \int_{\mathbb{R}^d} f_2^- dx\right)$

$= \int_{\mathbb{R}^d} f_1 \, dx + \int_{\mathbb{R}^d} f_2 \, dx$.

(2) If $f_1 \le f_2$ a.e. then

$$\int_{\mathbb{R}^d} f_1\,dx \le \underbrace{\int_{\mathbb{R}^d} f_1}_{} + \underbrace{\int_{\mathbb{R}^d}(f_2-f_1)}_{\ge 0} \overset{\text{Linearity}}{=} \int_{\mathbb{R}^d} f_2\,dx.$$

(3) $\int_{\mathbb{R}^d} f(x+y)\,dx = \int_{\mathbb{R}^d} f^+(x+y)\,dx - \int_{\mathbb{R}^d} f^-(x+y)\,dx$

$= \int_{\mathbb{R}^d} f^+(x)\,dx - \int_{\mathbb{R}^d} f^-(x)\,dx$

$= \int_{\mathbb{R}^d} f(x)\,dx.$

$\int_A f(x+y)\,dx = \int_{A+\{y\}} f(x)\,dx.$

Prop (Countable additivity w.r.t. domain) Def. $\mu_f(A) := \int_A f\,dx \longrightarrow$ "signed measure"

Let $A_i \cap A_j = \phi$. $A = \bigcup_{i=1}^{\infty} A_i$ Suppose $f \in L^1(A)$.

Then $\int_A f\,dx = \sum_{i=1}^{\infty} \int_{A_i} f\,dx.$ $\boxed{f \in L^1(A_i).\ A_i \subset A.}$

Proof: • $f \in L^1(A) \Rightarrow \int_A |f| < \infty \Rightarrow \int_{A_i} |f| = \int_A |f| \chi_{A_i} < \int_A |f| < \infty.$

$\Rightarrow f \in L^1(A_i)$.

• According to the "countable additivity w.r.t. domain" in non-negative theory.

$\int_A f^+\,dx = \sum_{i=1}^{\infty} \int_{A_i} f^+\,dx$

$\int_A f^-\,dx = \sum_{i=1}^{\infty} \int_{A_i} f^-\,dx$

$\Rightarrow \int_A f\,dx = \int_A (f^+ - f^-)\,dx = \sum\left(\int_{A_i} f^+ - \int_{A_i} f^-\right) = \sum_{i=1}^{\infty} \int_{A_i} f\,dx.$

In non-negative theory $\int_{\mathbb{R}^d} f\,dx = 0 \Leftrightarrow f = 0$ a.e.

this is no longer true for general $f \in L^1(\mathbb{R}^d)$

Prop: Let $A \in \mathcal{L}$. $f \in L^1(A)$ Then

$f = 0$ a.e on $A \Leftrightarrow \forall B \subset A\ B \in \mathcal{L}\ \int_B f\,dx = 0$

Proof. $\Rightarrow$ Obviously ($f^+ = 0$ a.e. $f^- = 0$. a.e).

$\Leftarrow$ Let $B_1 = \{x : f(x) > 0\}$. $B_2 = \{x : f(x) < 0\}$.

Then $\int_{B_1} f = \int_A f^+ = 0 \Rightarrow f^+ = 0$ a.e.

$\int_{B_2} f = -\int_A f^- = 0 \Rightarrow f^- = 0$ a.e.

$\Rightarrow f = 0$ a.e.

Prop. Let $f \in L^1(\mathbb{R}^d)$ and $\int_B f\,dx = 0$ for any box B, then $f = 0$ a.e.

Proof. We prove by contradiction.

WLOG, we assume $m(\{x: f(x) > 0\}) \geq a_0 > 0$.

- Note: $\{x: f(x) > 0\} = \bigcup_{n=1}^{\infty} \{x: f(x) > \frac{1}{n}\}$.

  $\{x: f(x) > \frac{1}{n}\} \subset \{x: f(x) > \frac{1}{n+1}\} \Rightarrow \lim_{n\to\infty} m(\{x: f(x) > \frac{1}{n}\}) = m(\{x: f(x) > 0\}) \geq a_0$.

Thus, $\exists\, n_0$ s.t. $m(\{x: f(x) > \frac{1}{n_0}\}) > \frac{1}{2} a_0$.

Take a closed set $F \subset \{x: f(x) > \frac{1}{n_0}\}$ s.t. $m(F) \geq \frac{a_0}{4}$.

Then $\int_F f(x)\,dx \geq \int_F \frac{1}{n_0}\,dx \geq \frac{a_0}{4n_0}$

- By condition, $\int_{\mathbb{R}^d} f(x)\,dx = 0$. (countable additivity).

Consider $U = \mathbb{R}^d \setminus F$ It's open.

By the structure theorem of opensets, $U = \bigcup_{n=1}^{\infty} B_n$.

where B_n are almost disjoint boxes

Let $U_0 = \bigcup_{n=1}^{\infty} \mathring{B}_n$ Then $m(U \setminus U_0) = m(\bigcup_{n=1}^{\infty} (B_n \setminus \mathring{B}_n)) = 0$.

Thus. $0 = \int_{\mathbb{R}^d} f(x)\,dx = \int_F f\,dx + \int_{U_0} f\,dx + \int_{U\setminus U_0} f\,dx$.

$= \underbrace{\int_F f\,dx}_{>0} + \underbrace{\sum_{i=1}^{\infty} \int_{\mathring{B}_i} f\,dx}_{U_0} + \int_{U\setminus U_0} f\,dx$. >0 Contradiction

Cor There is no measurable set $A \subset [0,1]$

s.t: $m(A \cap [a,b]) = \frac{b-a}{2}$ $\forall\, 0 \leq a \leq b \leq 1$.

Proof: Suppose such A exists. Then for $\forall [c,d] \subset \mathbb{R}$.

$\int_{[c,d]} (\chi_A - \frac{1}{2}\chi_{[0,1]}) = 0$.

$\Rightarrow \chi_A = \frac{1}{2}\chi_{[0,1]}$ a.e. Contradiction.

Thm Suppose $f \in L^1(A)$. Then $\forall \varepsilon > 0$. $\exists\, \delta > 0$ s.t:

$\forall\, B \subset A$ measurable, $m(B) < \delta \Longrightarrow \int_B |f| < \varepsilon$.

(Absolute continuity of Lebesgue integral.)

Proof: · We firstly assume $f \geq 0$.

By non-negative theory $\exists$ simple function φ

$0 \leq \varphi \leq f$ s.t. $\boxed{\int_A (f-\varphi)dx < \frac{\varepsilon}{2} \quad (\int_A f dx < +\infty)}$

Since φ is simple, non-negative and integrable.

One can find M s.t $\varphi \leq M$ a.e.

Take $\delta = \frac{\varepsilon}{2M}$.

$\int_B f dx = \int_B (f-\varphi)dx + \int_B \varphi dx < \frac{\varepsilon}{2} + M \cdot \frac{\varepsilon}{2M} = \varepsilon$.

· For general f. we apply the above results to f^+. f^-

i.e. $\exists \delta_1$ s.t: $m(B) < \delta_1 \Rightarrow \int_B f^+ < \varepsilon/2$

$\exists \delta_2$ s.t: $m(B) < \delta_2 \Rightarrow \int_B f^- < \varepsilon/2$.

Take $\delta = \min(\delta_1, \delta_2)$.

Then if $m(B) < \delta$. $\int_B |f| dx = \int_B f^+ dx + \int_B f^- dx < \varepsilon$.

3. Convergence.

$f_n \to f \not\Rightarrow \int f_n \to \int f$

Recall (Example) $f_n(x) = \begin{cases} n & 0 < x < \frac{1}{n} \\ 0 & x < 0 \quad x \geq \frac{1}{n}. \end{cases} \longrightarrow f(x) = 0$.

$\int_{\mathbb{R}} f_n dx = 1 \quad \not\longrightarrow \quad \int_{\mathbb{R}} f = 0$.

Lebesgue Dominated Convergence Theorem (LDCT)

Suppose f_n are measurable $f_n \to f$ a.e. ($f_n \to f$ in measure) still holds.

$\exists \; g \in L^1(A)$ s.t: $\underbrace{|f_n| \leq g}_{f_n \in L^1(A)}$ a.e. on A.

Then $f \in L^1(A)$ and

$$\lim_{n\to\infty} \int_A f_n(x) dx = \int_A f(x) dx$$

(In fact. $\lim_{n\to\infty} \int |f_n - f| dx = 0$ (i.e. $\|f_n - f\|_{L^1(\mathbb{R}^d)} \to 0$)

Lec 12

Proof: Let $A_0 = \{x : f_n(x) \not\to f(x)\}$.

$A_n = \{x : f_n(x) > g(x)\}$.

Then $m(A_0) = m(A_n) = 0$. $\Rightarrow A = \bigcup_{n=0}^{\infty} A_n \quad m(A) = 0$.

On A^c. we have $|f(x)| \le g(x)$. $\forall x \in A^c$.

$\Rightarrow \int_{\mathbb{R}^d} |f| dx = \int_{A^c} |f| dx \le \int_{A^c} g dx \underset{(=)}{\le} \int_{\mathbb{R}^d} g \, dx. < +\infty$.

$\Rightarrow f \in L^1(\mathbb{R}^d)$

- Apply Fatou Lemma to $f_n + g$. ← non-negative on A^c. we get

$\int_{\mathbb{R}^d} f dx + \int_{\mathbb{R}^d} g dx = \int_{A^c} f + g \, dx = \int_{A^c} \liminf_{n\to\infty} (f_n + g) dx. \le \liminf_{n\to\infty} \int_{A^c} (f_n + g) dx.$

$= \liminf_{n\to\infty} \int_{\mathbb{R}^d} f_n dx + \int_{\mathbb{R}^d} g \, dx$

$\Rightarrow \int_{\mathbb{R}^d} f dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^d} f_n dx$.

- Apply Fatou to $\{g - f_n\}$ on A^c.

$\int_{\mathbb{R}^d} g \, dx - \int_{\mathbb{R}^d} f dx = \int_{A^c} \liminf_{n\to\infty} (g - f_n) dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^d} (g - f_n) dx$

$\le \int_{\mathbb{R}^d} g dx - \limsup_{n\to\infty} \int_{\mathbb{R}^d} f_n \, dx$

$\Rightarrow \int_{\mathbb{R}^d} f dx \ge \limsup_{n\to\infty} \int_{\mathbb{R}^d} f_n dx$.

$\Rightarrow \lim_{n\to\infty} \int_{\mathbb{R}^d} f_n dx = \int_{\mathbb{R}^d} f \, dx$.

Cor || Under the same assumption, we have a stronger Conclusion

$\lim_{n\to\infty} \int_{\mathbb{R}^d} |f_n(x) - f(x)| dx = 0$.

Proof: Since $|f_n - f| \to 0$. a.e. and $|f_n - f| \le 2g$ a.e.

Apply LDCT to $\{|f_n - f|\}$ we get

$\int_{\mathbb{R}^d} |f_n - f| dx \to \int_{\mathbb{R}^d} 0 \, dx = 0$.

Recall

$\|f_n - f\|_{L^1(\mathbb{R}^d)}$

($f_n, f \in L^1(\mathbb{R}^d)$)

Def|| We say $f_n \to f$ in L^1 norm (or $f_n \to f$ in the mean)
if $\lim_{n\to\infty} \int_{\mathbb{R}^d} |f_n - f|\,dx = 0$. ($\|f_n - f\|_{L^1(\mathbb{R}^d)}$)

Prop.|| If $f_n, f \in L^1(\mathbb{R}^d)$ $f_n \to f$ on the norm.
Then $f_n \to f$ in measure. $\Leftarrow \forall \varepsilon > 0$. $\lim_{n\to\infty} m(\{x : |f_n - f| > \varepsilon\}) = 0$.

Proof: Use Markov inequality $\forall \lambda > 0$
$m(\{x : |f_n - f| > \lambda\}) \le \frac{1}{\lambda} \int_{\mathbb{R}^d} |f_n - f|\,dx \to 0$.

Therefore. in measure + dominance $\Longrightarrow$ in norm $L^1(\mathbb{R}^d)$.

Cor|| If $f_n \to f$ in L^1 norm. $\exists$ sequence $f_{n_k} \to f$ a.e.

Rmk: uniform convergence $\not\Rightarrow$ convergence in L^1-norm.
(None of convergence modes in Lec 7 implies convergence in L^1-norm)

e.g. $f_n = \frac{1}{n}\chi_{[0,n]} \to f = 0$ uniformly. $\int_{\mathbb{R}} |f_n - f|\,dx = 1 \not\to 0$

Thm|| (Completeness of $L^1(\mathbb{R}^d)$) (Riesz-Fischer)

"Cauchy sequence converges"
$\{x_n\}$ $\forall \varepsilon > 0$. $\exists N$ s.t.: $\forall n, m > N$. $|x_m - x_n| < \varepsilon$.

Def: A sequence $f_n \in L^1(\mathbb{R}^d)$ is a Cauchy sequence in $L^1(\mathbb{R}^d)$
if $\forall \varepsilon > 0$. $\exists N$ s.t.: $\forall n, m > N$.
$\|f_m - f_n\|_{L^1} = \int_{\mathbb{R}^d} |f_n(x) - f_m(x)|\,dx < \varepsilon$.
norm $\to$ distance

For any Cauchy sequence f_n in $L^1(\mathbb{R}^d)$, $\exists f \in L^1(\mathbb{R}^d)$ s.t.:
$f_n \to f$ in L^1-norm.

Proof: First we take a subsequence f_{n_k} s.t.:
$\|f_{n_{k+1}} - f_{n_k}\|_{L^1(\mathbb{R}^d)} < \frac{1}{2^k}$

Let $g_N(x) = \sum_{k=1}^{N} |f_{n_{k+1}}(x) - f_{n_k}(x)| + |f_{n_1}(x)|$ Then

g_1, g_2 —— are non-negative, increasing

$\Rightarrow \underline{g_N \to g \in L^1(\mathbb{R}^d)}$

$[\int_{\mathbb{R}^d} g\,dx = \lim_{N\to\infty} \int_{\mathbb{R}^d} g_N\,dx \le \int_{\mathbb{R}^d} |f_{n_1}|dx + \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty]$

Let $h_N(x) = f_{n_1}(x) + \sum_{k=1}^{N} (f_{n_{k+1}}(x) - f_{n_k}(x)) \; (= f_{n_{N+1}}(x))$.

Then $|h_N(x)| \le g(x)$

$\Rightarrow$ • $\forall x$. series $f_{n_1}(x) + \sum_{k=1}^{\infty} (f_{n_{k+1}}(x) - f_{n_k}(x))$ converges to some $f(x)$.

• $f(x) = \lim_{N\to\infty} (f_{n_1}(x) + \sum_{k=1}^{N} (f_{n_{k+1}}(x) - f_{n_k}(x))$ is measurable

$= \lim_{N\to\infty} h_N(x)$.

• Apply LDCT, $f \in L^1(\mathbb{R}^d)$, $\lim_{N\to\infty} \int_{\mathbb{R}^d} |f - h_N|\,dx = 0$.

m) $\Rightarrow \lim_{N\to\infty} \int_{\mathbb{R}^d} |f - f_{n_{N+1}}|\,dx = 0$.

i.e. $\forall \varepsilon > 0$ $\exists K$ s.t. $\forall N > K$.

$\|f_{n_{N+1}} - f\|_{L^1(\mathbb{R}^d)} < \frac{\varepsilon}{2}$.

By def of Cauchy sequence $\exists M$ s.t. $\forall n, m > M$.

$\|f_n - f_m\|_{L^1(\mathbb{R}^d)} < \frac{\varepsilon}{2}$

Take $L = \max\{K, M\}$. then $\forall n > L$.

$\|f_n - f\|_{L^1(\mathbb{R}^d)} \le \|f_n - f_{n_{L+1}}\|_{L^1(\mathbb{R}^d)} + \|f_{n_{L+1}} - f\|_{L^1(\mathbb{R}^d)} < \varepsilon$.

Proof of LDCT (in measure)

1st: By Riese, $\exists$ subseq $f_{n_k} \to f$ a.e.

• By LDCT $f \in L^1(\mathbb{R}^d)$ and $\|f_{n_k} - f\|_{L^1} \to 0$ (converges in L^1 norm)

• Suppose $\|f_n - f\|_{L^1} \not\to 0$. Then $\exists f_{m_k}$ s.t. $\|f_{m_k} - f\|_{L^1} \ge \varepsilon_0$. $\forall k$.

Note: $f_{m_k} \to f$ in measure $\Rightarrow \underset{\text{subseq}}{\exists} f_{m_k'}$ s.t. $\|f_{m_k'} - f\|_{L^1} \to 0$

. Contradiction

2^{nd} proof: ① $f_n - f \in L^1(\mathbb{R}^d)$.

② WANT $\int_{\mathbb{R}^d} |f_n - f| dx \to 0$. ($< \varepsilon$. $\forall \varepsilon > 0$).

$$= \int_{\underbrace{\{|f_n-f| > \varepsilon\}}_{\text{small}}} \overset{\le 2g}{|f_n - f|} dx + \int_{\{|f_n-f|<\varepsilon\}} \overset{\le \varepsilon}{|f_n - f|} dx \nearrow \int_{\{|f_n-f|<\varepsilon\} \cap B(0,n)} |f_n - f| dx + \int_{\{|f_n-f|<\varepsilon\} \cap B(0,n)^c} |f_n - f| dx$$

$$\int_{\mathbb{R}^d \cap B(0,n)} |f| dx \to \underbrace{\int_{\mathbb{R}^d} \overset{<\varepsilon}{|h|} dx}_{< +\infty}.$$

Lec 13

1 Countable additivity (w.r.t Domain and Function).

Prop. || Suppose $f \in L^1(A)$, $A = \bigcup_{i=1}^{\infty} A_i$. $A_i \cap A_j \neq \phi$ Then $\int_A f dx = \sum_{i=1}^{\infty} \int_{A_i} f$

2^{nd} proof: Let $\tilde{f_i} = f \chi_{A_i}$. $f_n = \sum_{i=1}^{n} \tilde{f_i}$ Then $f_n \to f$

$|f_n| \leq |f| \in L^1(A)$

By LDCT

$\int_A f dx = \lim_{n\to\infty} \int_A f_n dx = \lim_{n\to\infty} \sum_{i=1}^{n} \int_{A_i} f dx = \sum_{i=1}^{\infty} \int_{A_i} f dx.$

Prop: Let $f_n \in L^1(A)$ $\sum_{n=1}^{\infty} \int_A |f_n| dx < \infty$. ~~Then~~ $\exists f \in L^1(A)$ s.t. $\sum_{n=1}^{\infty} f_n$ converges to f a.e.

and $\int_A f dx = \sum_{n=1}^{\infty} \int_A f_n dx$.

Proof: By non-negative theory $\sum_{n=1}^{\infty} \int_A |f_n| dx = \int_A \sum_{n=1}^{\infty} |f_n| dx < +\infty$.

$\Rightarrow F = \sum_{n=1}^{\infty} |f_n| \in L^1$

$\Rightarrow F$ is a.e. finite

$\Rightarrow \sum_{n=1}^{\infty} f_n$ a.e. absolutely convergent

$\Rightarrow \sum_{n=1}^{\infty} f_n$ a.e converges to a function f.

Note: $f \overset{a.e.}{=} \lim_{N\to\infty} \sum_{n=1}^{N} f_n \to f$ is measurable.

Since $|f| \leq F \Rightarrow f \in L^1(A)$.

. Let $g_N = \sum_{n=1}^{N} f_n$. Then $g_N \to f$. a.e. and $|g_N| \leq F$

By LDCT.

$\int_A f dx = \lim_{N\to\infty} \int_A g_N(x) dx = \lim_{N\to\infty} \sum_{n=1}^{N} \int_A f_n dx = \sum_{n=1}^{\infty} \int_A f_n dx$..

2. Riemann integral v.s. Lebesgue integral

. Let $f: [a,b] \to R$ be a <u>bounded function</u>

Recall for any partition P $a = x_0 < x_1 < \cdots < x_n = b$ of $[a,b]$

$S_P(f) = \sum_{k=1}^{n} \sup_{x_{k-1} \leq x \leq x_k} f(x) \cdot (x_k - x_{k-1})$ —— Darboux upper sum

$\underline{S_P}(f) = \sum_{k=1}^{n} \inf_{x_{k-1} \leq x \leq x_k} f(x) \cdot (x_k - x_{k-1})$ —— Darboux lower sum

$\overline{\int_a^b} f\,dx = \inf_P \overline{S}_P(f)$ — Darboux upper integral

$\underline{\int_a^b} f\,dx = \sup_P \underline{S}_P(f)$ — Darboux lower integral

f is Riemann integral $\iff \overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx =: \int_a^b f\,dx$.

Now given a partition P. we can define step function

$$\overline{\Phi}_P(f) = \sum_{k=1}^{n} \sup_{[x_{k-1},x_k]} f \cdot \chi_{[x_{k-1},x_k]}. \quad (= \overline{S}_P(f))$$

$$\underline{\Phi}_P(f) = \sum_{k=1}^{n} \inf_{[x_{k-1},x_k]} f \cdot \chi_{[x_{k-1},x_k]} \quad (= \underline{S}_P(f))$$

By the trick of "common refinement", we can find P_n with $P_1 \le P_2 \le \cdots$ (P_{n+1} is a refinement of P_n)

S.t.: $\overline{\int_a^b} f\,dx = \lim_{n\to\infty} \overline{S}_{P_K}(f) = \lim_{K\to\infty} \int_{[a,b]} \overline{\Phi}_{P_K}(f)\,dx.$

$\underline{\int_a^b} f\,dx = \lim_{K\to\infty} \underline{S}_{P_K}(f) = \lim_{K\to\infty} \int_{[a,b]} \underline{\Phi}_{P_K}(f)\,dx.$

One can check

$$\left.\begin{aligned} \overline{\Phi}_{P_K}(f) &\searrow \overline{\Phi}(f) \\ \underline{\Phi}_{P_K}(f) &\nearrow \underline{\Phi}(f) \end{aligned}\right\} \text{measurable} \rightarrow \underline{\Phi}(f) \le f \le \overline{\Phi}(f)$$

Now we have

f is Riemann integral $\iff \lim_{K\to\infty} \int_{[a,b]} (\overline{\Phi}_{P_K}(f) - \underline{\Phi}_{P_K}(f))\,dx = 0.$

$\overset{}{\underset{\text{LDCT}}{\iff}} \int_{[a,b]} (\overline{\Phi}(f) - \underline{\Phi}(f))\,dx = 0.$

$\iff \overline{\Phi}(f) = \underbrace{\underline{\Phi}(f)}_{=f}$ a.e in $[a,b]$.

($\iff f$ is discontinuous only on a set measure 0)

Prop. If a $\boxed{\text{bounded function}}$ $f: [a,b] \to \mathbb{R}$ is Riemann integrable then $f \in \mathcal{L}^1([a,b])$ and $\int_{[a,b]} f\,dx = \int_a^b f(x)\,dx$.

Proof. By LDCT. $\underline{\Phi}(f) \in \mathcal{L}^1([a,b]) \Rightarrow f \in \mathcal{L}^1([a,b])$

and $\int_{[a,b]} f\,dx = \int_{[a,b]} \underline{\Phi}(f)\,dx = \lim_{K\to\infty} \int_{[a,b]} \underline{\Phi}_{P_K}(x)\,dx$

$= \int_a^b f(x)\,dx = \int_a^b f(x)\,dx.$

$\boxed{\text{Rmk}}$ There exists functions f so that the improper integral $\int_a^b f\,dx$ (in Riemann sense) exists, while f is not Lebesgue integrable.

$f: [0,+\infty) \to \mathbb{R}$ $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \chi_{(n-1,n)} \Rightarrow \int_0^{+\infty} f(x)dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\ln 2$

but $\int_{[0,+\infty)} |f|\,dx = +\infty.$

3. Integrals with parameters.

$$F(t) = \int_A f(t,x)\,dx.$$

Prop. Let $T \subset \mathbb{R}^{d_1}$, $A \subset \mathbb{R}^{d_2}$ be measurable sets, $f: T \times A \to \mathbb{R}$ be a function

Suppose ① $\forall t \in T$ $f_t(x) = f(t,x)$ is a measurable function on A.

② $\forall x \in A$ $f^{(x)}(t) := f(t,x)$ is a continuous function on T.

Moreover, suppose $\exists g \in L^1(A)$ s.t. $|f(t,x)| \le g(x)$ a.e. $x \in A$ $\forall t \in T$.

Then for $\forall t \in T$, $f_t \in L^1(A)$ and $F(t) := \int_A f(t,x)dx$ is continuous w.r.t. $t \in T$. Proof: Suppose $t_n \in T$ and $t_n \to t \in T$. Then

for $\forall x \in A$. $f(t_n,x) = f_{t_n}(x) \to f_{t_0}(x) = f(t_0,x)$ AND $|f(t_n,x)| \le g(x) \in L^1$.

By LDCT, we have $F(t_n) \to F(t_0)$.

Prop. Let $f: (a,b) \times A \to \mathbb{R}$ be a function s.t.

① $\forall t \in (a,b)$. $f_t(x) \in L^1(A)$

② $\forall x \in A$: $f^{(x)}(t)$ is differentiable.

Suppose $\exists g \in L^1(A)$ s.t. $\left|\frac{d}{dt} f(t,x)\right| \le g(x)$ $(\forall t \in (a,b))$ $x \in A$.

Then $\forall t \in (a,b)$

$$\frac{d}{dt}\int_A f(t,x)\,dx = \int_A \frac{d}{dt} f(t,x)\,dx.$$

Proof: For any $(t,x) \in (a,b) \times A$ one has

$$\frac{d}{dt} f(t,x) = \lim_{h_k \to 0} \frac{f(t+h_k,x) - f(t,x)}{h_k}.$$

20) By mean value theorem,

$$\left|\frac{f(t+h_k,x) - f(t,x)}{h_k}\right| \le g(x) \quad \forall x \in A. \quad h_k \to 0$$

by LDCT

$$\frac{d}{dt}\int_A f(t,x)\,dx = \lim_{k\to\infty} \int_A \frac{f(t+h_k,x) - f(t,x)}{h_k}\,dx$$

$$= \int_A \frac{d}{dt} f(t,x)\,dx.$$

4. The theorems of Fubini

Suppose $f=f(x,y)$ be a "2-variable" function on $A\times B\subset \mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$.

Question $\int_{A\times B} f(x,y)\,dxdy \overset{?}{=} \int_B\left(\int_A f(x,y)\,dx\right)dy$.

Example: $f(x,y)=\frac{x^2-y^2}{(x^2+y^2)^2}$ $\quad (x,y)\in[0,1]\times[0,1]$.

By using formulas $\int\frac{dx}{x^2+a^2}=\frac{1}{a}\arctan\frac{x}{a}+C$. $\int\frac{dx}{(x^2+a^2)^2}=\frac{x}{2a(x^2+a^2)}+\frac{1}{2a^2}\int\frac{dx}{x^2+a^2}$

One can show $\int_0^1\left(\int_0^1 f(x,y)\,dx\right)dy=\int_0^1\left(-\frac{1}{1+y^2}\right)dy=-\frac{\pi}{4}$

$\int_0^1\left(\int_0^1 f(x,y)\,dy\right)dx=\int_0^1\frac{1}{1+x^2}\,dx=\frac{\pi}{4}$.

$\boxed{\text{Reason}}$. $\int_0^1\left(\int_0^1|f(x,y)|\,dx\right)dy=+\infty$ ($\leftarrow$ Not integral).

