Algebraic Methods of Mathematical Physics

Lecture notes for MATH3103 and MATH7133

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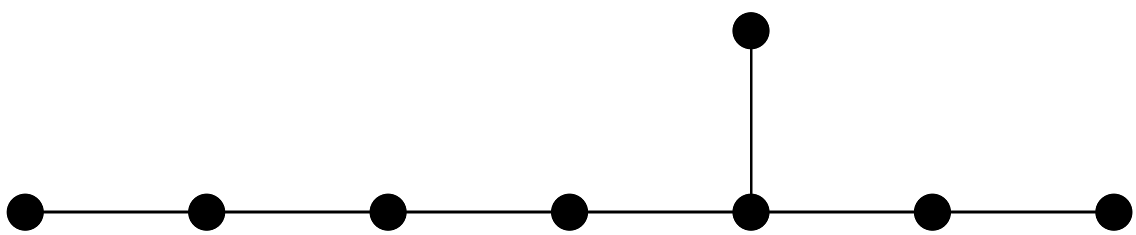

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Introduction

In the late 19th century, the Norwegian mathematician Marius Sophus **Lie** essentially created the theory of continuous symmetry and applied it to the study of geometry and differential equations. In the process, he introduced continuous transformation groups, now known as *Lie groups*. Their "linearisations" generate the associated infinitesimal transformations, whose algebraic structures, inherited from the groups, are encapsulated in the corresponding *Lie algebras* – the main topic of these lecture notes.

Contemporaneously and independently of Lie, Wilhelm Karl Joseph **Killing** developed the theory of Lie algebras well beyond Lie's work, albeit somewhat non-rigorously. This was resolved by Élie Joseph **Cartan** who significantly expanded Killing's work and seminal ideas and translated them into a mathematically rigorous theory. Curiously, the basic theory of Lie algebras is often attributed to Cartan, although most of it was discovered by Killing, except the so-called Killing form which was found by Cartan... In any case, both Killing and Cartan were instrumental in the early developments of the theory of Lie algebras.

Hermann Klaus Hugo **Weyl** is largely to credit for the introduction of Lie theory in quantum mechanics through work initiated in the 1920s. Much helped by this, Lie groups and Lie algebras have become household notions in mathematical and theoretical physics, and they continue to be indispensable tools for understanding the physical laws of nature. They also occupy a central place in pure mathematics where they often provide a bridge between seemingly disparate mathematical structures and notions. In fact, Lie theory itself is now widely regarded as one of the classical branches of mathematics.

Many other people have made prominent contributions to our understanding of the classical Lie theory, including Casimir, Chevalley, Coxeter, Dynkin, Engel, Freudenthal, Harish-Chandra, Serre, and Verma. In 1967, Victor Gershevich **Kac** and Robert Vaughan **Moody** independently introduced the Lie algebras now known as Kac-Moody algebras. This class of algebras not only includes all the classical ones, but also many infinite-dimensional Lie algebras of significance in mathematics and mathematical physics where they have applications to statistical physics, conformal field theory, and string theory.

Due to its enormous importance, Lie theory is still being studied by mathematicians and physicists alike and attracts a great number of researchers in many branches of the mathematical sciences. In July 2021, a simple google search for *Lie algebra* yielded 35,000,000 results – three times the number generated by a similar search for *Uluru*, but only a small fraction of that for *smartphone*. Although Lie theory may not have won over the entire internet, its natural beauty can rival nearly anything.

1 Lie algebras

As the name suggests, Lie algebras are examples of algebraic structures. We will not assume any extensive knowledge of abstract algebra theory, though. Instead, certain basic but possibly unfamiliar algebraic notions are reviewed in Appendix A.1. The reader should familiarise themselves with these notions, but may skip the more specialised material in Appendix A.2.

Lie algebras are defined over a field $\mathbb{F}$ (this may be a good opportunity to locate Appendix A.1). We will exclusively take $\mathbb{F}$ to be $\mathbb{R}$ or $\mathbb{C}$, the field of real or complex numbers, respectively. In fact, unless otherwise stated, all Lie algebras in these lecture notes may be taken over $\mathbb{C}$. Fields of so-called finite characteristic (see Appendix A.2) are also of interest, but will not be discussed in the lectures.

Lie algebras can be finite- or infinite-dimensional. However, in these lecture notes,

> *all Lie algebras are taken to be finite-dimensional.*

Moreover, with the exception of certain vector spaces in Section 3.6,

> *all vector spaces are assumed finite-dimensional.*

Accordingly, the representations considered in Theorems 3.11 and 4.6, for example, are all assumed finite-dimensional.

1.1 Definition

A **Lie algebra** over a field $\mathbb{F}$ is a vector space $\mathfrak{g}$ (over $\mathbb{F}$) endowed with a bilinear map, the **Lie bracket**

$$[\ ,\ ]:\ \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g},\qquad (x,y)\mapsto[x,y], \tag{1.1}$$

satisfying

$$[x,x]=0,\qquad \forall\, x\in\mathfrak{g}, \tag{1.2}$$

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0,\qquad \forall\, x,y,z\in\mathfrak{g}. \tag{1.3}$$

The condition (1.3) is known as the **Jacobi identity**, whereas the condition (1.2) implies the **antisymmetry** (or skew symmetry)

$$[x,y]=-[y,x],\qquad \forall\, x,y\in\mathfrak{g}. \tag{1.4}$$

The **dimension** d of the Lie algebra $\mathfrak{g}$ is defined as the dimension of $\mathfrak{g}$ as a vector space:

$$\mathrm{d}=\dim\mathfrak{g}. \tag{1.5}$$

To specify the field $\mathbb{F}$ over which the Lie algebra $\mathfrak{g}$ is defined, it is common to write $\mathfrak{g}(\mathbb{F})$.

Remark on notation

Unless otherwise stated, in Sections 1 to 4, $\mathfrak{g}$ will denote a Lie algebra over $\mathbb{F}$.

The Lie bracket can be seen as defining a multiplication on $\mathfrak{g}$, with $[x, y]$ being the **Lie product** of x and y. It readily follows from the antisymmetry (1.4) that Lie algebras are in general not commutative. Likewise, a Lie algebra is typically not associative. Indeed, the antisymmetry allows us to rewrite the Jacobi identity as

$$[x, [y, z]] - [[x, y], z] = [y, [x, z]] \tag{1.6}$$

whose righthand side need not be zero. In some sense, we can therefore view the Jacobi identity as a generalised notion of associativity.

For each $x \in \mathfrak{g}$, we introduce the **adjoint mapping**

$$\mathrm{ad}_x\colon\ \mathfrak{g} \to \mathfrak{g}, \qquad y \mapsto [x, y]. \tag{1.7}$$

These mappings will reappear in many places throughout the lecture notes.

Proposition 1.1. *For every pair $x, y \in \mathfrak{g}$, we have*

$$\mathrm{ad}_x \circ \mathrm{ad}_y - \mathrm{ad}_y \circ \mathrm{ad}_x = \mathrm{ad}_{[x,y]}, \tag{1.8}$$

where $\circ$ denotes the usual composition of maps.

Proof. For any $z \in \mathfrak{g}$, we have

$$\mathrm{ad}_x \circ \mathrm{ad}_y(z) = [x, [y, z]], \qquad \mathrm{ad}_y \circ \mathrm{ad}_x(z) = [y, [x, z]], \qquad \mathrm{ad}_{[x,y]}(z) = [[x, y], z]. \tag{1.9}$$

The Jacobi identity and the antisymmetry of the Lie bracket then imply that $\mathrm{ad}_x \circ \mathrm{ad}_y(z) - \mathrm{ad}_y \circ \mathrm{ad}_x(z) = \mathrm{ad}_{[x,y]}(z)$. ■

A **derivation** of $\mathfrak{g}$ is a linear map $\delta : \mathfrak{g} \to \mathfrak{g}$ that obeys the (Lie-algebraic) **Leibniz rule**

$$\delta([x, y]) = [\delta(x), y] + [x, \delta(y)], \qquad \forall\, x, y \in \mathfrak{g}. \tag{1.10}$$

A derivation δ is **nilpotent** if $\delta^n = 0$ for some $n \in \mathbb{N}$, where $\delta^n \equiv \underbrace{\delta \circ \cdots \circ \delta}_{n \text{ copies}}$.

Proposition 1.2. *For each $x \in \mathfrak{g}$, the adjoint mapping ad_x is a derivation of $\mathfrak{g}$.*

Proof. For all $y, z \in \mathfrak{g}$, we have

$$\mathrm{ad}_x([y, z]) = [x, [y, z]] = [[x, y], z] + [y, [x, z]] = [\mathrm{ad}_x(y), z] + [y, \mathrm{ad}_x(z)], \tag{1.11}$$

where the second equality follows from the Jacobi identity. ■

The **direct sum** $\mathfrak{g}_1 \boxplus \mathfrak{g}_2$ of two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ (over the same field) is the vector space (external) direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ with Lie product defined by

$$[x_1 + x_2, y_1 + y_2] := [x_1, y_1] + [x_2, y_2], \qquad \forall\, x_i, y_i \in \mathfrak{g}_i,\ i = 1, 2, \tag{1.12}$$

where the two Lie products on the righthand side are evaluated in $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively. It is straightforward to verify that $\mathfrak{g}_1 \boxplus \mathfrak{g}_2$ is indeed a Lie algebra and that its dimension is

$$\dim(\mathfrak{g}_1 \boxplus \mathfrak{g}_2) = \dim \mathfrak{g}_1 + \dim \mathfrak{g}_2. \tag{1.13}$$

Remark on notation

> We have introduced the symbol $\boxplus$ to indicate when a direct sum of two Lie algebras is viewed as a Lie algebra. If we are only concerned with the direct sum of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ as vector spaces, we use the more familiar notation $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. Such a distinction in notation is rarely employed in the literature. Instead, expressions like $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ are usually accompanied by a text specifying whether the direct sum is meant as a direct sum of Lie algebras or of vector spaces only. Furthermore, we have simplified the direct sum notation of vectors in $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ from $x_1 \oplus x_2$ to the more relaxed (and widespread) notation $x_1 + x_2$.

It is occasionally convenient to consider things with respect to a basis. Let $\{x_1, \ldots, x_\mathrm{d}\}$ be a **basis** for $\mathfrak{g}$. By definition, the basis is linearly independent and every element of $\mathfrak{g}$ can be written as a linear combination of the basis elements. Since $[x_a, x_b] \in \mathfrak{g}$, it follows that $[x_a, x_b]$ can be decomposed as

$$[x_a, x_b] = \sum_{c=1}^{\mathrm{d}} f_{ab}{}^c \, x_c, \qquad f_{ab}{}^c \in \mathbb{F}, \tag{1.14}$$

where $f_{ab}{}^c$ are the corresponding **structure constants**. In this notation, there are two lower indices: a and b (not to be mistaken for a single index given as the product ab). Of course, the structure constants depend on the chosen basis. In terms of the structure constants, the Lie algebra conditions (1.2) and (1.4) read

$$f_{aa}{}^c = 0, \qquad f_{ab}{}^c = -f_{ba}{}^c, \qquad a, b, c \in \{1, \ldots, \mathrm{d}\}. \tag{1.15}$$

Likewise, the Jacobi identity becomes

$$\sum_c \left(f_{ab}{}^c \, f_{cd}{}^e + f_{da}{}^c \, f_{cb}{}^e + f_{bd}{}^c \, f_{ca}{}^e \right) = 0, \tag{1.16}$$

where a summation like $\sum_c$ is meant to be over the entire range: $c = 1, \ldots, \mathrm{d}$. It is also useful to note that, to establish that a vector space $\mathfrak{g}$ with a bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra, it suffices to verify that (1.2) and (1.3) hold for all x, y, z in some basis for $\mathfrak{g}$.

Remark on notation and terminology

> In particular in the physics literature, the imaginary unit i is often included in the definition of the structure constants, in which case one writes $[x_a, x_b] = \sum_c \mathrm{i} f_{ab}{}^c \, x_c$. The rationale for doing this is that if the generators x_a are *hermitian*, $(x_a)^\dagger = x_a$, then the structure constants are real. Hermiticity will be discussed in Section 2.7.

The Lie algebra $\mathfrak{g}$ is said to be **generated** by the subset $X \subseteq \mathfrak{g}$ if every element of $\mathfrak{g}$ can be expressed as a linear combination of elements of X and repeated Lie products of the elements of X. A common notation indicating this is

$$\mathfrak{g} = \langle X \rangle. \tag{1.17}$$

Accordingly, the elements of X are then referred to as **generators** of $\mathfrak{g}$. A basis $\{x_1, \ldots, x_\mathrm{d}\}$ for $\mathfrak{g}$ is an obvious example of a set of generators:

$$\mathfrak{g} = \langle x_1, \ldots, x_\mathrm{d} \rangle = \langle x_a \,|\, a = 1, \ldots, \mathrm{d} \rangle. \tag{1.18}$$

1.2 Examples

Below follows a short preliminary list of Lie algebras. We will encounter many more examples as we go.

1. Any vector space V with zero Lie bracket (that is, $[u, v] = 0$ for all $u, v \in V$) is a Lie algebra. Such a Lie algebra is called an **abelian** Lie algebra, and it is both commutative and associative.

2. There is a unique (up to isomorphism, see Section 1.5) one-dimensional Lie algebra, here denoted by $\mathfrak{a}$ or $\mathfrak{a}(\mathbb{F})$ to indicate the field. Its single basis element x satisfies $[x, x] = 0$, so $\mathfrak{a}(\mathbb{F})$ is abelian.

3. The vector space $\mathbb{R}^3$ with Lie bracket
$$[u, v] := u \times v, \qquad \forall\, u, v \in \mathbb{R}^3, \tag{1.19}$$
is a Lie algebra. Here, $\times$ denotes the usual cross product.

4. The three-dimensional **Heisenberg algebra** is a Lie algebra whose three basis elements p, q, c satisfy
$$[p, q] = c, \qquad [c, p] = [c, q] = 0. \tag{1.20}$$
It follows that the algebra is generated by p, q.

5. Given an associative algebra A with multiplication $*$, one obtains a Lie algebra by defining the Lie bracket as
$$[x, y] := x * y - y * x, \qquad \forall\, x, y \in A. \tag{1.21}$$
This algebra is denoted by $\mathrm{Lie}(A)$ and is called the **Lie algebra of the associative algebra** A. The field $\mathbb{F}$ can thus be thought of as an abelian Lie algebra. Motivated by (1.21), the Lie product $[x, y]$ is often referred to as the **commutator** of x and y, even in a general Lie algebra.

6. Let A be an algebra over $\mathbb{F}$. A **derivation** of A is a linear map $\delta : A \to A$ that obeys the **Leibniz rule**
$$\delta(ab) = \delta(a)b + a\delta(b), \qquad \forall\, a, b \in A. \tag{1.22}$$
With Lie bracket $(\delta, \delta') \mapsto [\delta, \delta'] := \delta \circ \delta' - \delta' \circ \delta$, the set of all derivations of A forms a Lie algebra over $\mathbb{F}$, often denoted by $\mathrm{Der}(A)$.

7. The Lie algebra known as A_1 is a three-dimensional Lie algebra over $\mathbb{C}$ of fundamental importance in the general theory of abstract Lie algebras, as well as in applications. On the standard basis elements e, h, f, the Lie bracket is defined by
$$[h, e] := 2e, \qquad [h, f] := -2f, \qquad [e, f] := h. \tag{1.23}$$
We will encounter this Lie algebra repeatedly in these lecture notes. It also goes by the name $\mathfrak{sl}(2)$.

1.3 Lie subalgebras and ideals

Let X and Y be non-empty subsets of $\mathfrak{g}$ and define their **product** $[X,Y]$ to be the linear span of all Lie products $[x,y]$ where $x \in X$ and $y \in Y$. That is,

$$[X,Y] := \mathrm{span}_{\mathbb{F}}\big\{[x,y]\,\big|\,x \in X, y \in Y\big\}, \tag{1.24}$$

where the linear span ensures that this is a vector space over $\mathbb{F}$. In the special case $X = \{x\}$, we may write

$$[x,Y] = \mathrm{span}_{\mathbb{F}}\big\{[x,y]\,\big|\,y \in Y\big\}. \tag{1.25}$$

As the following proposition asserts, multiplication of subsets in a Lie algebra is commutative.

Proposition 1.3. *Let X and Y be non-empty subsets of $\mathfrak{g}$. Then, $[X,Y] = [Y,X]$.*

Proof. A general element $z \in [X,Y]$ is of the form

$$z = \sum_i a_i[x_i, y_i], \qquad a_i \in \mathbb{F},\ x_i \in X,\ y_i \in Y. \tag{1.26}$$

Since $x_i, y_i \in \mathfrak{g}$ for all i, we have

$$\sum_i a_i[x_i, y_i] = \sum_i (-a_i)[y_i, x_i] \in [Y,X], \tag{1.27}$$

so $[X,Y] \subseteq [Y,X]$. Likewise, every $z \in [Y,X]$ is seen to be an element of $[X,Y]$, so $[Y,X] \subseteq [X,Y]$. Hence, $[X,Y] = [Y,X]$. ∎

Let $\mathfrak{h}$ be a (vector) subspace of $\mathfrak{g}$. Then, $\mathfrak{h}$ is a **Lie subalgebra** of $\mathfrak{g}$ if

$$[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{h}, \tag{1.28}$$

that is, if

$$[x,y] \in \mathfrak{h}, \qquad \forall\, x,y \in \mathfrak{h}. \tag{1.29}$$

A Lie subalgebra of $\mathfrak{g}$ is thus a Lie algebra in its own right, with Lie bracket inherited from $\mathfrak{g}$. It follows from (1.2) that any nonzero element $x \in \mathfrak{g}$ generates a one-dimensional Lie subalgebra of $\mathfrak{g}$. Moreover, the zero vector space $\{0\}$ and the Lie algebra $\mathfrak{g}$ itself are Lie subalgebras of $\mathfrak{g}$; they are often referred to as the **trivial subalgebras** of $\mathfrak{g}$. A Lie subalgebra of $\mathfrak{g}$ different from $\mathfrak{g}$ itself is called a **proper subalgebra** of $\mathfrak{g}$.

Remark on notation and terminology

> In these lecture notes, as illustrated in (1.28), we write $A \subseteq B$ if all elements of the set A are elements of the set B. Note that A and B may thus coincide. To indicate that A is a **proper subset** of B, we write $A \subset B$ in which case $A \subseteq B$ but $A \neq B$. Accordingly, we have defined a *proper* Lie subalgebra of $\mathfrak{g}$ to be a Lie subalgebra of $\mathfrak{g}$ different from $\mathfrak{g}$ itself. It is noted, however, that some authors do not consider the trivial Lie subalgebra $\{0\}$ proper. A similar comment applies below to the trivial *ideal* $\{0\}$ of $\mathfrak{g}$.

Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$ and let $x \in \mathfrak{g}$ such that

$$[x, \mathfrak{h}] = \{0\} \qquad ([x, y] = 0 \text{ for all } y \in \mathfrak{h}). \tag{1.30}$$

We occasionally refer to this property by saying that x *commutes with* $\mathfrak{h}$ (or with every element y of $\mathfrak{h}$). More generally, we may say that the two subsets X and Y of $\mathfrak{g}$ commute if $[X, Y] = \{0\}$.

Remark on terminology

> It is emphasised that this commutativity terminology does not in general make sense since we have not defined the products xy and yx, let alone $x\mathfrak{h}$ and $\mathfrak{h}x$, but its use is nevertheless very common. In the case of matrix Lie algebras in Section 2, or when extending to the so-called universal enveloping algebra in Section 3.6, commutativity regains its usual meaning, as we will see.

Let $\mathfrak{i}$ be a (vector) subspace of $\mathfrak{g}$. Then, $\mathfrak{i}$ is an **ideal** of $\mathfrak{g}$ if

$$[\mathfrak{g}, \mathfrak{i}] \subseteq \mathfrak{i}, \tag{1.31}$$

that is, if

$$[x, y] \in \mathfrak{i}, \qquad \forall\, x \in \mathfrak{g},\ \forall\, y \in \mathfrak{i}. \tag{1.32}$$

Since $[y, x] = -[x, y]$, the condition could as well be written as $[\mathfrak{i}, \mathfrak{g}] \subseteq \mathfrak{i}$ or equivalently as $[y, x] \in \mathfrak{i}$ for all $x \in \mathfrak{g}$ and $y \in \mathfrak{i}$. Both the zero vector space $\{0\}$ and $\mathfrak{g}$ itself are ideals of $\mathfrak{g}$; they are often referred to as the **trivial ideals** of $\mathfrak{g}$. An ideal of $\mathfrak{g}$ different from $\mathfrak{g}$ itself is called a **proper ideal** of $\mathfrak{g}$. Note that an ideal of $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$:

$$\mathfrak{i} \text{ is an ideal of } \mathfrak{g} \quad \Rightarrow \quad \mathfrak{i} \text{ is a Lie subalgebra of } \mathfrak{g}. \tag{1.33}$$

The converse need not be true; there do exist Lie subalgebras which are not ideals. For example, the Lie subalgebra of A_1 generated by $\{h\}$ is not an ideal, see (1.23). Furthermore, if $\mathfrak{i}_1 \subseteq \mathfrak{i}_2$ are ideals of a Lie algebra, then $\mathfrak{i}_1$ is an ideal of $\mathfrak{i}_2$.

The Lie algebra $\mathfrak{g}$ is **simple** if

(i) it has no ideals other than $\{0\}$ and $\mathfrak{g}$;

(ii) it is not abelian.

In particular, a simple Lie algebra $\mathfrak{g}$ does not contain any element that has zero Lie product with $\mathfrak{g}$. That is, if $[x, \mathfrak{g}] = \{0\}$ for $x \in \mathfrak{g}$, then $x = 0$. We will have much more to say about simple Lie algebras in subsequent sections. Here, we demonstrate that A_1 introduced in Section 1.2 is a simple Lie algebra. To see this, let $\mathfrak{i} \neq \{0\}$ be an ideal of A_1 with

$$0 \neq x := ae + bh + cf \in \mathfrak{i}, \qquad a, b, c \in \mathbb{C}. \tag{1.34}$$

If $a = c = 0$, then $b \neq 0$ and

$$[x, e] = 2be \neq 0, \qquad [x, f] = -2bf \neq 0, \tag{1.35}$$

implying that $e, f \in \mathfrak{i}$ (since $\mathfrak{i}$ is an ideal). Hence, $\mathfrak{i} = A_1$ (since $h = x/b$). Otherwise, if $c \neq 0$, then

$$[[x, e], e] = [2be - ch, e] = -2ce \neq 0, \tag{1.36}$$

so $e \in \mathfrak{i}$. Hence,

$$[e, f] = h \in \mathfrak{i} \qquad \text{and} \qquad [h, f] = -2f \in \mathfrak{i}, \tag{1.37}$$

so $\mathfrak{i} = A_1$. Likewise, if $a \neq 0$, then $\mathfrak{i} = A_1$.

The next result relies on the notion of a *sum* of two subspaces of a given vector space, recalled in (A.15).

Proposition 1.4. *Let $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}$ be Lie subalgebras and $\mathfrak{i}_1, \mathfrak{i}_2, \mathfrak{i}$ ideals of $\mathfrak{g}$. Then,*

(i) $\mathfrak{h}_1 \cap \mathfrak{h}_2$ *is a Lie subalgebra of* $\mathfrak{g}$*;*

(ii) $\mathfrak{i}_1 \cap \mathfrak{i}_2$ *is an ideal of* $\mathfrak{g}$*;*

(iii) $\mathfrak{h} + \mathfrak{i}$ *is a Lie subalgebra of* $\mathfrak{g}$*;*

(iv) $\mathfrak{i}_1 + \mathfrak{i}_2$ *is an ideal of* $\mathfrak{g}$*;*

(v) $[\mathfrak{i}_1, \mathfrak{i}_2]$ *is an ideal of* $\mathfrak{g}$.

Proof.

(i) The intersection $\mathfrak{h}_1 \cap \mathfrak{h}_2$ is a subspace of $\mathfrak{g}$. Since both $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are Lie algebras in their own right,

$$[\mathfrak{h}_1 \cap \mathfrak{h}_2, \mathfrak{h}_1 \cap \mathfrak{h}_2] \subseteq [\mathfrak{h}_1, \mathfrak{h}_1] \subseteq \mathfrak{h}_1 \qquad \text{and} \qquad [\mathfrak{h}_1 \cap \mathfrak{h}_2, \mathfrak{h}_1 \cap \mathfrak{h}_2] \subseteq [\mathfrak{h}_2, \mathfrak{h}_2] \subseteq \mathfrak{h}_2. \tag{1.38}$$

It follows that

$$[\mathfrak{h}_1 \cap \mathfrak{h}_2, \mathfrak{h}_1 \cap \mathfrak{h}_2] \subseteq \mathfrak{h}_1 \cap \mathfrak{h}_2, \tag{1.39}$$

so $\mathfrak{h}_1 \cap \mathfrak{h}_2$ is a Lie subalgebra of $\mathfrak{g}$.

(ii) By (i), $\mathfrak{i}_1 \cap \mathfrak{i}_2$ is a Lie subalgebra of $\mathfrak{g}$. It is, in fact, an ideal, since

$$[\mathfrak{g}, \mathfrak{i}_1 \cap \mathfrak{i}_2] \subseteq [\mathfrak{g}, \mathfrak{i}_1] \cap [\mathfrak{g}, \mathfrak{i}_2] \subseteq \mathfrak{i}_1 \cap \mathfrak{i}_2. \tag{1.40}$$

(iii) By construction, $\mathfrak{h} + \mathfrak{i}$ is a subspace of $\mathfrak{g}$, satisfying

$$[\mathfrak{h} + \mathfrak{i}, \mathfrak{h} + \mathfrak{i}] \subseteq [\mathfrak{h}, \mathfrak{h}] + [\mathfrak{h}, \mathfrak{i}] + [\mathfrak{i}, \mathfrak{h}] + [\mathfrak{i}, \mathfrak{i}], \tag{1.41}$$

where $[\mathfrak{h}, \mathfrak{i}] = [\mathfrak{i}, \mathfrak{h}]$, by Proposition 1.3. Since $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{i}$ an ideal of $\mathfrak{g}$, we also have

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{i}] \subseteq \mathfrak{i}, \qquad [\mathfrak{i}, \mathfrak{i}] \subseteq \mathfrak{i}. \tag{1.42}$$

Hence,

$$[\mathfrak{h} + \mathfrak{i}, \mathfrak{h} + \mathfrak{i}] \subseteq \mathfrak{h} + \mathfrak{i}, \tag{1.43}$$

so $\mathfrak{h} + \mathfrak{i}$ is a Lie subalgebra of $\mathfrak{g}$.

(iv) By (iii), $\mathfrak{i}_1 + \mathfrak{i}_2$ is a Lie subalgebra of $\mathfrak{g}$. It is, in fact, an ideal, since

$$[\mathfrak{g}, \mathfrak{i}_1 + \mathfrak{i}_2] \subseteq [\mathfrak{g}, \mathfrak{i}_1] + [\mathfrak{g}, \mathfrak{i}_2] \subseteq \mathfrak{i}_1 + \mathfrak{i}_2. \tag{1.44}$$

(v) This is the content of Exercise 1.9. ■

The **centre** $Z(\mathfrak{g})$ of $\mathfrak{g}$ is defined as

$$Z(\mathfrak{g}) := \{x \in \mathfrak{g} \,\big|\, [x, \mathfrak{g}] = \{0\}\}. \tag{1.45}$$

Proposition 1.5.

(i) *$Z(\mathfrak{g})$ is an ideal of $\mathfrak{g}$.*

(ii) *$Z(\mathfrak{g}) = \mathfrak{g}$ if and only if $\mathfrak{g}$ is abelian.*

(iii) *$Z(\mathfrak{g}) = \{0\}$ for $\mathfrak{g}$ simple.*

Proof. This is the content of Exercise 1.13. ■

The **derived algebra** of $\mathfrak{g}$ is defined as

$$\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]. \tag{1.46}$$

An abelian Lie algebra $\mathfrak{g}$ is thus characterised by $\mathfrak{g}' = \{0\}$, whereas a simple Lie algebra $\mathfrak{g}$ is equal to its derived algebra: $\mathfrak{g}' = \mathfrak{g}$. The derived algebra will play an important role in Section 4.

Proposition 1.6. *The derived algebra of $\mathfrak{g}$ is an ideal of $\mathfrak{g}$.*

Proof. Since $\mathfrak{g}$ is an ideal of $\mathfrak{g}$, the statement is merely a special case of Proposition 1.4 (v) (setting $\mathfrak{i}_1 = \mathfrak{i}_2 = \mathfrak{g}$). ■

1.4 Cosets and quotient algebras

Let $W \subseteq V$ be vector spaces. A **coset** of W is a set of the form

$$v + W := \{v + w \,|\, w \in W\}, \qquad v \in V, \tag{1.47}$$

and can be thought of as the 'translation' of W by the vector v. Note that two such cosets can be equal:

$$v + W = v' + W \qquad \Longleftrightarrow \qquad v - v' \in W. \tag{1.48}$$

So unless $W = \{0\}$, the **representative** v of the coset $v + W$ is *not unique*. A coset can thus be thought of as an **equivalence class** of elements in V where two elements $v, v' \in V$ are said to be equivalent if $v - v' \in W$. The coset $v + W$ is then often denoted by $[v]$, in which case $[v] = [v']$ if and only if $v - v' \in W$.

The **quotient space**

$$V/W := \{v + W \,|\, v \in V\} \tag{1.49}$$

(usually pronounced "V modulo W" or simply "V mod W") is the set of all cosets of W in V. This becomes a vector space, with zero element $0 + W$, if addition and scalar multiplication are defined by

$$(v + W) + (v' + W) := (v + v') + W, \qquad a(v + W) := av + W \tag{1.50}$$

for all $v, v' \in V$ and $a \in \mathbb{F}$. For this construction to be well-defined, the operations in (1.50) must be *independent* of the representatives of the cosets. Verifying this is the content of Exercise 1.15. In terms of equivalence classes (with respect to W), we have

$$V/W = \{[v] \,|\, v \in V\}, \qquad [v] + [v'] = [v + v'], \qquad a[v] = [av], \tag{1.51}$$

with the zero element given by $[0]$.

Proposition 1.7. *For vector spaces $W \subseteq V$, the dimension of the quotient space V/W is*

$$\dim(V/W) = \dim V - \dim W. \tag{1.52}$$

Proof. This is the content of Exercise 1.11. ∎

Specialising to Lie algebras, let $\mathfrak{i}$ be an ideal of $\mathfrak{g}$. The corresponding cosets and quotient spaces are

$$x + \mathfrak{i} = \{x + z \,|\, z \in \mathfrak{i}\}, \qquad x \in \mathfrak{g}, \tag{1.53}$$

and

$$\mathfrak{g}/\mathfrak{i} = \{x + \mathfrak{i} \,|\, x \in \mathfrak{g}\}. \tag{1.54}$$

Proposition 1.8. *Let $\mathfrak{i}$ be an ideal of $\mathfrak{g}$. The quotient space $\mathfrak{g}/\mathfrak{i}$ is a Lie algebra with Lie product*

$$[x + \mathfrak{i}, y + \mathfrak{i}] := [x, y] + \mathfrak{i}, \qquad \forall\, x, y \in \mathfrak{g}, \tag{1.55}$$

where $[x, y]$ is the Lie product of x and y in $\mathfrak{g}$.

Proof. We must first show that the Lie product (1.55) is independent of the particular representatives x and y. That is, we must show that if $x+\mathfrak{i} = x'+\mathfrak{i}$ and $y+\mathfrak{i} = y'+\mathfrak{i}$ with $x, x', y, y' \in \mathfrak{g}$, then $[x, y] + \mathfrak{i} = [x', y'] + \mathfrak{i}$. The two assumptions imply that $x = x' + i_x$ and $y = y' + i_y$ for some $i_x, i_y \in \mathfrak{i}$. From this, it follows that

$$[x, y] + \mathfrak{i} = [x' + i_x, y' + i_y] + \mathfrak{i} = [x', y'] + [x', i_y] + [i_x, y'] + [i_x, i_y] + \mathfrak{i} = [x', y'] + \mathfrak{i}, \tag{1.56}$$

where the last equality is a consequence of $\mathfrak{i}$ being an ideal as this implies that $[x', i_y]$, $[i_x, y']$ and $[i_x, i_y]$ all lie in $\mathfrak{i}$. The Lie product is thus well-defined. Its antisymmetry follows from

$$[x + \mathfrak{i}, x + \mathfrak{i}] = [x, x] + \mathfrak{i} = 0 + \mathfrak{i}, \qquad \forall\, x \in \mathfrak{g}, \tag{1.57}$$

which is the zero element of the coset space. Finally, the Jacobi identity in $\mathfrak{g}/\mathfrak{i}$ follows from the Jacobi identity in $\mathfrak{g}$, as

$$\begin{aligned}
& [x + \mathfrak{i}, [y + \mathfrak{i}, z + \mathfrak{i}]] + [y + \mathfrak{i}, [z + \mathfrak{i}, x + \mathfrak{i}]] + [z + \mathfrak{i}, [x + \mathfrak{i}, y + \mathfrak{i}]] \\
= \;& [x + \mathfrak{i}, [y, z] + \mathfrak{i}] + [y + \mathfrak{i}, [z, x] + \mathfrak{i}] + [z + \mathfrak{i}, [x, y] + \mathfrak{i}] \\
= \;& [x, [y, z]] + \mathfrak{i} + [y, [z, x]] + \mathfrak{i} + [z, [x, y]] + \mathfrak{i} \\
= \;& 0 + \mathfrak{i}.
\end{aligned} \tag{1.58}$$

∎

The Lie algebra $\mathfrak{g}/\mathfrak{i}$ constructed in Proposition 1.8 is called a **quotient algebra** (or factor algebra). The next proposition implies that the derived algebra is the smallest ideal for which the corresponding quotient algebra is abelian.

Proposition 1.9. *Let* $\mathfrak{i}$ *be an ideal of* $\mathfrak{g}$*. Then,* $\mathfrak{g}/\mathfrak{i}$ *is abelian if and only if* $\mathfrak{g}' \subseteq \mathfrak{i}$*.*

Proof. The quotient algebra $\mathfrak{g}/\mathfrak{i}$ is abelian if and only if

$$[x + \mathfrak{i}, y + \mathfrak{i}] = [x, y] + \mathfrak{i} = 0 + \mathfrak{i}, \qquad \forall\, x, y \in \mathfrak{g}, \tag{1.59}$$

that is, if and only if $[x, y] \in \mathfrak{i}$ for all $x, y \in \mathfrak{g}$. Since $\mathfrak{i}$ is a vector space, this holds if and only if $\mathfrak{g}' \subseteq \mathfrak{i}$. ■

Corollary 1.10. *The quotient algebra* $\mathfrak{g}/\mathfrak{g}'$ *is an abelian Lie algebra.*

Proof. The statement is an immediate consequence of Proposition 1.9 applied to $\mathfrak{i} = \mathfrak{g}'$. ■

The derived algebra is in some sense a measure of how far $\mathfrak{g}$ is from being abelian: the 'smaller' $\mathfrak{g}'$ is, the 'larger' $\mathfrak{g}/\mathfrak{g}'$ is and the 'closer' $\mathfrak{g}$ is to being abelian. In the extreme case, $\mathfrak{g}' = \{0\}$ and $\mathfrak{g}$ *is* abelian. We will refine the notion of being 'almost abelian' when we discuss the so-called *solvable* Lie algebras in Section 4.1.

1.5 Homomorphisms

Let $\phi : V \to W$ be a linear map from the vector space V to the vector space W. Its **kernel** $\ker(\phi)$ and **image** $\operatorname{im}(\phi)$ are defined as

$$\ker(\phi) := \{v \in V \,|\, \phi(v) = 0\}, \qquad \operatorname{im}(\phi) := \{w \in W \,|\, w = \phi(v) \text{ for some } v \in V\}, \tag{1.60}$$

where $0 = 0_W$ is the zero element of W. Both sets are vector spaces, with $\ker(\phi) \subseteq V$ and $\operatorname{im}(\phi) \subseteq W$. The kernel of the linear map ϕ measures the degree to which ϕ fails to be injective. Indeed, the map is injective if and only if the kernel is the zero subspace: $\ker(\phi) = \{0\}$, where $0 = 0_V$.

A **homomorphism** from the Lie algebra $\mathfrak{g}_1$ to the Lie algebra $\mathfrak{g}_2$ is a linear map

$$\phi : \ \mathfrak{g}_1 \to \mathfrak{g}_2 \quad \text{such that} \quad \phi([x, y]) = [\phi(x), \phi(y)]. \tag{1.61}$$

It is noted that the Lie product in $\phi([x, y])$ is taken in $\mathfrak{g}_1$ whereas the Lie product $[\phi(x), \phi(y)]$ is taken in $\mathfrak{g}_2$.