Thm (Fubini) Let $A\subset\mathbb{R}^{d_1}$, $B\subset\mathbb{R}^{d_2}$ be measurable. $f=f(x,y)\in L^1(A\times B)$

Then (1) For a.e. $y\in B$. $f_y(x):=f(x,y)$, then $f_y\in L^1(A)$

(2) As a function of y $F(y):=\int_A f(x,y)\,dx$, then $F\in L^1(B)$

(3) $\int_{A\times B} f(x,y)\,dxdy=\int_B\left(\int_A f(x,y)\,dx\right)dy=\int_A\left(\int_B f(x,y)\,dy\right)dx.$

Rmk: • By zero extension, we may assume $A=\mathbb{R}^{d_1}$, $B=\mathbb{R}^{d_2}$

• For (3) it's enought to prove the first "="

Special $\to$ General

Proof: WLOG, we assume $A=\mathbb{R}^{d_1}$, $B=\mathbb{R}^{d_2}$.

For simplicity we let $\mathcal{F}=\{f\in L^1(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}):\ f$ satisfies (1), (2), (3)$\}$.

(Path of proof: $\chi_{Box}\rightsquigarrow\chi_{open}\rightsquigarrow\chi_{G_\delta}$, $\chi_{measure\,0}\rightsquigarrow\chi_{measurable}$ $\to f\in$ simple $\to f\in L^1$)

$\boxed{\text{Step 1}}$ box: Let $f=\chi_{B_1\times B_2}$, where $B_1\subset\mathbb{R}^{d_1}$. $B_2\subset\mathbb{R}^{d_2}$ are boxes

Then $f\in\mathcal{F}$.

To see this, (1) $f_y(x)=\chi_{B_2}(y)\cdot\chi_{B_1}(x)\in L^1(\mathbb{R}^{d_1})$

(2) $F(y)=\int_{\mathbb{R}^{d_1}} f(x,y)\,dx=\int_{\mathbb{R}^{d_1}}\chi_{B_2}(y)\chi_{B_1}(x)\,dx=\chi_{B_2}(y)\cdot m(B_1)\in L^1(\mathbb{R}^{d_2})$

(3) $\int_{\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}} f(x,y)\,dxdy=m(B_1\times B_2)=m(B_1)\times m(B_2)$

$=\int_{\mathbb{R}^{d_2}}\left(\int_{\mathbb{R}^{d_1}} f(x,y)\,dx\right)dy=\int_{\mathbb{R}^{d_2}} m(B_1)\chi_{B_2}(y)\,dy=m(B_1)\times m(B_2)$

4. The theorems of Fubini

Suppose $f=f(x,y)$ be a "2-variable" function on $A\times B\subset\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$.

Question $\int_{A\times B} f(x,y)\,dxdy \overset{?}{=} \int_B\left(\int_A f(x,y)\,dx\right)dy$.

Example: $f(x,y)=\frac{x^2-y^2}{(x^2+y^2)^2}$ $(x,y)\in[0,1]\times[0,1]$.

By using formulas $\int\frac{dx}{x^2+a^2}=\frac{1}{a}\arctan\frac{x}{a}+C$. $\int\frac{dx}{(x^2+a^2)^2}=\frac{x}{2a(x^2+a^2)}+\frac{1}{2a}\int\frac{dx}{x^2+a^2}$

One can show $\int_0^1\left(\int_0^1 f(x,y)\,dx\right)dy=\int_0^1\left(-\frac{1}{1+y^2}\right)dy=-\frac{\pi}{4}$

$\int_0^1\left(\int_0^1 f(x,y)\,dy\right)dx=\int_0^1\frac{1}{1+x^2}\,dx=\frac{\pi}{4}$.

$\boxed{\text{Reason}}$. $\int_0^1\left(\int_0^1|f(x,y)|\,dx\right)dy=+\infty$ ($\Rightarrow$ Not integral)

Thm (Fubini) Let $A\subset\mathbb{R}^{d_1}$, $B\subset\mathbb{R}^{d_2}$ be measurable. $f=f(x,y)\in L^1(A\times B)$

Then (1) For a.e. $y\in B$. $f_y(x):=f(x,y)$, then $f_y\in L^1(A)$

(2) As a function of y $F(y):=\int_A f(x,y)\,dx$, then $F\in L^1(B)$

(3) $\int_{A\times B} f(x,y)\,dxdy=\int_B\left(\int_A f(x,y)\,dx\right)dy\ \left(=\int_A\left(\int_B f(x,y)\,dy\right)dx.\right)$

Rmk: · By zero extension, we may assume $A=\mathbb{R}^{d_1}$. $B=\mathbb{R}^{d_2}$

· For (3) it's enought to prove the first "="

Special $\to$ General

Proof: WLOG, we assume $A=\mathbb{R}^{d_1}$, $B=\mathbb{R}^{d_2}$.

For simplicity we let $\mathcal{F}=\{f\in L^1(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}):$ f satisfies 1). 2). 3)$\}$

(Path of proof: $\chi_{Box}\rightsquigarrow\chi_{open}\rightsquigarrow\chi_{G_\delta}\rightsquigarrow\chi_{measure\,0}\rightsquigarrow\chi_{measurable}$
$\to f\in$ simple $\to f\in L^1$)

$\boxed{\text{Step 1}}$ box Let $f=\chi_{B_1\times B_2}$, where $B_1\subset\mathbb{R}^{d_1}$. $B_2\subset\mathbb{R}^{d_2}$ are boxes

Then $f\in\mathcal{F}$.

To see this: (1) $f_y(x)=\chi_{B_2}(y)\cdot\chi_{B_1}(x)\in L^1(\mathbb{R}^{d_1})$

(2) $F(y)=\int_{\mathbb{R}^{d_1}} f(x,y)\,dx=\int_{\mathbb{R}^{d_1}}\chi_{B_2}(y)\chi_{B_1}(x)\,dx=\chi_{B_2}(y)\cdot m(B_1)\in L^1(\mathbb{R}^{d_2})$

(3) $\int_{\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}} f(x,y)\,dxdy=m(B_1\times B_2)=m(B_1)\times m(B_2)$

$\underline{\underline{\quad}}\int_{\mathbb{R}^{d_2}}\left(\int_{\mathbb{R}^{d_1}} f(x,y)\,dx\right)dy=\int_{\mathbb{R}^{d_2}} m(B_1)\chi_{B_2}(y)\,dy=m(B_1)\times m(B_2)$

OURSTORYBEGINS

$\boxed{\text{Step 2}}$ If $f_1, f_2 \in \mathcal{F}$ then $c_1 f_1 + c_2 f_2 \in \mathcal{F}$

(If $f_1, \dots, f_n \in \mathcal{F}$, $\sum_{i=1}^{n} c_i f_i \in \mathcal{F}$.) finite case.

This follows from the linearity of Lebesgue integral.

$\boxed{\text{Step 3}}$ Suppose either if $f_n \nearrow f$, $f_n \in \mathcal{F}$; (a) $\to f \in \mathcal{F}$

or if $f_n \searrow f$, $f_n \in \mathcal{F}$; (b)

$f = f(x,y) \in L^1(A \times B)$.

Proof. enough to prove (a)

Consider $f_n - f_1 \nearrow f - f_1$ each $f_n - f_1 \geq 0$.

By monotone convergence thm $\quad f_1 \in L^1(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$

$$\int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} f_n(x,y)\,dx\,dy \to \int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} f(x,y)\,dx\,dy$$

(1a) For a.e. $y \in \mathbb{R}^{d_2}$: we have $(f_n)_y \in L^1(\mathbb{R}^{d_1})$ $(f_n)_y \nearrow f_y$.

Moreover $f_y \geq (f_1)_y$. By monotone convergence. (MC)

$$F_n(y) = \int_{\mathbb{R}^{d_1}} f_n(x,y)\,dx \to \int_{\mathbb{R}^{d_1}} f_y\,dx := F(y)$$ still need to prove $f_y \in L^1(\mathbb{R}^{d_1})$

(2) We have $F_n(y) \nearrow F(y)$, $F_n \in L^1(\mathbb{R}^{d_2})$

$$\int_{\mathbb{R}^{d_2}} F_n(y)\,dy \to \int_{\mathbb{R}^{d_2}} F(y)\,dy$$

It follows that $\int_{\mathbb{R}^{d_2}} F(y)\,dy = \int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} f(x,y)\,dx\,dy < +\infty$.

So $F \in L^1(\mathbb{R}^{d_2})$

(1b) So for a.e. y. $F(y)$ is finite $\Rightarrow$ for a.e. y, $f_y \in L^1(\mathbb{R}^{d_1})$

(3) $\int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} f(x,y)\,dx\,dy = \int_{\mathbb{R}^{d_2}} F(y)\,dy = \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f_y(x)\,dx \right) dy$.

$\boxed{\text{Step 4}}$ If E is a subset of the boundary of a $\boxed{\text{box}}$ ($\Rightarrow$ $\boxed{m(E) = 0}$) bounded

then $\chi_E \in \mathcal{F}$

$\boxed{\text{Step 5}}$ open set — If U is a bounded open set ($\chi_U \in L^1$) and $f = \chi_U$, then $f \in \mathcal{F}$.

We can write $U = \bigcup_{k=1}^{\infty} \bar{B}_k = \bigcup_{k=1}^{\infty} B_k$. Where $\bar{B}_k$'s are almost disjoint

boxes. and B_k's are disjoint s.t. $\mathring{\bar{B}}_k \subset B_k \subset \bar{B}_k$

By step 1. 2. 4. $\chi_{B_k} \in \mathcal{F}$.

By step 2 $\quad \chi_{\bigcup_{i=1}^{k} B_i} \in \mathcal{F}$.

By step 3 $\quad \chi_U \in \mathcal{F}$.

$\boxed{\text{Step 7}}$ For any measure 0 set E $\chi_E \in \mathcal{F}$.

Since $E = \bigcup_{n=1}^{\infty} (B(0,n) \cap E)$ while each $B(0,n) \cap E$ is bounded and of measure 0, So. one can apply step 3

$\boxed{\text{Step 8}}$ If $E \in \mathcal{L}$, $m(E) < \infty$. then $\chi_E \in \mathcal{F}$.

In fact, $E \in \mathcal{L}$. $m(E) < \infty$. $\Rightarrow E = G \setminus N$. Where $G = \bigcap_{n=1}^{\infty} U_n$ is a G_δ-set. each $m(U_n) < \infty$. and $m(N) = 0$.

Apply Step 5, 3, 7, 2. $\chi_E \in \mathcal{F}$.

$\boxed{\text{Step 9}}$ Any integral Simple function $\in \mathcal{F}$.

Apply step 2. 8.

$\boxed{\text{Step 10}}$ Any integrable function $f \in \mathcal{F}$

$f = f^+ - f^-$. $f^+, f^- \geq 0$. and integral

By Step 3. 9. f^+. $f^- \in \mathcal{F}$.

By step 2 $f \in \mathcal{F}$.

Lec 14.

d $\checkmark$ Tonelli: Suppose f is measurable and non-negative on $A\times B$.

Then (1) for a.e. y. $f(x,y)$ is measurable on A

symmetry (2) The function $F(y)=\int_A f(x,y)\,dx$ is measurable on B

(3) $\int_{A\times B} f(x,y)\,dx\,dy=\int_B(\int_A f(x,y)\,dx)\,dy \left(=\int_A(\int_B f(x,y)\,dy)\,dx\right)$

[may be $+\infty$]

Idea: Try to find a sequence of $f_n\in L^1(A\times B)\nearrow f$ (to use MCT

How? Cut both the domain and range.

Proof: For any n. we define $f_n(x)=\begin{cases} f(x) & \text{if } (x,y)\in B(0,n),\ f(x)<n. \\ n & \text{if } (x,y)\in B(0,n),\ f(x)\geq n \\ 0 & \text{if } (x,y)\notin B(0,n)\end{cases}$

Then $0\leq f_1\leq f_2\leq \cdots \nearrow f$

$f_n\in L^1(A\times B)$

- By Fubini (1). $\exists B_n$ with $m(B_n)=0$. s.t.: $f_n(x,y)\in L^1(A)$ for $y\in B\setminus B_n$

$\longrightarrow f_n(x,y)$ is measurable

$\Rightarrow$ for $y\in B\setminus(\bigcup_{i=1}^{\infty}B_i)$ $f_n(x,y)\nearrow f(x,y)\Rightarrow f(x,y)$ measurable $\boxed{(1)}$

- Since $f_n(\cdot,y)\nearrow f(\cdot,y)$ By MCT.

$F_n(\cdot,y)=\int_A f_n(x,y)\,dx\nearrow\int_A f(x,y)\,dx:=F(y)$ $\boxed{(2)}$

By Fubini (2) each F_n is measurable $\Rightarrow F$ is measurable.

By MCT for F. we get $\int_B(\int_A f_n(x,y)\,dx)\,dy\longrightarrow\int_B(\int_A f(x,y)\,dx)\,dy$

$\parallel$ Fubini (3) $\qquad\qquad \parallel$

By MCT for f_n w.r.t. (x,y) $\int_{A\times B} f_n(x,y)\,dx\,dy\rightarrow\int_{A\times B} f(x,y)\,dx\,dy$ $\checkmark$ $\boxed{(3)}$

Rmk: In applications, one may need to combine the Tonelli and Fubini:

i.e. First apply Tonelli to $|f|^{\geq 0}\rightarrow$ Compute $\int_{A\times B}|f|=\int_B(\int_A|f|\,dx)\,dy<\infty$?

If yes, the $f\in L^1(A\times B)$. So can apply Fubini to compute $\int_{A\times B} f\,d$

Cor || If $A \subset \mathbb{R}^{d_1}$. $B \subset \mathbb{R}^{d_2}$ are measurable, then $A \times B$ is measurable and $m(A \times B) = m(A) \times m(B)$ The proof is pure measure theoretic

Proof: Assuming first half, then apply Tonelli to $\chi_{A\times B}$ to get $m(A\times B) = \int_{\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}} \chi_{A\times B}\, dxdy$
$= \int_{\mathbb{R}^{d_2}} m(A)\chi_B(y)\, dy = m(A) \times m(B)$.

To prove the first half: $A = \bigcup_{n=0}^{\infty} A_n$. where $m(A_0) = 0$ and each A_n is closed.

$B = \bigcup_{m=0}^{\infty} B_m$. where $m(B_0) = 0$ and each B_n is closed

WANT: $A \times B = (\bigcup_{n=0}^{\infty} A_n) \times (\bigcup_{m=0}^{\infty} B_m) = \bigcup_{n,m=0}^{\infty} (A_n \times B_m)$

① For $m, n \geq 1$. A_n. B_m are closed $\Rightarrow A_n \times B_m$ is closed.

② It remains to prove $A_0 \times B_m$, $A_m \times B_0$ are measurable

[Observation: They should be measure 0].

Since $m(A_0) = 0$. one can find $A_0 \subset \bigcup_{i=1}^{\infty} C_i$, where C_i is ~~open~~ box, $\sum_{i=1}^{\infty} m(C_i) < \varepsilon$.

Similarly, $B_n \subset \bigcup_{j=1}^{\infty} D_j^{(n)}$ where $D_j^{(n)}$ are ~~open~~ box. s.t: $\sum_{j=1}^{\infty} m(D_j^{(n)}) < m(B_n) + \varepsilon$.

$\rightsquigarrow m(A_0 \times B_n) \leq m(\bigcup_{i,j} C_j \times D_j^{(n)}) \leq \sum_{i,j} m(C_i) \times m(D_j^{(n)}) = (\sum_i m(C_i))(\sum_j m(D_j^{(n)}))$
$< \varepsilon(\underbrace{m(B_n)}_{\text{need to be } < +\infty} + \varepsilon) \xrightarrow[\varepsilon \to 0]{} 0$

$A_n^{new} = (\bigcup_{j=1}^{n} A_j) \cap \overline{B(0,n)}$ closed. bounded $\bigcup A_n^{new} = \bigcup A_n = A$.

$B = \bigcup_{i=1}^{\infty} B_i^{new}$ the same way.

- Recall $f \geq 0$. measurabel. a.e. finite.

$\int_A f(x)dx = \underbrace{\int_0^{\infty} m(\{x : f(x) > t\})\, dt}_{\text{Riemann}}$

Prop: Suppose f is measurable, then for $1 \le p < \infty$,
$$\int_A |f(x)|^p dx = p \int_0^{+\infty} t^{p-1} m(\{x: |f(x)| > t\}) dt.$$

Proof: Let $F(t,x) = \chi_{\{x: |f(x)| > t\}}$.

$$\begin{aligned}\int_A |f(x)|^p dx &= \int_A \left(\int_0^{|f(x)|} p t^{p-1} dt \right) dx. \\ &= \int_A \int_0^{+\infty} p t^{p-1} F(t,x)\, dt\, dx. \\ &= \int_0^{+\infty} \left(\int_A p t^{p-1} F(t,x) dx \right) dt \\ &= p \int_0^{+\infty} t^{p-1} m(\{x: |f(x)| > t\}) dt.\end{aligned}$$

Def: Given two measurable function on $\mathbb{R}^d$. we define their <u>Convolution</u> to be
$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y) g(y) dy.$$
If the integral exists.

Rmk: $(g * f)(x) = \int_{\mathbb{R}^d} f(y) g(x-y) dy = \int_{\mathbb{R}^d} f(x-y) g(y) dy = (f * g)(x).$

Prop: If $f, g \in L^1(\mathbb{R}^d)$ Then
(1) for a.e $x \in \mathbb{R}^d$ $(f*g)(x)$ exists.
(2) $f * g \in L^1(\mathbb{R}^d)$
(3) $\int_{\mathbb{R}^d} |f*g|(x) dx \le \left(\int_{\mathbb{R}^d} |f| dx\right) \cdot \left(\int_{\mathbb{R}^d} |g| dx\right)$

Proof: we prove (3) ($\Rightarrow$ (2) $\Rightarrow$ (1))

· First assume $f, g \ge 0$ Then
$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-y) g(y) dy\, dx \underset{\text{Tonelli}}{=} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(x-y) dx \right) g(y) dy$$
$$= \int_{\mathbb{R}^d} g(y) \left(\int_{\mathbb{R}^d} f(x) dx \right) dy = \int_{\mathbb{R}^d} f(x) dx \cdot \int_{\mathbb{R}^d} g(y) dy$$

· For general, we only need to prove
$$|f * g| \le |f| * |g|.$$

To more properties of convolution, we need

Thm || If $f \in L^1(\mathbb{R}^d)$, then $\int_{\mathbb{R}^d} |f(x+h)-f(x)|\,dx \xrightarrow[h\to 0]{} 0$

(integral function is almost continuous in sense of mean)

Proof: Take K s.t: $\int_{\mathbb{R}^d \setminus B(0,K)} |f|\,dx < \varepsilon$ ($\Leftarrow \lim_{K\to\infty} \int_{B(0,K)} |f|\,dx = \int_{\mathbb{R}^d} |f|\,dx$.

Take $\underbrace{g \in L^1(\mathbb{R}^d)}_{\text{Continuous}}$ s.t: $\|g-f\|_{L^1(\mathbb{R}^d)} \le \varepsilon$

$$\Rightarrow \int_{\mathbb{R}^d} |f(x+h)-f(x)|\,dx = \int_{\mathbb{R}^d \setminus \underbrace{B(0,K+1)}_{\text{ensure } f(x+h)\in B(0,K)\ (h\to 0)}} |f(x+h)-f(x)|\,dx$$

$$+ \int_{B(0,K+1)} |f(x+h)-f(x)|\,dx$$

$$\le \varepsilon + \varepsilon + \|f(x+h)-g(x+h)+g(x+h)-g(x)+g(x)-f(x)\|_{L^1(B(0,K+1))}$$

$$\le 2\varepsilon + \|f(x+h)-g(x+h)\|_{L^1(\mathbb{R}^d)} + \|g(x+h)-g(x)\|_{L^1(B(0,K+1))} + \|g(x)-f(x)\|_{L^1}$$

$$\le 4\varepsilon + \|\underbrace{g(x+h)-g(x)}_{\text{uniformly continuous in } B(0,K+1)}\|_{L^1(B(0,K+1))} < 5\varepsilon.$$

Cor || $f \in L^1(\mathbb{R}^d)$. g is bounded, measurable, then

$f * g$ is uniformly continuous on $\mathbb{R}^d$

Proof: Let $M = \sup|g| < +\infty$.

$$|(f*g)(x+h)-(f*g)(x)| = \left|\int_{\mathbb{R}^d} f(x+h-t)g(t)\,dt - \int_{\mathbb{R}^d} f(x-t)g(t)\,dt\right|$$

$$\le \int_{\mathbb{R}^d} |f(x+h-t)-f(x-t)|\cdot|g(t)|\,dt$$

$$\le M\int_{\mathbb{R}^d} |f(x+h-t)-f(x-t)|\,dt$$

$$= M \underline{\int_{\mathbb{R}^d} |f(t+h)-f(t)|\,dt} \xrightarrow[h\to 0]{} 0.$$

Cor: || Suppose $f \in L^1(\mathbb{R}^d)$ $g \in C^k(\mathbb{R}^d)$ and g is $\underbrace{\text{compact supported}}_{\text{ensure } |g^k| \le M}$.

Then $f * g \in C^k(\mathbb{R}^d)$

Proof: $\frac{\partial}{\partial x_i}(f*g) = \frac{\partial}{\partial x_i}\int_{\mathbb{R}^d} f(y)\,g(x-y)\,dy$

$$\underset{\substack{\uparrow \\ \text{Prop last time}}}{=} \int_{\mathbb{R}^d} f(y)\frac{\partial}{\partial x_i} g(x-y)\,dy = f * \frac{\partial g}{\partial x_i}$$

bounded

Continuous.

$\exists\, \eta_h$ s.t. $f * \eta_h \in C^\infty$.

and $\|f*\eta_h - f\|_{L^1} \to 0$ $(h\to 0)$.

- $L: \mathbb{R}^d \to \mathbb{R}^d$ Linear $\Rightarrow m(L(A)) = |\det L| m(A)$.
- $f: \mathbb{R}^d \to \mathbb{R}$ non-negative, measurable $\Rightarrow \int_{\mathbb{R}^d} f(x)dx = m(\{(x,y) \in \mathbb{R}^{d+1} : 0 \le y \le f(x)\})$

Lec 15.

Mid-term

i.e. for subsets of X. we can "tell how large it is"

1 Measurable space.

Want: define measure (and then integral) on an abstract space
A set X.

We have seen:

If we want to keep nice property.

[e.g. countable additivity, transation-inv, $m([0,1]^d) = 1 \Rightarrow m$ on $\mathcal{L}$]

then one can't define measure on all sets of X.

$\rightarrow$ We only pick some nice subsets of X to define measure

Recall: In Lebesgue theory, we define measure for $A \in \mathcal{L}$.

(all Lebesgue measurable subsets on $\mathbb{R}^d$).

(1) $\emptyset \in \mathcal{L}$.

(2) $A \in \mathcal{L}$. then $A^c \in \mathcal{L}^c$.

(3) $A_1, A_2 \in \mathcal{L} \Rightarrow A_1 \cap A_2, A_1 \cup A_2, A_1 \setminus A_2, A_1 \Delta A_2 \in \mathcal{L}$.

(4) $A_1, A_2, \cdots \in \mathcal{L} \Rightarrow \bigcup_{i=1}^\infty A_i, \bigcap_{i=1}^\infty A_i \in \mathcal{L}$.

(5) open, closed, compact, $\underset{\text{countable } \cup}{F_\sigma}$. $G_\delta \cdots \in \mathcal{L}$. $\leftarrow$ drop it here. The conceptions are not defined on abstra set

Recall: $\cdot \bigcap_{i=1}^\infty A_i \in \mathcal{L} \Leftarrow (2), (4)$ 1st half $(\bigcap_{i=1}^\infty A_n) = (\bigcup_{i=1}^\infty A_n^c)^c$.

$\cdot$ Similarly $(3) \Leftarrow (2), (1)$, 1st half

Def: Let X be an abstract set

(1) A σ-algebra on X is a collection $\mathcal{F} \subset P(X)$ s.t:

(a) $\emptyset \in \mathcal{F}$. (b) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$. (c) $A_1, A_2 \cdots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^\infty A_i \in \mathcal{F}$.

(2) If $\mathcal{F}$ is a σ-algebra. We call $(X, \mathcal{F})$ a measurable space

Prop. Suppose $(X, \mathcal{F})$ is a measurable space. Then
(1) If $A_1, A_2 \in \mathcal{F}$, then $A_1 \cup A_2$, $A_1 \cap A_2$, $A_1 \setminus A_2$, $A_1 \triangle A_2 \in \mathcal{F}$.
(2) If $A_1, A_2 \cdots \in \mathcal{F}$, then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$.
(3) $X \in \mathcal{F}$.

Rmk: (1) We can replace (c) by (c'): $\bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$.
(2) If we replace (c) by a weaker condition
$(\tilde{c})$ $A_1, \cdots, A_n \in \mathcal{L} \Rightarrow \bigcup_{i=1}^{n} A_i \in \mathcal{F}$
Then the resulting $\mathcal{F}$ is called a Boolean algebra.

Example 1 $X = \mathbb{R}^d$. $\mathcal{F} \in \mathcal{L}$. (Lebesgue algebra)
2. (Null algebra) $X = \mathbb{R}^d$ $\mathcal{F} = \{A \subset \mathcal{L}: m(A) = 0 \text{ or } m(A^c) = 0\}$.
3. (Trivial algebra) $\mathcal{F} = \{\phi, X\}$.
4. (Discrete algebra) $\mathcal{F} = P(X)$
5. (Atomic algebra) $X = \bigcup_{\alpha \in I} A_\alpha$. ← disjoint union.
→ $\mathcal{F} = \{\bigcup_{\alpha \in J} A_\alpha, J \subset I\}$.
6. (Elementary algebra) $X = \mathbb{R}^d$
$\mathcal{F} = \{E \subset \mathbb{R}^d,\ E \text{ is elementary or } E^c \text{ is elementary}\}$.
Then $\mathcal{F}$ is a Boolean algebra on X but not a σ-algebra.
Similarly: Jordan algebra.

Prop. Let $\{\mathcal{F}_\alpha\}$ be a family of σ-algebra on X
Then $\mathcal{F} := \bigcap_\alpha \mathcal{F}_\alpha$ is a σ-algebra on X.

Proof. • $\phi \in \mathcal{F}_\alpha \Rightarrow \phi \in \mathcal{F}$.
• $A \in \mathcal{F} \Rightarrow A \in \mathcal{F}_\alpha\ \forall \alpha, \Rightarrow A^c \in \mathcal{F}_\alpha \Rightarrow A^c \in \mathcal{F}$.
• $A_n \in \mathcal{F} \Rightarrow A_n \in \mathcal{F}_\alpha\ \forall \alpha \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}_\alpha \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

Def. Let $\mathcal{X} \subset P(X)$ be any collection of subsets of X. We define $\mathcal{F} = \langle \mathcal{X} \rangle :=$ the intersection of all σ-algebra that contains $\mathcal{X}$
(= the smallest σ-algebra that contains A).
We call $\mathcal{F}$. the σ-algebra generated by $\mathcal{X}$.

Example Let $X = \mathbb{R}^d$

$\mathcal{X}$ = all open sets in $\mathbb{R}^d$

$\rightarrow$ $\langle \mathcal{X} \rangle$ = all Borel sets in $\mathbb{R}^d$ $:= \mathcal{B} \neq \mathcal{L}$. Borel algebra

σ-algebra generated by all open sets on $\mathbb{R}^d$

Rmk. Recall: $A \in \mathcal{L} \Leftrightarrow A = F \cup N$. where F is F_σ. N is null.
The Lebesgue algebra is generated by the Borel algebra and the Null algebra.