Remark on terminology

> A homomorphism is seen to be 'structure preserving' in the sense that applying the appropriate Lie bracket before or after applying the homomorphism yields the same result. The homomorphism thus preserves the Lie algebraic structure. It is exactly this structure preserving property which is the origin of the term *homomorphism*, as *homo* means "same" while *morphos* means "shape".

Remark on terminology

> To avoid confusion, one could refer to the homomorphisms (1.61) as *Lie algebra homomorphisms*. This may be sensible in certain contexts since similar notions of homomorphism arise naturally in the study of many other algebraic structures such as groups and rings.

A homomorphism may enjoy additional properties. In many cases, it then acquires a more specific name, as indicated in the following table.

Table of Lie algebra homomorphisms $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$

mono--morphism:	injective (into)
epi--morphism:	surjective (onto)
iso--morphism:	bijective (injective and surjective)
endo--morphism:	to $\mathfrak{g}_1$ itself ($\mathfrak{g}_2 = \mathfrak{g}_1$)
auto--morphism:	to $\mathfrak{g}_1$ itself ($\mathfrak{g}_2 = \mathfrak{g}_1$) & bijective

The Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are said to be **isomorphic** if there exists an isomorphism from $\mathfrak{g}_1$ to $\mathfrak{g}_2$. Because of the structure preserving nature of homomorphisms, two isomorphic Lie algebras are essentially the same and often not distinguished. We may write

$$\mathfrak{g}_1 \cong \mathfrak{g}_2 \tag{1.62}$$

to indicate that the two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic. For example, the one-dimensional abelian Lie algebra over the field $\mathbb{F}$ is isomorphic to the field itself:

$$\mathfrak{a}(\mathbb{F}) \cong \mathbb{F}. \tag{1.63}$$

Indeed, for any nonzero element $x \in \mathfrak{a}(\mathbb{F})$,

$$\mathfrak{a}(\mathbb{F}) \to \mathbb{F}, \qquad cx \mapsto c, \tag{1.64}$$

is a bijective homomorphism (where the Lie bracket on $\mathbb{F}$ is given by the commutator).

Let $\mathfrak{i}$ be an ideal of $\mathfrak{g}$. Then, the map

$$\phi_{\mathfrak{i}} : \ \mathfrak{g} \to \mathfrak{g}/\mathfrak{i}, \qquad x \mapsto x + \mathfrak{i}, \tag{1.65}$$

is a homomorphism, known as the corresponding **natural homomorphism** (or canonical homomorphism). It is indeed a homomorphism since

$$\phi_{\mathfrak{i}}([x, y]) = [x, y] + \mathfrak{i} = [x + \mathfrak{i}, y + \mathfrak{i}] = [\phi_{\mathfrak{i}}(x), \phi_{\mathfrak{i}}(y)], \qquad \forall\, x, y \in \mathfrak{g}. \tag{1.66}$$

It is furthermore surjective since any element of $\mathfrak{g}/\mathfrak{i}$ is of the form $x + \mathfrak{i}$ for some $x \in \mathfrak{g}$, that is, it is of the form $\phi_{\mathfrak{i}}(x)$. The natural homomorphism is therefore an example of an epimorphism. We also note that

$$\ker(\phi_{\mathfrak{i}}) = \mathfrak{i}. \tag{1.67}$$

To appreciate this, it is recalled that $0 + \mathfrak{i}$ is the zero element of $\mathfrak{g}/\mathfrak{i}$ and that $x + \mathfrak{i} = 0 + \mathfrak{i}$ if and only if $x \in \mathfrak{i}$.

Lemma 1.11 (Homomorphism lemma for Lie algebras). *Let $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a Lie algebra homomorphism. Then,*

(i) $\ker(\phi)$ *is an ideal of* $\mathfrak{g}_1$*;*

(ii) $\mathrm{im}(\phi)$ *is a Lie subalgebra of* $\mathfrak{g}_2$*.*

Proof. Since ϕ is linear, both $\ker(\phi)$ and $\mathrm{im}(\phi)$ are vector spaces.

(i) Let $x \in \ker(\phi)$ and $y \in \mathfrak{g}_1$. Then,

$$\phi([x, y]) = [\phi(x), \phi(y)] = [0, \phi(y)] = 0, \tag{1.68}$$

so $[x, y] \in \ker(\phi)$.

(ii) Let $x, y \in \mathrm{im}(\phi)$. Then, there exist $x', y' \in \mathfrak{g}_1$ such that $x = \phi(x')$ and $y = \phi(y')$, so

$$[x, y] = [\phi(x'), \phi(y')] = \phi([x', y']) \in \mathrm{im}(\phi). \tag{1.69}$$

■

Theorem 1.12 (The isomorphism theorems for Lie algebras)**.**

(i) *Let* $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ *be a Lie algebra homomorphism. Then,*

$$\mathfrak{g}_1/\ker(\phi) \cong \mathrm{im}(\phi). \tag{1.70}$$

(ii) *If* $\mathfrak{h}$ *is a Lie subalgebra and* $\mathfrak{i}$ *an ideal of* $\mathfrak{g}$*, then* $(\mathfrak{h} + \mathfrak{i})/\mathfrak{i} \cong \mathfrak{h}/(\mathfrak{h} \cap \mathfrak{i})$*.*

(iii) *If* $\mathfrak{i} \subseteq \mathfrak{j}$ *are ideals of* $\mathfrak{g}$*, then* $\mathfrak{j}/\mathfrak{i}$ *is an ideal of* $\mathfrak{g}/\mathfrak{i}$ *and* $(\mathfrak{g}/\mathfrak{i})/(\mathfrak{j}/\mathfrak{i}) \cong \mathfrak{g}/\mathfrak{j}$*.*

Proof.

(i) First, by Lemma 1.11, $\mathfrak{g}_1/\ker(\phi)$ and $\mathrm{im}(\phi)$ are both Lie algebras. Second, the map

$$\varphi : \ \mathfrak{g}_1/\ker(\phi) \to \mathrm{im}(\phi), \qquad x + \ker(\phi) \mapsto \phi(x), \tag{1.71}$$

is well-defined since, for all $x, y \in \mathfrak{g}_1$, the relation $x + \ker(\phi) = y + \ker(\phi)$ is equivalent to $x - y \in \ker(\phi)$, that is, $\phi(x - y) = 0$. Hence, $\phi(x) = \phi(y)$, so

$$\varphi(x + \ker(\phi)) = \phi(x) = \phi(y) = \varphi(y + \ker(\phi)). \tag{1.72}$$

Third, the map φ is clearly $\mathbb{F}$-linear and a homomorphism because

$$\begin{aligned}\varphi([x + \ker(\phi), y + \ker(\phi)]) &= \varphi([x, y] + \ker(\phi)) = \phi([x, y]) = [\phi(x), \phi(y)] \\ &= [\varphi(x + \ker(\phi)), \varphi(y + \ker(\phi))].\end{aligned} \tag{1.73}$$

Fourth, the map φ is injective since

$$\begin{aligned}\varphi(x + \ker(\phi)) = \varphi(y + \ker(\phi)) \ &\Rightarrow \ \phi(x) = \phi(y) \ \Rightarrow \ \phi(x - y) = 0 \ \Rightarrow \ x - y \in \ker(\phi) \\ &\Rightarrow \ x + \ker(\phi) = y + \ker(\phi).\end{aligned} \tag{1.74}$$

Fifth, the map φ is surjective since ϕ is surjective on its image. In conclusion, the map φ is an isomorphism, so $\mathfrak{g}_1/\ker(\phi) \cong \mathrm{im}(\phi)$.

(ii) This is part of Exercise 1.10.

(iii) Since
$$[\mathfrak{i},\mathfrak{g}] \subseteq \mathfrak{i} \qquad \Rightarrow \qquad [\mathfrak{i}, X] \subseteq \mathfrak{i}, \quad \forall\, X \subseteq \mathfrak{g}, \tag{1.75}$$
$\mathfrak{i}$ is an ideal of $\mathfrak{j}$, so $\mathfrak{j}/\mathfrak{i}$ is well defined. Furthermore, the quotient space $\mathfrak{j}/\mathfrak{i}$ is an ideal of $\mathfrak{g}/\mathfrak{i}$. Indeed,
$$[y+\mathfrak{i}, x+\mathfrak{i}] = [y,x] + \mathfrak{i} \in \mathfrak{j}/\mathfrak{i}, \qquad \forall\, y \in \mathfrak{j}, \quad \forall\, x \in \mathfrak{g}, \tag{1.76}$$
since $\mathfrak{j}$ is an ideal of $\mathfrak{g}$. To establish the isomorphism in (iii), we note that the map
$$\pi:\ \mathfrak{g}/\mathfrak{i} \to \mathfrak{g}/\mathfrak{j}, \qquad x+\mathfrak{i} \mapsto x+\mathfrak{j} \tag{1.77}$$
is linear. Since
$$\pi([x+\mathfrak{i}, y+\mathfrak{i}]) = \pi([x,y]+\mathfrak{i}) = [x,y]+\mathfrak{j} = [x+\mathfrak{j}, y+\mathfrak{j}] = [\pi(x+\mathfrak{i}), \pi(y+\mathfrak{i})], \tag{1.78}$$
it is also a Lie algebra homomorphism. So, by (i), $(\mathfrak{g}/\mathfrak{i})/\ker(\pi) \cong \mathrm{im}(\pi)$. Since
$$\ker(\pi) = \{x+\mathfrak{i} \in \mathfrak{g}/\mathfrak{i} \,|\, x+\mathfrak{j} = 0+\mathfrak{j}\} = \{x+\mathfrak{i} \in \mathfrak{g}/\mathfrak{i} \,|\, x \in \mathfrak{j}\} = \mathfrak{j}/\mathfrak{i}, \tag{1.79}$$
while $\mathrm{im}(\pi) = \mathfrak{g}/\mathfrak{j}$ (π is readily seen to be surjective), it follows that $(\mathfrak{g}/\mathfrak{i})/(\mathfrak{j}/\mathfrak{i}) \cong \mathfrak{g}/\mathfrak{j}$. ∎

The set of all automorphisms of $\mathfrak{g}$ is denoted by $\mathrm{Aut}(\mathfrak{g})$. With multiplication given by composition of maps, $\mathrm{Aut}(\mathfrak{g})$ is a group with unit element given by the identity map. $\mathrm{Aut}(\mathfrak{g})$ is often referred to as the **automorphism group** of $\mathfrak{g}$.

Lemma 1.13 (Extended Leibniz rule)**.** *Let δ be a derivation of $\mathfrak{g}$. Then,*
$$\delta^n[x,y] = \sum_{k=0}^{n} \binom{n}{k} [\delta^k x, \delta^{n-k} y], \qquad \forall\, x,y \in \mathfrak{g}. \tag{1.80}$$

Proof. This is the content of Exercise 1.8. ∎

Proposition 1.14. *Let δ be a nilpotent derivation of $\mathfrak{g}$. Then,* $\exp(\delta) \in \mathrm{Aut}(\mathfrak{g})$.

Proof. Since δ is nilpotent, we have $\delta^n = 0$ for some $n \in \mathbb{N}$ and therefore
$$\exp(\delta) = \sum_{\ell=0}^{n-1} \frac{\delta^\ell}{\ell!}. \tag{1.81}$$

To establish that $\exp(\delta) : \mathfrak{g} \to \mathfrak{g}$ is a homomorphism, for general $x, y \in \mathfrak{g}$, we consider
$$\begin{aligned}
\exp(\delta)[x,y] &= \sum_{\ell=0}^{n-1}\sum_{k=0}^{\ell} \frac{1}{\ell!}\binom{\ell}{k} [\delta^k x, \delta^{\ell-k} y] = \sum_{\ell=0}^{n-1}\sum_{k=0}^{n-1} \frac{1}{(\ell-k)!\,k!}[\delta^k x, \delta^{\ell-k} y] \\
&= \sum_{\ell=0}^{n-1}\sum_{k=0}^{n-1} \frac{1}{\ell!\,k!}[\delta^k x, \delta^\ell y] = [\sum_{k=0}^{n-1} \frac{\delta^k}{k!}\,x, \sum_{\ell=0}^{n-1} \frac{\delta^\ell}{\ell!}\,y] \\
&= [\exp(\delta)x, \exp(\delta)y].
\end{aligned} \tag{1.82}$$

Similarly, $\exp(-\delta) : \mathfrak{g} \to \mathfrak{g}$ is a homomorphism. Furthermore, for each $x \in \mathfrak{g}$, we have

$$\begin{aligned}\exp(\delta)\exp(-\delta)x &= \exp(\delta)\sum_{k=0}^{n-1}\frac{(-\delta)^k}{k!}\,x = \sum_{\ell=0}^{n-1}\sum_{k=0}^{n-1}\frac{1}{\ell!\,k!}\,\delta^\ell(-\delta)^k x = \sum_{m=0}^{n-1}\sum_{k=0}^{m}\frac{1}{(m-k)!\,k!}\,(-1)^k\delta^m x\\ &= \sum_{m=0}^{n-1}\frac{1}{m!}\Big(\sum_{k=0}^{m}\binom{m}{k}(-1)^k\Big)\delta^m x = \sum_{m=0}^{n-1}\frac{1}{m!}\,\delta_{m,0}\,\delta^m x\\ &= x,\end{aligned} \tag{1.83}$$

so $\exp(\delta)\exp(-\delta)$ is the identity map. It follows that $\exp(\delta) : \mathfrak{g} \to \mathfrak{g}$ is an automorphism. ■

The exponential map plays an important role in Lie theory as it provides a crucial link between Lie algebras and Lie groups. However, as our focus is on the algebraic side of things, we will not delve into this.

1.6 *Exercises*

Exercise 1.1.

Verify that the condition (1.2) implies the antisymmetry (1.4). Show that the converse is also true if $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Is this converse statement true for all fields $\mathbb{F}$?

Exercise 1.2.

Show that no Lie algebra (over $\mathbb{R}$ or $\mathbb{C}$) can have a unit element.

Exercise 1.3.

Show that a two-dimensional Lie algebra cannot be simple.

Exercise 1.4.

Verify that the Jacobi identity becomes (1.16) when expressed in terms of structure constants.

Exercise 1.5.

Verify that $\mathbb{R}^3$ with $[u, v] := u \times v$ is a Lie algebra. It may be helpful to recall that the three standard basis vectors e_1, e_2 and e_3 satisfy

$$e_1 \times e_2 = e_3, \quad e_2 \times e_3 = e_1, \quad e_3 \times e_1 = e_2. \tag{1.84}$$

Exercise 1.6.

Verify that an associative algebra with Lie bracket (1.21) is a Lie algebra.

Exercise 1.7.

Prove or give a counterexample to the following statement: If $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are Lie subalgebras of $\mathfrak{g}$, then so is $\mathfrak{h}_1 \cup \mathfrak{h}_2$.

Exercise 1.8.

Verify the extended Leibniz rule (1.80).

Exercise 1.9.

Let $\mathfrak{i}_1$ and $\mathfrak{i}_2$ be ideals of $\mathfrak{g}$. Show that $[\mathfrak{i}_1, \mathfrak{i}_2]$ is an ideal of $\mathfrak{g}$.

Exercise 1.10.

Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{i}$ an ideal of $\mathfrak{g}$. Show that

(i) $\mathfrak{i}$ is an ideal of $\mathfrak{h} + \mathfrak{i}$,

(ii) $\mathfrak{h} \cap \mathfrak{i}$ is an ideal of $\mathfrak{h}$,

(iii) $(\mathfrak{h} + \mathfrak{i})/\mathfrak{i} \cong \mathfrak{h}/(\mathfrak{h} \cap \mathfrak{i})$.

Exercise 1.11.

Let V, W be vector spaces such that $W \subseteq V$. Show that the dimension of the corresponding quotient space V/W is

$$\dim(V/W) = \dim V - \dim W. \tag{1.85}$$

Exercise 1.12.

Verify that the set of all derivations of the $\mathbb{F}$-algebra A forms a Lie algebra with Lie bracket $(\delta, \delta') \mapsto [\delta, \delta'] = \delta \circ \delta' - \delta' \circ \delta$. Also, show that the composition $\delta \circ \delta'$ of two derivations δ and δ' need not be a derivation.

Exercise 1.13.

Show that the centre $Z(\mathfrak{g})$ of $\mathfrak{g}$ is an ideal of $\mathfrak{g}$. Show that $Z(\mathfrak{g}) = \mathfrak{g}$ if and only if $\mathfrak{g}$ is abelian. Show that the centre of a simple Lie algebra is trivial.

Exercise 1.14.

Let $\mathfrak{g}$ be a non-abelian Lie algebra. Show that $\dim(Z(\mathfrak{g})) \leqslant \dim(\mathfrak{g}) - 2$.

Exercise 1.15.

Verify that the quotient space construction (1.50) is well-defined.

Exercise 1.16.

Show that a subset of a Lie algebra is an ideal if and only if it is the kernel of a Lie algebra homomorphism.

Exercise 1.17.

Let $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a surjective Lie algebra homomorphism. Show that $\phi(Z(\mathfrak{g}_1)) \subseteq Z(\mathfrak{g}_2)$.

Exercise 1.18.

Show that, up to isomorphism, the three-dimensional Heisenberg algebra is the only Lie algebra $\mathfrak{g}$ satisfying $\dim \mathfrak{g} = 3$, $\dim \mathfrak{g}' = 1$, and $\mathfrak{g}' \subseteq Z(\mathfrak{g})$.

Exercise 1.19.

Let a Lie algebra be spanned by the generators x, y, z subject to the relations

$$[x, y] = z, \qquad [y, z] = [z, x] = x. \tag{1.86}$$

Show that the dimension of the Lie algebra is at most one.

Exercise 1.20.
Let $\{e, h, f\}$ and $\{x, y, z\}$ denote bases for A_1 with Lie products

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h \tag{1.87}$$

and

$$[x, y] = 2x + 2y, \qquad [y, z] = 2y + 2z, \qquad [z, x] = 2z + 2x. \tag{1.88}$$

Construct an automorphism of A_1 where

$$x \mapsto g_x(e, h), \qquad y \mapsto g_y(h), \qquad z \mapsto g_z(h, f), \tag{1.89}$$

for some functions g_x, g_y and g_z.

Exercise 1.21.
Let Y be a (vector) subspace of $\mathfrak{g}$. The **normaliser** of Y in $\mathfrak{g}$ is defined as

$$N_{\mathfrak{g}}(Y) := \{x \in \mathfrak{g} \,|\, [x, Y] \subseteq Y\}. \tag{1.90}$$

(i) Show that $N_{\mathfrak{g}}(Y)$ is a Lie subalgebra of $\mathfrak{g}$.

Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$.

(ii) Show that $\mathfrak{h}$ is an ideal of $N_{\mathfrak{g}}(\mathfrak{h})$.

(iii) Show that $N_{\mathfrak{g}}(\mathfrak{h})$ is the largest Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$ as an ideal.

Exercise 1.22.
Let Y be a (vector) subspace of $\mathfrak{g}$. The **centraliser** of Y in $\mathfrak{g}$ is defined as

$$C_{\mathfrak{g}}(Y) := \{x \in \mathfrak{g} \,|\, [x, Y] = \{0\}\}. \tag{1.91}$$

(i) Show that $C_{\mathfrak{g}}(Y)$ is a Lie subalgebra of $\mathfrak{g}$.

(ii) Is $C_{\mathfrak{g}}(Y)$ necessarily an ideal of $\mathfrak{g}$?

(iii) Determine the centraliser of $Y = \{\begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \,|\, z \in \mathbb{C}\}$ in $\mathfrak{sl}(2, \mathbb{C})$.

Exercise 1.23.
Classify, up to isomorphism, all two-dimensional Lie algebras over $\mathbb{F}$.

2 Matrix Lie algebras

As we will discuss in later sections, with five exceptions, every finite-dimensional simple Lie algebra over $\mathbb{C}$ is isomorphic to one of the **classical Lie algebras**

$$\begin{aligned} &A_r \cong \mathfrak{sl}(r+1) \quad (r \geqslant 1), \qquad && B_r \cong \mathfrak{so}(2r+1) \quad (r \geqslant 2), \\ &C_r \cong \mathfrak{sp}(2r) \qquad (r \geqslant 3), \qquad && D_r \cong \mathfrak{so}(2r) \qquad (r \geqslant 4), \end{aligned} \tag{2.1}$$

where $r \in \mathbb{N}$. The first notation is used to denote the abstract Lie algebras; the second notation is usually reserved for the corresponding matrix Lie algebras discussed in the following. If a classical Lie algebra is over the field $\mathbb{C}$, it is then customary to leave out the dependence on the field: $\mathfrak{sl}(r+1) = \mathfrak{sl}(r+1, \mathbb{C})$ and so on. The five **exceptional Lie algebras** are denoted by E_6, E_7, E_8, F_4, and G_2; they will be discussed in Section 6. The constraints on the integer label r in (2.1) are introduced to avoid duplications in the list of algebras, as

$$C_1 \cong B_1 \cong A_1, \qquad C_2 \cong B_2, \qquad D_3 \cong A_3, \tag{2.2}$$

and to exclude

$$D_1 \cong \mathfrak{a} \ \text{ (abelian)}, \qquad D_2 \cong A_1 \boxplus A_1 \ \text{ (direct sum)}, \tag{2.3}$$

which are non-simple (for the reasons indicated). The isomorphisms in (2.2) and (2.3) will be explored in the exercises and in Section 6.9.

The partial classification of Lie algebras alluded to above is for *complex* Lie algebras. It relies on the fact that the field $\mathbb{C}$ is algebraically closed, where a field $\mathbb{F}$ is said to be **algebraically closed** if every con-constant polynomial $p(x)$ with coefficients in $\mathbb{F}$ has a root (or zero) in $\mathbb{F}$. This is true for the field $\mathbb{C}$ of complex numbers, but not for the field $\mathbb{R}$ of real numbers, as illustrated by the polynomial $p(\lambda) = \lambda^2 + 1$. Consequently, a well-known result from linear algebra says that every $n \times n$ matrix over $\mathbb{C}$ has an eigenvector, while the similar statement does not hold for matrices over $\mathbb{R}$. This is basically the reason why the treatment of complex Lie algebras is simpler than that of real Lie algebras. Indeed, the polynomial $p(\lambda)$ arises as the characteristic polynomial of the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.4}$$

The eigenvalues then follow from

$$0 = p(\lambda) = \lambda^2 + 1 = (\lambda + \mathrm{i})(\lambda - \mathrm{i}) \tag{2.5}$$

and are given by the non-real values $-\mathrm{i}$ and $+\mathrm{i}$. Corresponding eigenvectors are read off from

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix} = -\mathrm{i} \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\mathrm{i} \\ 1 \end{pmatrix} = \mathrm{i} \begin{pmatrix} -\mathrm{i} \\ 1 \end{pmatrix}, \tag{2.6}$$

and are likewise non-real.

2.1 Linear maps

Let V denote a vector space (over $\mathbb{F}$) of dimension n. The set of all linear maps $V \to V$ is denoted by $\mathrm{End}(V)$ and is itself a vector space of dimension n^2 (to appreciate this, one may think of the linear maps as $n \times n$ matrices). It is rather obviously a ring, called the **endomorphism ring** of V, with multiplication defined by the usual composition $\circ$ of maps. However, it also admits a Lie bracket structure, setting

$$[x, y] := x \circ y - y \circ x, \qquad \forall\, x, y \in \mathrm{End}(V). \tag{2.7}$$

Indeed, with this, $\mathrm{End}(V)$ becomes a Lie algebra, as verified in Exercise 2.1. To distinguish this new algebraic structure from the original one with multiplication defined by composition of maps, we write $\mathfrak{gl}(V)$ for $\mathrm{End}(V)$ viewed as a Lie algebra and refer to it as the corresponding **general linear algebra**. In fact, the general linear algebra is merely the Lie algebra associated with the associative algebra $\mathrm{End}(V)$, introduced in Section 1.2. That is,

$$\mathfrak{gl}(V) = \mathrm{Lie}(\mathrm{End}(V)). \tag{2.8}$$

Since any Lie algebra $\mathfrak{g}$ is a vector space, it makes sense to talk about the endomorphism ring $\mathrm{End}(\mathfrak{g})$ and the corresponding general linear algebra $\mathfrak{gl}(\mathfrak{g})$. The adjoint mappings, for example, are elements of $\mathfrak{gl}(\mathfrak{g})$: $\mathrm{ad}_x \in \mathfrak{gl}(\mathfrak{g})$ for each $x \in \mathfrak{g}$. They are the images of the **adjoint map** ad (note the slight change in terminology),

$$\mathrm{ad}:\ \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), \qquad x \mapsto \mathrm{ad}_x. \tag{2.9}$$

According to Proposition 1.1, the adjoint map is a Lie algebra homomorphism.

Proposition 2.1. *The kernel of the adjoint map of $\mathfrak{g}$ is given by the centre of $\mathfrak{g}$:*

$$\ker(\mathrm{ad}) = Z(\mathfrak{g}). \tag{2.10}$$

Proof. $\ker(\mathrm{ad})$ consists of all $x \in \mathfrak{g}$ for which $\mathrm{ad}_x = 0$, that is, for which $[x, y] = 0$ for all $y \in \mathfrak{g}$. This is exactly the characterisation of the elements of the centre of $\mathfrak{g}$. ■

Corollary 2.2. *Any simple Lie algebra is isomorphic to a linear Lie algebra.*

Proof. Let $\mathfrak{g}$ be simple. By Proposition 1.5, $Z(\mathfrak{g}) = \{0\}$ so $\ker(\mathrm{ad}) = \{0\}$ and ad is a monomorphism. The first isomorphism theorem (Theorem 1.12 (i)) then implies that $\mathfrak{g} \cong \mathrm{im}(\mathrm{ad})$ which is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$. ■

2.2 General linear algebras

We write $M_n(\mathbb{F}) = M_{n\times n}(\mathbb{F})$ for the vector space of all $n \times n$ matrices with entries from $\mathbb{F}$. This becomes a Lie algebra with Lie bracket

$$(x, y) \mapsto [x, y] := xy - yx, \qquad \forall\, x, y \in M_n(\mathbb{F}), \tag{2.11}$$

where xy and yx are usual matrix products of the matrices x and y. This Lie algebra is known as the **general linear algebra** of $n \times n$ matrices over $\mathbb{F}$ and is denoted by $\mathfrak{gl}(n, \mathbb{F})$.

The Lie algebra $\mathfrak{gl}(n, \mathbb{F})$ has a natural basis consisting of the **matrix units** E_{ij}, $1 \leqslant i, j \leqslant n$, where E_{ij} is the $n \times n$ matrix whose only nonzero entry is a 1 in the (i, j) position. The corresponding structure constants readily follow from the easily verified commutation relation

$$[E_{ij}, E_{k\ell}] = \delta_{j,k} E_{i\ell} - \delta_{i,\ell} E_{kj}, \tag{2.12}$$

where

$$\delta_{i,j} := \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{2.13}$$

is the **Kronecker delta** function (or symbol). As the dimension of $\mathfrak{gl}(n, \mathbb{F})$ is given by the cardinality of any basis for $\mathfrak{gl}(n, \mathbb{F})$, we have

$$\dim \mathfrak{gl}(n, \mathbb{F}) = n^2. \tag{2.14}$$

Let I_n denote the $n \times n$ **identity matrix**

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{2.15}$$

Proposition 2.3. *The centre of $\mathfrak{gl}(n, \mathbb{F})$ is a one-dimensional abelian Lie algebra generated by I_n. Consequently,*

$$Z(\mathfrak{gl}(n, \mathbb{F})) \cong \mathfrak{a}(\mathbb{F}). \tag{2.16}$$

Proof. This is the content of Exercise 2.9. ■

For each matrix $s \in M_n(\mathbb{F})$, we define

$$\mathfrak{gl}_s(n, \mathbb{F}) := \{x \in M_n(\mathbb{F}) \,|\, x^t s = -sx\}, \tag{2.17}$$

where x^t denotes the matrix transpose of x, and endow it with the multiplication (2.11). It readily follows that

$$\mathfrak{gl}_{-s}(n, \mathbb{F}) = \mathfrak{gl}_s(n, \mathbb{F}). \tag{2.18}$$

Proposition 2.4. *For each $s \in M_n(\mathbb{F})$, $\mathfrak{gl}_s(n, \mathbb{F})$ is a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{F})$.*

Proof. $\mathfrak{gl}_s(n, \mathbb{F})$ is seen to be a vector space. It is invariant under the Lie bracket operation since, for all $x, y \in \mathfrak{gl}_s(n, \mathbb{F})$,

$$[x, y]^t s = (y^t x^t - x^t y^t)s = y^t(-sx) - x^t(-sy) = syx - sxy = -s[x, y], \tag{2.19}$$

where we have used that $(xy)^t = y^t x^t$ (which is true for all pairs $x, y \in M_n(\mathbb{F})$). ■

Proposition 2.5. *Let $s \in M_n(\mathbb{F})$ and let $g \in M_n(\mathbb{F})$ be an invertible matrix such that $g^t s g = s$. Then, $x \mapsto gxg^{-1}$ defines an automorphism of $\mathfrak{gl}_s(n, \mathbb{F})$.*

Proof. On $M_n(\mathbb{F})$, the map is clearly linear, and it is an endomorphism since

$$g[x,y]g^{-1} = gxyg^{-1} - gyxg^{-1} = gxg^{-1}gyg^{-1} - gyg^{-1}gxg^{-1} = [gxg^{-1}, gyg^{-1}] \tag{2.20}$$

for all $x, y \in M_n(\mathbb{F})$. Its inverse map is obtained by replacing g by g^{-1}, so it is an automorphism of $M_n(\mathbb{F})$. To see that it is also an automorphism of $\mathfrak{gl}_s(n,\mathbb{F})$, it suffices to show that it maps $\mathfrak{gl}_s(n,\mathbb{F})$ back to itself. For any $x \in \mathfrak{gl}_s(n,\mathbb{F})$, we have

$$(gxg^{-1})^t s = (g^{-1})^t x^t g^t s = (g^{-1})^t x^t s g^{-1} = (g^{-1})^t(-sx)g^{-1} = -sgxg^{-1}, \tag{2.21}$$

so $gxg^{-1} \in \mathfrak{gl}_s(n,\mathbb{F})$. ■

2.3 Special linear algebras

The **trace** of a square matrix is the sum of the entries on its diagonal. Correspondingly, tr is the linear map

$$\mathrm{tr}: \ M_n(\mathbb{F}) \to \mathbb{F}, \qquad x \mapsto \sum_{k=1}^{n} x_{kk}. \tag{2.22}$$

Since

$$\mathrm{tr}([x,y]) = \mathrm{tr}(xy - yx) = \mathrm{tr}(xy) - \mathrm{tr}(yx) = 0 = [\mathrm{tr}(x), \mathrm{tr}(y)], \qquad \forall\, x, y \in M_n(\mathbb{F}), \tag{2.23}$$

this is a Lie algebra homomorphism from $\mathfrak{gl}(n,\mathbb{F})$ to the abelian Lie algebra $\mathbb{F}$. In establishing this, we used the linearity of tr and the familiar **cyclicity** property

$$\mathrm{tr}(xy) = \mathrm{tr}(yx). \tag{2.24}$$

It also follows from the cyclicity that two **similar** matrices, x and $s^{-1}xs$ with s an invertible $n \times n$ matrix, have the same trace:

$$\mathrm{tr}(s^{-1}xs) = \mathrm{tr}(x). \tag{2.25}$$

It is recalled that two distinct but similar matrices represent the same linear map, just expressed in different bases. For x and $s^{-1}xs$, s is thus the matrix governing the corresponding change of basis.

The set of traceless $n \times n$ matrices is given by

$$\{x \in M_n(\mathbb{F}) \,|\, \mathrm{tr}(x) = 0\} \tag{2.26}$$

and is seen to be a vector space. For $n \geqslant 2$, a convenient basis for this vector space is given by

$$\{E_{11} - E_{22},\, E_{22} - E_{33}, \ldots,\, E_{n-1,n-1} - E_{nn}\} \sqcup \{E_{ij} \,|\, i \neq j;\ i, j = 1, \ldots, n\}. \tag{2.27}$$

It is stressed that the identity matrix I_n is *not* an element of the vector space (2.26).

Proposition 2.6. *With the Lie bracket (2.11), $\{x \in M_n(\mathbb{F}) \,|\, \mathrm{tr}(x) = 0\}$ is a Lie algebra over $\mathbb{F}$.*

Proof. We need to show that the trace of the commutator of two traceless $n \times n$ matrices is zero. However, this is merely a special case of the stronger statement in (2.23) that $\mathrm{tr}([x, y]) = 0$ for all $x, y \in M_n(\mathbb{F})$. ■

The Lie algebra in Proposition 2.6 is denoted by $\mathfrak{sl}(n, \mathbb{F})$ and is called the **special linear algebra** of $n \times n$ matrices over $\mathbb{F}$. It is clearly a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{F})$. By construction, the kernel of the homomorphism $\mathrm{tr} : \mathfrak{gl}(n, \mathbb{F}) \to \mathbb{F}$ is $\mathfrak{sl}(n, \mathbb{F})$, and tr is readily seen to be surjective. According to the first isomorphism theorem 1.12, we thus have

$$\mathfrak{gl}(n, \mathbb{F})/\mathfrak{sl}(n, \mathbb{F}) \cong \mathbb{F}, \tag{2.28}$$

where a coset of the form $x + \mathfrak{sl}(n, \mathbb{F})$ consists of the $n \times n$ matrices whose trace is $\mathrm{tr}(x)$:

$$x + \mathfrak{sl}(n, \mathbb{F}) = \{y \in M_n(\mathbb{F}) \,|\, \mathrm{tr}(y) = \mathrm{tr}(x)\}. \tag{2.29}$$

Proposition 2.7. $\dim \mathfrak{sl}(n, \mathbb{F}) = n^2 - 1$.

Proof. The dimension is simply given by the cardinality of the basis (2.27). Alternatively, by Proposition 1.7 and (2.28), we have

$$1 = \dim \mathbb{F} = \dim(\mathfrak{gl}(n, \mathbb{F})/\mathfrak{sl}(n, \mathbb{F})) = \dim \mathfrak{gl}(n, \mathbb{F}) - \dim \mathfrak{sl}(n, \mathbb{F}) = n^2 - \dim \mathfrak{sl}(n, \mathbb{F}), \tag{2.30}$$

from which $\dim \mathfrak{sl}(n, \mathbb{F}) = n^2 - 1$ immediately follows. ■

An element of the three-dimensional Lie algebra $\mathfrak{sl}(2, \mathbb{F})$ is of the form

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \qquad a, b, c \in \mathbb{F}. \tag{2.31}$$

It follows that a basis is given by

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{2.32}$$

in accordance with (2.27). As one easily verifies, the corresponding nontrivial commutation relations are

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h. \tag{2.33}$$

These are recognised as the defining relations of the Lie product (1.23) on the abstract Lie algebra A_1. The ensuing isomorphism $A_1 \cong \mathfrak{sl}(2)$ is merely the first of the many such isomorphisms listed in (2.1).

A basis for the eight-dimensional Lie algebra $\mathfrak{sl}(3, \mathbb{F})$ is likewise given by

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_\theta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{2.34}$$

$$f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad f_\theta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

It is straightforward to verify that the corresponding nontrivial commutation relations are

$$\begin{array}{llll} [h_i, e_j] = A_{ij} e_j, & [h_i, f_j] = -A_{ij} f_j, & [h_i, e_\theta] = e_\theta, & [h_i, f_\theta] = -f_\theta, \\ [e_1, e_2] = e_\theta, & [f_1, f_2] = -f_\theta, & [e_i, f_j] = \delta_{i,j} h_j, & [e_\theta, f_\theta] = h_1 + h_2, \\ [e_1, f_\theta] = -f_2, & [e_2, f_\theta] = f_1, & [e_\theta, f_1] = -e_2, & [e_\theta, f_2] = e_1, \end{array} \tag{2.35}$$

for all $i, j \in \{1, 2\}$, where

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{2.36}$$

It is noted that

$$[h_i, h_j] = [e_i, e_\theta] = [f_i, f_\theta] = 0, \qquad \forall\, i, j \in \{1, 2\}. \tag{2.37}$$

The matrix A in (2.36) is the so-called *Cartan matrix* of $\mathfrak{sl}(3)$. Cartan matrices are discussed more generally in Section 6.4, and their role in Lie bracket relations like (2.35) will be elucidated in Section 6.5.