2. The measure space

Recall: The Lebesgue measure $m: \boxed{\mathcal{L} \to [0,+\infty]}$ satisfies

(1) $m(\phi) = 0$

(2) If $A_1, A_2, \cdots$ disjoint, then $m(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m(A_n)$

(3) For any $A_1, A_2, \cdots$, we have $m(\bigcup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} m(A_n)$ from m^* sub-additivity

(4) If $A_1 \subset A_2 \subset \cdots$ then $m(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} m(A_n)$

(5) If $A_1 \supset A_2 \supset \cdots$, $m(A_1) < +\infty$, then $m(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} m(A_n)$.

(6) $m(A + \{x\}) = m(A)$ $\leftarrow$ not define on abstract set

(7) $A_1 \subset A_2 \Rightarrow m(A_1) \le m(A_2)$

(8) (Fatou) If $A_1, A_2 \cdots \in \mathcal{L}$. then $m(\liminf_{n\to\infty} A_n) \le \liminf_{n\to\infty} m(A_n)$.

(9) If $A_1, A_2 \cdots \in \mathcal{L}$. $m(\bigcup_{n=1}^{\infty} A_n) < \infty$. then $m(\limsup_{n\to\infty} A_n) \ge \limsup_{n\to\infty} m(A_n)$

(10) (Dominant Convergence) Suppose $\{A_n\} \to A$.
($\chi_{A_n} \to \chi_A$ pointwise)
$\exists F \in \mathcal{L}$. $F \supset A_n$ for all n. $m(F) < \infty$. then $m(A) = \lim_{n\to\infty} m(A_n)$

(11) If $m(A) = 0$. $B \subset A$ then $m(B) = 0$
$\boxed{\forall B \subset A}$ $B \in \mathcal{L}$

Def Let $(X, \mathcal{F})$ be a measurable space

1) A map $\mu: \mathcal{F} \to [0, +\infty]$ is called a measure if

(a) $\mu(\phi) = 0$

(b) $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ for disjoint $A_n \in \mathcal{F}$.

2) We call $(X, \mathcal{F}, \mu)$ a measure space

Prop: || Let $(X, \mathcal{F}, \mu)$ be a measurable space
then (3), (4), (5), (7), (8), (9), (10) hold.

Rmk · We say $(X, \mathcal{F}, \mu)$ a finite measure space if $\underline{\mu(X) < +\infty}$.

· We say $(X, \mathcal{F}, \mu)$ a σ-finite measure space if
$\underline{\exists A_n \in X.\ m(A_n) < +\infty.\ \text{s.t. } X = \bigcup_{n=1}^{\infty} A_n}$.

· We say $(X, \mathcal{F}, \mu)$ a Probability measure space if $\underline{\mu(X) = 1}$

Example 1 (Lebesgue measure) $X = \mathbb{R}^d$. $\mathcal{F} = \mathcal{L}$. $\mu = m$.

2. For any non-negative measurable function f on $\mathbb{R}^d$. one can define a measure $\mu_f: \mathcal{L} \to [0, +\infty)$
by $\mu_f(A) := \int_A f\,dx$.

3. (Dirac measure) Fix $x_0 \in X$. we define
$\delta_{x_0}: \mathcal{F} \to [0, +\infty]$. by $\delta_{x_0}(A) = \chi_A(x_0) = \begin{cases} 1 & x_0 \in A \\ 0 & x_0 \notin A \end{cases}$.

Example 4. (The counting measure) For any $A \in \mathcal{F}$.

$$\#A = \begin{cases} \text{number of elements in } A, & \text{if } A \text{ is a finite set} \\ \infty & \text{if } A \text{ is an infinite set} \end{cases}$$

More generally, For any nonnegative function f on X. one can define $\#_f A = \sup_{(a_n) \subset A} \sum_n f(a_n)$
(for A is an uncountable set).

5. Suppose $(X, \mathcal{F}, \mu)$ is a measure space $Y \subset X$.
$\longrightarrow \mathcal{F}_Y = \{A \cap Y . A \in \mathcal{F}\}$. is a σ-algebra on Y.

- If $Y \in \mathcal{F}$. then $\mu_Y(A \cap Y) := \mu(A \cap Y)$ is a measure on $(Y, \mathcal{F}_Y)$.

6. Suppose $\mu_1, \mu_2, \cdots$ are measures on $(X, \mathcal{F})$
Then for any $c_1, c_2, \cdots \geq 0$. the function $\mu := \sum_{n=1}^{\infty} \underbrace{c_n \mu_n}_{\geq 0}$
is still a measure on $(X, \mathcal{F})$.

☆ Property (11) doesn't hold for abstract measure space $(X, \mathcal{F}, \mu)$.
Suppose $A \in \mathcal{F}$, and $\mu(A) = 0$.
Q: Do we have $\mu(B) = 0$ for $\forall B \subset A$? (i.e. $B \in \mathcal{F}$)
Problem: It's possible that $A \in \mathcal{F}$. $\mu(A) = 0$. $B \subset A$ But $B \notin \mathcal{F}$.
e.g. Consider Dirac measurable on $(\mathbb{R}^d, \mathcal{L})$

$\forall A \in \mathcal{L}$, $x_0 \notin A \Rightarrow \delta_{x_0}(A) = 0$.

But there exists non-measurable set $\tilde{A}$ that doesn't contain x_0 Since $\tilde{A} \notin \mathcal{L}$. So we don't have $\delta_{x_0}(\tilde{A}) = 0$.

Def: We say a measure space $(X, \mathcal{F}, \mu)$ is complete if it satisfies $\mu(A) = 0$. $\underbrace{B \subset A}_{(\text{on } X)}$. $\Rightarrow B \in \mathcal{F}$ ($\xRightarrow{\text{and}}$ $\mu(B) = 0$).

Fact: Borel algebra is not complete
$\exists$ measure zero Borel set A s.t. $\exists B \subset A$, B is not Borel.
$\rightarrow (X, \mathcal{B}, \underset{m}{\overset{\shortparallel}{\mu}})$

Prop: Let $(X, \mathcal{F}, \mu)$ be a measure space.
$\mathcal{N} = \{N \in \mathcal{F} : \mu(N) = 0\}$.
Let $\bar{\mathcal{F}} = \{A \cup B : A \in \mathcal{F}$. and $\exists N \in \mathcal{N}$ s.t. $B \subset N\}$.
Then $\bar{\mathcal{F}}$ is a σ-algebra on X and $\bar{\mu} : \bar{\mathcal{F}} \to [0, +\infty]$

defined by $\bar{\mu}(A \cup B) = \mu(A)$.
is a complete measure on $(X, \bar{\mathcal{F}})$

Lec 16

1 Measurable function

Def: A function $f: X \to R$ (or $[-\infty, +\infty]$) is measurable ($\mathcal{F}$-measurable).
If $\forall t \in R$ the set $\{x: f(x) > t\} \in \mathcal{F}$.
$\Leftrightarrow$ $<, \leq, \geq$
$\Leftrightarrow$ $\forall$ open set $U \subset R$ the set $f^{-1}(U) \in \mathcal{F}$.
closed - Borel

Rmk: Let $(X, \mathcal{F})$, $(Y, \mathcal{G})$ be measurable space
A map $\varphi: X \to Y$ is called measurable if
$\varphi^{-1}(B) \in \mathcal{F}$ for any $B \in \mathcal{G}$.

So, A function $f: X \to R$ is measurable means the $\boxed{\text{map } f}$
$\underset{\uparrow(\text{map})}{f}: (X, \mathcal{F}) \to (R, \mathcal{B})$ is measurable
↑ Borel σ-algebra

In particular, a function $f: R^d \to R$ is measurable in Lebesgue theory
means; f is a measurable map $f: (R^d, \mathcal{L}) \longrightarrow (R, \mathcal{B})$
Not $f: (R^d, \mathcal{L}) \to (R, \mathcal{L})$ is measurable. ($\exists A \in \mathcal{L} \cap R$ $f^{-1}(A) \notin \mathcal{L} \cap R^d$)

• Prop: Let $(X, \mathcal{F})$ be a measurable space

(1) $A \in \mathcal{F} \Leftrightarrow \chi_A$ is measurable function

(2) If f_1, f_2 are measurable, then $c_1 f_1 + c_2 f_2$, $f_1 \cdot f_2$ are measurable

(3) If $\varphi: R \to R$ is continuous, $f: X \to R$ is measurable.
then $\varphi \circ f$ is measurable.

(4) f_n are measurable $\Rightarrow$ $\limsup f_n$, $\liminf f_n$, $\sup f_n$, $\inf f_n$
is measurable

Simple function: $f(x) = \sum_{i=1}^{n} c_i \chi_{A_i}$ $A_i \in \mathcal{F}$

Prop: || Let $f: X \to [0, +\infty] \geq 0$. measurable. Then there exists $0 \leq g_1 \leq g_2 \cdots \nearrow f$, where each g_i is a simple function on $(X, \mathcal{F})$ (Moreover, the converge is uniform on any set where f's bounded). $\to$ a set A $\underline{\forall x \in A}$ Convergent uniformly.

Proof: $f_n(x) = \begin{cases} \frac{k-1}{2^n} & \text{if } \frac{k-1}{2^n} \leq f(x) \leq \frac{k}{2^n} \quad k = 1, 2, \cdots, n2^n \\ n & \text{if } f(x) \geq n \end{cases}$

MCT Let $(X, \mathcal{F}, \mu)$ be a measure space. and $0 \leq f_1 \leq f_2 \leq \cdots$

Then $\int_X \lim_{n\to\infty} f_n \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu$

$\to$ Linearity, countability I: $\int_X \sum_{n=1}^{\infty} f \, d\mu = \sum_{n=1}^{\infty} \int_X f \, d\mu$

Corollaries of MCT II: $\int_{\bigcup_{n=1}^{\infty} A_n} f \, d\mu = \sum_{n=1}^{\infty} \int_{A_n} f \, d\mu$

$A_n \in \mathcal{F}$, disjoint

2. Theory of integration: Non-negative case.

Δ Let $g(x) = \sum_{i=1}^{n} c_i \chi_{A_i}$ be a non-negative simple function

$\int_X g \, d\mu = \sum_{i=1}^{\infty} c_i \mu(A_i)$

Again: need to check that the integral is independent of the representation of g.

Δ Let $(X, \mathcal{F}, \mu)$ be a measure space

Let $f: X \to [0, +\infty]$ be a non-negative measurable function

Def: || $\int_X f \, d\mu := \sup \{ \int_X g \, d\mu : 0 \leq g \leq f, \ g \text{ is simple} \}$.

Countable additivity III Let $(X, \mathcal{F})$ be a measurable space
$f: X \to [0, \infty]$ be a non-negative measurable function. Then for any sequence $\mu_1, \mu_2, \cdots$ of measure on $\mathcal{F}$.

$$\int_X f \, d\left(\sum_{n=1}^{\infty} \mu_n\right) = \sum_{n=1}^{\infty} \int_X f \, d\mu_n$$

Proof. · $f = \chi_A$, $A \in \mathcal{F}$ $LHS = \sum_{n=1}^{\infty} \mu_n(A) = RHS$
· Linearity → simple function
· MCT → all functions ≥ 0.

Example. Let $X = \mathbb{N} \{1, 2, 3 \cdots\}$
$\mathcal{F} = P(X)$
$\mu = \#$ "counting measure"

Note. Any sequence of numbers $a_1, a_2, a_3 \cdots$ can be viewed as a function $a: \mathbb{N} \to \mathbb{R}$, $a(n) := a_n$.
It's $\mathcal{F}$-measurable

Suppose $a_n \geq 0$ for all n. Then

$$\int_X a \, d\mu = \lim_{N \to \infty} \int_X a^N d\mu = \lim_{N \to \infty} \sum_{n=1}^{N} a(n) = \sum_{n=1}^{\infty} a_n$$

$$a^N(n) = \begin{cases} a(n) & n \leq N \\ 0 & n > N \end{cases}$$

"Infinite series = the integral over $\mathbb{N}$ w.r.t. the counting measure"

Countable additivity I: View $f_1, f_2 \cdots \geq 0$ as one function on
$f: X \times \mathbb{N} \to [0, \infty]$ $f(x, n) := f_n(x)$

Then "Tonelli" $\int_X \left(\int_{\mathbb{N}} f(x, n) \, d\#_n\right) d\mu_x = \int_{\mathbb{N}} \left(\int_X f(x, n) \, d\mu_x\right) d\#_n$
$\left(\int_X \sum f_n \, d\mu_x = \sum \int_X f_n \, d\mu_x\right)$

3. Integration: absolute convergence theory

Def. Let $(X, \mathcal{F}, \mu)$ be a measure space, f a measurable function
We say f is absolutely integral if

$$\|f\|_{L^1(X, \mathcal{F}, \mu)} := \int_X |f| \, d\mu < \infty$$

le

For such f, define
$$\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu.$$
$\rightarrow L^1(X, \mathcal{F}, \mu)$ or $L^p(X, \mathcal{F}, \mu)$

e.g. $L^1(N, P(X), \#) = \{(a_n) : \sum_{n=1}^{\infty} |a_n| < +\infty\}$

$L^p(N, P(X), \#) = \{(a_n) : \sum_{n=1}^{\infty} |a_n|^p < +\infty\}$

$\rightarrow$ Monotonicity. Linearity. countable additivity. $\boxed{\text{absolute Continuity}}$

Given $f \in L^1(X, \mathcal{F}, \mu)$ $\forall \varepsilon > 0\ \exists \delta$ s.t

$\forall A.\ \mu(A) < \delta.\ \int_A |f| d\mu < \varepsilon$

D.C.T: $f_n \to f$ a.e (or in measure)

$\exists g \in L^1(X, \mathcal{F}, \mu)$ s.t: $|f_n| \le g$

$\Rightarrow$ $f \in L^1(X, \mathcal{F}, \mu)$ $\int_X |f_n - f| d\mu \to 0$

and $f_n \to f$ in $L^1(X, \mathcal{F}, \mu)$ norm

$(\Rightarrow \int_X f_n d\mu \to \int_X f d\mu)$

$\rightarrow$ Completeness of L^1. (L^p)

What could be different ?

- $(X, \mathcal{F})$. $(Y, \mathcal{G})$ be measurable spaces

$\rightsquigarrow (X \times Y, \mathcal{F} \otimes \mathcal{G})$

the σ-algebra generated by sets of the form $A \times B$.
$A \in \mathcal{F}$. $B \in \mathcal{G}$.

Fact | Suppose $(X, \mathcal{F}, \mu_X)$ $(Y, \mathcal{G}, \mu_Y)$ be two measure space.

Then $\exists$ a unique measure $\mu_{X \times Y}$ on $\mathcal{F} \otimes \mathcal{G}$ s.t:

$$\mu_{X \times Y}(A \times B) = \mu_X(A) \cdot \mu_Y(B).$$

σ-finite

i.e. $X = \bigcup_{n=1}^{\infty} X_n$

$\mu(X_n) < \infty$

Torelli Thm (Incomplete Version)

Let $(X, \mathcal{F}, \mu_X)$ $(Y, \mathcal{G}, \mu_Y)$ be two σ-finite measure space

$f: X \times Y \to [0, +\infty]$ be a $\underline{\mathcal{F} \otimes \mathcal{G} \text{ measurable}}$ function. Then

(1) $f(\cdot, y)$ is measurable on X $\underline{(\forall y \text{ on } Y)}$.

(2) $y \to \int_X f(x,y) d\mu_X$ is measurable on Y.

(3) $\int_{X \times Y} f(x,y)\, d\mu_X d\mu_Y = \int_Y \left(\int_X f(x,y) d\mu_X\right) d\mu_Y = \int_X \left(\int_Y f(x,y) d\mu_Y\right) d\mu_X.$

Example: △ $X = [0,1]$ $\mathcal{F} = P(X)$, $\mu = \#$ → NOT σ-finite
and
△ $Y = [0,1]$ $\mathcal{F} = \mathcal{L}$, $\mu = m$

$$f(x,y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$$

$$\int_Y \left(\int_X f(x,y)\, d\#_X \right) dm_Y = \int_Y 1\, dm = 1$$

$$\int_X \left(\int_Y f(x,y)\, dm_Y \right) d\#_X = \int_X 0\, d\# = 0$$

$$\boxed{\begin{array}{ll} \text{Difference:} & \mathcal{L}^{k_1+k_2} \neq \underline{\mathcal{L}^{k_1} \otimes \mathcal{L}^{k_2}} \\ & \text{FACT } \mathcal{L}^{k_1+k_2} \neq \mathcal{L}^{k_1} \otimes \mathcal{L}^{k_2} \\ \text{But} & \mathcal{B}^{k_1+k_2} = \mathcal{B}^{k_1} \otimes \mathcal{B}^{k_2} \end{array}}$$

Lec 17

Recall: Elementary → Jordan ⇒ Lebesgue outer measure m^*
⇒ Lebesgue measure m

I. Outer measure.

The difference between m^* and m.

- m^* is defined for all sets. — m is defined for $A \in \mathcal{L}$ and $m(A) = m^*(A)$.
- Subadditive (countable) — Additive (countable)

$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^*(A_n)$ — $m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n)$ for disjoint A_n's.

Def. For any set X. an outer measure is a map $\mu^*: P(X) \to [0,+\infty]$
s.t.

(1) $\mu^*(\phi) = 0$.

(2) If $A_1 \subset A_2$, then $\mu^*(A_1) \leq \mu^*(A_2)$ (monotonicity)

(3) $\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$ (countable subadditivity)

Recall: For abstract measure, additivity + non-negative $\to$ monotonicity
For outer measure, subadditivity + non-negative $\not\to$ monotonicity

To construct measure (and σ-algebra):
① Construct outer measure $\to$ ② construct σ-algebra and ③ Construct outer measure on it

① Prop: Let $\mathcal{E}$ be any collection of subsets of X. (i.e $\mathcal{E} \subset P(X)$).
Such that $\bigcup_{E \in \mathcal{E}} E = X$, $\phi \in \mathcal{E}$.

Let $\rho: \mathcal{E} \to [0, +\infty]$ be any function s.t: $\rho(\phi) = 0$.
(Cause Contradiction?)
For any $A \subset X$. let
$$\mu^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \rho(E_n) \,\middle|\, A \subset \bigcup_{n=1}^{\infty} E_n,\ E_n \in \mathcal{E} \right\}.$$
(if A can't be covered by countable E_n's. then def $\mu^*(A) = +\infty$)

Then $\mu^*: P(X) \to [0, +\infty]$ is an outer measure.

Proof: · Obviously $\mu^*(\phi) = 0$. $\mu^*(A_1) \le \mu^*(A_2)$ if $A_1 \subset A_2$.
· For countable sub additivity: If $\exists A_n$ s.t: $\mu^*(A_n) = +\infty$.
then $\mu^*(\bigcup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} \mu^*(A_n)$.
Next suppose each $\mu^*(A_n) < +\infty$.
By def, $\exists E_m^{(n)} \in \mathcal{E}$. s.t. $A_n \subset \bigcup_{m=1}^{\infty} E_m^{(n)}$ and
$\sum_{m=1}^{\infty} \rho(E_m^{(n)}) \le \mu^*(A_n) + \varepsilon \cdot \frac{1}{2^n}$.
$\Rightarrow \bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} E_m^{(n)}$

$$\mu^*(\bigcup_{n=1}^{\infty} A_n) \le \cancel{\mu^*(\bigcup_{n,m} E_m^{(n)})} \le \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon. \quad (\forall \varepsilon > 0).$$
$$\Rightarrow \varepsilon \to 0 \quad \mu^*(\bigcup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} \mu^*(A_n)$$

From outer measure $\rightsquigarrow$ measure [i.e. find $\mathcal{F}$]
Recall: $A \in \mathcal{L} \iff \forall \varepsilon > 0\ \exists$ open set $U \supset A$ s.t: $m^*(U \setminus A) < \varepsilon$
$\iff \forall \varepsilon > 0\ \exists$ closed set $F \subset A$ s.t: $m^*(A \setminus F) < \varepsilon$

$\Leftrightarrow$ $A = G \setminus N$, $G = G_\delta$. N = null set

$\Leftrightarrow$ $A = F \cup N$, $F = F_\sigma$ N = null set

Caratheodory $\Leftrightarrow$ $\forall$ box B $|B| = m^*(B \cap A) + m^*(B \setminus A)$. $\leftarrow$ If $A \subset B$ (bounded) it's enough to check this for B.

$\Leftrightarrow$ $\forall T \subset \mathbb{R}^d$ $m^*(T) = m^*(T \cap A) + m^*(T \setminus A)$.

Def: Let μ^* be an outer measure on a set X. We say a subset $A \subset X$ is (Cathedory) measurable if

$$\mu^*(T) = \mu^*(T \cap A) + \underbrace{\mu^*(T \cap A^c)}_{\mu^*(T \setminus A)} \quad \forall T \subset X. \quad (*)$$

Rmk: By subadditivity we already has

$$\mu^*(T) \le \mu^*(T \cap A) + \mu^*(T \setminus A).$$

$\boxed{\text{So it's enough to check } \geq}$

Lemma: If $\mu^*(A) = 0$. then A is measurable.

Proof: $\mu^*(T) \geq \underbrace{\mu^*(T \cap A)}_{0} + \mu^*(T \setminus A)$

Thm (Carathedory extension thm) Suppose $\mu^* : P(X) \to [0, +\infty]$ is any outer measure, Let $\mathcal{F} = \{A \subset X : A \text{ satisfies } (*)\}$. Then (1) $\mathcal{F}$ is a σ-algebra, and (2) $\mu = \mu^*|_{\mathcal{F}}$ is a measure. In fact it's a complete measure

Proof: (1) · $\emptyset \in \mathcal{F}$.

· $A \in \mathcal{F} \Rightarrow \mu^*(T) = \mu^*(T \cap \underbrace{A}_{(A^c)^c}) + \mu^*(T \cap A^c) \Rightarrow A^c \in \mathcal{F}$. $\forall T \subset X$.

· Now Suppose $A_1, A_2, \cdots \in \mathcal{F}$.

– First prove $A_1 \cup A_2 \in \mathcal{F}$:

$$\mu^*(T) = \mu^*(T \cap A_1) + \mu^*(T \cap A_1^c)$$

$$= \boxed{\mu^*((T \cap A_1) \cap A_2) + \mu^*((T \cap A_1) \cap A_2^c)} + \underline{\mu^*((T \cap A_1^c) \cap A_2^c)} + \boxed{\mu^*((T \cap A_1^c) \cap A_2)}$$

$T \cap (A_1 \cup A_2) = (T \cap (A_1 \cap A_2)) \cup (T \cap (A_1 \cap A_2^c)) \cup (T \cap (A_1^c \cap A_2))$

$$\geq \mu^*(T \cap (A_1 \cup A_2)) + \underline{\mu^*(T \cap A_1^c \cap A_2^c)}$$

- $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

- Write $B = \bigcup_{n=1}^{\infty} A_n$. $B_N = \bigcup_{n=1}^{N} A_n \in \mathcal{F}$.

$\boxed{\text{WANT}: \mu^*(T) \geq \mu^*(T \cap B) + \mu^*(T \cap B^c)}$

NOTE: $\mu^*(T \cap B_N^c) \geq \mu^*(T \cap B^c)$

$\rightarrow \lim_{N\to\infty} \mu^*(T \cap B_N^c) \geq \mu^*(T \cap B^c)$.

? $\mu^*(T \cap B_{N+1}) = \mu^*(T \cap B_N) + \mu^*(T \cap (B_{N+1} \setminus B_N))$

$\Rightarrow \lim_{N\to\infty} \mu^*(T \cap B_{N+1}) = \sum_{n=1}^{\infty} \mu^*(T \cap (B_{N+1} \setminus B_N))$

$\Rightarrow \mu^*(T) \geq \sum_{N=1}^{\infty} \mu^*(T \cap (B_{N+1} \setminus B_N)) + \mu^*(T \cap B^c)$

$\geq \mu^*(T \cap B) + \mu^*(T \cap B^c)$

(2) $\cdot\ \mu(\phi) = \mu^*(\phi) = 0$.

$\cdot$ Let $A_n \in \mathcal{F}$ disjoint. WANT: $\mu^*(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu^*(A_n)$

- $\mu^*(A_1 \cup A_2) = \mu^*(A_1 \cup A_2 \cap A_2) + \mu^*(A_1 \cup A_2 \cap A_2^c) = \mu^*(A_1) + \mu^*(A_2)$

- $\mu^*(A_1 \cup A_2 \cup \cdots \cup A_N) = \sum_{n=1}^{N} \mu^*(A_n)$

- $\mu^*(\bigcup_{n=1}^{\infty} A_n) \geq \mu^*(\bigcup_{n=1}^{N} A_n) = \sum_{n=1}^{N} \mu^*(A_n) \xrightarrow{N\to\infty} \mu^*(\bigcup_{n=1}^{\infty} A_n) \geq \sum_{n=1}^{\infty} \mu^*(A_n)$

$\Rightarrow \mu^*(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu^*(A_n)$. $\qquad \leq \sum_{n=1}^{\infty} \mu^*(A_n)$.

2. Premeasure $\leftarrow$ How to construct a "nice" outer measure μ^*

Recall: Elementary or Jordan $\longrightarrow$ Lebesgue

Q: finite additivity μ_0 on a Boolean algebra $\mathcal{B}$ $\longrightarrow$ Countable additivity μ on a σ-algebra $\mathcal{F}$

A. In general, no.

Example. $X = \mathbb{N}$ $\quad \mathcal{B} = P(X)$ $\quad \mathcal{F} = P(X)$ $\quad \mu_0(A) = \begin{cases} 0 & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is not finite} \end{cases}$

finite additivity

$\mu_0(\mathbb{N}) = \infty \neq \sum_{n=1}^{\infty} \mu_0(\{n\})$.

Def: A premeasure is a finite additive measure μ_0 on a Boolean algebra $\mathcal{B}$. (ϕ. A^c. $\bigcup_{n=1}^{N} A_n$) For disjoint A_n $\mu_0(\bigcup_{n=1}^{N} A_n) = \sum_{n=1}^{N} \mu_0(A_n)$.
which satisfies an additional property:
If $A_1, A_2, \ldots$ disjoint, and if $\bigcup_{n=1}^{\infty} A_n \in \mathcal{B}$.
then $\mu_0(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu_0(A_n)$.

Example: Let $(X. \mathcal{F}. \mu_X)$ and $(Y. \mathcal{G}. \mu_Y)$ be measure spaces.
Let $\mathcal{B} = \{ (A_1 \times B_1) \cup (A_2 \times B_2) \cup \cdots \cup (A_k \times B_k) : A_i \in \mathcal{F}. B_i \in \mathcal{G} \}$. Can suppose $A_i \times B_i$'s are disjoint
Can check: $\mathcal{B}$ is a Boolean algebra.
Define: $\mu_0 (\bigcup_{i=1}^{k} (A_i \times B_i)) := \sum_{i=1}^{k} \mu_X(A_i) \times \mu_Y(B_i)$.
One can check that μ_0 is a premeasure.
e.g. A simple case $\overset{\in \mathcal{B}}{A \times B} = \bigcup_{n=1}^{\infty} \overset{\in \mathcal{B}}{(A_n \times B_n)}$ $A_n \times B_n$'s are disjoint
$\Rightarrow$ $\chi_A(x) \chi_B(y) = \chi_{A \times B} = \sum_{n=1}^{\infty} \chi_{A_n \times B_n} = \sum_{n=1}^{\infty} \chi_{A_n}(x) \cdot \chi_{B_n}(y)$
$\Rightarrow$ $\underbrace{\int_X \chi_A(x) \chi_B(y) \, d\mu_X}_{= \chi_B(y) \cdot \mu_X(A)} = \sum_{n=1}^{\infty} \mu_X(A_n) \cdot \chi_{B_n}(y)$
$\Rightarrow$ $\mu_X(A) \mu_Y(B) = \sum_{n=1}^{\infty} \mu_X(A_n) \cdot \mu_Y(B_n)$.

Thm (Hahn-Kolmogorov) Let $\mathcal{B}$ be a Boolean algebra on X. μ_0 is a premeasure on $\mathcal{B}$. Then there exists a σ-algebra $\mathcal{F} \supset \mathcal{B}$. and a measure μ on $\mathcal{F}$ s.t: $\mu = \mu_0$ for $A \in \mathcal{B}$
(i.e. $(\mathcal{F}. \mu)$ extends $(\mathcal{B}. \mu_0)$) $\mathcal{F} \supset \mathcal{M} = \langle \mathcal{B} \rangle$

$\rightarrow$ We get $\cancel{\mu_X \times \mu_Y}$ ($\mu_{X \times Y}$) on $X \times Y$.

Proof: We have seen
$\mu^*(A) := \inf \{ \sum_{n=1}^{\infty} \mu_0(E_n) : E_n \in \mathcal{B}. A \subset \bigcup_{n=1}^{\infty} E_n \}$.
— outer measure on X.
$\mathcal{F} = \{ A \subset X : \mu^*(T) = \mu^*(T \cap A) + \mu^*(T \cap A^c), \forall T \subset X \}$ $\rightarrow$ σ-algebra
Remains to check: $\mu = \mu^*|_{\mathcal{F}}$ is a measure

1) $\mathcal{B} \subset \mathcal{F}$

2) For $A \in \mathcal{B}$, we have $\mu^*(A) = \mu_0(A)$.

(1) Let $A \in \mathcal{B}$. Need $\mu^*(T) \geq \mu^*(T \cap A) + \mu^*(T \setminus A)$.

If $\mu^*(T) = +\infty$ ✓.

Now assume $\mu^*(T) < +\infty$. $\Rightarrow \exists E_n \in \mathcal{B}$. $T \subset \bigcup_{n=1}^{\infty} E_n$.

$\sum_{n=1}^{\infty} \mu_0(E_n) \leq \mu^*(T) + \varepsilon$.

- $T \cap A \subset \bigcup_{n=1}^{\infty} \underbrace{E_n \cap A}_{\in \mathcal{B}}$. $\Rightarrow \mu^*(T \cap A) \leq \sum_{n=1}^{\infty} \mu^*(E_n \cap A) \leq \sum_{n=1}^{\infty} \mu_0(E_n \cap A)$.
- $\mu^*(T \setminus A) = \mu^*(T \cap A^c) \leq \sum_{n=1}^{\infty} \mu_0(E_n \cap A^c)$

$\mu^*(T \setminus A) + \mu^*(T \cap A) \leq \sum_{n=1}^{\infty} \left(\mu_0(E_n \cap A) + \mu_0(E_n \cap A^c)\right) = \sum_{n=1}^{\infty} \mu_0(E_n)$

$\leq \mu^*(T) + \varepsilon$.

(2) By def. only need $\mu^*(A) \geq \mu_0(A)$.

Suppose $A \subset \bigcup_{n=1}^{\infty} E_n$. $E_n \in \mathcal{B}$. $\mu^*(A) \geq \sum_{n=1}^{\infty} \mu_0(E_n) - \varepsilon$. $A \subset \bigcup_{n=1}^{\infty} \widehat{E_n}$

Let $\widehat{E_n} = \left(\bigcup_{i=1}^{n} E_i \setminus \bigcup_{i=1}^{n-1} E_i\right) = E_n \setminus \bigcup_{i=1}^{n-1} E_i$ Then $\widehat{E_n}$ disjoint $= \bigcup_{n=1}^{\infty} E_n$.

$\Rightarrow \mu_0(A) = \mu_0\left(\bigcup_{n=1}^{\infty} (\widehat{E_n} \cap A)\right) = \sum_{n=1}^{\infty} \mu_0(\widehat{E_n} \cap A) \leq \sum_{n=1}^{\infty} \mu_0(\widehat{E_n}) \leq \sum_{n=1}^{\infty} \mu_0(E_n)$

$\leq \mu^*(A) + \varepsilon$.

The extension μ of μ_0 is unique when μ is σ-finite (on $\mathcal{A}$)

bra

:

usure

Lec 18 Metric v.s. measure

1. Metric Outer measure

In R^d, we have open sets, closed sets, compact sets.
They all come from the "metric structure"
the distance between two points.

Def: Let X be a set. A distance function d is a function
$d: X \times X \to [0, +\infty)$
s.t. (1) $d(x,y) = d(y,x)$
(2) $d = 0 \Leftrightarrow x = y$.
(3) $d(x,y) + d(y,z) \geq d(x,z)$ $\forall x, y, z \in X$. (Triangle inequality)
We say (X, d) is a metric space.

Example. $X = R^d$. $d(x,y) = \|x - y\| = \sqrt{\sum_{i=1}^{d} (x_i - y_i)^2}$.

Example. X any set $d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$

Example. $X = L^1(R^d)$. (If $f = g$ a.e. then we regard f and g as the same element in $L^1(R^d)$).
$d(f,g) = \int_{R^d} |f - g| \, dx$.

Given a metric space (X,d), we can define
- Open ball: $B_r(x) = \{y \in X.\ d(x,y) < r\}$.
- Open set U: $\forall x \in U\ \exists r > 0$. s.t: $B_r(x) \subset U$.
- Closed set F: $\Leftrightarrow F^c$ is open.
- Compact set K: $\Leftrightarrow$ ~~K is closed and bounded~~. not hold for example 2.
~~$\forall x_0 \in X.\ \exists M$ s.t: $d(x_0, y) < M, \forall y \in X$.~~

$\Leftrightarrow$ For $K \subset \bigcup_\alpha U_\alpha$. U_α is open. one can find finitely many $U_1, \dots, U_N$ s.t: $K \subset \bigcup_{n=1}^{N} U_n$.

- F_σ: = Countable union for closed sets.
- G_δ: = Countable intersection for open sets
- Borel set: the σ-algebra generated by all open sets (or closed sets ($\mathcal{B}_X$).

Def. Any measure μ on $\mathcal{B}_X$ is called Borel measure.

e.g. $m|_{\mathcal{B}}$ is a Borel measure. ($\Leftarrow \mathcal{L} \supset \mathcal{B}$)
(Lebesgue)

To prove this: • $\mathcal{L}$ is a σ-algebra.
• $\mathcal{L}$ contains all open/closed sets.

We used (Pset 2 – Part 1) $d(A,B)>0 \Rightarrow m^*(A\cup B) = m^*(A) + m^*(B)$

Def: We say an outer measure μ^* on a metric space (X,d) is a metric outer measure if
$d(A,B)>0 \Rightarrow \mu^*(A\cup B) = \mu^*(A) + \mu^*(B)$

RmK: $A, B \subset X$.
$d(A,B) = \inf\{d(x,y): x\in A, y\in B\}$.
This is NOT a distance function on $P(X)$.
because $A\cap B \neq \emptyset \Rightarrow d(A,B)=0$.

Thm || Let μ^* be a metric outer measure on (X,d)
Then all Borel sets are measurable w.r.t. μ^*.

$\rightarrow$ the induced measure $\underset{\mu^*|_{\mathcal{B}}}{\mu}$ is a Borel measure

Proof: $\mu^* \to \mathcal{F}$ σ-algebra of all measurable set.
Want: $\mathcal{B}_X \subset \mathcal{F}$.

So: It's enough to prove: all closed sets $F \in \mathcal{F}$.
i.e: $\mu^*(T) \underset{(?)}{=} \mu^*(T\cap F) + \mu^*(T\setminus F)$.

W.L.O.G. we assume $\mu^*(T) < \infty$.

For each n. let

$T_n = \{x \in T \setminus F : d(x, F) \geq \frac{1}{n}\} \subset T \setminus F$. $\quad d(x,F) = 0.\ x \in F$

Then $T \setminus F = \bigcup_{n=1}^{\infty} T_n$.

Since μ^* is metric outer measure. $\mu^*(T) \geq \mu^*((T \cap F) \cup T_n)$.

$\overset{*}{=} \mu^*(T \cap F) + \mu^*(T_n)$.

Fact: $\lim_{n \to \infty} \mu^*(T_n) = \mu^*(T \setminus F)$.

$\Rightarrow \mu^*(T) \geq \mu^*(T \cap F) + \mu^*(T \setminus F)$.

Prop: Suppose μ^* (μ) is a metric outer measure. $\boxed{\text{(Borel measure)}}$

Suppose $\mu(B_x(r)) < \infty$. $\forall x \in X.\ \forall r \in \mathbb{R}$.

(Then $\forall$ Borel set A. $\forall \varepsilon > 0$. $\exists$ open $U \supset A$. closed $F \subset A$.

s.t: $\mu(U \setminus F) < \varepsilon$. $\Rightarrow \mu(U \setminus A) < \varepsilon$. $\mu(A \setminus F) < \varepsilon$) Regularity

2. Hausdorff measure

$\mathbb{R}^d$, $\|x - y\|$.

Given any $A \subset \mathbb{R}^d$ bounded. we define the diameter

$\mathrm{diam}(A) = \sup\{d(x,y).\ x, y \in A\}$.

Def: For any $\alpha \geq 0$. $\delta > 0$. define

$\downarrow$ by δ $(\uparrow)$

$h^*_{\alpha,\delta}(A) = \inf\{\sum_{k=1}^{\infty} (\mathrm{diam}\, A_k)^\alpha : A \subset \bigcup_{k=1}^{\infty} A_k,\ \mathrm{diam}(A_k) < \delta\}$.

and let

$h^*_\alpha(A) = \lim_{\delta \to 0} h^*_{\alpha,\delta}(A)$

Prop: h^*_α is a metric ② outer measure ① on $\mathbb{R}^d$

$\rightarrow$ We get a Borel measure $\mathcal{H}^\alpha(A) = h^*_\alpha(A)$ $\quad A \in \mathcal{B}$.

α- dimensional Hausdorff measure.

proof: $\cdot\ h_\alpha^*(\phi)=0$

$\cdot\ A_1 \subset A_2 \Rightarrow h_\alpha^*(A_1) \le h_\alpha^*(A_2)$.

$\cdot\ A_1, A_2, \cdots \Rightarrow h_\alpha^*(\bigcup_{n=1}^\infty A_n) \le \sum_{n=1}^\infty h_\alpha^*(A)$.

it's enough to prove $h_{\alpha,\delta}^*(\bigcup_{n=1}^\infty A_n) \le \sum_{n=1}^\infty h_{\alpha,\delta}^*(A_n)$. $(\forall \delta)$.

Take $F_{j,k}$ with $\operatorname{diam}(F_{j,k}) < \delta$.

$A_j \subset \bigcup_{k=1}^\infty F_{j,k}$.

$\sum_{k=1}^\infty \operatorname{diam}(F_{jk})^\alpha \le h_{\alpha,\delta}^*(A_j) + \frac{\varepsilon}{2^j}$.

$\Rightarrow \bigcup_{j=1}^\infty A_j \subset \bigcup_{j,k} F_{j,k}$.

$\Rightarrow h_{\alpha,\delta}^*(\bigcup_{k=1}^\infty A_j) \le \sum_{j=1}^\infty h_{\alpha,\delta}^*(A_j) + \varepsilon$. let $\varepsilon \to 0$. $\delta \to 0$.

Want: $h_\alpha^*(A_1 \cup A_2) = h_\alpha^*(A_1) + h_\alpha^*(A_2)$
$(\ge)$.

$\cdot$ Now let $d(A_1, A_2) > 0$.

Take $\delta < d(A_1, A_2)$

Let $F_1, F_2, \cdots$ $\operatorname{diam}(F_i) < \delta$. $A_1 \cup A_2 \subset \bigcup_{i=1}^\infty F_i$

Let $F_1', F_2' \cdots$ be those that intersects A_1.

$F_1'', F_2'' \cdots$ ——— A_2.

Then $\{F_1', F_2' \cdots, F_1'', F_2'' \cdots\} \subset \{F_1, F_2 \cdots\}$.

$\Rightarrow \sum_j \operatorname{diam}(F_j')^\alpha + \sum_i \operatorname{diam}(F_i'')^\alpha \le \sum_k (\operatorname{diam} F_k)^\alpha$.

$\Rightarrow h_\alpha^*(A_1 \cup A_2) \ge h_\alpha^*(A_1) + h_\alpha^*(A_2)$

Therefore $h_\alpha^*(A_1 \cup A_2) = h_\alpha^*(A_1) + h_\alpha^*(A_2)$

Prop: (1) If $\mathcal{H}^\alpha(A) < \infty$. then $\forall \beta > \alpha$. $\mathcal{H}^\beta(A) = 0$

(2) If $\mathcal{H}^\beta(A) > 0$. then $\forall \alpha < \beta$, $\mathcal{H}^\alpha(A) = +\infty$.

Proof: Suppose $\alpha < \beta$. $\operatorname{diam} F < \delta$.

$\Rightarrow (\operatorname{diam} F)^\beta = (\operatorname{diam} F)^{\beta-\alpha} (\operatorname{diam} F)^\alpha < \underset{\delta \to 0}{\delta^{\beta-\alpha}} (\operatorname{diam} F)^\alpha$.

Def: For any Borel set A, the Hausdorff dimension of A.

is $\dim_H A = \sup\{\beta : \mathcal{H}^\beta(A) = \infty\}$.

$= \inf\{\alpha : \mathcal{H}^\alpha(A) = 0\}$.

Lec 19 Riesz representation theorem

· Recall Lebesgue outer measure satisfies

· $\mu(A) = \inf\{\mu(U) : U \supset A.\ U \text{ open}\}$. (i)

→ outer regularity

· $\mu(A) = \sup\{\mu(K) : K \subset A\ \ K \text{ compact}\}$ (ii)

Regular Measure: (for Borel Measure).

Def. Let: $\mu : \mathcal{B}_X \to [0, +\infty]$ be a Borel measure.

We call μ is ① outer regular if it satisfies (i) for all $A \in \mathcal{B}_X$

② inner regular - - - - - - - (ii) - - - - - -

③ regular if it is both outer and inner regular

About Compact Sets:

· Now Recall: A set K is compact if for any open covering $\{U_\alpha\}$ of K. (i.e. each U_α is open and $K \subset \bigcup_\alpha U_\alpha$) there exists a finite subcovering $\{U_1, \dots, U_N\}$

(i.e. $U_i \in \{U_\alpha\}$ $1 \le i \le N$. and $K \subset \bigcup_{i=1}^{N} U_i$)

Example. Let $K \subset \mathbb{R}^d$ be a bounded closed set

Then K is compact

e.g. $[0,1]$.

Fact 1 If K is compact, $F \subset K$ is closed, then F is compact

Proof. Take $\{U_\alpha\}$ an open covering of F.

Consider $\{U_\alpha, F^c\}$ is an open covering of K.

$\Rightarrow \exists \{U_1, \dots, U_N, F^c\}$ covers K $\underset{F \subset K}{\Rightarrow}$ covers F.

$\Rightarrow \{U_1, \dots, U_N\}$ covers F.

Fact 2 If f is a continuous function, K is compact, then $\exists$ $x_0, y_0 \in K$ s.t. $f(x_0) = \underset{K}{\inf}\{f(x) : x \in K\}$ (min), $f(y_0) = \underset{K}{\sup}\{f(x), x \in K\}$ (max).

Proof Suppose NOT. $\Rightarrow \exists \{x_n\} \subset K$. $f(x_1) > f(x_2) > \cdots \searrow \inf\{f(x)\}$ & $f(K)$.
Let $U_k = \{x : f(x) > f(x_k)\}$. Then $\{U_k\}$ cover K.
And $U_1 \subsetneq U_2 \subsetneq \cdots$
$\Rightarrow$ No finite Subcovering

Prop: Let (X,d) be a compact metric space.
Let $\mu: B_X \to [0,+\infty]$ be a (Borel measure) metric outer measure.
Suppose $\mu(X) < +\infty$.
Then μ is regular

proof. By Last Problem set, problem 1 (P Set 9 Part 2).
- $\forall \varepsilon > 0$. $\exists U$ s.t: $\mu(U \setminus A) < \varepsilon \Rightarrow \mu(U) < \mu(A) + \varepsilon$ for $\forall A \subset B_X$.
- $\exists F$ closed., $F \subset X$ compact $\Rightarrow F$ compact s.t: $\mu(A \setminus F) < \varepsilon$. $\mu(A) < \mu(F) + \varepsilon$.

Thm (Partition of unity Simplest version).
Let (X,d) be a $\boxed{\text{compact}}$ (metric space). Let $U_1, \dots, U_N$ be open sets in X. $K \subset U_1 \cup \cdots \cup U_N$ is a compact set, Then $\exists$ continuous functions $f_1, \dots, f_N$ on X s.t.

1. $f_i(x) \geq 0$. $\forall i\ \forall x$.
2. $\sum_{i=1}^{N} f_i(x) \leq 1$. $\forall x \in X$.
3. $\text{supp}(f_i) \subset U_i$ $\quad (\text{Supp} f) = \overline{\{x : f(x) \neq 0\}}$). Compact in condition of thm. closed and $\subset X$ (compact)
4. $\sum_{i=1}^{N} f_i(x) = 1$. $\forall x \in K$.

Proof: [proof of Lemma
Write $U = \bigcup_{i=1}^{N} U_i$ open $\Rightarrow U^c$ is closed thus compact
$\Rightarrow d(K, U^c) = \delta > 0$.

$d(K, U^c) = \inf\{d(x,y) : x \in K, y \in U^c\}$
$= \inf\{d(x, U^c) : x \in K\}$.

Lemma: X compact metric space. $U \subset X$ is open. $K \subset U$ compact. $\exists V$ open s.t: $K \subset V \subset \overline{V} \subset U$.

Consider $g(x) = d(x, U^c)$ Then g is continuous on K.

Let $\mathcal{V} = \{V : V \text{ open } \exists i \text{ s.t. } \bar{V} \subset U_i\}$.

By lemmaA $\mathcal{V}$ is an open covering of $K \Rightarrow K = V_1 \cup \cdots \cup V_l$.

Let $K_i = \bigcup_{\bar{V}_j \subset U_i} \bar{V}_j$. Then $K_i \subset U_i$ compact and $\bigcup_{i=1} K_i \supset K$.

$\Rightarrow d(K, U^c) = \inf\{g(x) : x \in K\} = \underline{d(x_0, U^c) > 0}$ (only need U^c is closed)

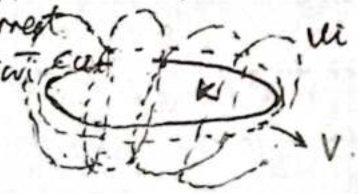
Let $V = B(K, \frac{\delta}{2}) = \bigcup_{x \in K} B(x, \frac{\delta}{2})$

Then V is open, $K \subset V \subset \bar{V} \subset U$]

[Let $V_i = U_i \cap V$. Then V_i open. $K \subset V_1 \cup \cdots \cup V_n$.]

↑ In order to ensure ③ ↑ But not correct

- K_i, $K_i \subset W_i \subset \bar{W_i} \subset U_i$, $\exists W_i$ open



Let $\boxed{f_i(x) = \dfrac{d(x, W_i^c)}{d(x,K) + \sum_{i=1}^{n} d(x, W_i^c)}}$ $\neq 0 \ (>0)$

$\sum f_i(x) = \dfrac{\sum d(x, V_i^c)}{d(x,K) + \sum d(x, V_i^c)} \leq 1$.

$x \in K \Rightarrow d(x,K) = 0 \Rightarrow \sum f_i = 1$.

for ③: $f_i(x) = 0$ if $x \in V_i^c$

$\Rightarrow \operatorname{Supp}(f_i) \subset \bar{V_i} \subset U_i$.

Now suppose (X, d) is a compact metric space.

$\mu : \mathcal{B}_X \to [0, +\infty]$ is a Borel measure, $\boxed{\mu(X) < +\infty}$

For any $\underline{f \in C(X)} \Rightarrow f$ attains max/min values

$\Rightarrow \exists M$ s.t. $|f(x)| < M$.

Define $\forall f \in C(X)$.

$I_\mu(f) = \int_X f \, d\mu \in \mathbb{R}$.

We have (1) $I_\mu(c_1 f_1 + c_2 f_2) = c_1 I_\mu(f_1) + c_2 I_\mu(f_2)$.

(2) If $f \geq 0$. Then $I_\mu(f) \geq 0$.

Where $I_\mu : C(X) \to \mathbb{R}$.

Def: We call $\ell : C(X) \to \mathbb{R}$ is a <u>linear functional</u>

if $\ell(c_1 f_1 + c_2 f_2) = c_1 \ell(f_1) + c_2 \ell(f_2)$

↓ function of function

We say <u>linear functional ℓ is positive</u> if

$f \geq 0 \Rightarrow \ell(f) \geq 0$.

Thm (Riesz representation thm) (Simplest version)

Let (X, d) be a ~~compact~~ locally compact, σ compact metric space. and ℓ be a positive linear functional on $C(X)$. Then $\exists$ a unique Borel measure regular locally finite μ s.t. $\ell(f) = \int_X f \, d\mu \quad \forall f \in C(X)$.

Moreover, $\mu(X) < +\infty$.

Rmk: we have seen that such μ must be a regular Borel measure.

Idea:
- we need a metric outer measure
- $\Uparrow$ we need an outer measure
- $\Uparrow$ we need $\mu(U)$. U open

$\boxed{\text{Suppose we get } \mu:}$ $\mu(U) = \int_X \chi_U \, d\mu \neq \ell(\chi_U)$. not continuous

But we can find $f_n \nearrow \chi_U$. $\quad f_n(x) = \begin{cases} 0 & x \in U^c \\ n\,d(x, U^c) & x \in U \ d(x,U^c) < \frac{1}{n} \\ 1 & x \in U. \ d(x, U^c) \geq \frac{1}{n}. \end{cases}$

Then $0 \leq f_n \leq 1$. $\operatorname{supp} f_n \subset U$.

$\boxed{\text{Then}}$ $\mu(U) = \int_X \chi_U \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu = \lim_{n\to\infty} \ell(f_n)$

Proof. Step 1 (Uniqueness). We have seen that if μ_1, μ_2 are two Borel measure satisfy the theorem, then for any open set U. $\mu_1(U) = \lim_{n\to\infty} \ell(f_n) = \mu_2(U)$.

By outer regularity

$$\mu_1(A) = \mu_2(A) \quad \forall A \in \mathcal{B}_X.$$

Step 2 For any open $U \subset X$. let

$$\mu^*(U) = \sup\{\ell(f) : 0 \leq f \leq 1, \ \operatorname{supp}(f) \subset U\}.$$

and for any $A \subset X$. let

$$\mu^*(A) = \inf\{\mu^*(U). \ U \supset A \ U \text{ open}\}.$$

Claim: μ^* is an outer measure

- $\mu^*(\phi) = 0$ ✓
- $A_1 \subset A_2 \Rightarrow \mu^*(A_1) \leq \mu^*(A_2)$ ✓
- $\mu^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$ ...

First: If U, V open then $\mu^*(U \cup V) \le \mu^*(U) + \mu^*(V)$.

For any f: $0 \le f \le 1$. $K = \text{supp}(f) \subset U \cup V$.

By P.O.U. $\exists \varphi, \psi$ s.t: $\varphi + \psi = 1$ on K.

$0 \le \varphi, \psi \le 1$. $\text{supp}(\varphi) \subset U$. $\text{supp}(\psi) \subset V$.

$\Rightarrow f = (\varphi + \psi) f = \varphi f + \psi f$ $\quad$ $0 \le \varphi f \le 1$ $\text{Supp}(\varphi f) \subset U$

$\Rightarrow l(f) = l(\varphi f + \psi f)$ $\quad$ $0 \le \psi f \le 1$ $\text{Supp}(\psi f) \subset V$.

$\le \mu^*(U) + \mu^*(V)$

$\Rightarrow \mu^*(U \cup V) \le \mu^*(U) + \mu^*(V)$.

Then $\rightarrow$ countable additivity for $\{A_n\}$ (WLOG. $\forall A_n$ $\mu^*(A_n) < +\infty$).

Take open $U_n \supset A_n$ s.t: $\mu^*(U_n) \le \mu^*(A_n) + \frac{\varepsilon}{2^i}$

Let $U = \bigcup_{n=1}^{\infty} U_n$. $\forall f \in C(X)$. $0 \le f \le 1$. $\text{supp}(f) = K \subset U$.

By compactness $\exists U_1, \cdots, U_N$ s.t: $K \subset \bigcup_{i=1}^{N} U_i$.

$\Rightarrow l(f) \le \mu^*(\bigcup_{n=1}^{N} U_n) \le \sum_{n=1}^{N} \mu^*(U_n) \le \sum_{n=1}^{N} \mu^*(A_n) + \varepsilon$.

$\le \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon$.

$\Rightarrow \mu^*(\bigcup_{n=1}^{\infty} A_n) \le \mu^*(\bigcup_{n=1}^{\infty} U_n) \le \sup l(f) \le \sum_{n=1}^{\infty} \mu^*(A_n)$.

Step 3. μ^* is metric outer measure.

Assume $d(A,B) > 0$. WANT: $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.

Take $U \supset A \cup B$ s.t: $\mu^*(U) \le \mu^*(A \cup B) + \varepsilon$.

Let $U_1 = U \cap B(A, \frac{r}{3})$ $U_2 = U \cap B(B, \frac{r}{3})$.

$\Rightarrow U_1, U_2$ open $U \supset U_1 \cup U_2 \supset A \cup B$. $U_1 \cap U_2 = \phi$ $d(U_1, U_2) > \frac{r}{3}$.

Now suppose $f_i \in C(X)$ $0 \le f_i \le 1$. $\text{supp}(f_i) \subset U_i$.

$l(f_i) \ge \mu^*(U_i) - \varepsilon$.

Let $f = f_1 + f_2$ Then $0 \le f \le 1$. $\text{supp}(f) \subset U_1 \cup U_2$.

$\Rightarrow \mu^*(A \cup B) \ge \mu^*(U) - \varepsilon \ge \mu^*(U_1 \cup U_2) - \varepsilon$

$\ge l(f) - \varepsilon = l(f_1) + l(f_2) - \varepsilon$

$\ge \mu^*(U_1) + \mu^*(U_2) - 3\varepsilon$.

$\ge \mu^*(A) + \mu^*(B) - 3\varepsilon$

$\Rightarrow \mu^*$ is metric outer measure $\Rightarrow \mu^*|_{\mathcal{B}}$ is a Borel measure.

Step 4 X is open $\forall 0 \le f \le 1$. $L(1-f) \ge 0 \Rightarrow l(1) \ge l(f)$.

$\mu^*(X) \le l(1) < +\infty$.

$(l = I_\mu)$

Step 5 $l(f) = \int_X f \, d\mu$. the proof works for more general version of Riesz rep th

$l(f) \le \int_X f \, d\mu$. $\forall f$ ($l(-f) \le \int_X (-f) d\mu \Rightarrow l(f) \ge \int_X f d\mu$).

$f \in C(X) \Rightarrow a_0 = \min f$. $b_0 = \max f$. on $\boxed{K = supp(f)}$.

Take $a = y_0 < y_1 < y_2 \cdots < y_{n-1} < y_n = b$ s.t: $y_i - y_{i-1} < \varepsilon$

Let $E_i = \{x: y_i < f(x) \le y_{i+1}\} \in \mathcal{B}_X$. $E_i \cap E_j = \phi$.