The Lie algebra $\mathfrak{sl}(2)$ appears as a Lie subalgebra of $\mathfrak{sl}(3)$. In fact, we can identify more than one copy of $\mathfrak{sl}(2)$ inside $\mathfrak{sl}(3)$. Examples are

$$\mathfrak{sl}(2) \cong \langle e_1, h_1, f_1 \rangle \cong \langle e_2, h_2, f_2 \rangle \cong \langle e_\theta, h_1 + h_2, f_\theta \rangle \cong \langle \sqrt{2}(e_1 + e_2), 2(h_1 + h_2), \sqrt{2}(f_1 + f_2) \rangle, \tag{2.38}$$

where the generators $e_1, h_1, \ldots$ are the $\mathfrak{sl}(3)$ basis elements used in (2.35). More generally, if $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then there exists a monomorphism $\mathfrak{h} \to \mathfrak{g}$, known as an **embedding** of $\mathfrak{h}$ in $\mathfrak{g}$. However, as illustrated by the $\mathfrak{sl}(2) \subset \mathfrak{sl}(3)$ scenario, such an embedding need not be unique. So instead of just indicating the pair $(\mathfrak{g}, \mathfrak{h})$, it may be important to specify *which* embedding of $\mathfrak{h}$ in $\mathfrak{g}$ one is interested in.

Theorem 2.8. *For $n \geqslant 2$, $\mathfrak{sl}(n, \mathbb{C})$ is a simple Lie algebra.*

Proof. This is the content of Exercise 2.10. ∎

2.4 Triangular and diagonal matrices

For $n \in \mathbb{N}$, we define the set of **upper-triangular** $n \times n$ matrices as

$$\mathfrak{t}(n, \mathbb{F}) := \{x \in M_n(\mathbb{F}) \mid x_{ij} = 0 \text{ for all } 1 \leqslant j < i \leqslant n\} \tag{2.39}$$

and the set of **strictly upper-triangular** $n \times n$ matrices as

$$\mathfrak{n}^+(n, \mathbb{F}) := \{x \in M_n(\mathbb{F}) \mid x_{ij} = 0 \text{ for all } 1 \leqslant j \leqslant i \leqslant n\}, \tag{2.40}$$

and endow them with the Lie bracket (2.11). The set $\mathfrak{n}^-(n, \mathbb{F})$ of **strictly lower-triangular** $n \times n$ matrices is defined similarly. Likewise, the set of **diagonal** $n \times n$ matrices is denoted by $\mathfrak{d}(n, \mathbb{F})$ and endowed with the same Lie bracket. It is straightforward to check that each of these sets is a vector space closed under this Lie bracket, so they are all Lie subalgebras of $\mathfrak{gl}(n, \mathbb{F})$. In fact, $\mathfrak{n}^+(n, \mathbb{F})$ and $\mathfrak{n}^-(n, \mathbb{F})$ are Lie subalgebras of $\mathfrak{sl}(n, \mathbb{F})$. As vector spaces, we clearly have

$$\mathfrak{t}(n, \mathbb{F}) = \mathfrak{d}(n, \mathbb{F}) \oplus \mathfrak{n}^+(n, \mathbb{F}), \qquad \mathfrak{gl}(n, \mathbb{F}) = \mathfrak{n}^-(n, \mathbb{F}) \oplus \mathfrak{d}(n, \mathbb{F}) \oplus \mathfrak{n}^+(n, \mathbb{F}). \tag{2.41}$$

Decompositions like these will play an important role in Section 5. It is also noted that

$$\mathfrak{t}(1, \mathbb{F}) = \mathfrak{d}(1, \mathbb{F}) \cong \mathfrak{a}(\mathbb{F}), \qquad \mathfrak{n}^+(1, \mathbb{F}) = \mathfrak{n}^-(1, \mathbb{F}) = \{0\}. \tag{2.42}$$

2.5 Symplectic algebras

For $n = 2r$ even ($r \in \mathbb{N}$), we introduce the $n \times n$ matrix

$$s = \begin{pmatrix} 0_r & I_r \\ -I_r & 0_r \end{pmatrix}, \tag{2.43}$$

where 0_r is the $r \times r$ **zero matrix** (whose entries are all 0), and use it to define the so-called **symplectic algebra** as

$$\mathfrak{sp}(2r, \mathbb{F}) := \{x \in M_{2r}(\mathbb{F}) \,|\, x^t s = -sx\} \tag{2.44}$$

endowed with (2.11). According to Proposition 2.4, this is a Lie subalgebra of $\mathfrak{gl}(2r, \mathbb{F})$. Since $s^{-1} = -s$, we see that the condition $x^t s = -sx$ is equivalent to $x^t = sxs$. Symplectic algebras are not defined for $n \times n$ matrices for n odd.

Proposition 2.9. *$x \in \mathfrak{sp}(2r, \mathbb{F})$ if and only if x is of the form*

$$x = \begin{pmatrix} m & p \\ q & -m^t \end{pmatrix}, \qquad m, p, q \in M_r(\mathbb{F}), \tag{2.45}$$

with p and q symmetric.

Proof. A general element $x \in M_{2r}(\mathbb{F})$ can be written as

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, b, c, d \in M_r(\mathbb{F}). \tag{2.46}$$

It follows that

$$x^t s = \begin{pmatrix} -c^t & a^t \\ -d^t & b^t \end{pmatrix}, \qquad -sx = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}. \tag{2.47}$$

The requirement $x^t s = -sx$ is thus equivalent to imposing

$$b = b^t, \qquad c = c^t, \qquad d = -a^t \tag{2.48}$$

on (2.46). The form (2.45) readily follows. ∎

Corollary 2.10. *$\mathfrak{sp}(2r, \mathbb{F})$ is a Lie subalgebra of $\mathfrak{sl}(2r, \mathbb{F})$, with $\dim \mathfrak{sp}(2r, \mathbb{F}) = 2r^2 + r$.*

Proof. Using the characterisation (2.45), we have

$$\operatorname{tr}(x) = \operatorname{tr}(m) + \operatorname{tr}(-m^t) = 0, \tag{2.49}$$

implying that $\mathfrak{sp}(2r,\mathbb{F})$ is a Lie subalgebra of $\mathfrak{sl}(2r,\mathbb{F})$. It also follows that the number of independent entries of a generic element $x \in \mathfrak{sp}(2r,\mathbb{F})$ is given by the sum of the numbers of independent entries of m, $p = p^t$ and $q = q^t$, that is,

$$r^2 + \tfrac{1}{2}r(r+1) + \tfrac{1}{2}r(r+1) = 2r^2 + r. \tag{2.50}$$

■

In Exercise 2.13, the reader is encouraged to work out a basis for $\mathfrak{sp}(2r,\mathbb{F})$.

Theorem 2.11. *For $r \geqslant 1$, $\mathfrak{sp}(2r,\mathbb{C})$ is a simple Lie algebra.*

Proof. We refer to the literature for a proof of this. ■

Remark on applications

> The symplectic Lie algebras over the real numbers, $\mathfrak{sp}(2n,\mathbb{R})$, appear very naturally in classical mechanics as they describe infinitesimal canonical transformations in a $2n$-dimensional phase space that leave the form of the Hamilton equations invariant.

2.6 Orthogonal algebras

An $n \times n$ matrix x is said to be **anti-symmetric** if $x^t = -x$. The commutator of two anti-symmetric matrices x, y is again anti-symmetric:

$$[x,y]^t = (xy)^t - (yx)^t = y^t x^t - x^t y^t = -[x^t, y^t] = -[x,y]. \tag{2.51}$$

For each $n \in \mathbb{N}$, we can thus define the so-called **orthogonal algebra** as the Lie algebra of anti-symmetric $n \times n$ matrices,

$$\mathfrak{so}(n,\mathbb{F}) := \{x \in M_n(\mathbb{F}) \,|\, x^t = -x\}, \tag{2.52}$$

with Lie bracket (2.11). In fact, $\mathfrak{so}(n,\mathbb{F})$ is equal to $\mathfrak{gl}_s(n,\mathbb{F})$ defined in (2.17) with $s = I_n$. Since $\operatorname{tr}(x^t) = \operatorname{tr}(x)$ for any $n \times n$ matrix, it follows that $\operatorname{tr}(x) = 0$ for all $x \in \mathfrak{so}(n,\mathbb{F})$, so $\mathfrak{so}(n,\mathbb{F})$ is a Lie subalgebra of $\mathfrak{sl}(n,\mathbb{F})$. It is a straightforward consequence of the characterisation $x^t = -x$ of the elements of $\mathfrak{so}(n,\mathbb{F})$ that

$$\{E_{ij} - E_{ji} \,|\, 1 \leqslant i < j \leqslant n\} \tag{2.53}$$

is a basis for the orthogonal algebra $\mathfrak{so}(n,\mathbb{F})$. Hence,

$$\dim \mathfrak{so}(n,\mathbb{F}) = \tfrac{1}{2}n(n-1). \tag{2.54}$$

A convenient basis for the three-dimensional Lie algebra $\mathfrak{so}(3,\mathbb{F})$ is given by

$$R_1 = -(E_{23} - E_{32}), \qquad R_2 = E_{13} - E_{31}, \qquad R_3 = -(E_{12} - E_{21}), \tag{2.55}$$

that is,

$$R_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad R_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.56}$$

As one easily verifies, the corresponding nontrivial commutation relations are

$$[R_1, R_2] = R_3, \qquad [R_2, R_3] = R_1, \qquad [R_3, R_1] = R_2. \tag{2.57}$$

Theorem 2.12. *For $n = 3$ or $n \geqslant 5$, $\mathfrak{so}(n, \mathbb{C})$ is a simple Lie algebra.*

Proof. We refer to the literature for a proof of this. ■

The Lie algebra $\mathfrak{so}(4, \mathbb{C})$ is *not* simple. Instead, as indicated in (2.3) and shown in Exercise 2.24, it is a direct sum of two simple Lie algebras: $\mathfrak{so}(4, \mathbb{C}) \cong \mathfrak{so}(3, \mathbb{C}) \boxplus \mathfrak{so}(3, \mathbb{C})$.

Remark on applications

The orthogonal Lie algebras over the real numbers, $\mathfrak{so}(d, \mathbb{R})$, appear very naturally in physics as they describe infinitesimal rotations in d dimensions. In three dimensions, for example, the three matrices

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \quad R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \quad R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

rotate vectors by an angle θ about the x, y and z-axis, respectively. For small θ,

$$R_x(\theta) = I_3 + \theta R_1 + \mathcal{O}(\theta^2), \qquad R_y(\theta) = I_3 + \theta R_2 + \mathcal{O}(\theta^2), \qquad R_z(\theta) = I_3 + \theta R_3 + \mathcal{O}(\theta^2),$$

where the matrices R_1, R_2 and R_3 are the basis elements (2.56) of $\mathfrak{so}(3, \mathbb{R})$.

For $p, q \in \mathbb{N}_0$, let $I_{p,q}$ denote the square matrix

$$I_{p,q} = \begin{pmatrix} I_p & 0_{p\times q} \\ 0_{q\times p} & -I_q \end{pmatrix}, \tag{2.58}$$

where $0_{k\times\ell}$ is the $k \times \ell$ zero matrix. We now define $\mathfrak{so}(p, q, \mathbb{F})$ as the matrix Lie algebra $\mathfrak{gl}_s(n, \mathbb{F})$ for which $s = I_{p,q}$ and $n = p + q$:

$$\mathfrak{so}(p, q, \mathbb{F}) := \{x \in M_{p+q}(\mathbb{F}) \,|\, x^t I_{p,q} = -I_{p,q} x\} \tag{2.59}$$

with Lie bracket (2.11). It follows from (2.18) that the definition of $\mathfrak{so}(p, q, \mathbb{F})$ is unchanged if $s = I_{p,q}$ is replaced by $-I_{p,q}$ in (2.59), implying that $\mathfrak{so}(n, 0, \mathbb{F}) \cong \mathfrak{so}(0, n, \mathbb{F})$. The algebra $\mathfrak{so}(n, 0, \mathbb{F})$ is recognised as the orthogonal algebra $\mathfrak{so}(n, \mathbb{F})$, so

$$\mathfrak{so}(n, 0, \mathbb{F}) \cong \mathfrak{so}(0, n, \mathbb{F}) \cong \mathfrak{so}(n, \mathbb{F}). \tag{2.60}$$

In fact, as shown in Exercise 2.26, the algebras $\mathfrak{so}(p, q, \mathbb{C})$ with the *same* value for $p + q$ are all isomorphic to each other:

$$\mathfrak{so}(p, q, \mathbb{C}) \cong \mathfrak{so}(p + q, \mathbb{C}). \tag{2.61}$$

The similar statement for $\mathbb{F} = \mathbb{R}$ is *not true*. The generalisation from $s = I_{p+q}$ in the definition of $\mathfrak{so}(p+q, \mathbb{F})$ to $s = I_{p,q}$ in $\mathfrak{so}(p, q, \mathbb{F})$ is thus only worthwhile if $\mathbb{F} = \mathbb{R}$. In that case, one usually writes $\mathfrak{so}(p, q) = \mathfrak{so}(p, q, \mathbb{R})$. For $p, q > 0$, the matrix Lie algebras $\mathfrak{so}(p, q)$ are known as **indefinite orthogonal algebras**. Since $I_{p,q}$ is invertible (with $I_{p,q}^{-1} = I_{p,q}$), the cyclicity of the trace operator shows that the relation $x^t I_{p,q} = -I_{p,q} x$ implies $\mathrm{tr}(x) = 0$. It follows that $\mathfrak{so}(p, q)$ is a Lie subalgebra of $\mathfrak{sl}(p+q, \mathbb{R})$. As the next proposition shows, the dimension of $\mathfrak{so}(p, q)$ only depends on $n = p + q$. It is thus the same for all indefinite orthogonal algebras generated by $n \times n$ matrices.

Proposition 2.13. *For $p, q \in \mathbb{N}_0$, the dimension of $\mathfrak{so}(p, q)$ is given by*

$$\dim \mathfrak{so}(p, q) = \tfrac{1}{2}(p+q)(p+q-1). \tag{2.62}$$

Proof. This is the content of Exercise 2.27. ■

Although the statement in (2.61) does not apply if $\mathbb{C}$ is replaced by $\mathbb{R}$, the family of indefinite orthogonal algebras do enjoy some isomorphisms.

Proposition 2.14. *For $p, q \in \mathbb{N}_0$, $\mathfrak{so}(p, q) \cong \mathfrak{so}(q, p)$.*

Proof. If $p = 0$ or $q = 0$, the isomorphism is covered in (2.60). For $p, q > 0$, let $n = p + q$ and define the $n \times n$ matrix σ_n by

$$\sigma_n := \begin{pmatrix} 0 & \dots & 0 & 1 \\ \vdots & \iddots & \iddots & 0 \\ 0 & \iddots & \iddots & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix}. \tag{2.63}$$

It satisfies

$$\sigma_n I_{p,q} \sigma_n^{-1} = -I_{q,p}. \tag{2.64}$$

Mimicking the proof of Proposition 2.5, it then follows that $\mathfrak{gl}_{I_{p,q}}(n, \mathbb{R})$ and $\mathfrak{gl}_{-I_{q,p}}(n, \mathbb{R})$ are isomorphic. Since $\mathfrak{gl}_{-I_{q,p}}(n, \mathbb{R}) \cong \mathfrak{gl}_{I_{q,p}}(n, \mathbb{R})$, we have thus established that $\mathfrak{gl}_{I_{p,q}}(n, \mathbb{R}) \cong \mathfrak{gl}_{I_{q,p}}(n, \mathbb{R})$, that is, $\mathfrak{so}(p, q) \cong \mathfrak{so}(q, p)$. ■

Remark on applications

> The indefinite orthogonal Lie algebras $\mathfrak{so}(d-1, 1)$ appear naturally in relativistic physics as they describe infinitesimal Lorentz transformations in d-dimensional Minkowski spacetime.

2.7 Unitary Lie algebras

The **conjugate transpose** (or hermitian transpose) of an $m \times n$ matrix A with complex entries is the $n \times m$ matrix $A^\dagger$ obtained by taking the transpose of A and the complex conjugate of each entry. That is,

$$(A^\dagger)_{ij} = \overline{A_{ji}} \qquad \text{for all} \qquad 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m, \tag{2.65}$$

where the bar denotes complex conjugation. Compactly, this is written

$$A^\dagger = \overline{A^t} \qquad \text{or} \qquad A^\dagger = (\bar{A})^t. \tag{2.66}$$

A square matrix is said to be **hermitian** (or self-adjoint) if it equals its own conjugate transpose,

$$A^\dagger = A. \tag{2.67}$$

Similarly, it is called **anti-hermitian** (or skew-hermitian) if

$$A^\dagger = -A. \tag{2.68}$$

The set of anti-hermitian $n \times n$ matrices is given by $\{x \in M_n(\mathbb{C}) \,|\, x^\dagger = -x\}$ and is seen to be a vector space over $\mathbb{R}$. However, since $\mathrm{i}x$ is hermitian if x is anti-hermitian, it is *not* a vector space over $\mathbb{C}$. As verified in Exercise 2.16, endowed with the Lie bracket (2.11),

$$\mathfrak{u}(n) := \{x \in M_n(\mathbb{C}) \,|\, x^\dagger = -x\} \tag{2.69}$$

is a Lie algebra. It is known as the **unitary Lie algebra** of $n \times n$ matrices. There is also a **special unitary Lie algebra** consisting of the elements of $\mathfrak{u}(n)$ with zero trace, defined as

$$\mathfrak{su}(n) := \{x \in \mathfrak{u}(n) \,|\, \mathrm{tr}(x) = 0\}. \tag{2.70}$$

This is readily seen to be a Lie subalgebra of $\mathfrak{sl}(n, \mathbb{C})$. Although the matrices making up the elements of $\mathfrak{u}(n)$ and $\mathfrak{su}(n)$ have complex entries, the Lie algebras $\mathfrak{u}(n)$ and $\mathfrak{su}(n)$ are, as already indicated, *real Lie algebras*. This illustrates that the elements of a real matrix Lie algebra need not be real!