$K = \bigcup_{i=1}^n E_i$.

$\exists \widetilde{U}_1, \dots, \widetilde{U}_n$ open s.t: $\widetilde{U}_i \supset E_i$. $\mu(\widetilde{U}_i) < \mu(E_i) + \frac{\varepsilon}{n}$

Let $U_i = \widetilde{U}_i \cap f^{-1}(-\infty, y_i + \varepsilon)$. Then U_i open. $E_i \subset U_i$, $\mu(U_i) < \mu(E_i) + \frac{\varepsilon}{n}$, $\boxed{\sup_{U_i} \le y_i}$

(partition)

$\bigcup_{i=1}^n U_i = K \Rightarrow \exists \phi_1, \dots, \phi_n$ s.t: $\phi_i \ge 0$. $supp(\phi_i) \subset U_i$

$\sum_{i=1}^n \phi_i = 1$ on $K \Rightarrow f = f \cdot \sum_{i=1}^n \phi_i = \sum_{i=1}^n f \cdot \phi_i$.

and $\phi_i f \le \phi_i (y_i + \varepsilon)$ $\forall i = 1, \dots, n$.

Note: $\sum_{i=1}^n \phi_i = 1$ on $K \Rightarrow \mu(K) \le l(\sum_{i=1}^n \phi_i) = \sum_{i=1}^n l(\phi_i)$

This implies

$$
\begin{aligned}
l(f) &= \sum_{i=1}^n l(\phi_i f) \le \sum_{i=1}^n (y_i + \varepsilon) l(\phi_i). \\
&= \sum_{i=1}^n (y_i + |a| + \varepsilon) l(\phi_i) - |a| \sum_{i=1}^n l(\phi_i) \\
&\le \sum_{i=1}^n (y_i + |a| + \varepsilon) \mu(U_i) - |a| \mu(K) \\
&\le \sum_{i=1}^n (y_i + |a| + \varepsilon)(\mu(E_i) + \tfrac{\varepsilon}{n}) - |a| \mu(K) \qquad \mu(K) = \sum_{i=1}^n \mu(E_i) \\
&= \sum_{i=1}^n (y_i + \varepsilon) \mu(E_i) + (y_i + |a| + \varepsilon) \frac{\varepsilon}{n}. \\
&\le \sum_{i=1}^n (y_i - \varepsilon) \mu(E_i) + \varepsilon (2\mu(K) + b + |a| + \varepsilon) \\
&\le \int_X f \, d\mu + \varepsilon (2\mu(K) + b + |a| + \varepsilon).
\end{aligned}
$$

Let $\varepsilon \to 0$. $l(f) \le \int_X f \, d\mu = I_\mu(f)$

$\mathcal{L}^1$ space

Lec 20

1. Spaces and Structures

· Spaces $\Rightarrow$ { · a set contains objects that we want to study
  · there are "structures" that describes relations among those objects

· We have learned the following structures on "Space V" (elements $v, w, \ldots$)

(1) Linear structure: $\left.\begin{array}{l} v \to av \\ v, w \to v+w \end{array}\right\} \to$ linear space (vector space). e.g. $\mathbb{R}^d$, $\mathcal{L}^1(\mathbb{R}^d)$, $C_c(X)$

(2) Multiplication Structure

$v, w \to v \cdot w \in V$ (x. convolution) ...

(innerproduct is NOT a multiplication)

e.g. $\mathbb{R}$. $\mathbb{R}^3$. $(\vec{v}, \vec{w} \rightsquigarrow \vec{v} \times \vec{w})$

$C_c(X)$.

(3) Norm structure. A function $\|\cdot\|: V \to \mathbb{R}$ s.t. (vector space (Require))

$$\begin{cases} (1) \ \|0\| = 0 \quad \|v\| > 0 \text{ for } v \neq 0. \\ (2) \ \|av\| = |a| \|v\| \\ (3) \ \|v+w\| \leq \|v\| + \|w\| \end{cases}$$

V	$\|x\|$
e.g. $\mathbb{R}^d$,	$\sqrt{x_1^2 + \cdots + x_d^2}$
$\mathbb{R}^d$,	$\|x_1\| + \cdots + \|x_d\|$
$\mathbb{R}^d$,	$\sup(\|x_1\|, \ldots, \|x_n\|)$
$\mathcal{L}^1(\mathbb{R}^d)$.	$\int_{\mathbb{R}^d} \|f\| dx$.

(4) Metric Structure. A function $d: X \times X \to \mathbb{R}$. s.t.

(Not require X to be a vector space)

$$\begin{cases} (1) \ d(x,x) = 0. \quad d(x,y) > 0 \text{ for } x \neq y \\ (2) \ d(x,y) = d(y,x) \\ (3) \ d(x,z) \leq d(x,y) + d(y,z) \end{cases}$$

Note. Norm structure induces a metric structure by $d(v,w) = \|v-w\|$.

(5) Topological Structure "define open sets". $\mathcal{T} \subset P(X)$.

- $\phi \in \mathcal{F}$.
- $x_1, \cdots, x_n \in \mathcal{F} \Rightarrow \bigcap_{i=1}^{n} x_n \in \mathcal{F}$. (finite)
- $x_\alpha \in \mathcal{F} \Rightarrow \bigcup_\alpha x_\alpha \in \mathcal{F}$.

$\rightarrow$ closed set
Compact set (by covering)
Convergence
Continuous function

Note: For any metric space, one can study the "metric topology" i.e. the topology generated by <u>open balls</u>

For space of functions, one of the very important aspect in topology is "how functions converges".

(6) <u>Inner product</u> (on <u>Vector space</u>) $\langle \cdot, \cdot \rangle : V \times V \to R$ (or C).

(1) $\langle av_1 + bv_2, w \rangle = a\langle v_1, w\rangle + b\langle v_2, w\rangle$.

(2) $\langle v, w \rangle = \langle w, v \rangle$

(3) $\langle 0, 0 \rangle = 0$. $\langle v, v \rangle > 0$. $\forall v \neq 0$.

Note: $\cdot \langle \cdot, \cdot \rangle \rightarrow \|v\| = \sqrt{\langle v, v \rangle}$

- we have "angles": $\angle(v,w) = \arccos \frac{\langle v, w \rangle}{\|v\| \cdot \|w\|}$

(7) Measure Structure.

(8) Order Structure

(9) Symmetry

(10) Duality

Given V. find another V^* s.t. $\forall l \in V^* \ \forall v \in V$.

"pairing" $\langle\langle l, v \rangle\rangle \in R$. function.

e.g. · inner product

- measure $\leftrightarrow$ function. $\langle\langle \mu, f \rangle\rangle := \int_X f \, d\mu$
- linear functional $\leftrightarrow$ function. $\langle\langle l, f \rangle\rangle = l(f)$

e.g. Riesz rep. thm

2. $\mathcal{L}^p$ Spaces

- Let $X \subset \mathbb{R}^d$ measurable, or more generally $(X, \mathcal{F}, \mu)$ be a measure space.

Def. (1) For any measurable function f, the $\boxed{\mathcal{L}^p\text{-norm}}$ of f is $\|f\|_{\mathcal{L}^p(X)} := \left(\int_X |f|^p dx\right)^{1/p}$

(2) The $\boxed{\text{space of } \mathcal{L}^p \text{ functions}}$ $= \mathcal{L}^p(X) = \{f : f \text{ is measurable } \|f\|_{\mathcal{L}^p} < +\infty\}$.

Example. $\mathcal{L}^p(\mathbb{R}^d)$

Example: $X = \mathbb{N}$. $\mathcal{F} = P(X)$ $\mu = \#$. : counting measure

Then
- a function $\longleftrightarrow$ a sequence $(a) = (a_1, a_2, \cdots)$
- Any function is measurable.
- $\mathcal{L}^p$-norm: $\|a\|_{\mathcal{L}^p} = \left(\int_N |a|^p d\#\right)^{1/p} = \left(\sum_{i=1}^{\infty} |a_i|^p\right)^{1/p}$

$\mathcal{L}^p$-Space $\ell^p = \{(a) : (\sum |a_i|^p)^{1/p} < \infty\}$

Rmk: We always regard a.e. equal functions as the same function. In other words $\mathcal{L}^p$-space is
$\mathcal{L}^p(X) = \{f : f \text{ measurable}, \|f\|_{\mathcal{L}^p} < \infty\} / \sim$.
$\sim$: $f_1 \sim f_2 \Leftrightarrow f_1 = f_2$ a.e.

$\mathcal{L}^p$-Vector Space

Prop. 1) For each $0 < p < +\infty$. $\mathcal{L}^p(X)$ is a Vector Space

Proof: $\|f\|_{\mathcal{L}^p} < \infty \Rightarrow \|af\|_{\mathcal{L}^p} = \left(\int_X |af|^p d\mu\right)^{1/p} = |a| \|f\|_{\mathcal{L}^p} < +\infty$.

i.e.: $f \in \mathcal{L}^p \Rightarrow a \cdot f \in \mathcal{L}^p$.

- We need $f, g \in \mathcal{L}^p \Rightarrow f + g \in \mathcal{L}^p$.

$|f+g|^p \le (2\max(|f|, |g|))^p \le 2^p(|f|^p + |g|^p)$

$\Rightarrow \|f+g\|_{\mathcal{L}^p}^p \le 2^p(\|f\|_{\mathcal{L}^p}^p + \|g\|_{\mathcal{L}^p}^p) < +\infty$

$\|f+g\|_{\mathcal{L}^p} < +\infty$.

$\mathcal{L}^p$ $(p\geq 1)$ — normed space

OURSTORYBEGINS

$\boxed{p\geq 1}$

Thm: The $\mathcal{L}^p$-norm $\|\cdot\|_{\mathcal{L}^p}$ is a norm on $\mathcal{L}^p(X)$.

i.e: (1) $\|0\|_{\mathcal{L}^p}=0$. $\|f\|_{\mathcal{L}^p}>0$ for $\forall f\neq 0$

(2) $\|af\|_{\mathcal{L}^p}=|a|\,\|f\|_{\mathcal{L}^p}$

(3) $\|f+g\|_{\mathcal{L}^p}\leq\|f\|_{\mathcal{L}^p}+\|g\|_{\mathcal{L}^p}$ (Minkovski ineq.) | Triangle inequ | $\|\cdot\|_{\mathcal{L}^p}$ is a conv function on $\mathcal{L}^p$

→ Metric structure → topological structure → Convergence in $\mathcal{L}^p$-norm

Lemma: $\forall s>0,\ p>0$. $\|f^s\|_{\mathcal{L}^p}=\|f\|^s_{\mathcal{L}^{ps}}$

Proof: $RHS=\left(\int_X|f|^{ps}d\mu\right)^{\frac{s}{ps}}=\left(\int_X|f^s|^p d\mu\right)^{\frac{1}{p}}=\|f^s\|_{\mathcal{L}^p}=LHS$.

Thm (Hölder's inequality) || Suppose $p,q>0$ $(p>r)$ $r>0$. $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$.
Suppose $f\in\mathcal{L}^p$. $g\in\mathcal{L}^q$
Then $fg\in\mathcal{L}^r$ and $\|fg\|_{\mathcal{L}^r}\leq\|f\|_{\mathcal{L}^p}\|g\|_{\mathcal{L}^q}$

Proof: WLOG. assume $f,g\geq 0$, $A=\|f\|_{\mathcal{L}^p}>0$. $B=\|g\|_{\mathcal{L}^q}>0$.

First assume: $r=1$. $\left(\frac{1}{p}+\frac{1}{q}=1\right)$ $\quad \underbrace{\frac{1}{p}}_{\in(0,1)}+\underbrace{\frac{1}{q}}_{\in(0,1)}=1.\ p>1$ ☆

need: (Young's inequality) $a,b\geq 0\Rightarrow ab\leq\frac{1}{p}a^p+\frac{1}{q}b^q$

$$\Rightarrow\frac{\|f\cdot g\|_{\mathcal{L}^1}}{AB}=\int_X\frac{f}{A}\cdot\frac{g}{B}\,d\mu\leq\int_X\frac{1}{p}\left(\frac{f}{A}\right)^p+\frac{1}{q}\left(\frac{g}{B}\right)^q d\mu$$

$$=\frac{1}{pA^p}\underbrace{\int_X f^p d\mu}_{A^p}+\frac{1}{qB^q}\underbrace{\int_X g^q d\mu}_{B^q}=\frac{1}{p}+\frac{1}{q}=1.$$

For general r. $\|fg\|_{\mathcal{L}^r}=\|(fg)^{r\frac{1}{r}}\|_{\mathcal{L}^r}=\|f^rg^r\|^{\frac{1}{r}}_{\mathcal{L}^1}$

$$\leq\|f^r\|^{\frac{1}{r}}_{\mathcal{L}^{p/r}}\ \|g^r\|^{1/r}_{\mathcal{L}^{q/r}}=\|f\|_{\mathcal{L}^p}\cdot\|g\|_{\mathcal{L}^q}.\quad \frac{p}{r}>1,\ \frac{r}{p}+\frac{r}{q}=1$$

Proof: Note $p=1$ ✓. WLOG $f,g\geq 0$.

$p>1$, Let q s.t: $\frac{1}{p}+\frac{1}{q}=1$. $\Rightarrow p+q=pq\Rightarrow p=q(p-1)$

$$\|f+g\|^p_{\mathcal{L}^p}=\int_X f(f+g)^{p-1}d\mu+\int_X g(f+g)^{p-1}d\mu.$$

$$\leq\|f\|_{\mathcal{L}^p}\cdot\|(f+g)^{p-1}\|_{\mathcal{L}^q}+\|g\|_{\mathcal{L}^p}\cdot\|(f+g)^{p-1}\|_{\mathcal{L}^q}.$$

$$=\left(\|f\|_{\mathcal{L}^p}+\|g\|_{\mathcal{L}^p}\right)\|(f+g)\|^{p-1}_{\mathcal{L}^{q(p-1)}}$$

$$=\left(\|f\|_{\mathcal{L}^p}+\|g\|_{\mathcal{L}^p}\right)\|f+g\|^{p-1}_{\mathcal{L}^p}.$$

L^p normed vector space $\rightarrow$ metric space.

Lec 21

- Recall. we have many space $L^p(X)$.

  Those spaces are not non-intersecting

  In last pset If $\boxed{\mu(X)<\infty}$ then

  $p_1 < p_2 \Rightarrow L^{p_2}(X) \subset L^{p_1}(X) \leftarrow \|\cdot\|_{L^{p_1}}$ is a norm on $L^{p_2}(X)$.

  $\|f\|_{L^p} \longrightarrow \|f\|_{L^{p_0}}$ for $p \to p_0^-$ (Q: what if $p \to p_0^+$?)

  $\exists$ example s.t. $\|f\|_{L^p} \not\to \|f\|_{L}$

1. Q: what happens if $p \to +\infty$?

   Want: $\|f\|_{L^\infty} = \lim\limits_{p\to\infty} \|f\|_{L^p}$

   e.g: Take $X=[0,4]$ $f = \begin{cases} 8 & x=0 \\ 4 & 0<x\le 1 \\ -2 & 1<x\le 3 \\ 1 & 3<x\le 4 \end{cases}$

   $\|f\|_{L^p} = \left(\int_{[0,4]} |f|^p dx\right)^{1/p} = (4^p + 2\cdot 2^p + 1\cdot 1^p)^{1/p} \xrightarrow[p\to\infty]{} 4$

   The maximal value of f ignoring the value of f on a null set

$p=+\infty$? L^∞ space

$\boxed{\text{Def:}}$ Let $(X, \mathcal{F}, \mu)$ be a measure space, f is a measurable function

(1) We say f is essentially bounded if $\exists M$ s.t. $|f(x)| \le M$ a.e $x\in X$.

(2) The L^∞-norm of f is $\|f\|_{L^\infty} = \inf\{M : |f(x)| \le M \text{ a.e } x \in X\}$.
$\sup|f(x)|$

(3) The L^∞-space

$L^\infty(X) = \{f : \underbrace{\|f\|_{L^\infty} < +\infty}_{\text{essentially bounded}}\}$.

Thm. $L^\infty(X)$ is a normed vector space and $\|\cdot\|_{L^\infty}$ is a norm.

Proof.
- $a\in\mathbb{R}$. $f\in L^\infty(X) \Rightarrow \exists M$ s.t. $|f| \le M$ a.e.

  $\Rightarrow |af| \le |a| M$ a.e. $\quad \|af\|_{L^\infty} < +\infty$. $\quad \|af\|_{L^\infty} = |a|\cdot\|f\|_{L^\infty}$

- $f, g \in L^\infty(X) \Rightarrow \exists M_1, M_2$ s.t. $|f| \le M_1$ a.e. $|g| \le M_2$ a.e.

  $\Rightarrow |f+g| \le M_1 + M_2$ a.e. $\|f+g\|_{L^\infty} < +\infty$.

  $\|f+g\|_{L^\infty} \le \|f\|_{L^\infty} + \|g\|_{L^\infty}$

- $\|f\|_{L^\infty} \ge 0$. $\|f\|_{L^\infty} = 0 \Rightarrow f \overset{a.e.}{=} 0$

Minkowski For $1 \le p \le \infty$. $\|f+g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}$.

Hölder For $0 < p, q, r \le \infty$ ($p>1, q>1$), $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ $\Rightarrow$ $\|fg\|_{L^r} \le \|f\|_{L^p} \cdot \|g\|_{L^q}$.

Proof: enough to prove $p=\infty$ $q=r$ case.

$\|f \cdot g\|_{L^r} = (\int_X |f g|^r d\mu)^{1/r} \le \|f\|_{L^\infty} (\int_X |g|^r d\mu)^{1/r} = \|f\|_{L^\infty} \|g\|_{L^r}$

Rmk: One can easily down Minkowski and Hölder's inequality for more than 2 functions.

- $1 \le p \le \infty$. $\|f_1 + \cdots + f_n\|_{L^p} \le \|f_1\|_{L^p} + \cdots + \|f_n\|_{L^p}$.
- $0 < p_1, \cdots, p_n, r \le \infty$, $\underbrace{\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n}}_{\frac{1}{r_2}} = \frac{1}{r}$ $\Rightarrow$ $\|f_1 \cdots f_n\|_{L^r} \le \|f_1\|_{L^{p_1}} \cdots \|f_n\|_{L^{p_n}}$

Rmk: For $X = \mathbb{N}$. $\mathcal{F} = P(X)$ $\mu = \#$. $\rightarrow L^\infty(X, \mu) = \ell^\infty$.

$\ell^\infty = \{(a_1, a_2, \cdots) : \sup_{n \ge 1} |a_n| < \infty\}$ = the space of all bounded seqs.

$\|(a_1, a_2, \cdots)\|_{\ell^\infty} := \sup_{n \ge 1} |a_n|$

X. L^p — metric Space

2. L^p – Space as metric Space

- For $1 \le p \le \infty$. define L^p – metric on $L^p(X, \mu)$

$d_{L^p(X)}(f, g) := \|f - g\|_{L^p}$.

Then (1) $d_{L^p(X)}(f, g) \ge 0$ "$=0$" $\Leftrightarrow f = g$ a.e. $\Leftrightarrow f = g$ in $L^p(X)$.

(2) $d_{L^p(X)}(f, g) = d_{L^p(X)}(g, f)$

(3) $d_{L^p(X)}(f, h) \le d_{L^p(X)}(f, g) + d_{L^p(X)}(g, h)$ $\forall f, g, h \in L^p(X)$.

So $(L^p(X), d_{L^p(X)})$ is a metric space.

Def: We say f_n converges to f in L^p-norm if $\lim_{n\to\infty} \|f_n - f\|_{L^p} = 0$.

Convergence and Completeness

Def: we say a sequence $f_n \in L^p(X)$ is a <u>Cauchy sequence</u> if $\forall \varepsilon > 0$ $\exists N$ s.t.: $\forall n, m > N$. $\|f_n - f_m\|_{L^p} < \varepsilon$.

Def. A metric space is complete if any Cauchy sequence converges.
Def A normed space V is call a Banach Space if it's complete.
Thm. (Riesz-Fischer) For $1 \le p \le +\infty$. $L^p(X)$ is a Banach space.

Proof: (Similar to lecture 12)
Let $\{f_n\}$ is a cauchy sequence in $L^p(X)$.
We need to find $f \in L^p(X)$ s.t: $\|f_n - f\|_{L^p} \to 0$.

Idea: $f \stackrel{?}{=} f_1 + \underbrace{\sum_{i=2}^{\infty} (f_i - f_{i-1})}_{\text{Converge?}}$
then $f_n \to f$. $g = |f_1| + \sum_{i=2}^{k} \underbrace{|f_i - f_{i-1}|}_{|f_{n_i} - f_{n_{i-1}}|}$

First assume $1 \le p < +\infty$.

Step 1. Pick a subsequence $f_{n_1}, f_{n_2}, \cdots$.
s.t: $\|f_{n_k} - f_{n_{k-1}}\|_{L^p} < \frac{1}{2^k}$

Step 2. Construct g.
$$g_k = |f_{n_1}| + \sum_{i=2}^{k} |f_{n_i} - f_{n_{i-1}}|$$
By monotone $g_k \to g = |f_{n_1}| + \sum_{i=2}^{\infty} |f_{n_i} - f_{n_{i-1}}|$.
$\Rightarrow g \in L^p(X)$.
($\|g_k\|_{L^p} \le \|f_{n_1}\|_{L^p} + \sum_{i=1}^{k} \|f_{n_i} - f_{n_{i-1}}\|_{L^p} \le \|f_{n_1}\|_{L^p} + 1 < \infty$)

Moreover, $|f_{n_k}| = |f_{n_1} + \sum_{i=2}^{k} (f_{n_i} - f_{n_{i-1}})| \le g_k \le g$.

Step 3: Let $f = f_{n_1} + \sum_{i=2}^{\infty} (f_{n_i} - f_{n_{i-1}}) = \lim_{k\to\infty} f_{n_k}$.
$\Rightarrow$ ~~$f \in L^p$~~ $\|f_{n_k} - f\|_{L^p} \to 0$.
$|f_{n_k}| \le g$, $|f| \le g$ $\Rightarrow |f_{n_k} - f| \le 2g \Rightarrow f_{n_k} - f \to 0 \Rightarrow |f_{n_k} - f|^p \le \underbrace{2^p g^p}_{\text{integrable}}$
$(f_{n_k} - f)^p \to 0$
By LDCT.

Step 4: Use triangle inequality to show $\|f_n - f\|_{L^p} \to 0$.

Now consider $p = +\infty$.

Step 1. Same. Step 2. $f = f_{n_1} + \sum_{i=2}^{\infty} (f_{n_i} - f_{n_{i-1}})$
$\|f\|_{L^\infty} \le \|f_{n_1}\|_{L^\infty} + 1$ a.e. $f \in L^\infty(X)$
$\|f_{n_N} - f\|_{L^\infty} = \|\sum_{i=N}^{\infty} f_{n_i} - f_{n_{i-1}}\|_{L^\infty} \le \frac{1}{2^{N-1}} \to 0$

Step 3. Same.

L^p is Separable.

$\boxed{\text{Def.}}$ Let (V, d) be a metric space.

We say V is separable if there exists a countable set $V_0 = \{v_1, v_2, \cdots\}$ s.t. $\overline{V_0} = V$ i.e. $\forall v \in V$. $\forall \varepsilon > 0$. $\exists i$ s.t. $|v_i - v| < \varepsilon$.
(V_0 dense in V).

e.g. $\mathbb{R}^d \supset \mathbb{Q}^d$

$\boxed{\text{Thm}}$ For $1 \leq p < \infty$. $L^p(\mathbb{R}^d)$ is separable.

$\boxed{\text{More generally}}$ Suppose X satisfies: ($\exists$ countable many open sets $U_1, U_2 \cdots$ s.t. ① $\mu(U_i) < +\infty$

② For any open set $U \subset X$. exist $U_{n_1}, U_{n_2} \cdots$ s.t. $U = \bigcup_{i=1}^{\infty} U_{n_i}$)

then $L^p(X, \mu)$ is separable.

NOTE: Not true for $p = +\infty$.

Lemma For $0 < p < \infty$. the set
$\mathcal{S} = \{\varphi = \sum_{i=1}^{k} a_i \chi_{A_i} : \mu(A_i) < \infty.\ k \in \mathbb{N}\}$. is dense in $L^p(X)$.

[In particular $(\mathcal{S}, \|\cdot\|_{L^p})$ is NOT a Banach Space].

Proof: $\boxed{\text{Step 1}}$ $\mathcal{B} = \{f \in L^p.\ \exists M \text{ s.t. } |f(x)| \leq M\}$. is dense in L^p.

i.e. $\forall f \in L^p$ $\exists \{f_n\} \subset \mathcal{B}$ s.t. $\|f_n - f\|_{L^p} \to 0$.

Let $f_n = \begin{cases} f(x) & |f(x)| \leq n \\ n. & |f(x)| > n \end{cases}$ Then $f_n \in L^p$. $f_n \to f$.
$f_n \in \mathcal{B}$.

Use LDCT. $|f_n| \leq |f|$. $\Rightarrow |f_n - f| \leq 2|f|$ $|f_n - f|^p \to 0$.
$\Rightarrow \|f_n - f\|_{L^p} \to 0$. $|f_n - f|^p \leq 2^p |f|^p \in L^1$

$\boxed{\text{Step 2}}$ $\mathcal{BB} = \{f \in \mathcal{B}.\ \mu(\text{Supp} f) < \infty\}$ is dense in $\mathcal{B}$.

Note $\int_X |f|^p d\mu = A < \infty$ $\Rightarrow \mu(\{x : |f(x)| > \frac{1}{m}\}) \leq A m^p$

Let $f_m = \begin{cases} f(x) & \text{if } |f(x)| > \frac{1}{m}. \\ 0 & \text{if } |f(x)| < \frac{1}{m}. \end{cases}$

$\forall f \in \mathcal{B}$

Then $f_m \in \mathcal{BB}$, $|f_m| \leq |f|$. $|f_m - f|^p \to 0$. $|f_m - f|^p \leq 2^p |f|^p$

Step 3. $\mathcal{S}$ is dense in $\mathcal{BB}$.

Let $f \in \mathcal{BB}$ Let $A_i = \{x : \frac{i}{2^k} \le f(x) < \frac{i+1}{2^k}\}$.

Let $\varphi_k = \underset{\text{finite}}{\sum_{i}} \frac{i}{2^k} \chi(A_i) \in \mathcal{S}$. $\mu(A_i) < \infty$.

$\|\varphi_k - f\|_{L^p} \to 0$.

Step 4. $\forall f \in L^p(X)$ find $f_1 \in \mathcal{B}$ s.t. $\|f_1 - f\|_{L^p} < \varepsilon/4$.

find $f_2 \in \mathcal{BB}$ s.t. $\|f_2 - f_1\|_{L^p} < \frac{\varepsilon}{4}$. find $\varphi \in \mathcal{S}$ s.t. $\|\varphi - f_2\|_{L^p} < \varepsilon/4$.

Applying Triangle inequality

Prove: $L^p(X)$ is separable. (in thm's condition)

$V_0 = \{\sum_I a_I \chi_{U_I},\ a_I \in \mathbb{Q}.\ U_I = V_{i_1} \cup \cdots \cup V_{i_k}\}$. $I = \{i_1, \dots, i_k\}$.

$L^{\boxed{2}}$ Space

Lec 22

L^2-spaces as inner product spaces

- Consider $p=2$ Then
  - $\|f_1 + f_2\|_{L^2} \le \|f_1\|_{L^2} + \|f_2\|_{L^2}$
  - $\|f_1 \cdot f_2\|_{L^1} \le \|f_1\|_{L^2} \cdot \|f_2\|_{L^2}$ (Hölder)
  - $L^2(X)$ is complete
  - $L^2(\mathbb{R}^n)$ or $\ell^2 (= L^2(\mathbb{N}), \mu = \#)$ are separable.

Recall In $\mathbb{R}^d$, we have inner product

$\langle \cdot, \cdot \rangle \quad \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$

$\langle \vec{v}, \vec{w} \rangle \to \sum_{i=1}^{d} v_i w_i$

In ℓ^2, we can define an inner product

$\langle \cdot, \cdot \rangle : \ell^2 \times \ell^2 \to \mathbb{R}$

$\langle (a), (b) \rangle = \sum_{i=1}^{\infty} a_i b_i = \int_{\mathbb{N}} a \cdot b \, d\mu$.

Hölder $\Rightarrow$ convergence

Similarly, we can define an inner product on $L^2(X)$.

$\langle\cdot,\cdot\rangle : L^2(X)\times L^2(X)\to R$

$\langle f,g\rangle := \int_X f\cdot g\,d\mu < +\infty$.

Rmk: (1) By Hölder

$f,g\in L^2(X) \Rightarrow f\cdot g\in L^1(X)$ $\int_X |f\cdot g|\,d\mu < +\infty$.

(2) If f,g are complex-valued, then

$\langle f,g\rangle := \int_X f\cdot\bar{g}\,d\mu$.

Prop: The inner product $\langle\cdot,\cdot\rangle$ on $L^2(X)$ satisfies.

(1) $\langle f,f\rangle\geq 0$. and "$=0$" $\Leftrightarrow$ $f=0$ a.e. in $L^2(X)$. $f=0$, zero function.

(2) $\langle a_1f_1+a_2f_2,g\rangle = a_1\langle f_1,g\rangle + a_2\langle f_2,g\rangle$.

(3) $\langle f,g\rangle = \overline{\langle g,f\rangle}$.

($\langle f,g\rangle = \langle g,f\rangle$)

Def: For any vector space V, if a function $\langle\cdot,\cdot\rangle : V\times V\to R$ satisfies (1).(2).(3) then we say $(V,\langle\cdot,\cdot\rangle)$ is an inner product space

$\boxed{\text{Cauchy-Schwarz. inequality}}$

By Hölder $|\langle f_1,f_2\rangle| \leq \|f_1\|_{L^2}\cdot\|f_2\|_{L^2}$. in $L^2(X)$.

$(\int_X f_1f_2\,d\mu)^2 \leq \int_X f_1^2\,d\mu\cdot\int_X f_2^2\,d\mu$.

Special space

$X=\{1,\cdots,n\}$. $\mu=\#$ $\left|\sum_{i=1}^n a_ib_i\right| \leq \left(\sum_{i=1}^n a_i^2\right)^{\frac12}\left(\sum_{i=1}^n b_i^2\right)^{\frac12}$

Let $(V,\langle\cdot,\cdot\rangle)$ be any inner product space.

Then we define a norm on V by $\|v\| = \sqrt{\langle v,v\rangle}$.

One can check: $\cdot\ \|\cdot\|$ is a norm on V

$\cdot$ One always has

$|\langle v,w\rangle| \leq \|v\|\cdot\|w\|$ (Cauchy-Schwarz)

Def || If an inner product space $(V, \langle\cdot,\cdot\rangle)$ is complete we say $(V, \langle\cdot,\cdot\rangle)$ is a $\boxed{\text{Hilbert space}}$

By def. Any Hilbert Space is a Banach Space

Examples: ① $\mathbb{R}^d$ ← $\boxed{X=\{1,2,\dots,d\}.\ \mu=\#.\ \Rightarrow L^2(X)=\mathbb{R}^d}$

② $L^2(X)$.

Example: $V = C([0,1]) \subset L^2([0,1])$.

$\langle f, g\rangle := \int_0^1 f\cdot g\,dx$.

Then $(V, \langle\cdot,\cdot\rangle)$ is inner product space.

But NOT Complete.

Now let $(V, \langle\cdot,\cdot\rangle)$ be an inner-product space.

$v, w \in V. \Rightarrow \|v+w\|^2 = \langle v+w, v+w\rangle = \langle v,v\rangle + \langle v,w\rangle + \langle w,v\rangle + \langle w,w\rangle$

$= \|v\|^2 + \|w\|^2 + 2\langle v,w\rangle.$

Def: || The angle between v, w is

$$\theta = \arccos \frac{\langle v,w\rangle}{\|v\|\cdot\|w\|}$$

In particular $v \perp w \Leftrightarrow \langle v,w\rangle = 0$.

Prop. || (Parallelogram rule) In $(V, \langle\cdot,\cdot\rangle)$.

$\|v+w\|^2 + \|v-w\|^2 = 2(\|v\|^2 + \|w\|^2)$

Proof: $\|v\pm w\|^2 = \|v\|^2 + \|w\|^2 \pm 2\langle v,w\rangle$

Can check: In $L^p(X)$ $(p\neq 2)$. one can find f, g s.t:

$\|f+g\|_{L^p}^2 + \|f-g\|_{L^p}^2 \neq 2(\|f\|_{L^p}^2 + \|g\|_{L^p}^2)$

$\Rightarrow$ In L^p $(p\neq 2)$ one can't define an inner product $\langle\cdot,\cdot\rangle$ s.t: $\langle f, f\rangle = \|f\|_{L^p}^2$

If a norm satisfies: $\|v+w\|^2 + \|v-w\|^2 = 2(\|v\|^2 + \|w\|^2)$

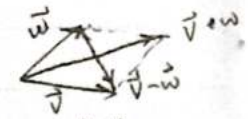
$\langle v,w\rangle = \frac{1}{4}(\|v+w\|^2 - \|v-w\|^2)$ is an inner product.

Thm ($\boxed{\text{Existence of minimizer}}$) Let H be a Hilbert Space $K \subset H$ is a nonempty, closed, convex subset. Then for

$H \ni \boxed{\begin{array}{l}\forall z_n \in K \\ z_n \to y \in K\end{array}}$ $\boxed{\begin{array}{l}\forall z_1, z_2 \in K \\ \forall 0<t<1.\ tz_1+(1-t)z_2 \in K\end{array}}$

$\forall x \in H$, there exists a unique $y \in K$ s.t:

$$\|x-y\| = \inf\{\|x-z\| : z \in K\}$$



Proof. (Uniqueness). If y_1, y_2 are two minimizers $y_1 \neq y_2$.

$y_1, y_2 \in K. \Rightarrow (y_1+y_2)/2 \in K$

By Parallelogram rule, $\|x - \frac{y_1+y_2}{2}\|^2 + \|\underbrace{\frac{y_1-y_2}{2}}_{>0}\|^2 = \|x-y_1\|^2$

Contradiction.

want: $\{z_n\}$ is a Cauchy seq.

(Existence) Let $z_n \in K$ s.t: $\|x-z_n\| \to d \triangleq \inf\{\|x-z\| : z \in K\}$.

Then $(z_n+z_m)/2 \in K$.

$\Rightarrow d^2 \le \|x - \frac{z_n+z_m}{2}\|^2 \le \|x - \frac{z_n+z_m}{2}\|^2 + \|\frac{z_n-z_m}{2}\|^2 = \frac{1}{2}\|x-z_n\|^2 + \frac{1}{2}\|x-z_m\|^2$.

$\Rightarrow \|z_n - z_m\|^2 \to 0$. as $n.m \to \infty$.

$\Rightarrow \{z_n\}$ is a Cauchy seq. By closeness $y \in K$.
$\to y$ in H.

Rmk: For $1<p<\infty$. the "existence of minimizer" still holds in $L^p(X)$, But fails for $L^1(X)$. $L^\infty(X)$.

$\boxed{\text{Cor}}$ Let H be a Hilbert Space. $V \subset H$ is a closed vector space of H; then $\forall x \in H$. $\exists$ unique $x_V \in V$ and
$x_{V^\perp} \in V^\perp = \{w \in H.\ \langle v, w\rangle = 0, \forall v \in V\}$.
s.t: $x = x_V + x_{V^\perp}$
Moreover x_V is the vector s.t: $\|x - x_V\| = \inf\{\|x-z\|, z \in V\}$

proof: Only need to prove $x_{V^\perp} : x - x_V \in V^\perp$

Suppose $x_{V^\perp} \notin V^\perp$, i.e. $\exists v \in V$ s.t. $\langle x_{V^\perp}, v\rangle \neq 0$.

$\Rightarrow \exists\ v \in V$ s.t: $\langle x_{V^\perp}, v\rangle = 1$.

Now. $\|x - (x_V + \varepsilon v)\|^2 = \|x_{V^\perp} - \varepsilon v\|^2 = \|x_{V^\perp}\|^2 + \varepsilon^2\|v\|^2 - 2\varepsilon$
$= \|x - x_V\|^2 \underbrace{- 2\varepsilon + \varepsilon^2\|v\|^2}_{<0\ \ \varepsilon \to 0}$.

$\Rightarrow \|x-(x_v+\varepsilon v)\| < \|x-x_v\|$ Contradiction.

$H = V \oplus V^{\perp}$

Recall: $\mathbb{R}^d$.

$L: \mathbb{R}^d \to \mathbb{R}$ is linear functional

$\Leftrightarrow Lx = a_1x_1 + \cdots + a_dx_d$.

$\updownarrow$ $a = (a_1, \cdots, a_d) \in \mathbb{R}^d$.

Now let H be a Hilbert space. For any $v \in H$. we define a linear functional $L_v: H \to \mathbb{R}$, $w \longmapsto \langle v, w \rangle$

Note: L_v is continuous

$|L_v(w-w_i)| = |\langle v, w-w_i \rangle| \le \|v\| \cdot \|w-w_i\|$.

Thm (Riesz representation thm)

Let H be a Hilbert space

$L: H \to \mathbb{R}$ is any continuous linear functional

Then there exists a unique $v \in H$

s.t. $L_v = L$

$\{$Space of all continuous linear functional on $H\} \cong H$.

Proof: (Uniqueness)

If $L_{v_1} = L_{v_2} = L$

$\Rightarrow L_{v_1}(v_1-v_2) = L_{v_2}(v_1-v_2)$

$\Rightarrow \langle v_1, v_1-v_2 \rangle = \langle v_2, v_1-v_2 \rangle$

$\Rightarrow \|v_1-v_2\| = 0$.

(Existence) WLOG, we assume $L \neq 0$.

Let $V = \ker L = \{x \in H : L(x) = 0\} = L^{-1}(0)$ closed.

$\Rightarrow \underline{V^{\perp} \neq 0}$.

Take $w \in V^{\perp}$ s.t: $\|w\| = 1$. $\Rightarrow L(w) \neq 0$.

Let $v = L(w)\, w$. Then.

$$L_v(x) = \langle L(w)w, x\rangle = \langle L(w)w,\ \underbrace{x - \frac{L(x)}{L(w)}w}_{V} + \underbrace{\frac{L(x)}{L(w)}w}_{V^{\perp}}\rangle = \langle L(w)w, \frac{L(x)}{L(w)}w\rangle$$

$= L(x)$. $\forall x \in H$.

Lec 23

Today: linear functionals on $L^p(X)$

(function defined on ~~abstract space~~ (function space))

1. Linear functionals on normed vector spaces.

- We have seen.

$\boxed{\text{Riesz representation thm for } L^2(X)}$ $L: L^2(X) \to \mathbb{R}$.

Countinous linear functional

$\Rightarrow \exists!\ g \in L^2(X)$ s.t: $L(f) = \int_X f \cdot g\, d\mu$. $\forall f \in L^2(X)$.

Note: If we use complex-valued functions. i.e.

$L^2(X, \mathbb{C})$, then $\exists!\ g \in L^2(X, \mathbb{C})$ s.t: $L(f) = \int_X f \bar{g}\, d\mu$.

$\boxed{\text{Riesz representation thm for } C_c(X)}$

$L: C_c(X) \to \mathbb{R}$ be bounded linear functional

$\Rightarrow \exists$ Borel measure μ_1, μ_2 on X s.t:

$$L(f) = \int_X f\, d\mu_1 - \int_X f\, d\mu_2 = \int_X f\, \underbrace{(d\mu_1 - d\mu_2)}_{:= d\mu}$$

signed measure.

Note: We proved this for positive linear functionals in 19 ($\mu_2 = 0$) and then in PSet 10-1 for general L. (with X compact)

Note: assumption on X is: X is locally compact metric space, and is σ-compact.

only need: Hausdorff.

• Justification of assumptions / definitions

(1) The spaces $L^2(X)$. $C_c(X)$ and $L^p(X)$ are normed vector spaces:

Spaces: $L^2(X)$: $\|f\|_{L^2} = \left(\int_X |f|^2 \, d\mu\right)^{1/2}$

$C_c(X)$: $\|f\|_{C_c(X)} = \sup_X |f|$

$L^p(X)$: $\|f\|_{L^p(X)} = \left(\int_X |f|^p \, d\mu\right)^{1/p}$.

(2) Let V be a normed vector space: $\ell: V \to R$ linear functional.

Then $\underbrace{\ell \text{ is continuous}}_{\substack{\forall \text{ open set } U \subset R \\ \ell^{-1}(U) \text{ is open} \\ (a)}} \iff \underbrace{\ell \text{ is continuous at } 0}_{\substack{\forall \varepsilon > 0 \ \ell^{-1}((-\varepsilon,\varepsilon)) \text{ contains} \\ \text{an open set (centered on } 0 \in V) \\ (b)}} \iff \underbrace{\ell \text{ is bounded}}_{\substack{\exists C \text{ s.t.:} \\ |\ell(v)| \le C\|v\| \\ (c)}}$

Reason: • $(a) \Rightarrow (b)$, $(c) \Rightarrow (b)$

• We also have $(b) \Rightarrow (a)$: Let $v \in \ell^{-1}(U) \Rightarrow \exists \varepsilon$ s.t.: $(\ell(v)-\varepsilon, \ell(v)+\varepsilon) \subset U$. $\Rightarrow$ By triangle inequality $\underbrace{v + \ell^{-1}((-\varepsilon,\varepsilon))}_{\text{open } \subset \ell^{-1}(U)}$

• Finally $(a) \Rightarrow (c)$ By (a) $\ell^{-1}((-1,1))$ is open.

Since $0 \in \ell^{-1}((-1,1))$, So $\exists r > 0$ s.t.: $B_r(0) \subset \ell^{-1}((-1,1))$.

i.e. $\ell(B_r(0)) \subset (-1,1) \Rightarrow |\ell(v)| = \frac{\|v\|}{r} \left|\ell\left(\frac{v r}{\|v\|}\right)\right| \le \frac{\|v\|}{r}$

(3) Any $\overset{f \ge 0 \Rightarrow \ell(f) \ge 0}{\text{positive linear functional}}$, $\ell: C(X) \to R$ is bounded if X is compact.

Reason: Let $C_0 = |\ell(1)|$. for $\forall f \in C(X)$ and $|f| \le 1$.

We have $1 \pm f \ge 0 \Rightarrow C_0 \pm \ell(f) \ge 0 \Rightarrow |\ell(f)| \le C_0$

So for any $f \in C(X)$.

$|\ell(f)| = \left|\ell\left(\frac{f}{\|f\|} \cdot \|f\|\right)\right| = \|f\| \cdot \left|\ell\left(\frac{f}{\|f\|}\right)\right| \le \|f\| \cdot C_0$

Note: By the same arguement one can prove that if X is locally compact metric (or Hausdorff) space then for any compact set $K \subset X$ $\exists C_K$ s.t.:

$|\ell(f)| \le C_K \|f\|$ $\forall f \in C_c(X)$ with $\text{supp}(f) \subset K$

for positive linear functional ℓ.

2. Linear functionals on $L^p(X)$.

- Now consider the space $L^p(X)$ $p \geq 1$.
  As in the case of $L^2(X)$. one can start with a function g. and define $L_g: L^p(X) \to \mathbb{R}$ $f \mapsto \int_X f \cdot g \, d\mu$.
  Note that for L_g to be well-defined, we need $f \cdot g \in L^1(X)$. for all $f \in L^p(X)$. This is the case if $g \in L^q(X)$ where $\frac{1}{q} + \frac{1}{p} = 1$.

Prop: for any $g \in L^q(X)$, $L_g: L^p(X) \to \mathbb{R}$ is a Continuous linear functional.

Proof: We have seen that L_g is well-defined.
It's obviously linear i.e. $L_g(c_1 f_1 + c_2 f_2) = c_1 L_g(f_1) + c_2 L_g(f_2)$.
To prove L_g is continuous, it's enough to prove L_g is bounded, which follows from the Hölder's inequality
$|L_g(f)| \leq \|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q} = C\|f\|_{L^p}$. $C = \|g\|_{L^q}$.

The main theorem we want to prove today is

Thm (Riesz representation thm for $L^p(X)$)
Suppose μ is a σ-finite measure on X.
(i.e. $X = \bigcup_{n=1}^{\infty} X_n$ and $\mu(X_n) < \infty$. $\forall n$).
Let $1 \leq p < \infty$. and $q \geq 1$ s.t: $\frac{1}{p} + \frac{1}{q} = 1$.
Then for any continuous linear functional $\ell: L^p(X) \to \mathbb{R}$. there exists a unique $g \in L^q(X)$ s.t: $\ell = L_g$.

vector space
$V^* =$ all continuous linear functional on V

$(L^p)^* = L^q$

$\ell - L_g \sim g$.

$\|L_g\|_{(L^p)^*} = \|g\|_{L^q}$

Rmk: • The thm fails for $p = +\infty$.
In other words, for any $g \in L^1(X)$. g defines a continuous linear functional $L_g: L^\infty(X) \to \mathbb{R}$. however, there are more continuous linear functionals on $L^\infty(X)$ than these L_g's.

• The main ingredient in the proof is the following theorem that we will prove later

Radon-Nikodym Thm:

Let μ be a σ-finite measure on $(X, \mathcal{F})$

Let ν be a signed measure on $(X, \mathcal{F}) \to$ i.e. $\nu: \mathcal{F} \to \mathbb{R}$ s.t.: $\cdot \nu(\phi) = 0$. $\cdot \nu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \nu(A_n)$ disjoint

s.t.: $\mu(A) = 0 \Rightarrow \nu(A) = 0$.

Then $\exists f \in L^1(X)$ s.t.: $\nu(A) = \int_A f \, d\mu$

$l(\chi_A) = \int_X \chi_A f \, d\mu$.

- Proof of Riesz representation thm for $L^p(X)$

$\boxed{\text{Step 1}}$ (Uniqueness) Let $g_1, g_2 \in L^q(X)$ s.t. $L_{g_1} = L_{g_2}$

$\Rightarrow L_{g_1 - g_2} = 0$ on $L^p(X)$. Let $g = g_1 - g_2 \in L^q(X)$

Suppose $g \neq 0$ in $L^q(X)$. then $\exists$ measurable set $A \subset X$ with $0 < \mu(A) < \infty$ s.t: $g > 0$ on A or $g < 0$ on A.

Let $f = \chi_A \in L^p(X)$, Then $L_g(f) = \int_X f \cdot g \, d\mu \neq 0$. Contradiction.

$\boxed{\text{Step 2}}$ Suppose $\mu(A) < \infty$.

[Idea: linear functional $\to$ signed measure $\xrightarrow{R-N}$ a function $g \in L^1$.
$g \to L_g$.]

- Since $\mu(X) < \infty$. for any measurable set $A \in \mathcal{F}$. the function $\chi_A \in L^p(X)$. So we can define $\nu: \mathcal{F} \to \mathbb{R}$ by

$$\nu(A) := l(\chi_A)$$

This is a signed measure since

① $\nu(\phi) = l(0) = 0$

② Let $A_1, A_2 \cdots \in \mathcal{F}$ be disjoint Then $\chi_{\bigcup_{i=1}^{n} A_i} \to \chi_{\bigcup_{i=1}^{\infty} A_n}$ pointwise. Also $|\chi_{\bigcup_{i=1}^{k} A_i}| \leq 1 \in L^p(X)$. $\forall k$.

So by dominated convergence theorem $\chi_{\bigcup_{i=1}^{k} A_i} \to \chi_{\bigcup_{i=1}^{\infty} A_i}$ in $L^p(X)$

Since l is continuous, we get

$$\sum_{i=1}^{k} l(\chi_{A_i}) = l(\chi_{\bigcup_{i=1}^{k} A_i}) \to l(\chi_{\bigcup_{i=1}^{\infty} A_i}).$$

i.e. $\nu(\bigcup_{i=1}^{\infty} A_i) = \lim_{k \to \infty} \sum_{i=1}^{k} \nu(A_i)$.

• By Radon-Nikodym theorem, $\exists\, g \in L^1(X)$ s.t.

$\nu(A) = \int_A g \, d\mu \quad \forall A \in \mathcal{F}$ Note: $\mu(A)=0 \Rightarrow \chi_A = 0$ in $L^p(X)$ $\Rightarrow \nu(A) = \ell(\chi_A) = 0$

We want to prove $\ell = L_g$ on $L^p(X)$.

By def. for any $A \in \mathcal{F}$. one has $\ell(\chi_A) = \nu(A) = \int_A g\, d\mu = L_g(\chi_A)$.

By linearity. $\ell = L_g$ for all simple function.

In Lec. 21. we have shown that the space of simple functions is dense in $L^p(X)$.

Reason: Assume $g \geq 0$. otherwise consider g_+ and g_-
Let $f \in L^p(X)$. Assume $f \geq 0$. Otherwise consider f_+ and f_-

Then $\exists$ simple functions $\varphi_1, \varphi_2 \cdots \nearrow f$ in $L^p(X)$

Assume $|f| \leq c$. Then f By dominated convergence,

$L_g(f) = \lim_{n\to\infty} L_g(\varphi_n) = \lim_{n\to\infty} \ell(\varphi_n) = \ell(f) < \infty \Rightarrow L_g$ well defin

For unbounded $f \in L^p(X)$ $f \geq 0$, we let $f_N = \min(f, N)$. $f_N \to f$ in L^p, by $\|\,\|_{L^p}$

Then by monotone convergence, $L_g(f) = \lim_{N\to\infty} L_g(f_N)$

$= \lim_{N\to\infty} \ell(f_N) = \ell(f) < \infty$.

• It remains to prove $g \in L^q$. WLOG, we assume $g \geq 0$

Since $\ell = L_g$ is continuous, it's a bounded linear functional.

So $\exists\, c > 0$ s.t: $|L_g(f)| \leq c \|f\|_{L^p(X)} \quad \forall f \in L^p(X)$.

Case 1: $1 < p < \infty$.

Idea: Want $\|g\|_{L^q} < \infty$. So we take $f = g^{q-1}$

so that formally $\|g\|_{L^q}^q = |L_g(f)| \leq c \cdot \|f\|_{L^p}$

and $\|f\|_{L^p} = \left(\int_X |g|^{(q-1)p} d\mu\right)^{1/p}$

$= \left(\int_X |g|^q d\mu\right)^{1-\frac{1}{q}} = \|g\|_{L^q}^{q-1}$

$\Rightarrow \|g\|_{L^q} \leq c$. But this arguement is NOT true since we don't know $f = g^{q-1} \in L^p$ yet

(x)

Let $f_N = \min(g, N)^{q-1}$. Then $f_N \in L^p(X)$.

since $\mu(X) < +\infty$. We have $L_g(f_N) = \int_X g \cdot f_N \, d\mu$. $c \|f_N\|_{L^p} \geq$

$\geq \int_X \min(g, N)^q d\mu = \|\min(g, N)\|_{L^q}^q$.

$\|f_N\|_{L^p} = \left(\int_X \min(g,N)^q d\mu\right)^{1/p} = \|\min(g,N)\|_{L^q}^{q-1}$

$\Rightarrow \|\min(g,N)\|_{L^q} \le C.$

Letting $N \to \infty$. using monotone convergence, we get

$$\|g\|_{L^q} \le C.$$

Case 2: $p=1$

Let $f_N = \chi_{g>N}$. Then $f_N \in L^p(X)$ $(L^1(X))$. So.

$N \int_{g>N} 1 \, d\mu \le \int_{g>N} g \, d\mu = \int_X g f_N d\mu$

$\le C \|f_N\|_{L^1} = C \int_{g>N} 1 \, d\mu.$

$\Rightarrow$ For $N > C$. $\mu(\{g>N\}) = 0$. So $\|g\|_{L^\infty} \le C$.

$\boxed{\text{Step 3}}$ Suppose $\mu(X) = +\infty$

By assumption $\exists$ $X_1 \subset X_2 \subset \cdots \subset X$. $\mu(X_n) < \infty$.

s.t. $X = \bigcup_{n=1}^{\infty} X_n$.

By step 2. one can find $g_n \in L^q(X_n)$ s.t. $l = L g_n$ on $L^p(X_n)$.

By step 1. for $m<n$ one has $g_m = g_n$ on X_m. Thus

one get one function g defined on X s.t.

$$g = g_n \text{ on } X_n$$

In step 2. we showed that $\|g_n\|_{L^q(X_n)} \le C$. for <u>same C</u>, $\forall n$

So by monotone convergence arguement

$$g \in L^q(X) \text{ and } \|g\|_{L^q} \le C.$$

Lec 24

1. Signed measure

- Recall (Lec 10.11.16)

Countable additivity $\int_{\bigcup_{n=1}^{\infty} A_n} f d\mu = \sum_{n=1}^{\infty} \int_{A_n} f d\mu$ A_n's disjoint

- if f is non-negative, then "$\mu_f = f d\mu$" is a measurable, where $\mu_f(A) = \int_A f d\mu$
- if f is absolutely integrable, then "$\mu_f = f d\mu$" is a "signed measure", where $\mu_f(A) = \int_A f d\mu$.

Rmk. · Measure could take value $+\infty$

· Signed measure could take value $+\infty$ or $-\infty$.

So we don't really need to assume $f \in L^1$. to construct μ_f

A problem. If $A_1, A_2 \subset X$. disjoint and $\mu(A_1) = -\infty$ $\mu(A_2) = +\infty$

Q: what is $\mu(A_1 \cup A_2)$?

Need to assume μ can not attain both $+\infty$ and $-\infty$.

Def: A signed measure on a measurable space $(X, \mathcal{F})$ is a function $\mu: \mathcal{F} \to [-\infty, +\infty]$.

S.t. (1) $\mu(\phi) = 0$.

(2) $\mu(\mathcal{F}) \subset [-\infty, +\infty)$ or $(-\infty, +\infty]$

(3) $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ for disjoint $A_n \in \mathcal{F}$.

Moreover, if $\mu(\bigcup_{n=1}^{\infty} A_n) \neq \pm\infty$ we require the series $\sum_{n=1}^{\infty} \mu(A_n)$ converges absolutely

Example. (1) If μ_1, μ_2 are measures. either μ_1 or μ_2 is finite.

Then $\mu = \mu_1 - \mu_2$ is a signed measure.

(2) Let f be a measurable function on X.

Suppose either $f_+ \in L^1(X, \mu)$ or $f_- \in L^1(X, \mu)$

then "$\mu_f = f\,d\mu$" is a signed measure
$\mu_f(A) = \int_A f\,d\mu = \int_A f_+\,d\mu - \int_A f_-\,d\mu$.
Integrable in the extended sense.

Def: Let μ be a signed measure on $(X, \mathcal{F})$
(1) We say $A \subset \mathcal{F}$ is a positive set if $\forall$ measurable $B \subset A$. $\mu(B) \geq 0$.
(2) .... negative set ....
.... $\mu(B) \leq 0$
(3) .... null set ....
.... $\mu(B) = 0$.