A basis for the one-dimensional Lie algebra $\mathfrak{u}(1)$ is given by the 1×1 matrix (i), where i is the imaginary unit. For $n \geqslant 2$, a basis for $\mathfrak{u}(n)$ is given by

$$\{\mathrm{i}E_{ii} \,|\, i = 1, \ldots, n\} \sqcup \{E_{ij} - E_{ji}, \mathrm{i}(E_{ij} + E_{ji}) \,|\, 1 \leqslant i < j \leqslant n\}, \tag{2.71}$$

where it is emphasised that the vector space $\mathfrak{u}(n)$ is over $\mathbb{R}$. Mimicking the basis (2.27) for $\mathfrak{sl}(n, \mathbb{F})$, a basis for $\mathfrak{su}(n)$ for $n \geqslant 2$ is then seen to be given by

$$\{\mathrm{i}(E_{11}-E_{22}), \mathrm{i}(E_{22}-E_{33}), \ldots, \mathrm{i}(E_{n-1,n-1}-E_{nn})\} \sqcup \{E_{ij}-E_{ji}, \mathrm{i}(E_{ij}+E_{ji}) \,|\, 1 \leqslant i < j \leqslant n\}. \tag{2.72}$$

Proposition 2.15. *The dimensions of the unitary Lie algebras are*

$$\dim \mathfrak{u}(n) = n^2, \qquad \dim \mathfrak{su}(n) = n^2 - 1. \tag{2.73}$$

Proof. The dimensions of the unitary Lie algebras are equal to the cardinalities of the bases (2.71) and (2.72):

$$\mathfrak{u}(n):\ n + 2 \times \tfrac{1}{2}n(n-1) = n^2, \qquad \mathfrak{su}(n):\ (n-1) + 2 \times \tfrac{1}{2}n(n-1) = n^2 - 1. \tag{2.74}$$

■

It is noted that the trace of a unitary matrix is purely imaginary (or zero), that is, $\mathrm{tr}(x) \in \mathrm{i}\mathbb{R}$ for all $x \in \mathfrak{u}(n)$. It follows that i times the trace map, $\mathrm{i} \cdot \mathrm{tr} : \mathfrak{u}(n) \to \mathbb{C}$, is a Lie algebra homomorphism with image $\mathrm{im}(\mathrm{i}\cdot\mathrm{tr}) = \mathbb{R}$ and kernel $\ker(\mathrm{i}\cdot\mathrm{tr}) = \mathfrak{su}(n)$. As in the similar analysis of the general and special linear algebras in (2.28), the first isomorphism theorem thus implies that

$$\mathfrak{u}(n)/\mathfrak{su}(n) \cong \mathbb{R}. \tag{2.75}$$

A basis for the three-dimensional Lie algebra $\mathfrak{su}(2)$ can be read off from (2.72). After a rescaling of these basis elements, a particularly convenient basis is given by

$$\{u_k = -\tfrac{\mathrm{i}}{2}\sigma_k \mid k = 1, 2, 3\}, \tag{2.76}$$

where the **Pauli matrices** are defined as

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.77}$$

The nontrivial commutation relations are

$$[u_1, u_2] = u_3, \qquad [u_2, u_3] = u_1, \qquad [u_3, u_1] = u_2. \tag{2.78}$$

In contrast to the anti-hermitian generators (2.76) of $\mathfrak{su}(2)$, the Pauli matrices are *hermitian.*

Remark on notation

A very common notation for the one-dimensional (abelian) Lie algebra $\mathfrak{a}$ is $\mathfrak{u}(1)$. However, this can be confusing since $\mathfrak{u}(1)$ is a *real* Lie algebra, while $\mathfrak{a}$ need not be. Indeed, as stated in (1.63), $\mathfrak{a}(\mathbb{F}) \cong \mathbb{F}$, while $\mathfrak{u}(1) \cong \mathbb{R}$.

Remark on applications

The Pauli matrices play an important role in quantum mechanics where they are used, for instance, to describe the interaction of the spin of a particle with an external electromagnetic field. In the physics literature, one often, somewhat sloppily, regards the Pauli matrices σ_k, rather than $-\frac{\mathrm{i}}{2}\sigma_k$, as the generators of $\mathfrak{su}(2)$, see also Remark on notation and terminology following (1.14)-(1.16).

Remark on applications

In quantum mechanics, physical observables are usually represented by hermitian operators on some Hilbert space. For simplicity, we may think of them as matrices acting on a vector space. The requirement of hermiticity ensures that the eigenvalues are real, allowing us to relate them to measurements of physical quantities.

2.8 Complexification and real forms

Let V be a vector space over $\mathbb{R}$. The **complexification** of V, denoted by $V_{\mathbb{C}}$, is the space of formal linear combinations of the form

$$v + \mathrm{i}w, \qquad v, w \in V. \tag{2.79}$$

This is clearly a real vector space and becomes a vector space over $\mathbb{C}$ if we define

$$\mathrm{i}(v + \mathrm{i}w) := -w + \mathrm{i}v. \tag{2.80}$$

We may view it as the direct sum of two copies of V:

$$V_{\mathbb{C}} = V \oplus \mathrm{i}V. \tag{2.81}$$

The (complex) dimension of $V_{\mathbb{C}}$ equals the (real) dimension of V. Conversely, if we view a complex vector space as a vector space over $\mathbb{R}$, its dimension is twice the complex dimension.

Extending this to Lie algebras, the **complexification** $\mathfrak{g}_{\mathbb{C}}$ of the real Lie algebra $\mathfrak{g}$ is the complexification of $\mathfrak{g}$ as a vector space

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus \mathrm{i}\mathfrak{g}, \tag{2.82}$$

with the Lie bracket of $\mathfrak{g}$ extended by linearity to $\mathbb{C}$. Examples of complexifications are

$$\mathfrak{gl}(n,\mathbb{R})_{\mathbb{C}} \cong \mathfrak{gl}(n,\mathbb{C}), \quad \mathfrak{sl}(n,\mathbb{R})_{\mathbb{C}} \cong \mathfrak{sl}(n,\mathbb{C}), \quad \mathfrak{sp}(2n,\mathbb{R})_{\mathbb{C}} \cong \mathfrak{sp}(2n,\mathbb{C}), \quad \mathfrak{so}(n,\mathbb{R})_{\mathbb{C}} \cong \mathfrak{so}(n,\mathbb{C}) \tag{2.83}$$

and

$$\mathfrak{u}(n)_{\mathbb{C}} \cong \mathfrak{gl}(n,\mathbb{C}), \qquad \mathfrak{su}(n)_{\mathbb{C}} \cong \mathfrak{sl}(n,\mathbb{C}). \tag{2.84}$$

Conversely, we can associate a real Lie algebra $\hat{\mathfrak{g}}_{\mathbb{R}}$ to a given complex Lie algebra $\hat{\mathfrak{g}}$ by removing the imaginary parts of the elements.

That $\mathfrak{gl}(n,\mathbb{C})$ is the complexification of $\mathfrak{gl}(n,\mathbb{R})$, in particular, merely means that a complex matrix can be written as $x + \mathrm{i}y$ where x and y are real matrices. Likewise, that $\mathfrak{gl}(n,\mathbb{C})$ is the complexification of $\mathfrak{u}(n)$ comes about by writing a given complex matrix z uniquely as $x + \mathrm{i}y$ with x and y anti-hermitian (implying that $\mathrm{i}y$ is hermitian). This is done by setting $x = \frac{1}{2}(z - z^\dagger)$ and $y = \frac{1}{2\mathrm{i}}(z + z^\dagger)$.

It is noted that the isomorphisms $\mathfrak{gl}(n,\mathbb{R})_{\mathbb{C}} \cong \mathfrak{gl}(n,\mathbb{C})$ and $\mathfrak{u}(n)_{\mathbb{C}} \cong \mathfrak{gl}(n,\mathbb{C})$ imply the isomorphism $\mathfrak{gl}(n,\mathbb{R})_{\mathbb{C}} \cong \mathfrak{u}(n)_{\mathbb{C}}$. However, the two real Lie algebras $\mathfrak{gl}(n,\mathbb{R})$ and $\mathfrak{u}(n)$ are *not* isomorphic (except if $n = 1$). Instead, the real Lie algebras $\mathfrak{gl}(n,\mathbb{R})$ and $\mathfrak{u}(n)$ are examples of *real forms* of the complex Lie algebra $\mathfrak{gl}(n,\mathbb{C})$. In general, a **real form** of a complex Lie algebra $\mathfrak{g}$ is a real Lie algebra whose complexification is isomorphic to $\mathfrak{g}$. Accordingly, the real Lie algebras $\mathfrak{sl}(n,\mathbb{R})$ and $\mathfrak{su}(n)$ are real forms of the complex Lie algebra $\mathfrak{sl}(n,\mathbb{C})$.

In general, a complex Lie algebra may contain several non-isomorphic real forms. This is illustrated by the orthogonal algebra $\mathfrak{so}(n,\mathbb{C})$ whose set of non-isomorphic real forms includes $\mathfrak{so}(p,n-p)$, $p = 0,1,\ldots,\lfloor\frac{n}{2}\rfloor$. The so-called **compact** real form is given by (the *definite* orthogonal algebra) $\mathfrak{so}(n,\mathbb{R})$, while the so-called **split** (or normal) real form is (the *indefinite* orthogonal algebra) $\mathfrak{so}(r,r)$ for $n = 2r$ and $\mathfrak{so}(r,r+1) \cong \mathfrak{so}(r+1,r)$ for $n = 2r+1$.

2.9 *Exercises*

Exercise 2.1.

Verify that imposing the Lie product (2.7) on $\mathrm{End}(V)$ turns the latter into a Lie algebra. This is a special case of Exercise 1.6.

Exercise 2.2.

Argue that, up to isomorphism, $\mathfrak{sl}(2,\mathbb{C})$ is the only simple complex Lie algebra of dimension 3.

Exercise 2.3.

Let $\{e_{-1}, e_0, e_1\}$ be a basis for the complex vector space V, and let

$$[e_i, e_j] := \begin{cases} (j-i)e_{i+j}, & i+j \in \{-1,0,1\}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.85}$$

define a bilinear operation on V. Show that this operation defines a Lie bracket on V and that the ensuing Lie algebra is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

Exercise 2.4.

Verify the commutation relations (2.35) for the $\mathfrak{sl}(3, \mathbb{F})$ basis generators given in (2.34).

Exercise 2.5.

Let V and W be finite-dimensional vector spaces, and let

$$\mu : V \times V \to W \tag{2.86}$$

be bilinear. Show that

$$\{x \in \mathfrak{gl}(V) \mid \mu(xu, v) + \mu(u, xv) = 0 \text{ for all } u, v \in V\} \tag{2.87}$$

is a Lie subalgebra of $\mathfrak{gl}(V)$.

Exercise 2.6.

Let

$$f : M_n(\mathbb{F}) \to \mathbb{F} \tag{2.88}$$

be a linear map such that $f([x, y]) = 0$ for all $x, y \in M_n(\mathbb{F})$. Show that f is a scalar multiple of the trace map. That is, show that there exists $a \in \mathbb{F}$ such that $f(x) = a\,\mathrm{tr}(x)$ for all $x \in M_n(\mathbb{F})$.

Exercise 2.7.

Find a matrix Lie algebra isomorphic to the two-dimensional Lie algebra with basis $\{x, y\}$ and Lie product $[x, y] = x$.

Exercise 2.8.

Let $x \in \mathfrak{gl}(n, \mathbb{F})$ have n distinct eigenvalues, denoted by $\lambda_1, \ldots, \lambda_n$. Prove that the n^2 eigenvalues of ad_x are given by $\lambda_i - \lambda_j$ where $1 \leqslant i, j \leqslant n$. (Note that the scalars $\lambda_i - \lambda_j$ need not be distinct.)

Exercise 2.9.

Show that $Z(\mathfrak{gl}(n, \mathbb{F})) \cong \mathfrak{a}(\mathbb{F})$.

Exercise 2.10.

Show that $\mathfrak{sl}(n, \mathbb{C})$ is simple for $n \geqslant 2$.

Exercise 2.11.

Show that the real Lie algebras $\mathfrak{so}(3, \mathbb{R})$ and $\mathfrak{su}(2)$ are isomorphic.

Exercise 2.12.

As is easily verified, the Lie algebras $\mathfrak{sl}(2, \mathbb{F})$, $\mathfrak{so}(3, \mathbb{F})$, $\mathfrak{t}(2, \mathbb{F})$ and $\mathfrak{n}(3, \mathbb{F})$ are all three-dimensional. In each of the cases $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$, work out how the Lie algebras are related by isomorphisms.

Exercise 2.13.

(i) For $r \geqslant 2$, work out an explicit basis for $\mathfrak{sp}(2r, \mathbb{C})$.

(ii) With respect to the basis obtained under (i), determine the structure constants of $\mathfrak{sp}(4, \mathbb{C})$.

Exercise 2.14.
Show that $[\mathfrak{d}(n, \mathbb{F}), \mathfrak{n}^+(n, \mathbb{F})] = \mathfrak{n}^+(n, \mathbb{F})$ and $[\mathfrak{t}(n, \mathbb{F}), \mathfrak{t}(n, \mathbb{F})] = \mathfrak{n}^+(n, \mathbb{F})$.

Exercise 2.15.
Let $r \in \mathbb{N}$.

(i) Show that $\mathfrak{so}(r, r) \cong \{x \in M_{2r}(\mathbb{R}) \mid x^t s = -sx\}$ where $s = \begin{pmatrix} 0_r & I_r \\ I_r & 0_r \end{pmatrix}$.

(ii) Show that $\mathfrak{so}(r+1, r) \cong \{x \in M_{2r+1}(\mathbb{R}) \mid x^t s = -sx\}$ where $s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0_r & I_r \\ 0 & I_r & 0_r \end{pmatrix}$.

Exercise 2.16.
Verify that $\mathfrak{u}(n)$ endowed with the commutator bracket is a Lie algebra.

Exercise 2.17.
Let $\mathfrak{h} = \{x \in M_2(\mathbb{C}) \mid x^\dagger s = -sx\}$, where $s = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

(i) Find a basis for $\mathfrak{h}$.

(ii) Show that $\mathfrak{h}$ is a Lie algebra if we set $[x, y] = xy - yx$ for all $x, y \in \mathfrak{h}$.

(iii) Determine whether $\mathfrak{h}$ is simple.

Exercise 2.18.
Let $s = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

(i) Find a basis for $\mathfrak{gl}_s(3, \mathbb{F})$.

(ii) Show that $\mathfrak{gl}_s(3, \mathbb{F})$ is not simple.

Exercise 2.19.

(i) Show that $\mathfrak{gl}(n, \mathbb{F})$ contains a one-dimensional abelian ideal $\mathfrak{a}$ consisting of all scalar multiples of the identity matrix. As vector spaces, this simply means that

$$\mathfrak{gl}(n, \mathbb{F}) \cong \mathfrak{sl}(n, \mathbb{F}) \oplus \{aI_n \mid a \in \mathbb{F}\}, \qquad n \geqslant 2. \tag{2.89}$$

(ii) Let $\mathfrak{a}$ be as in (i). Show that

$$\mathfrak{gl}(n, \mathbb{F})/\mathfrak{a} \cong \mathfrak{sl}(n, \mathbb{F}), \qquad n \geqslant 2. \tag{2.90}$$

Exercise 2.20.
Show that $\mathfrak{sp}(2r, \mathbb{C}) \cap \mathfrak{su}(2r)$ is a real Lie algebra.

Exercise 2.21.
Show that

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} \mathrm{i}a & z \\ -\bar{z} & -\mathrm{i}a \end{pmatrix} \,\middle|\, a \in \mathbb{R},\ z \in \mathbb{C} \right\}. \tag{2.91}$$

Exercise 2.22.
Verify the isomorphisms (of complex Lie algebras) listed in (2.83) and (2.84).

Exercise 2.23.
Show that the two real forms $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(2)$ of the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ are non-isomorphic Lie algebras.

Exercise 2.24.
Show that $D_2 \cong A_1 \boxplus A_1$.

Exercise 2.25.
Identify an $\mathfrak{sl}(3)$ subalgebra of $\mathfrak{sl}(4)$.

Exercise 2.26.
For $p, q \in \mathbb{N}_0$, show that $\mathfrak{so}(p,q,\mathbb{C}) \cong \mathfrak{so}(p+q,\mathbb{C})$.

Exercise 2.27.
For $p, q \in \mathbb{N}_0$, show that $\dim \mathfrak{so}(p,q) = \frac{1}{2}(p+q)(p+q-1)$.

Exercise 2.28.
Following the definition of the indefinite orthogonal algebras in Section 2.6, define the corresponding **indefinite unitary Lie algebras** $\mathfrak{u}(p,q)$ and $\mathfrak{su}(p,q)$.

Exercise 2.29.
Establish the isomorphism $\mathfrak{su}(n)_{\mathbb{C}} \cong \mathfrak{sl}(n,\mathbb{C})$.

Exercise 2.30.

(i) Show that the Pauli matrices satisfy $\mathrm{tr}(\sigma_i\sigma_j) = 2\delta_{i,j}$ for all $i, j \in \{1,2,3\}$.

(ii) Verify that $\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 3I_2$.

Exercise 2.31.
Consider the **Gell-Mann matrices**

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{2.92}$$

and

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{2.93}$$

(i) Show that $\mathrm{tr}(\lambda_i \lambda_j) = 2\delta_{i,j}$ for all $i, j \in \{1, \dots, 8\}$.

(ii) Show that $\{u_k = -\frac{\mathrm{i}}{2}\lambda_k \mid k = 1, \dots, 8\}$ is a basis for $\mathfrak{su}(3)$.

(iii) Work out the commutation relations $[u_k, u_\ell]$ for $k, \ell \in \{1, \dots, 8\}$.

(iv) Verify that

$$\sum_{k=1}^{8} \lambda_k^2 = \tfrac{16}{3} I_3. \tag{2.94}$$

It is remarked that the Lie algebra $\mathfrak{su}(3)$ *plays an important role in the description of quarks in Quantum ChromoDynamics (also known as QCD).*

3 Representation theory

Let A be an algebra (or group or ring or similar). Basically, the (so-called linear) **representation theory** of A is the study of the ways in which A may act on a vector space. More generally, it can be viewed as the science of mathematically encoding or describing symmetries, here generated or governed by A. Loosely speaking, a (linear) representation ρ of $\mathfrak{g}$ on a vector space V assigns to each $x \in \mathfrak{g}$ an operator $\rho(x)$ acting on V, in such a way that linearity and the Lie bracket are preserved. The precise definition is given below.

3.1 Representations and modules

A **representation** of $\mathfrak{g}$ on the vector space V (over the same field $\mathbb{F}$ as $\mathfrak{g}$) is a Lie algebra homomorphism

$$\rho: \ \mathfrak{g} \to \mathfrak{gl}(V), \qquad x \mapsto \rho(x). \tag{3.1}$$

The vector space V is then referred to as the corresponding **representation space** and its dimension is known as the **dimension** of the representation:

$$\dim(\rho) := \dim V. \tag{3.2}$$

The element $\rho(x) \in \mathfrak{gl}(V)$ acts on vectors $v \in V$, suggesting the notation $\rho(x)(v)$ or even $(\rho(x))(v)$. This is unnecessarily cumbersome, so we will adopt a leaner notation and simply write $\rho(x)v$. This notation fits well with the fact that $\rho(x)$ can be identified with a matrix (simply because it is a linear map between vector spaces).