Thm Hahn decomposition theorem: Let μ be a signed measure on $(X, \mathcal{F})$. Then $\exists$ positive set X_+ and negative set X_- for μ s.t.: $X = X_+ \cup X_-$, $X_+ \cap X_- = \phi$. Moreover if X'_+, X'_- is another such decomposition, then $\underset{X_- \Delta X'_-}{\overset{\parallel}{X_+ \Delta X'_+}}$ is a null set

Proof: WLOG assume $\mu(A) < +\infty$.?
(Idea: Pick X_+ to be a positive set with maximal measure)
Let $m_+ = \sup\{\mu(A).\ A \text{ is a positive set}\} < +\infty$.
Let $A_1, A_2, \ldots$ be a sequence of positive sets s.t.: $\mu(A_n) \to m_+$.
Let $A = \bigcup_{n=1}^{\infty} A_n$ Then A is a positive set and
$\mu(A) = \lim_{n\to\infty} \mu(A_n) = m_+$.
Let $X_+ = A$ $X_- = A^c$ Need: X_- is a negative set.
By contradiction: Suppose $\exists B_1 \subset X_-$, s.t: $\mu(B_1) > 0$.
• If B_1 is a positive set, then $\underbrace{\mu(X_+ \cup B_1)}_{\text{positive set}} > \mu(X_+)$
if $A_1 \subset A_2 \subset \cdots$ then $\mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$.

- If B_1 is NOT a positive set. Then $\exists B_2 \subset B_1$. s.t:
$\mu(B_2) > \mu(B_1)$. $\Rightarrow \exists n_1$ s.t: $\mu(B_2) > \mu(B_1) + \frac{1}{n}$.
Greedy Choice Take B_2, n_1 s.t: n_1 is as small as possib
  Again: If B_2 is positive, then we have done.
→ We get a sequence B_2, n_1; B_3, n_2; B_4, n_3; --- s.t:
$B_1 \supset B_2 \supset B_3$ --- $\mu(B_{K+1}) > \mu(B_K) + \frac{1}{n_K}$ n_K is the smallest possib
Let $B = \bigcap_{i=1}^{\infty} B_i$ Then $+\infty > \mu(B) = \lim_{n\to\infty} \mu(B_n) = \sum_{i=1}^{\infty} \frac{1}{n_i}$

So $\lim_{i\to\infty} n_i = +\infty$

Claim: B is a positive set
  If NOT: $\exists C \subset B$ s.t: $\mu(C) > \mu(B)$.
  $\Rightarrow \exists m_0 > 0$ and $N_0 > 0$. s.t: $\forall K > N_0$. $\mu(C) - \mu(B_K) > \frac{1}{m_0}$
  Take k s.t: $n_K > m_0$.
  $\mu(C) > \mu(B_K) + \frac{1}{m_0}$ contradictions with the choice of B_{K+1}!

Suppose $X = X_+ \cup X_- = X_+' \cup X_-'$
  Then $\underbrace{X_+ \Delta X_+'}_{\text{positive}} = \underbrace{X_- \Delta X_-'}_{\text{negative.}} \longrightarrow$ Null.

e) Thm (Jordan decomposition theorem)
Let μ be a signed measure on $(X, \mathcal{F})$. then $\boxed{\exists \text{ unique}}$ measures μ_+. μ_- s.t: $\mu = \mu_+ - \mu_-$ $\mu_+ \perp \mu_-$ i.e. $\mu_+|_{X_-} = 0$ $\mu_-|_{X_+} = 0$. $X_- \amalg X_+$
Moreover if $X = X_+ \cup X_-$ is a Hahn decomposition, then
$\mu_+ = \mu|_{X_+}$. $\mu_- = -\mu|_{X_-}$
i.e. $\forall A \in \mathcal{F}$, $\mu_+(A) = \mu(A \cap X_+)$
$\mu_-(A) = -\mu(A \cap X_-)$

Proof. $\cdot$ $\mu_+(A)$. $\mu_-(A)$.
  One can check. $\underbrace{\mu_+ . \mu_-}_{\text{one is finite.}}$ are measure and $\mu = \mu_+ - \mu_-$

· Suppose we also have $\mu = \nu_+ - \nu_-$ and $\nu_+|_{X_-'} = 0$ $\nu_-|_{X_+'} = 0$
Then $A \subset X_+'$, we have $\mu(A) = \nu_+(A)$.
$\Rightarrow \forall A \in \mathcal{F}$.

$\mu_+(A) = \mu(A \cap X_+) = \mu(A \cap X_+') = \nu_+(A \cap X_+') = \nu_+(A)$.
Similarly $\mu_-(A) = \nu_-(A)$.

Rmk. • In the Jordan decomposition. $\mu = \mu_+ - \mu$.
We call $\mu_+ (\mu_-)$ the positive(negative) ~~part~~ of μ.
or positive(negative) variation of μ.

def 1: We can define the absolute value (or Total variation) of μ to be $|\mu| = \mu_+ + \mu$

def 2. · We call a signed measure μ is finite or σ-finite if $|\mu|$ is finite or σ-finite.

def 3. · In general. we say two signed measure μ, ν are $\boxed{\text{mutually singular}}$
$\exists X = X_1 \amalg X_2$. s.t: $\underbrace{\mu|_{X_2} = 0}_{\text{i.e. } \forall A \subset X_2,\ \mu(A)=0}$. $\nu|_{X_1} = 0$. Notation $\mu \perp \nu$.
$\Leftrightarrow \mu$ supported on X_1. ν supported on X_2

def 4. · We say ν is absolutely continuous w.r.t μ
if every μ-null set is also a ν-null set
Notation $\nu \ll \mu$.

Example: · Jordan decomposition $\mu = \mu_+ - \mu_-$, $\mu_+ \perp \mu_-$
· Dirac measure $m_{x_0} \perp$ lebesgue measure m
· Let f be a measurable function which is integrable in extended sense.
$\rightarrow$ "$\mu_f = f\,d\mu$" is a signed measure and $\mu_f \ll \mu$

Prop. Let ν be a finite signed measure, and let μ be a (positive) measure on $(X, \mathcal{F})$. Then $\nu \ll \mu$ if and only if $\forall \varepsilon > 0$. $\exists \delta > 0$ s.t: $\forall A \in \mathcal{F}$. if $\mu(A) < \delta$ then $|\nu(A)| < \varepsilon$.

Lebesgue-Radon-Nikodym Theorem

Let μ be a σ-finite measure, and ν a σ-finite signed measure. Then there exists unique signed measures μ_r, μ_s s.t: $\nu = \mu_r + \mu_s$ where $\mu_s \perp \mu$. $\mu_r << \mu$

Rmk: If ν is measure, then so are μ_r, μ_s.
if ν is finite, then μ_r is finite

Moreover $\exists$ (extended sense) integrable function f on X s.t: $d\mu_r = f d\mu$.

μ_r is finite $\iff f \in L^1(X, \mu)$.
μ_r is a measure $\iff f \geq 0$.

Cor: (Radon-Nikodym Theorem)

Suppose $\nu << \mu$. ($\mu(A) = 0 \Rightarrow \nu(A) = 0$).
Then $\exists$ integrable function f s.t: $d\nu = f d\mu$.
Moreover, if ν is finite, then $f \in L^1(X)$

Proof: if $\nu \leq \mu$, $\nu \perp \mu$ then $\nu = 0$.

We call f the Radon-Nikodym derivative of ν w.r.t. μ.

$\underline{d\nu = f d\mu}$ $\quad f = \frac{d\nu}{d\mu}$.

Proof: Case 1 ν is finite measure

. Define $\mathcal{M} = \{f : X \to [0, +\infty] : \int_A f d\mu \leq \nu(A)\ \forall A \in \mathcal{F}\}$.

Then $\mathcal{M} \neq \emptyset$ since $0 \in \mathcal{M}$

. If $f, g \in \mathcal{M}$, then $\max(f, g) \in \mathcal{M}$.

Reason: Let $B = \{f > g\}$. Then $\forall A \in \mathcal{F}$.

$$\int_A \max(f,g)\, d\mu = \int_{A\cap B} f d\mu + \int_{A \cap B^c} g d\mu \leq \nu(A\cap B) + \nu(A \cap B^c) \leq \nu(A)$$

Now set $m := \sup_{f \in \mathcal{M}} \int_A f d\mu \leq \nu(A) < +\infty$

Let $f_n \in \mathcal{M}$ s.t: $\int_A f_n d\mu \to m$.

Let $g_n = \sup(f_1, \dots, f_n) \in \mathcal{M}$. Then $g_n \nearrow f := \sup_n f_n$

$$\Rightarrow \quad m = \lim_{n\to\infty} \int_A f_n d\mu \leq \lim_{n\to\infty} \int_A g_n d\mu \leq m.$$

by monotone convergence $\int_A f\,d\mu = m$.
$\Rightarrow$ f is a.e. finite
WLOG, assume $f < +\infty$
Note: $m = \int_A f\,d\mu \le \nu(A)$ i.e. $f \in \mathcal{M}$.
Let $\mu_s = \nu - \underbrace{f\,d\mu}_{\mu_r}$.

Need: μ_s is singular w.r.t. μ.

Lemma: Let μ, ν be finite measures on $(X, \mathcal{F})$
Then either $\mu \perp \nu$, or $\exists\, \varepsilon > 0$ and $A \in \mathcal{F}$,
s.t. $\nu(A) > 0$ and A is a positive set w.r.t. $\mu - \varepsilon\nu$.

Proof. By lemma $\exists\, \varepsilon > 0$, $A \in \mathcal{F}$ s.t. $\mu(A) > 0$
$\forall B \subset A$ $\mu_s(B) > \varepsilon\mu(B)$
In particular, $\forall C \in \mathcal{F}$.
$\int_C (f + \varepsilon\chi_A)\,d\mu = \int_C f\,d\mu + \varepsilon\mu(A \cap C)$
$\le \int_C f\,d\mu + \nu(A \cap C) - \int_{A\cap C} f\,d\mu$.
$= \int_{C \cap A^c} f\,d\mu + \nu(A \cap C)$
$\le \nu(C \cap A^c) + \nu(A \cap C) = \nu(C)$.
i.e. $f + \varepsilon\chi_A \in \mathcal{M}$.
i.e. $m \ge \int (f + \varepsilon\chi_A)\,d\mu = \int f\,d\mu + \varepsilon\int \chi_A\,d\mu = m + \varepsilon\mu(A) > m$.
contradiction.

Thus $\mu_s \perp \mu$.
Let $\mu_r = f\,d\mu$. Then $\nu = \mu_r + \mu_s$ $\mu_r \ll \mu$, $\mu_s \perp \mu$.
Moreover $f \ge 0$, and $f \in L^1(X, \mu)$

Uniqueness $\nu = \mu_r' + \mu_s' = \mu_r + \mu_s$.
$\Rightarrow \mu_r' - \mu_r = \mu_s - \mu_s'$
$\mu_r' - \mu_r \ll \mu$. $\mu_s - \mu_s' \perp \mu$
$\Rightarrow \mu_r' - \mu_r = \mu_s - \mu_s' = 0$.

Lec 25 Fundamental theorem of Calculus (Version 1)

Case of dimension 1

Recall. $F:(a,b)\to R$. $\to$ we call a function f the derivative of F if: $\underbrace{f(x)=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}}_{\text{def of derivative}}$.

We learned in Calculus

If f is a continuous function, we can define $F(x)=\int_0^x f(t)\,dt$. Then $F'(x)=f(x)$.

In other words, we have

$$f(x)=\lim_{h\to 0^+}\frac{1}{h}\int_x^{x+h} f(t)\,dt=\lim_{h\to 0^+}\frac{1}{h}\int_{x-h}^{x} f(t)\,dt$$

Thm (Lebesgue differentiation theorem dim=1 case).

Take $\mu=m$.

Suppose $f\in\mathcal{L}^1(R)$ then

$$\lim_{h\to 0^+}\frac{1}{h}\int_{[x,x+h]} f(t)\,dt=f(x) \quad \text{a.e. } x\in R$$

$$\lim_{h\to 0^+}\frac{1}{h}\int_{[x-h,x]} f(t)\,dt=f(x). \quad \text{a.e. } x\in R$$

i.e. if we let $F(x)=\int_{[-\infty,x]} f(t)\,dt$, then $F'(x)=f(x)$. a.e. $x\in R$

Idea.

Density argument

We want to prove some property say. $\lim_{n\to\infty} L_n(f)=g$.

for a class of functions f

Step 1: Prove that the formula holds for a dense sub-class of nice functions

step 2: Controlled the difference $|L_n(f_1)-L_n(f_2)|\le$ some norm of $f-g$

step 3: Combine 2 steps to get the full proof

Thm ((One-side) Hardy-Littlewood maximal inequality) for R^d

Suppose $f\in\mathcal{L}^1(R^d)$ then for $\forall\alpha>0$.

$$m(\{x\in R^d: \sup_{h>0}\frac{1}{h}\int_{[x,x+h]}|f(t)|\,dt>\alpha\})\le\frac{1}{\alpha}\int_R|f(t)|\,dt. \quad (*)$$

(higher dimension)

$$m(\{x\in R^d: \underbrace{\boxed{\sup_{r>0}\frac{1}{m(B(x,r))}\int_{B(x,r)}|f(t)|\,dt}}_{f^*(x)\ \text{maximal function}}>\alpha\})\le\frac{3^d}{\alpha}\int_{R^d}|f(t)|\,dt$$

~~$\left|\frac{1}{h}\int_{[x,x+h]} f(t)dt - f(x)\right| = \left|\frac{1}{h}\int_{[x,x+h]} f(t)-f(x)\,dt\right|$~~

~~$\int |f(t)-f(x)| \le \int |f(t)-g(t)| + |g(t)-g(x)| + |g(x)-f(x)|\,dt$~~

Proof. We only need to prove (1).

We have known that (1) holds for continuous f.

[PSet 6-I problem 2(3)] $\forall f\in L^1(\mathbb{R})$ $\forall \varepsilon>0$ $\exists$ g continuous with compact support s.t.: $\|f-g\|_{L^1} = \int_{\mathbb{R}} |f(x)-g(x)|\,dm(x) \le \varepsilon$.

$\boxed{\text{Fixed any } \alpha,\ \varepsilon}$ By (*) $m\left(\left\{x : \sup_{h>0} \frac{1}{h}\int_{[x,x+h]} |f(t)-g(t)|\,dt > \alpha\right\}\right) \le \frac{\varepsilon}{\alpha}$.

By Markov:

$$m(\{x : |f-g| > \alpha\}) \le \frac{\varepsilon}{\alpha}.$$

So $\exists E$ with $m(E) \le \frac{2\varepsilon}{\alpha}$ s.t. $\forall x\in \mathbb{R}\setminus E$.

$$\frac{1}{h}\int_{[x,x+h]} |f(t)-g(t)|\,dt \le \alpha.$$

$$|f(x)-g(x)| \le \alpha \to \frac{1}{h}\int_{[x,x+h]} |f(x)-g(x)|\,dt \le \alpha$$

By calculus $\frac{1}{h}\int_{[x,x+h]} |g(t)-g(x)|\,dt \le \alpha$.

$$\left|\frac{1}{h}\int_{[x,x+h]} g(t)\,dt - g(x)\right| \le \alpha \text{ for all } h \text{ small enough.}$$

By triangle inequality, for all h small enough.

$$\left|\frac{1}{h}\int_{[x,x+h]} f(t)\,dt - f(x)\right| \le 3\alpha.$$

$$\Rightarrow \limsup_{h\to 0^+} \left|\frac{1}{h}\int_{[x,x+h]} f(t)\,dt - f(x)\right| \le 3\alpha$$

Keeping α fixed. let $\varepsilon \to 0$.

$$\limsup_{h\to 0^+} \left|\frac{1}{h}\int_{[x,x+h]} f(t)\,dt - f(x)\right| \le 3\alpha \quad x\in\mathbb{R} \text{ a.e.} \qquad A_\alpha.\ m(A_\alpha^c) = 0$$

Let $\alpha = \frac{1}{h} \to 0$.

$$m\left(\bigcup_n A_{\frac{1}{n}}^c\right) = 0$$

$$\frac{1}{h}\int_{[x,x+h]} f(t)\,dt = f(x) \quad \text{a.e. } x.$$

$\mathbb{R} \to \mathbb{R}^d$. Suppose $f\in L^1_{loc}(\mathbb{R}^d)$

(1) $\lim_{r\to 0} \frac{1}{m(B_r(x))}\int_{B_r(x)} f(t)\,dt = f(x)$ a.e. $x\in\mathbb{R}^d$

$\Uparrow$

$$\lim_{h\to 0^+} \frac{1}{2h}\int_{[x-h,x+h]} f(t)\,dt = f(x)$$

(2) Moreover, $\lim_{r\to 0} \frac{1}{m(B(x,r))}\int_{B(x,r)} |f(x)-f(t)|\,dt = 0$ a.e. $x\in\mathbb{R}^d$

Note: In dim 1. $\lim_{h\to 0^+}\frac{1}{2h}\int_{[x-h,x+h]} f(t)\,dt = F'(t)$.

So one can think of

$$\text{"}\underbrace{\lim_{r\to 0}\frac{1}{m(B(x,r))}\int_{B(x,r)} f(t)\,dt}_{= \frac{\mu(B(x,r))}{m(B(x,r))}}\text{"} \text{ as a "derivative"}$$

Let $d\mu = f\,dm$

"deriavative" of μ w.r.t. m.

Rmk: The theorem holds for $f\in L^1_{loc}(\mathbb{R}^d)$.

i.e. $\forall x.\ \exists r.$ s.t: $f\in L^1(B(x,r))$

Notation: $Mf := \sup_{r>0}\frac{1}{m(B(x,r))}\int_{B(x,r)}|f(t)|\,dt$

"Vitali-type Covering lemma" Let $\mathcal{U}=\{B(x_\alpha,r_\alpha)\}$ be a family of open balls, s.t: $U\subset\bigcup_\alpha B(x_\alpha,r_\alpha)$. Then $\forall c<m(U)$ one can find disjoint balls $B(x_1,r_1)$ — $B(x_n,r_n)$ in $\mathcal{U}$ s.t:

$$\sum_{j=1}^n m(B(x_j,r_j)) > 3^{-d}c.$$

$=0$

Proof of Hardy-Littlewood

Let $U=\{x\in\mathbb{R}^d : \underline{Mf^*} > \alpha\}$.

For any $x\in U$. $\exists r(x)$ s.t: $\int_{B(x,r(x))}|f(t)|\,dt > \alpha\cdot m(B(x,r(x)))$

Then $U\subset\bigcup_{x\in U}B(x,r(x))$

By Vitali. $\forall c<m(U)$. $\exists B(x_1,r_1),\dots,B(x_n,r_n)$ in this family

s.t: $c<3^d\sum_{j=1}^n m(B(x_j,r(x_j))) < 3^d\frac{1}{\alpha}\sum_j \underbrace{\int_{B(x_j,r(x_j))}}_{\text{disjoint}}|f(t)|\,dt$

$\le \frac{3^d}{\alpha}\int_{\mathbb{R}^d}|f(t)|\,dt$

Proof of Vitali properties in Radon measure

For any $c < m(\mathcal{U})$ one can find K compact s.t: $m(K) > c$. $K \subset \mathcal{U}$. Since $K \subset \bigcup_{\alpha} B(x_\alpha, r_\alpha)$, one can find $B(y_1, s_1), \ldots, B(y_m, s_m)$ in the family $\mathcal{U}$ s.t:

$$K \subset \bigcup_{i=1}^{m} B(y_i, s_i)$$

WLOG assume $s_1 \geq s_2 \geq \cdots \geq s_m$.

"Greedy algorithm": Let $A_1 = B(y_1, s_1)$.

Suppose we have got $A_1, \ldots, A_{j-1}$, Let A_j be the ball in the family $\{B(y_i, s_i)\}$.

(Stop until we can't find more) s.t: $A_j \cap (A_1 \cup \cdots \cup A_{j-1}) = \phi$. with largest volumes

Then we get n-disjoint balls $\underset{B(x_1, r_1)}{\underset{\shortparallel}{A_1}}, \ldots, \underset{B(x_n, r_n)}{\underset{\shortparallel}{A_n}}$.

For $x \in B(y_k, s_k)$, then we can find A_j s.t: $A_j \cap B(y_k, s_k) = \phi$.

Take the first $A_j = B(x_j, r_j)$. Then $r_j \geq s_k$. ($\underbrace{B(y_k, s_k) \cap (A_1 \cup \cdots A_{j-1})}_{=\phi}$).

Then $B(y_k, s_k) \subset \underline{B(x_j, 3r_j)}$. metric property.

Since $K \subset \bigcup_{k} B(y_k, s_k) \subset \bigcup_{j} B(x_j, 3r_j)$

$$\Rightarrow \quad m(K) \leq \sum_{j=1}^{n} m(B(x_j, 3r_j)) = 3^d \sum_{j=1}^{n} m(B(x_j, r_j))$$

additional property on Lebesgue measure

Rmk: The Vitali Lemma holds for Radon measure μ which satisfies "doubling property"

$$\mu(B(x, 2r)) \leq C \cdot m(B(x, r))$$

$\forall x. \forall r.$ C is constant.

"Lebesgue differential" V.S. "Radon–Nikodym derivative"

$$d\mu = f\,dm \quad = \lim_{r\to 0} \frac{\mu(B(x,r))}{m(B(x,r))} = f(x) \qquad f = \frac{d\mu}{dm} \rightsquigarrow d\mu = f\,dm.\ (\mu \ll m)$$

not required in left side

$\forall$ compact K. $\mu(K) < \infty$. $\mu(A) = \inf_{\text{open } U \supset A} \mu(U)$

Thm: Let μ be a locally finite, outer regular Borel measure on $\mathbb{R}^d$.

Let $\mu = \mu_r + \mu_s$. $\mu_r \ll m$. $\mu_s \perp m$. $\quad \mu = \mu_s + \mu_r$, $d\mu_r = f\,dm$. $\frac{d\mu_r}{dm}$

Then for a.e. $x \in \mathbb{R}^d$. $\lim_{r\to 0} \frac{\mu(B(x,r))}{m(B(x,r))} = \frac{d\mu_r}{dm}(x)$.

Proof: By Lebesgue differentiation theorem, for a.e. $x \in \mathbb{R}^d$

$$\lim_{r\to 0} \frac{\mu_r(B(x,r))}{m(B(x,r))} = \lim_{r\to 0} \frac{\int_{B(x,r)} f(t)\,dt}{m(B(x,r))} = f(x) = \frac{d\mu_r}{dm}(x).$$

It remains to prove

$$\boxed{\lim_{r\to 0} \frac{\mu_s(B(x,r))}{m(B(x,r))} = 0.}$$

Let $A \in \mathcal{B}$ s.t. $\mu_s(A) = 0$. $m(A^c) = 0$.

For $k \in \mathbb{N}$ let $E_k = \{x \in A : \limsup_{r\to 0} \frac{\mu_s(B(x,r))}{m(B(x,r))} > \frac{1}{k}\} \subset A$

Need: $m(E_k) = 0$ $\forall k$.

By outer regularity. $\exists U_\varepsilon \supset A$ s.t. U_ε open.

$\mu_r(U_\varepsilon) + \mu_s(U_\varepsilon) = \mu(U_\varepsilon) \le \mu(A) + \varepsilon = \mu_r(A) + \varepsilon$.

$\Rightarrow \mu_s(U_\varepsilon) \le \varepsilon$.

For any $x \in E_k$. $\exists$ open ball $B_x \subset U_\varepsilon$ s.t. $\mu_s(B_x) > \frac{1}{k} m(B_x)$.

Let $E_k \subset V_\varepsilon = \bigcup_{x \in E_k} B_x \subset U_\varepsilon$. Then $\forall\, c < m(V_\varepsilon)$

$\exists$ finite balls $B_{x_1}, \dots, B_{x_n}$ disjoint, s.t.

$$c < 3^d \sum_{j=1}^n m(B_{x_j}) \le 3^d k \sum \mu_s(B_{x_j}) \le 3^d k\, \mu_s(V_\varepsilon) \le 3^d k \varepsilon.$$

$\Rightarrow m(E_k) = 0$

. Both Lebesgue differential and thm above hold if we replac

$\{B(x,r)\}$ by any family $\{E(x,r)\}$ that "shrinks nicely" to each point". i.e. $x \in E(x,r)$. s.t: ① $E(x,r) \subset B(x,r)$.
② $\exists \alpha$ indep of x, r
s.t: $m(E(x,r)) > \alpha\, m(B(x,r))$,

Recall: $F \in C^1(\mathbb{R})$. F' is Riemann integral
$\int_a^b F'(x)\,dx = F(b) - F(a)$

Lec 26 Differentiability of functions

1 The case of monotone function

Note: Consider $\mu_f = f\,dm$, $f \geq 0$ and f is integrable
Then $F(x) = \int_{(-\infty,x]} f\,dt = \mu_f((-\infty,x])$
We have seen F' exists a.e. (Lebesgue differentiation)
In general, for any bounded measure μ.
We can define $F(x) = \mu((-\infty,x])$.

Example: $\cdot\ \mu = \chi_{[0,1]}\,dm + \delta_0 + \delta_1$

$\rightarrow F(x) = \begin{cases} 0 & x<0 \\ 1+x & 0 \le x < 1 \\ 3 & x \ge 1. \end{cases}$ $\quad F' = \chi_{[0,1]}$ a.e.

WANT: F. $\int_{[a,b]} F'(t)\,dt = F(a) - F(b)$. let $F'(t) = f$ $\quad F = \int_{[-\infty,x]} f(t)\,dt$ → which kind of F:
$F = \int_{[-\infty,x]} f\,dt = \mu([-\infty,x])$.

Note: These functions $F(x) = \mu((-\infty,x])$.
are monotone increasing, right-continuous.
(Let $x_n \to x_0^+$ Then $(-\infty, x_n] \to (-\infty, x_0]$
$x_n \to x_0^-$ $\quad (-\infty, x_n] \to (-\infty, x_0)$.
apply monotone convergence of measure)

Conversely, in PSet 9-2-2, for any increasing, right continuous function $F: \mathbb{R} \to \mathbb{R}$, there exists a metric measure (⇒ Borel measure) μ_F s.t: $\mu_F((a,b]) = F(b) - F(a)$
(PSet-10-1, ⇒ μ_F is Radon ⇒ regular)

Thm (Monotone differentiation thm)
Any monotone function F on $\mathbb{R}$ is a.e. differentiable

Proof: WLOG. we assume F is monotone increasing.

Let $G(x) = F(x+)$. right continuous.

Observations: F is discontinuous at most countable many points.

Reason: x_α discontinuous $(F(x_\alpha^-), F(x_\alpha^+)) \Rightarrow \bigcup_\alpha (F(x_\alpha^-), F(x_\alpha^+))$ open. disjoint $\Longrightarrow$ PSet 2-1-2. countable many

$\uparrow$ monotonicity

$\Rightarrow G = F$ a.e.

$E(x,r)$ Shrinks nicely.

Consider $\lim\limits_{h\to 0^+} \dfrac{G(x+h)-G(x)}{h} = \lim\limits_{h\to 0^+} \dfrac{\mu_G((x, x+h])}{m((x, x+h])} = \dfrac{d(\mu_G)_r}{dm}(x)$. a.e. x

$\lim\limits_{h\to 0^+} \dfrac{G(x)-G(x-h)}{h} = \lim\limits_{h\to 0^+} \dfrac{\mu_G((x-h, x])}{m((x-h, x])} = \dfrac{d(\mu_G)_r}{dm}(x)$ a.e. x

$\Rightarrow G'(x)$ exists for a.e. x.