The notation simplifies even further when describing the action of $\mathfrak{g}$ on V as the action on a module. To be specific, a (left) **$\mathfrak{g}$-module** is a vector space V endowed with an operation

$$\mathfrak{g} \times V \to V, \qquad (x, v) \mapsto xv, \tag{3.3}$$

satisfying

$$\begin{aligned} (\mathrm{M1}): \qquad (ax + by)v &= a(xv) + b(yv) \\ (\mathrm{M2}): \qquad x(av + bw) &= a(xv) + b(xw) \\ (\mathrm{M3}): \qquad [x, y]v &= x(yv) - y(xv) \end{aligned} \tag{3.4}$$

for all $x, y \in \mathfrak{g}$, $a, b \in \mathbb{F}$ and $v, w \in V$. Conditions (M1) and (M2) say that the operation must preserve the linearity in $\mathfrak{g}$ and V, respectively, while condition (M3) ensures compatibility with the Lie bracket. In fact, given a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, we may view V as a $\mathfrak{g}$-module via the action $xv = \rho(x)v$. Conversely, given a $\mathfrak{g}$-module V, the relation $\rho(x)v = xv$ defines a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. Thus, studying modules or representations essentially amounts to the same thing.

By the homomorphism lemma for Lie algebras, Lemma 1.11, the kernel of a representation ρ of $\mathfrak{g}$ is an ideal of $\mathfrak{g}$ while the image of ρ is a Lie subalgebra of $\mathfrak{gl}(V)$. By working with ρ instead of $\mathfrak{g}$ itself, we will in general lose information about $\mathfrak{g}$. If, however, the kernel is zero, or equivalently ρ is a monomorphism, then no information is lost. In this case, the representation is said to be **faithful**.

Since the adjoint map $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a Lie algebra homomorphism, it provides a representation of $\mathfrak{g}$ on itself, known as the **adjoint representation**. According to Proposition 2.1, the kernel of the adjoint map is given by $Z(\mathfrak{g})$, so

$$\boxed{\textit{the adjoint representation is faithful} \quad \Longleftrightarrow \quad Z(\mathfrak{g}) = \{0\}.}$$

For example, this is the case for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{F})$, but not for $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{F})$ since

$$Z(\mathfrak{gl}(2, \mathbb{F})) = \mathrm{span}_{\mathbb{F}}\Big\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\Big\}. \tag{3.5}$$

As already indicated, for a representation ρ of $\mathfrak{g}$, we can identify $\rho(x) \in \mathfrak{gl}(V)$ with a matrix. However, this identification is basis dependent. With respect to an ordered basis $\{v_1, \ldots, v_n\}$ for the n-dimensional vector space V, we denote by $R_\rho(x)$ the matrix associated with $\rho(x)$. To describe the adjoint representation in terms of matrices, we thus use an ordered basis for the d-dimensional Lie algebra itself, such as $\{x_1, \ldots, x_{\mathrm{d}}\}$. The entries of the corresponding matrix representation of x_a are then given in terms of the associated structure constants,

$$\big(R_{\mathrm{ad}}(x_a)\big)_{cb} = f_{ab}{}^c, \tag{3.6}$$

where c is the row label and b the column label. For example, in the ordered basis $\{e, h, f\}$, the adjoint representation of $\mathfrak{sl}(2, \mathbb{F})$, with relations given in (2.33), is given by

$$R_{\mathrm{ad}}(e) = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad R_{\mathrm{ad}}(h) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \qquad R_{\mathrm{ad}}(f) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}. \tag{3.7}$$

Let $\mathfrak{g}(V)$ denote the image of the $\mathfrak{g}$-module V under the action of $\mathfrak{g}$,

$$\mathfrak{g}(V) := \{v \in V \,|\, v = xw \text{ for some } x \in \mathfrak{g},\ w \in V\}. \tag{3.8}$$

A **submodule** of V is a $\mathfrak{g}$-invariant subspace U of V, that is, $\mathfrak{g}(U) \subseteq U$. Thus, U is a $\mathfrak{g}$-module in its own right. In particular, $\{0\}$ and V are both submodules of V. A **proper submodule** of V is a submodule of V different from V.

An **irreducible** (or simple) module for $\mathfrak{g}$ is a $\mathfrak{g}$-module V whose only submodules are $\{0\}$ and V. A one-dimensional $\mathfrak{g}$-module is an almost trivial example of an irreducible module. Physicists occasionally refer to the corresponding one-dimensional representation as a **singlet representation**. A $\mathfrak{g}$-module which is not irreducible is called **reducible**.

Let V be a $\mathfrak{g}$-module. The **submodule generated by** $v \in V$, denoted by $\mathfrak{g}(v)$, is defined to be the subspace of V spanned by v itself and all vectors reachable from v by the action of $\mathfrak{g}$. That is,

$$\mathfrak{g}(v) := \mathrm{span}_{\mathbb{F}}\big\{u \in \{v, x_1 \cdots x_n v\} \,|\, x_1, \ldots, x_n \in \mathfrak{g},\ n \in \mathbb{N}\big\}, \tag{3.9}$$

where the action of $\mathfrak{g}$ on v is **right-nested** in the sense that

$$x_1 x_2 \cdots x_n v := x_1(x_2(\cdots(x_n v)\cdots)). \tag{3.10}$$

It is straightforward to verify that $\mathfrak{g}(v)$ is indeed a submodule of V. If $x_1 = \cdots = x_n$, we may simplify the notation even further and write $x_1^n v$, for example.

Proposition 3.1. *The $\mathfrak{g}$-module V is irreducible if and only if $V \subseteq \mathfrak{g}(v)$ for each nonzero $v \in V$.*

Proof. The proof is by contradiction, showing that V is reducible if and only if there exists nonzero $v_0 \in V$ such that $V \nsubseteq \mathfrak{g}(v_0)$. First, if V is reducible, then there exists a nonzero proper submodule $U \subset V$. For any $u \in U$, $\mathfrak{g}(u) \subseteq U$, so $V \nsubseteq \mathfrak{g}(u)$. Setting $v_0 = u$ for any given nonzero $u \in U$ thus completes the "only if" part of the proof. To establish the "if" part, we let nonzero $v_0 \in V$ such that $V \nsubseteq \mathfrak{g}(v_0)$. Since $v_0 \neq 0$, $\mathfrak{g}(v_0) \neq \{0\}$, and since $V \nsubseteq \mathfrak{g}(v_0)$, $\mathfrak{g}(v_0) \neq V$. Since $\mathfrak{g}(v_0)$ is a submodule of V, it is therefore a proper submodule of V. Hence, V is reducible. ■

3.2 Intertwiners

Let V_1 and V_2 be $\mathfrak{g}$-modules with corresponding representations denoted by ρ_1 and ρ_2. A linear map $\phi : V_1 \to V_2$ is called an **intertwiner** if

$$\phi \circ \rho_1(x) = \rho_2(x) \circ \phi, \qquad \forall\, x \in \mathfrak{g}. \tag{3.11}$$

That is, it 'commutes' with or **intertwines** the action of $\mathfrak{g}$. The associated diagram

$$\begin{tikzcd} V_1 \arrow[r, "\rho_1(x)"] \arrow[d, "\phi"] & V_1 \arrow[d, "\phi"] \\ V_2 \arrow[r, "\rho_2(x)"] & V_2 \end{tikzcd}$$

is thus an example of a **commutative diagram**. In such a diagram, all (allowed) paths between a given ordered pair of objects correspond to the *same* combined map. In the example above, the two paths from the top-left V_1 to the bottom-right V_2 do indeed yield the same map.

If the intertwiner ϕ in (3.11) is invertible, then

$$\rho_1(x) = \phi^{-1} \circ \rho_2(x) \circ \phi \tag{3.12}$$

and the two modules V_1 and V_2 are said to be **isomorphic** as $\mathfrak{g}$-modules. They are thus indistinguishable, not only as vector spaces, but also as $\mathfrak{g}$-modules, and are therefore considered **equivalent**. From the perspective of the Lie algebra, the two modules are simply the 'same'. The corresponding representations ρ_1 and ρ_2 are likewise said to be equivalent. Usually, one is interested in representations only up to equivalence. Accordingly, a typical problem in representation theory is to determine all *inequivalent* irreducible representations of a given Lie algebra.

Lemma 3.2. *Let $\phi : V \to W$ be an intertwiner of the $\mathfrak{g}$-modules V and W. Then,*

(i) $\ker(\phi)$ *is a $\mathfrak{g}$-invariant subspace of V;*

(ii) $\mathrm{im}(\phi)$ *is a $\mathfrak{g}$-invariant subspace of W.*

Proof. This is the content of Exercise 3.7. ■

Theorem 3.3 (Schur's lemma for Lie algebras)**.** *Let V and W be irreducible $\mathfrak{g}$-modules and $\phi : V \to W$ an intertwiner. Then, $\phi = 0$ or ϕ is invertible.*

Proof. By Lemma 3.2, the kernel $\ker(\phi)$ is an invariant subspace of V. Since V is irreducible, the invariant subspace $\ker(\phi)$ is either $\{0\}$ or V itself. It follows that ϕ is either injective (if $\ker(\phi) = \{0\}$) or zero (if $\ker(\phi) = V$). Likewise, the image $\operatorname{im}(\phi)$ is an invariant subspace of W. Since W is irreducible, $\operatorname{im}(\phi) = \{0\}$ or $\operatorname{im}(\phi) = W$, so ϕ is either zero (if $\operatorname{im}(\phi) = \{0\}$) or surjective (if $\operatorname{im}(\phi) = W$). Taken together, ϕ is either zero or a bijection. ∎

Corollary 3.4. *Let $\mathfrak{g}$ be a complex Lie algebra and V and W irreducible $\mathfrak{g}$-modules.*

(i) *If $\varphi : V \to V$ is an intertwiner, then $\varphi = \lambda \operatorname{id}_V$ for some $\lambda \in \mathbb{C}$.*

(ii) *Let ϕ_1 and ϕ_2 be nonzero intertwiners $V \to W$. Then, $\phi_1 = \lambda\phi_2$ for some $\lambda \in \mathbb{C}^\times$.*

Proof.

(i) Since we are working over the complex numbers, the linear map φ has at least one eigenvalue $\lambda \in \mathbb{C}$. The linear map $\varphi - \lambda \operatorname{id}_V$ is then an intertwiner from V to V, since, for the representation ρ corresponding to the $\mathfrak{g}$-module V, we have

$$(\varphi - \lambda \operatorname{id}_V) \circ \rho(x) = \varphi \circ \rho(x) - \lambda\rho(x) = \rho(x) \circ \varphi - \rho(x)\lambda = \rho(x) \circ (\varphi - \lambda \operatorname{id}_V) \tag{3.13}$$

for all $x \in \mathfrak{g}$. Let $v \in V$ denote an eigenvector corresponding to λ (recall that all eigenvectors are nonzero). Then, $(\varphi - \lambda \operatorname{id}_V)v = 0$, so $\varphi - \lambda \operatorname{id}_V$ is not invertible (since the kernel is nontrivial). By Schur's lemma, $\varphi - \lambda \operatorname{id}_V = 0$, that is, $\varphi = \lambda \operatorname{id}_V$.

(ii) By Schur's lemma, ϕ_1 and ϕ_2 are invertible. It follows that ϕ_2^{-1} is an intertwiner from W to V. Let ρ_V and ρ_W denote the representations corresponding to the $\mathfrak{g}$-modules V and W, respectively, and let $x \in \mathfrak{g}$. Then,

$$\begin{aligned}(\phi_2^{-1} \circ \phi_1) \circ \rho_V(x) &= \phi_2^{-1} \circ (\phi_1 \circ \rho_V(x)) = \phi_2^{-1} \circ (\rho_W(x) \circ \phi_1) \\ &= (\phi_2^{-1} \circ \rho_W(x)) \circ \phi_1 = (\rho_V(x) \circ \phi_2^{-1}) \circ \phi_1 \\ &= \rho_V(x) \circ (\phi_2^{-1} \circ \phi_1),\end{aligned} \tag{3.14}$$

meaning that the composition $\phi_2^{-1} \circ \phi_1$ is an intertwiner from V to V. By (i), it is of the form $\phi_2^{-1} \circ \phi_1 = \lambda \operatorname{id}_V$ for some $\lambda \in \mathbb{C}$, so $\phi_1 = \lambda\phi_2$. Since $\phi_1 \neq 0$, it follows that $\lambda \in \mathbb{C}^\times$.

∎

3.3 Dual representations

Given a vector space V over $\mathbb{F}$, the **dual space** V^* is the set of all linear maps $f : V \to \mathbb{F}$. The dual space becomes a vector space if we endow it with the natural addition and scalar multiplication rules

$$(f + g)(v) = f(v) + g(v), \qquad (af)(v) = a(f(v)), \qquad v \in V,\ a \in \mathbb{F}, \tag{3.15}$$

for all $f, g \in V^*$. Let $\{v_1, \ldots, v_n\}$ be a basis for the vector space V. We then define the **dual basis** for V as the set $\{f^1, \ldots, f^n\}$ of linear maps satisfying

$$f^i(v_j) = \delta^i_j, \qquad 1 \leqslant i, j \leqslant n, \tag{3.16}$$

where δ^i_j is the Kronecker delta symbol. In fact, this set provides a basis for V^*, implying that

$$\dim V^* = \dim V. \tag{3.17}$$

Because of this, it may be tempting to identify the two vector spaces. This is often safe enough, but may occasionally lead to confusion.

Dual spaces can be used to fabricate new Lie algebra representations from existing ones. Indeed, let V be a module over $\mathfrak{g}$. The dual space V^* becomes a $\mathfrak{g}$-module, called the **dual module** (or contragredient module), if the action of $x \in \mathfrak{g}$ on $f \in V^*$ is given by the map $xf \in V^*$ defined by

$$xf: \; V \to \mathbb{F}, \qquad v \mapsto -f(xv). \tag{3.18}$$

To establish that this is indeed a $\mathfrak{g}$-module, we first note that conditions (M1) and (M2) in (3.4) are respected, essentially by construction. To verify condition (M3), we must show that

$$([x,y]f)(v) = (x(yf) - y(xf))(v) \tag{3.19}$$

for all $x, y \in \mathfrak{g}$, $f \in V^*$ and $v \in V$. Using (3.18) repeatedly, we have

$$\begin{aligned} ([x,y]f)(v) &= -f([x,y]v) = -f(x(yv) - y(xv)) = -f(x(yv)) + f(y(xv)) \\ &= (xf)(yv) - (yf)(xv) = -y(xf)(v) + x(yf)(v) = (x(yf) - y(xf))(v). \end{aligned} \tag{3.20}$$

Equivalently, let $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ on the vector space V. Then, the **dual representation** (or contragredient representation) π^* of $\mathfrak{g}$ is given by

$$\pi^*: \; \mathfrak{g} \to \mathfrak{gl}(V^*), \qquad x \mapsto -(\pi(x))^t, \tag{3.21}$$

where the transpose is taken of the matrix representative $R_\pi(x)$ in some ordered basis.

3.4 Direct sums

Let V and W be $\mathfrak{g}$-modules. The corresponding **direct sum module** $V \oplus W$ is constructed by letting $\mathfrak{g}$ act as

$$x(v \oplus w) = (xv) \oplus (xw) \tag{3.22}$$

for all $x \in \mathfrak{g}$, $v \in V$, and $w \in W$. It is convenient to simplify this notation and write

$$x(v + w) = xv + xw. \tag{3.23}$$

That $V \oplus W$ with this action is indeed a $\mathfrak{g}$-module is verified in Exercise 3.4. The dimension of the direct sum module is

$$\dim(V \oplus W) = \dim V + \dim W. \tag{3.24}$$

A $\mathfrak{g}$-module is **completely reducible** (or fully reducible) if it can be written as a direct sum of irreducible $\mathfrak{g}$-modules. A $\mathfrak{g}$-module is **indecomposable** if it cannot be written as a direct sum of two nonzero submodules. If it can, then it is called **decomposable**.

In terms of the corresponding representations $\rho_V : \mathfrak{g} \to \mathfrak{gl}(V)$ and $\rho_W : \mathfrak{g} \to \mathfrak{gl}(W)$, we have the **direct sum representation**

$$\rho_{V\oplus W}: \; \mathfrak{g} \to \mathfrak{gl}(V \oplus W), \qquad x \mapsto \rho_{V\oplus W}(x), \tag{3.25}$$

where $\rho_{V\oplus W}(x)$ acts on $v+w=v\oplus w$ as

$$\rho_{V\oplus W}(x)(v+w)=(\rho_V(x)v)\oplus(\rho_W(x)w)=\rho_V(x)v+\rho_W(x)w. \tag{3.26}$$

In a basis ordered as $B_V\sqcup B_W$, the matrix associated with $\rho_{V\oplus W}(x)$ is the block-diagonal matrix

$$R_{\rho_{V\oplus W}}(x)=\begin{pmatrix} R_{\rho_V}(x) & 0_{V\times W} \\ 0_{W\times V} & R_{\rho_W}(x)\end{pmatrix}, \tag{3.27}$$

where the matrix $0_{V\times W}$ is the $\dim V\times\dim W$ zero matrix. Accordingly, the dimension of the direct sum representation is given by

$$\dim(\rho_{V\oplus W})=\dim(\rho_V)+\dim(\rho_W)=\dim V+\dim W. \tag{3.28}$$

3.5 Tensor products

Let V and W be $\mathfrak{g}$-modules. The corresponding **tensor product module** $V\otimes W$ is constructed by letting $\mathfrak{g}$ act as

$$x(v\otimes w)=(xv)\otimes w+v\otimes(xw) \tag{3.29}$$

for all $x\in\mathfrak{g}$, $v\in V$, and $w\in W$, extended linearly to all of $V\otimes W$. That $V\otimes W$ with this action is indeed a $\mathfrak{g}$-module is verified in Exercise 3.4. The dimension of the tensor product module is

$$\dim(V\otimes W)=(\dim V)(\dim W). \tag{3.30}$$

In terms of the corresponding representations $\rho_V:\mathfrak{g}\to\mathfrak{gl}(V)$ and $\rho_W:\mathfrak{g}\to\mathfrak{gl}(W)$, we have the **tensor product representation**

$$\rho_{V\otimes W}:\ \mathfrak{g}\to\mathfrak{gl}(V\otimes W),\qquad x\mapsto\rho_{V\otimes W}(x), \tag{3.31}$$

where $\rho_{V\otimes W}(x)$ acts on $v\otimes w$ as

$$\rho_{V\otimes W}(x)(v\otimes w)=(\rho_V(x)v)\otimes w+v\otimes(\rho_W(x)w). \tag{3.32}$$

This tensor product structure is associative in the sense that

$$(\rho_1\otimes\rho_2)\otimes\rho_3\cong\rho_1\otimes(\rho_2\otimes\rho_3), \tag{3.33}$$

where $\rho_i=\rho_{V_i}$ and $\rho_i\otimes\rho_j=\rho_{V_i\otimes V_j}$. The dimension of the tensor product representation is given by

$$\dim(\rho_{V\otimes W})=\dim(\rho_V)\dim(\rho_W)=(\dim V)(\dim W). \tag{3.34}$$

3.6 Universal enveloping algebra

In Section 1.2, we associated a Lie algebra to any given associative algebra A and denoted it by $\mathrm{Lie}(A)$. In the following, we go the other way by associating an associative algebra to any given Lie algebra $\mathfrak{g}$. It is referred to as the *universal enveloping algebra* of $\mathfrak{g}$ and is denoted by $\mathfrak{U}(\mathfrak{g})$.