Let $H = G - F$ We prove $H' = 0$ a.e.

Let $\{x_n\}$ be discontinuous points of F s.t. $H(x_n) \neq 0$.

By monotonicity, $H(x_n) > 0$. Moreover,

$$\sum_{x_n \in [-N,N]} H(x_n) \le F(N) - F(-N)$$

$\Rightarrow$ $\nu = \sum H(x_n)\,\delta_{x_n}$ is locally finite measure.

AND $\nu \perp m$.

By same arguement $H'(x) = 0$ a.e. $\Rightarrow F' = G'$ a.e.

$F' = \lim\limits_{h\to 0} \dfrac{F(x+h)-F(x)}{h} = \lim\limits_{h\to 0} \dfrac{\mu_F(x, x+h)}{m((x, x+h))} = \dfrac{d(\mu_F)_r}{dm}$

Rough idea: Non-negative theory. absolute integrable theory

$f \geq 0$ $\qquad$ $f = f^+ - f^-$

μ measure $\qquad$ Signed measure $\mu = \mu_+ - \mu_-$

F monotone increasing $\qquad$ difference of two monotone increasing

??

Which kind of F?

Note: if F is a bounded increasing function. Then for any $x_1 < x_2 < \cdots < x_n$, we have $\sum_{i=1}^{n-1} |F(x_{i+1}) - F(x_i)| < F(+\infty) - F(-\infty)$

Def: The $\boxed{\text{total variation}}$ of F is
$$\|F\|_{TV} = \sup\left\{ \sum_{i=1}^{n-1} |F(x_{i+1}) - F(x_i)| : \forall n.\ \forall x_1 < x_2 < \cdots < x_n \right\}.$$
We say F is a BV (bounded variation) function if $\|F\|_{TV} < +\infty$.

e.g. If F is monotone and bounded, then F in BV.
or the difference of two bounded and monotone functions

Rmk: If f is BV then F is bounded.

Thm: A function $F: \mathbb{R} \to \mathbb{R}$ is a BV function if and only if it's the difference of two bounded monotone increasing functions
($\Rightarrow$ any BV function is a.e. differentiable)

Proof: Define "positive variation function",
$$F^+(x) = \sup\left\{ \sum_{i=1}^{\infty} \max\left(F(x_{i+1}) - F(x_i), 0\right) .\ \forall n.\ \forall x_1 < x_2 < \cdots < x_n \le x \right\}$$
Then $0 \le F^+(x) \le \|F\|_{TV}$, and $F^+(x) \nearrow$.

Let $F^-(x) = F^+(x) - F(x)$. Then F^- bounded and $F = F^+ - F^-$

Remains to prove: $F_-(x) \nearrow$.

Let $a < b$ Want: $F_-(a) \le F_-(b)$ i.e.
$$F^+(b) \ge F^+(a) + F(b) - F(a). \quad (*)$$
If $F(b) \le F(a)$ $\quad$ $(*)$ holds

If $F(b) > F(a)$ $\quad$ For any partition
$$\begin{cases} x_1 < x_2 < \cdots < x_n \le a \\ x_1 < x_2 < \cdots < x_n < x_{n+1} = a < x_{n+2} = b. \end{cases}$$
By def of F^+, $F^+(b) - F^+(a) \ge F(b) - F(a)$.

$\forall$ locally BV f. f is a.e. differentiable and $f' \in L^1_{loc}(\mathbb{R})$

WANT $\displaystyle\int_a^b F'(x)\,dx = F(b) - F(a) \quad (\star)$

Note: Even for monotone function $(\star)$ may fail.

Note Generally: $F(x) = F(a) + \displaystyle\int_a^x F'(t)\,dt.$

OURSTORYBEGINS

In Lec 11. Lebesgue integral is "absolutely continuous"

$f \in L' \Rightarrow \forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall A$ with $m(A) < \delta$. we have $\int_A |f| dx < \varepsilon$.

$f = F'$

Take $\underbrace{A = \overset{n}{\underset{i=1}{\bigcup}} (a_i, b_i)}_{<\delta}$ disj. and $\rightarrow \left|\int_A \underset{= F'(t)}{f}\right| \le \sum_{i=1}^{n} \left|\int_{(a_i,b_i)} F'(t) dt\right| = \sum_{i=1}^{n} \left(F(b_i) - F(a_i)\right)$
$\le \int_A |f| < \varepsilon$.

Def: || A function f is absolutely continuous ($\boxed{AC}$) $\forall \varepsilon. \exists \delta$.
If for any disjoint (a_i, b_i) $1 \le i \le n$. with $\sum_{i=1}^{n} (b_i - a_i) < \delta$.
We have $\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \varepsilon$.

Example. $f \in L' \Rightarrow F(x) = \int_{(-\infty, x]} f(t) dt$ is AC.

Prop 1 || If F is AC on $[a,b]$ then F is BV.

Prop 2 || If F is AC on $[a,b]$ and $F'(x) = 0$ a.e. then F = constant
Only need to prove $F(a) = F(b)$.

Proof. $F'(x) = 0. \Rightarrow \exists (a_i, b_i) \ni x$. s.t. $|F(b_i) - F(a_i)| \le \varepsilon (b_i - a_i)$
can take (a_i, b_i) arbitrarily small

Let $A = \{x \mid F'(x) = 0\}. \Rightarrow m(A) = b - a$.

Vitali covering of $A \Rightarrow \exists (a_i, b_i)$ s.t. $\sum_{i=1}^{M} |b_i - a_i| > (b-a) - \delta$

$[a,b] \setminus \bigcup_{i=1}^{M} (a_i, b_i) = \sum_{k=1}^{N} (\alpha_k, \beta_k) \rightsquigarrow \sum_{k=1}^{N} |\beta_k - \alpha_k| < \delta$.

Finally, $|F(b) - F(a)| \le \varepsilon$. Const.

Prop 3 || Let F be a monotone $\boxed{\text{increasing}}$ function
Then $\int_{[a,b]} F'(x) dx \le F(b) - F(a)$.
In particular $F' \in L'([a,b])$.

Proof. $LHS = \int_{[a,b]} \lim \frac{F(x+\frac{1}{n}) - F(x)}{\frac{1}{n}} dx$.

$\le \liminf_{n \to \infty} \int_{[a,b]} \frac{\overbrace{F(x+\frac{1}{n}) - F(x)}^{\ge 0}}{\frac{1}{n}} dx$.
Fatou

$= \liminf_{n \to \infty} \frac{1}{n} \left[\int_{[b, b+\frac{1}{n}]} F(x) dx - \int_{[a, a+\frac{1}{n}]} F(x) dx\right] \rightarrow F(b) - F(a)$

$F \in L'([a,b])$

The fundamental theorem of calculus (Version 2)

Let $F : [a,b] \to \mathbb{R}$ be AC. Then $\int_{[a,b]} F'(x)\,dt = F(b) - F(a)$

$F(x) = F(a) + \int_{[a,x]} F'(t)\,dt$

Proof: $AC \Rightarrow BV \Rightarrow F'$ exists a.e.

Moreover, F is $BV \Rightarrow F = F_1 - F_2$, F_i bounded monotone

$\Rightarrow F' \in L^1([a,b])$.

Let $G(x) = \int_{[a,x]} F'(t)\,dt + F(a)$.

Then by a.c. of Lebesgue integral. we know G is AC.

$\Rightarrow F - G$ is AC.

By Fundamental Theorem of Calculus (Version 1).

we know $G'(x) = F'(x)$ a.e.

$\Rightarrow (F-G)' = 0$ a.e.

By Prop 2 $F - G = \text{const}$, $F(a) = G(a) \Rightarrow F = G$.

Rmk: $AC \Rightarrow F(x) = F(a) + \int_a^x F'(t)\,dt$

$\Leftarrow$ If and $F' \in L^1$

Lec. 27.

1 Some applications

Thm (Integration by parts) Let $f \cdot g$ be AC functions. on $[a,b]$
Then $\int_{[a,b]} f'(t) g(t) dt = f(t)g(t)\Big|_a^b - \int_{[a,b]} f(t) g'(t) dt$

proof. By pset 14-1-3 $f \cdot g$ is AC. Since

$$(fg)' = f'g + fg' \quad a.e.$$

$$\Rightarrow f(b)g(b) - f(a)g(a) = \int_{[a,b]} (fg)' dt = \int_{[a,b]} f'g\, dt = \int_{[a,b]} fg' dt.$$

Thm Let f be AC function on $[a,b]$. Then $\|f\|_{TV([a,b])}$
$= \int_{[a,b]} |f'(t)| dt = \|f'\|_{L^1([a,b])}$.

Proof: • For any partition $a = x_1 < x_2 < \cdots < x_n = b$. we have

$$\sum_{i=1}^{n-1} |f(x_{i+1}) - f(x_i)| = \sum_{i=1}^{n-1} \left| \int_{[x_i, x_{i+1}]} f'(t) dt \right| \le \sum_{i=1}^{n} \int_{[x_i, x_{i+1}]} |f'(t)| dt = \int_{[a,b]} |f'(t)| dt$$
$$= \|f'\|_{L^1([a,b])}$$

• Want: $\|f'\|_{L^1} \le \|f\|_{TV} + \varepsilon$

By Pset 6-1-2. $\because f' \in L^1([a,b])$. $\exists$ step function g on $[a,b]$
s.t: $\|h\|_{L^1} < \varepsilon$, where $h = f' - g$.

Let $G(x) = \int_{[a,x]} g(t) dt \quad H(x) = \int_{[a,x]} h(t) dt$.

$\Rightarrow f(x) = f(a) + \int_{[a,x]} f'(t) dt = f(a) + G(x) + H(x)$

Since $H' = h \in L^1$. H is AC. $\Rightarrow \|H\|_{TV} \le \|h\|_{L^1} < \varepsilon$.

By triangle $\quad \|f\|_{TV} \ge \|G\|_{TV} - \|H\|_{TV} \ge \|G\|_{TV} - \varepsilon \ge \|f'\|_{L^1} - 2\varepsilon$

Write $g = \sum_{i=1}^{n-1} c_i \chi_{[x_i, x_{i+1}]}$, where $a = x_1 < x_2 < \cdots < x_n = b$.

$$\Rightarrow \|G\|_{TV} \ge \sum_{i=1}^{n-1} |G(x_{i+1}) - G(x_i)| = \sum_{i=1}^{n-1} \left| \int_{[x_i, x_{i+1}]} c_i\, dt \right|$$
$$= \int_{[a,b]} \sum_{i=1}^{n-1} |c_i| \chi_{[x_i, x_{i+1}]} dt = \|g\|_{L^1}.$$

Rmk. These theorem holds for $\mathbb{C}$-valued functions.

A geometric application: rectifable curves.

2. Lipschitz Countinuity

Def: A function $f: \mathbb{R} \to \mathbb{R}$ is Lipschitz Continuous if $\exists M > 0$. s.t: $|f(x) - f(y)| \leq M |x - y|$ $\forall x, y$.

$\rightarrow$ Lipschitz Continuous function between metric space

$f: \underset{\mathbb{R}^d}{X} \to \underset{\mathbb{R}^n}{Y}$ $\quad |f(x_1) - f(x_2)| \leq M |x_1 - x_2|$.

Def: $Lip(f) = \sup\limits_{x \neq y} \dfrac{|f(x) - f(y)|}{|x - y|}$

Prop: If f is AC. and $|f'| \leq M$ a.e. then f is Lipschitz continuous, and $Lip(f) \leq M$.

Proof: $|f(x) - f(y)| = \left| \int_{[x, y]} f'(t)\, dt \right|$
$\leq \int_{[x,y]} |f'(t)|\, dt \leq M |x - y|$

Thm: Let $F: [a,b] \to \mathbb{R}$ be Lipschitz Continuous with $Lip(F) = M$. Then $\exists f \in L^{\infty}([a,b]) \subset L^1([a,b])$ with $\|f\|_{L^\infty} \leq M$ s.t: $F(x) - F(a) = \int_{[a,x]} f(t)\, dt$

Cor: If $F: \mathbb{R} \to \mathbb{R}$ Lipschitz continuous, $Lip(F) = M$. then F is AC, and $|F'| \leq M$ a.e.

Cor: Any Lipschitz continuous function $F: \mathbb{R} \to \mathbb{R}$ is a.e. differentiable

Proof: We define a linear functional on the space of step function by $L\left(\sum_{i=1}^{n-1} a_i \chi_{[x_i, x_{i+1}]}\right) = \sum_{i=1}^{n-1} a_i \left(F(x_{i+1}) - F(x_i)\right)$

For any $h = \sum_{i=1}^{n-1} a_i \chi_{[x_i, x_{i+1}]}$ (step function) we have

$|L(h)| = \left|\sum_{i=1}^{n-1} a_i (F(x_{i+1}) - F(x_i))\right| \le \sum_{i=1}^{n-1} |a_i| \cdot M |x_{i+1} - x_i|$

$= M \|h\|_{L^1([a,b])}$.

$\Rightarrow$ L is bounded on the space of step functions which is dense in $L^1([a,b])$

$\rightsquigarrow$ L extends a continuous linear functional on $L^1([a,b])$.

By Riesz representation $\exists f \in L^\infty([a,b])$. s.t: $L(g) = \int_{[a,b]} f(t) g(t)\,dt$.

Take $g = \chi_{[a,x]}$ $x \in [a,b]$.

we get $F(x) - F(a) = L(g) = \int_{[a,b]} f(t) \chi_{[a,x]}(t)\,dt$

$= \int_{[a,x]} f(t)\,dt$.

- Now consider functions $f: \mathbb{R}^d \to \mathbb{R}$

Recall: Directional derivative of f in the direction $v \in \mathbb{R}^d$ at x_0, $D_v f(x_0) = \lim_{\mathbb{R} \ni h \to 0} \frac{f(x_0 + hv) - f(x_0)}{h}$.

f is differentiable at x_0 if $\exists$ a linear map $L: \mathbb{R}^d \to \mathbb{R}$

s.t: $\lim_{\mathbb{R}^d \ni h \to 0} \frac{|f(x_0+h) - f(x_0) - L(h)|}{|h|} = 0$.

$\uparrow$ depends on x_0.

Rmk: If f is differentiable, then $D_v f(x_0) = L(v)$.

In this case we can denote $L(e_i) = D_{e_i} f(x_i) = \frac{\partial f}{\partial x_i} = \partial_i f$.

Then we write $\nabla f = \langle \partial_1 f, \cdots, \partial_d f \rangle$

$\Rightarrow L(v) = \nabla f \cdot v$

It's possible that all $D_v f(x_0)$ exists, but f is not differentiable at x_0, [One need: continuous of partial derivatives].

Thm (Rademacher differentiability theorem) Let: $F: \mathbb{R}^d \to \mathbb{R}^{(n)}$ be a Lipschitz coutinuous function, then F is a.e. differentiable.

Proof: Since F is continuous, $D_v F(x_0)$ exists if and only if
$$\limsup_{\mathbb{Q} \ni h \to 0} \frac{F(x_0+hv)-F(x_0)}{h} = \liminf_{\mathbb{Q} \ni h \to 0} \frac{F(x_0+hv)-F(x_0)}{h}.$$

$\Rightarrow \forall v \in \mathbb{R}^d$ the set

$E_v := \{x \in \mathbb{R}^d : D_v f(x_0)$ does not exist$\}$ is a Borel set

Moreover, $D_v F$ is a measurable function on E_v^c.

and $|D_v F| \le C$.

Fact: $m(E_v) = 0$

Proof: Obvious for $v = 0$.

For $v \neq 0$ apply a linear transformation $v \rightsquigarrow e_1$

So WLOG, we need to show $\partial_1 F$ exists a.e.

We split $\mathbb{R}^d = \underset{\overset{\downarrow}{x^\circ}}{\mathbb{R}} \times \underset{\overset{\downarrow}{y^\circ}}{\mathbb{R}^{d-1}}$

Fix any (x°, y°)

$\partial_1 F(x^\circ, y^\circ)$ exists $\Leftrightarrow$ $\underbrace{x \to F(x, y^\circ)}_{\text{Lipschitz}}$ is differentiable at x°.

$\Rightarrow$ $E_v^{y_0} = \{x^\circ \in \mathbb{R} : (x^\circ, y^\circ) \in E_v\}$ is a null set.

By Fubini (Tonelli).

$$m(E_v) = \iint \chi_{E_v}\, dx\, dy = \int_{\mathbb{R}^{d-1}} \underbrace{\int_{\mathbb{R}} \chi_{E^{y_0}}(x)\, dx}_{\forall y_0}\, dy = 0.$$

$\Rightarrow m\left(\bigcup_{v \in \mathbb{Q}^d} E_v\right) = 0.$

i.e. For a.e. $x \in \mathbb{R}^d$. $D_v F(x)$ exists for all $v \in \mathbb{Q}^d$.

Let $g \in \underset{\text{compact supp}}{C_0^\infty}(\mathbb{R}^d)$ Then

$$\int_{\mathbb{R}^d} D_v F(x)\, g(x)\, dx = \int_{\mathbb{R}^d} \lim_{\mathbb{Q} \ni h \to 0} \frac{F(x+hv)-F(x)}{h} g(x)\, dx$$
$$\overset{(\mathrm{LDCT})}{=} \lim_{\mathbb{Q} \ni h \to 0} \int_{\mathbb{R}^d} \frac{F(x+hv)-F(x)}{h} g(x)\, dx$$

$$= \lim_{0 \neq h \to 0} \int_{\mathbb{R}^d} F(x) \underbrace{\frac{g(x+hv) - g(x)}{h}}_{\text{compact supp}} dx \xlongequal{\text{LDCT}} \int_{\mathbb{R}^d} F(x) D_v g(x)\, dx.$$

$\Rightarrow$ LHS is linear in v

So if we let $v = \sum_{i=1}^{d} v_i e_i$ then

$$\int (D_v F)(x)\, g(x) = \sum v_i \int (\partial_i F)(x)\, g(x)\, dx.$$

$$\Rightarrow \int_{\mathbb{R}^d} (D_v F - v \cdot \nabla F)\, g(x)\, dx = 0. \quad \forall g \in C_0^\infty(\mathbb{R}^d)$$

$$\Rightarrow D_v F = v \cdot \nabla F \text{ a.e } x. \quad \forall v \in \mathbb{Q}^d.$$

- Let $A = \{x \in \mathbb{R}^d : D_v F(x) = v \cdot \nabla F(x). \ \forall v \in \mathbb{Q}^d\}$

Claim: F is differentiable on A.

Define $\widetilde{F}(h) = F(x_0+h) - F(x_0) - h \cdot \nabla F(x_0)$ $x_0 \in A$.

Want $\lim_{h \to 0} \frac{|\widetilde{F}(h)|}{|h|} = 0.$ $\uparrow$ $\widetilde{F}(0) = 0.$

$\widetilde{F}$ is Lipschitz

Let $\varepsilon > 0$. $h = ru$, $r = |h|$ $u \in S^{d-1}$

Take a finite set $V_\varepsilon \subset \mathbb{Q}^d \cap S^{d-1}$, s.t: $\exists v \in V_\varepsilon$.

with $|u - v| < \varepsilon$.

x°. $\Rightarrow \frac{|\widetilde{F}(rv) - \widetilde{F}(0)|}{r} < \varepsilon \quad \forall v \in V_\varepsilon \quad r$ small.

$\Rightarrow |\widetilde{F}(rv)| \leq r\varepsilon.$

Since $\widetilde{F}$ is Lipschitz

$$|\widetilde{F}(h) - \widetilde{F}(rv)| = |\widetilde{F}(ru) - \widetilde{F}(rv)| \leq \mathrm{Lip}(\widetilde{F}) \cdot |r| \cdot |u - v| \leq \mathrm{Lip}(\widetilde{F}) \cdot \varepsilon \cdot r$$

$$\Rightarrow |\widetilde{F}(h)| \leq (\mathrm{Lip}(\widetilde{F}) + 1)\, |h|\, \varepsilon$$

$$\Rightarrow \lim_{h \to 0} \frac{|\widetilde{F}(h)|}{|h|} = 0.$$

Pset 14-1.

1. ④ $\forall a=x_0<\cdots<x_{n+1}=b=y_0<\cdots<y_{m+1}=c$.

$\|F\|_{TV([a,c])} \geq \sum|F(x_{k+1})-F(x_k)|+\sum|F(y_{k+1})-F(y_k)|$

右边取 sup $\Rightarrow \|F\|_{TV([a,c])} \geq \|F\|_{TV([a,b])}+\|F\|_{TV([b,c])}$.

另一方面 $\forall \varepsilon>0\ \exists\ a\leq x_0\cdots<x_{n+1}\leq b<y_0<\cdots<y_{m+1}\leq c$.

$\|F\|_{TV([a,c])}-\varepsilon \leq \sum|F(x_{k+1})-F(x_k)|+\sum|F(y_{k+1})-F(y_k)|+|y_0-x_{n+1}|$

$\leq \sum|F(x_{k+1})-F(x_k)|+|b-x_{n+1}|+\sum|F(y_{k+1})-F(y_k)|+|y_0-b|$

$\leq \|F\|_{TV([a,b])}+\|F\|_{TV([b,c])}$

2. ① $D^+F=\limsup\limits_{h\to 0}\dfrac{F(x+h)-F(x)}{h}$.

$G=\limsup\limits_{\substack{h\to 0\\ h\in\mathbb{Q}}}\dfrac{F(x+h)-F(x)}{h}$.

只证 $D^+F=G$. $\forall\varepsilon>0\ \forall h>0$. 取 $h_1<h<h_2$, $h_1,h_2\in\mathbb{Q}$.

$\dfrac{h_1}{h}>1-\varepsilon$, $\dfrac{h_2}{h}<1+\varepsilon$.

则 $(1-\varepsilon)\dfrac{F(x+h_1)-F(x)}{h_1}\leq\dfrac{F(x+h_1)-F(x)}{h}\leq\dfrac{F(x+h)-F(x)}{h}\leq\dfrac{F(x+h_2)-F(x)}{h_2}(1+\varepsilon)$

取 $\limsup\limits_{h\to 0^+}\Rightarrow (1-\varepsilon)G(x)\leq D^+F(x)\leq(1+\varepsilon)G(x)$.

② $F\nearrow\Rightarrow D^+F=F'$ a.e.

$m(\{D^+F>\alpha\})=m(\{F'>\alpha\})\leq\frac{1}{\alpha}\int_a^b F'dx\leq\frac{1}{\alpha}(F(b)-F(a))$

3. ② $\forall\varepsilon>0.\ \exists\delta>0.\ \exists[a_i,b_i]\ A\subset\bigcup[a_i,b_i]\ \underbrace{\sum(b_i-a_i)<\delta}$. $\cancel{\sum|F(b_i)-F(a_i)|}<\varepsilon$

取 $a_i'=\operatorname{argmin}\limits_{x\in[a_i,b_i]}F(x)$ $b_i'=\operatorname{argmax}\limits_{x\in[a_i,b_i]}F(x)$

$\sum(b_i'-a_i')<\delta$.

则 $\underbrace{[a_i',b_i']\subset[a_i,b_i]}$ $F([a_i,b_i])=[F(a_i'),F(b_i')]$

$m(F(A))\leq\sum m(F[a_i,b_i])\leq\underbrace{\sum|F(b_i')-F(a_i')|}<\varepsilon$.

4. $F_j\geq 0$. $F_j\nearrow$. $F=\sum F_j<\infty$. pf. $F'=\sum\limits_{j=0}^{\infty}F_j'$.

pf: 不妨设 F_j 右连续. $F_j(x)=F_j(x+)$ $F(x)=F(x+)$

$\mu_F=\sum\mu_{F_j}$ 设 $\underset{\ll m}{(\mu)_{ac}}$. $\underset{\perp m}{(\mu)_s}$

$\mu_F = \Sigma \mu_{F_j} = \underbrace{\Sigma (\mu_{F_j})_{ac}}_{\ll m} + \underbrace{\Sigma(\mu_{F_j})_s}_{\perp m}$ 由分解唯一 $\begin{cases}(\mu_F)_{ac} = \Sigma(\mu_{F_j})_{ac}\\ (\mu_F)_s = \Sigma(\mu_F)_s\end{cases}$

$\forall A \in \mathcal{B}(\mathbb{R})$.

$(\mu_F)_{ac}(A) = \Sigma(\mu_{F_j})_{ac}(A) = \Sigma\int_A F_j'\,dm \xrightarrow{MCT} \int_A \Sigma F_j'\,dm$.

$= \int_A F'\,dm \Rightarrow F' = \Sigma F_j'$

Pset 14-2

1. $\{u_\alpha\}_{\alpha\in I}$ Lip. Lip$(u_\alpha)\le L$.

(1) $U = \sup u_\alpha$. 若在 x_0 点有限. pf: Lip$(U)\le L$

(2) $u = \inf u_\alpha$ ------- pf: Lip$(u)\le L$.

Pf. (1) $\forall x\in X$. $\forall\alpha$

$|u_\alpha(x) - u_\alpha(x_0)| \le L\,d(x,x_0)$

$U(x) = \sup_\alpha u_\alpha(x) \le \sup_\alpha u_\alpha(x_0) + L\,d(x,x_0) < \infty$.

$\Rightarrow U$ well-defined

$U(x) = \sup_\alpha u_\alpha(x) \le \sup_\alpha(u_\alpha(x)-u_\alpha(y)) + \sup_\alpha u_\alpha(y)$
$\le L\,d(x,y) + U(y)$.

2. $A\subset X$ $\forall$ Lip $f: A\to R$, Lip$(f)=L$ 令 $\hat f(x) = \inf_{y\in A}(f(y) + L\,d(x,y))$

Proof: Lip$(\hat f)\le L$.

Pf. 令 $g_y(x) = f(y) + L\,d(x,y)$.

$|g_y(x) - g_y(x')| = L|d(x,y) - d(x',y)| \le L\,d(x,x')$

$\Rightarrow$ Lip$(g_y)\le L$.

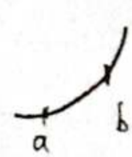
$\forall x_0$. $|f(y) - f(x_0)| \le L\,d(x_0,y)$.

$\Rightarrow \hat f(x_0) \ge f(y) + L\,d(x_0,y) \ge f(x_0) > \infty$. 由1可得.

3 f 凸 Prove:

① f 可微 a.e.

② f' ↑.



① $\forall a<x<b$. $x=(1-t)a+tb$ $\quad f(x)\le(1-t)f(a)+tf(b)$

$\Rightarrow \frac{f(x)-f(a)}{x-a} \le \frac{f(b)-f(a)}{b-a} \le \frac{f(b)-f(x)}{b-x}$.

OURSTORYBEGINS

④ 用LDT证 $\int_{\mathbb{R}^d} h\varphi = 0$ for $\forall \varphi \in C_0^\infty$. 则 $h = 0$ a.e. $h \in L^1_{loc}(\mathbb{R}^d)$

μ, ν signed, $\mu \perp \nu \Rightarrow |\mu| \perp \nu$. $\mu|_{X_2} = 0$, $\nu|_{X_1} = 0$.

(反证) 若 $\mu|_{X_2} = 0$ 但 $|\mu||_{X_2} \neq 0$. 则 $\exists A \subset X_2$ s.t.:

$|\mu|(A) > 0$. $X = X_+ \sqcup X_-$. $\mu_+|_{X_-} = 0$, $\mu|_{X_+} = 0$.

assume >0

$\underline{\mu_+(A \cap X_+)} + \mu_-(A \cap X_-) > 0$.

$A \subset X_2$. $A \cap X_+ \subset X_2$. $\mu(A \cap X_+) = \mu_+(A \cap X_+) > 0$.