First, we introduce the so-called tensor algebra of the vector space V. To this end, we define the following tensor powers of V:

$$V^{\otimes 0} := \mathbb{F}, \qquad V^{\otimes n} := \underbrace{V \otimes \cdots \otimes V}_{n \text{ factors}}, \qquad n \in \mathbb{N}. \tag{3.35}$$

$V^{\otimes n}$ is thus a vector space of dimension

$$\dim V^{\otimes n} = (\dim V)^n. \tag{3.36}$$

We also define the infinite direct sum of these vector spaces as

$$T_V := \bigoplus_{n=0}^{\infty} V^{\otimes n}, \tag{3.37}$$

whose elements can be expressed as finite sums of terms, each of which is an element of some $V^{\otimes n}$. The space T_V admits a natural multiplication $*$ given by the tensor product

$$(u_1 \otimes \cdots \otimes u_m) * (v_1 \otimes \cdots \otimes v_n) = u_1 \otimes \cdots \otimes u_m \otimes v_1 \otimes \cdots \otimes v_n, \tag{3.38}$$

for $u_i, v_j \in V$, linearly extended to all of T_V. With respect to this multiplication, we have

$$V^{\otimes m} \otimes V^{\otimes n} \subseteq V^{\otimes (m+n)}. \tag{3.39}$$

More generally, it yields a map

$$T_V \times T_V \to T_V, \tag{3.40}$$

whereby T_V becomes an algebra, known as the **tensor algebra** of V. Multiplication by an element of $V^{\otimes 0}$ is equivalent to scalar multiplication, so the element $1 \in V^{\otimes 0}$ is the (multiplicative) identity element of T_V.

Viewing $\mathfrak{g}$ as a vector space, we can construct the tensor algebra $T_{\mathfrak{g}}$ of $\mathfrak{g}$. To implement the Lie algebraic structure of $\mathfrak{g}$, we consider the (infinite) subset

$$\mathfrak{i}_{\mathfrak{g}} := \operatorname{span}\{a \otimes (x \otimes y - y \otimes x - [x,y]) \otimes b \,|\, x, y \in \mathfrak{g};\ a, b \in T_{\mathfrak{g}}\} \tag{3.41}$$

of $T_{\mathfrak{g}}$, consisting of all linear combinations of elements of the form

$$a \otimes (x \otimes y - y \otimes x - [x,y]) \otimes b, \qquad x, y \in \mathfrak{g}, \quad a, b \in T_{\mathfrak{g}}. \tag{3.42}$$

It is noted that the core part, $x \otimes y - y \otimes x - [x,y]$, is composed as a difference of an element of $\mathfrak{g}^{\otimes 2}$ (namely $x \otimes y - y \otimes x$) and an element of $\mathfrak{g}^{\otimes 1}$ (namely $[x,y]$). As the notation $\mathfrak{i}_{\mathfrak{g}}$ suggests, this subset is an ideal of $T_{\mathfrak{g}}$. In fact, it is a two-sided ideal of $T_{\mathfrak{g}}$, where a **two-sided** ideal $\mathfrak{i} \subseteq A$ of an associative algebra A satisfies $a * y \in \mathfrak{i}$ and $y * a \in \mathfrak{i}$ for all $a \in A$ and $y \in \mathfrak{i}$, that is, $a * \mathfrak{i} \subseteq \mathfrak{i}$ and $\mathfrak{i} * a \subseteq \mathfrak{i}$. The quotient algebra

$$\mathfrak{U}(\mathfrak{g}) := T_{\mathfrak{g}}/\mathfrak{i}_{\mathfrak{g}} \tag{3.43}$$

with multiplication

$$(a + \mathfrak{i}_{\mathfrak{g}}) \cdot (b + \mathfrak{i}_{\mathfrak{g}}) := a \otimes b + \mathfrak{i}_{\mathfrak{g}}, \qquad \forall\, a, b \in T_{\mathfrak{g}}, \tag{3.44}$$

is then a unital associative algebra over $\mathbb{F}$, known as the **universal enveloping algebra** of $\mathfrak{g}$. That the multiplication rule (3.44) is well-defined is verified in Exercise 3.11. The explicit addition of $\mathfrak{i}_{\mathfrak{g}}$ is usually dropped, in which case the commutator of $u, v \in \mathfrak{U}(\mathfrak{g})$ simply reads

$$[u, v] = u \otimes v - v \otimes u. \tag{3.45}$$

In particular, as an element of $\mathfrak{U}(\mathfrak{g})$,

$$x \otimes y - y \otimes x - [x, y] = 0, \qquad x, y \in \mathfrak{g}. \tag{3.46}$$

More generally, as elements of $\mathfrak{U}(\mathfrak{g})$,

$$x_1 \otimes \cdots \otimes x_j \otimes (y_1 \otimes y_2) \otimes x_{j+3} \otimes \cdots \otimes x_n - x_1 \otimes \cdots \otimes x_j \otimes (y_2 \otimes y_1) \otimes x_{j+3} \otimes \cdots \otimes x_n \in \mathfrak{g}^{\otimes n} \tag{3.47}$$

should be identified with

$$x_1 \otimes \cdots \otimes x_j \otimes [y_1, y_2] \otimes x_{j+3} \otimes \cdots \otimes x_n \in \mathfrak{g}^{\otimes (n-1)}. \tag{3.48}$$

Any element $x \in \mathfrak{g}$ can be viewed as an element of $T_{\mathfrak{g}}$. This provides a canonical map ψ from $\mathfrak{g}$ into $\mathfrak{U}(\mathfrak{g})$,

$$\psi : \ \mathfrak{g} \to \mathfrak{U}(\mathfrak{g}), \tag{3.49}$$

and justifies the term *enveloping* algebra. It is very common to leave out ψ and simply write x instead of $\psi(x)$; we will adopt this practice here. The embedding map ψ also has a so-called universality property ensuring that

$$\boxed{\textit{the representation theory of } \mathfrak{U}(\mathfrak{g}) \textit{ is essentially the same as that of } \mathfrak{g}.}$$

Although we will not go into details, we stress that this is true despite $\mathfrak{U}(\mathfrak{g})$ being infinite-dimensional while $\mathfrak{g}$ need not be. In fact, one of our standing assumptions in these lecture notes is that $\mathfrak{g}$ is finite-dimensional. The following theorem offers an explicit prescription for constructing a basis for $\mathfrak{U}(\mathfrak{g})$.

Theorem 3.5 (Poincaré-Birkhoff-Witt theorem)**.** *Let* $\{x_1, \ldots, x_{\mathrm{d}}\}$ *be an ordered basis for* $\mathfrak{g}$*. Then, the elements*

$$x_1^{r_1} \cdots x_{\mathrm{d}}^{r_{\mathrm{d}}}, \qquad r_1, \ldots, r_{\mathrm{d}} \in \mathbb{N}_0, \tag{3.50}$$

form a basis for $\mathfrak{U}(\mathfrak{g})$*.*

Proof. We refer to the literature [1, 7, 8] for a proof of this. ■

The general yet explicit nature of the Poincaré-Birkhoff-Witt basis makes it a very powerful tool in the representation theory of Lie algebras.

3.7 Classification of $\mathfrak{sl}(2)$-modules

We recall the standard basis for the Lie algebra $\mathfrak{sl}(2) = \mathfrak{sl}(2, \mathbb{C})$,

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{3.51}$$

and the corresponding nontrivial commutation relations:

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h. \tag{3.52}$$

Proposition 3.6. *Let V be an $\mathfrak{sl}(2)$-module and $v \in V$ an eigenvector of h with eigenvalue λ. Then,*

(i) *the vector ev is either zero or an eigenvector of h with eigenvalue $\lambda + 2$;*

(ii) *the vector fv is either zero or an eigenvector of h with eigenvalue $\lambda - 2$.*

Proof. We have

$$h(ev) = [h, e]v + e(hv) = 2ev + e(\lambda v) = (\lambda + 2)ev \tag{3.53}$$

and

$$h(fv) = [h, f]v + f(hv) = -2fv + f(\lambda v) = (\lambda - 2)fv. \tag{3.54}$$

■

Applying e (or f) to an eigenvector of h thus has the effect of increasing (respectively decreasing) the eigenvalue in increments of 2 (or to produce zero). Correspondingly, the generator e is often referred to as a **raising operator**, while f is referred to as a **lowering operator**. Together, they are known as **ladder operators**.

To describe the irreducible representations of $\mathfrak{sl}(2)$, it is convenient to first study the space $\mathbb{C}[X, Y]$ of polynomials in the two variables X and Y with complex coefficients. For each $d \in \mathbb{N}_0$, we thus let V_d be the space of homogeneous polynomials in X and Y of degree d,

$$V_d := \operatorname{span}_{\mathbb{C}}\{X^k Y^{d-k} \mid k = 0, 1, \ldots, d\}, \tag{3.55}$$

here written as the span of a basis consisting of monomials. The space V_d is clearly a vector space of (complex) dimension $d + 1$.

For each $d \in \mathbb{N}_0$, we now introduce the linear map

$$\phi_d : \ \mathfrak{sl}(2) \to \mathfrak{gl}(V_d), \qquad x \mapsto \phi_d(x), \tag{3.56}$$

whose action on the elements of the basis $\{e, h, f\}$ for $\mathfrak{sl}(2)$ is defined by

$$\phi_d(e) := X\frac{\partial}{\partial Y}, \qquad \phi_d(h) := X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}, \qquad \phi_d(f) := Y\frac{\partial}{\partial X}. \tag{3.57}$$

Proposition 3.7. *For each $d \in \mathbb{N}_0$, the linear map ϕ_d is a representation of $\mathfrak{sl}(2)$.*

Proof. By construction, ϕ_d is linear, so it suffices to verify that

$$\phi_d([x,y]) = [\phi_d(x), \phi_d(y)], \qquad \forall\, x,y \in \{e,h,f\}. \tag{3.58}$$

We do this by showing that the two sides have identical actions on any given basis vector X^aY^b of V_d (where $a+b=d$, $a,b \in \mathbb{N}_0$). For $x=h$ and $y=e$, we have

$$\phi_d([h,e])(X^aY^b) = \phi_d(2e)(X^aY^b) = 2X\frac{\partial}{\partial Y}(X^aY^b) = 2bX^{a+1}Y^{b-1} \tag{3.59}$$

and

$$\begin{aligned}
[\phi_d(h), \phi_d(e)](X^aY^b) &= \phi_d(h)(\phi_d(e)(X^aY^b)) - \phi_d(e)(\phi_d(h)(X^aY^b)) \\
&= \phi_d(h)(bX^{a+1}Y^{b-1}) - \phi_d(e)((a-b)X^aY^b) \\
&= b(a-b+2)X^{a+1}Y^{b-1} - (a-b)bX^{a+1}Y^{b-1} \\
&= 2bX^{a+1}Y^{b-1}.
\end{aligned} \tag{3.60}$$

Verifying (3.58) for $x=h$ and $y=f$ is done similarly. Finally, for $x=e$ and $y=f$, we find

$$\phi_d([e,f]) = \phi_d(h)(X^aY^b) = (a-b)X^aY^b \tag{3.61}$$

and

$$\begin{aligned}
[\phi_d(e), \phi_d(f)](X^aY^b) &= \phi_d(e)(aX^{a-1}Y^{b+1}) - \phi_d(f)b(X^{a+1}Y^{b-1}) \\
&= a(b+1)X^aY^b - b(a+1)X^aY^b \\
&= (a-b)X^aY^b.
\end{aligned} \tag{3.62}$$

∎

Every V_d is thus an $\mathfrak{sl}(2)$-module. It is noted that the $d+1$ basis vectors X^kY^{d-k} for V_d in (3.55) are eigenvectors of $\phi_d(h)$ with distinct eigenvalues,

$$\phi_d(h)(X^kY^{d-k}) = (2k-d)X^kY^{d-k}, \qquad k = 0,1,\ldots,d. \tag{3.63}$$

This means that all the eigenspaces of $\phi_d(h)$ are one-dimensional (separately spanned by a basis vector of the form X^kY^{d-k}). Hence, $\phi_d(h)$ acts diagonalisably on V_d. Also, for $d \neq d'$, the modules V_d and $V_{d'}$ have different dimensions and can therefore not be isomorphic. However, according to the following theorem, they are all irreducible $\mathfrak{sl}(2)$-modules. To simplify the notation in the following, we will occasionally write $\mathfrak{g}$ for $\mathfrak{sl}(2)$.

Theorem 3.8. *For each $d \in \mathbb{N}_0$, the $\mathfrak{sl}(2)$-module V_d is irreducible.*

Proof. The theorem follows from Proposition 3.1 if we can show that any vector $u \in V_d$ can be reached from any given nonzero vector $v \in V_d$, that is, $V_d \subseteq \mathfrak{g}(v)$ for each nonzero $v \in V_d$. We do this by establishing that

(i) $X^d \in \mathfrak{g}(v)$ for all nonzero $v \in V_d$;

(ii) $u \in \mathfrak{g}(X^d)$ for all $u \in V_d$.

To verify property (i), we write the nonzero vector $v \in V_d$ as

$$v = \sum_{k=0}^{d} \nu_k X^k Y^{d-k} = \sum_{k=k_0}^{d} \nu_k X^k Y^{d-k}, \tag{3.64}$$

where k_0 denotes the minimal value of k for which $\nu_k \neq 0$. It follows that

$$X^d = \frac{1}{\nu_{k_0}(d-k_0)!} \left(\phi_d(e)\right)^{d-k_0} v \in \mathfrak{g}(v). \tag{3.65}$$

To verify property (ii), we reach the arbitrary but given vector

$$u = \sum_{k=0}^{d} \mu_k X^k Y^{d-k} \in V_d \tag{3.66}$$

from X^d as

$$u = \sum_{k=0}^{d} \frac{\mu_k\, k!}{d!} \left(\phi_d(f)\right)^{d-k} X^d \in \mathfrak{g}(X^d), \tag{3.67}$$

thus concluding the proof. ■

So far, everything in this section could as well have been developed for $\mathfrak{sl}(2,\mathbb{R})$. This is not the case for the following lemma as it relies on the field being algebraically closed.

Proposition 3.9. *Let V be a nonzero $\mathfrak{sl}(2)$-module. Then, V contains an eigenvector v of h satisfying $ev = 0$.*

Proof. As we are working over $\mathbb{C}$, the linear map $h : V \to V$ has at least one eigenvalue, λ, with at least one corresponding eigenvector, w. Consider the infinite sequence of vectors

$$w,\ ew,\ e^2w, \ldots . \tag{3.68}$$

If they are all nonzero, then, according to Proposition 3.6, they form an infinite sequence of h-eigenvectors with distinct eigenvalues. Since eigenvectors with distinct eigenvalues are linearly independent, V would be infinite-dimensional, in contradiction with our standing assumption that all vector spaces are finite-dimensional. It follows that there exists $k \in \mathbb{N}_0$ such that $e^k w \neq 0$ but $e^{k+1}w = 0$. The nonzero vector $v = e^k w$ is an eigenvector of h:

$$hv = he^k w = (\lambda + 2k)e^k w = (\lambda + 2k)v. \tag{3.69}$$

Using Proposition 3.6, this readily follows by induction on k. ■

A nonzero vector v of the type considered in Proposition 3.9, characterised by

$$ev = 0, \qquad hv = \lambda v, \qquad \lambda \in \mathbb{C}, \tag{3.70}$$

is called a **highest-weight vector**. The corresponding h-eigenvalue λ is called the **highest weight**. The vector $X^d \in V_d$ is thus a highest-weight vector of highest weight d. Similarly, a nonzero vector u satisfying

$$fu = 0, \qquad hu = \lambda u, \qquad \lambda \in \mathbb{C}, \tag{3.71}$$

is called a **lowest-weight vector**. Its h-eigenvalue λ is known as the corresponding **lowest weight**. The vector $Y^d \in V_d$ is thus a lowest-weight vector of lowest weight $-d$. It is straightforward to modify the proof of Proposition 3.9 to show that V not only contains a highest-weight vector, but also a lowest-weight vector.

Theorem 3.10 (Classification of irreducible $\mathfrak{sl}(2)$-modules)**.** *Let V be an irreducible $\mathfrak{sl}(2)$-module of dimension $d+1$, where $d \in \mathbb{N}_0$. Then, $V \cong V_d$.*

Proof. By Proposition 3.9, V contains a highest-weight vector v with $hv = \lambda v$ for some $\lambda \in \mathbb{C}$. Following the proof of Proposition 3.9 (with $w = v$), the sequence of vectors

$$v,\ fv,\ f^2 v, \ldots \tag{3.72}$$

is seen to terminate in the sense that there exists $k \in \mathbb{N}_0$ such that $f^k v \neq 0$ but $f^{k+1} v = 0$. The proof of the theorem is now completed in four steps, by showing that

(i) $\{v, fv, \ldots, f^k v\}$ is a basis for a submodule of V;

(ii) $\{v, fv, \ldots, f^d v\}$ is a basis for V;

(iii) $\lambda = d$;

(iv) $V \cong V_d$.

The four steps are established as follows.

(i) The vectors in $\{v, fv, \ldots, f^k v\}$ are linearly independent since, by Proposition 3.6, they are h-eigenvectors with distinct eigenvalues. The vectors thus form a basis for a submodule if their span is invariant under the action of the Lie algebra generators. By construction, the span of the vectors is invariant under h and f. To see that it is invariant under e, we use induction on j to show that

$$e(f^j v) \in \operatorname{span}\{f^i v \,|\, 0 \leqslant i < j\}, \qquad 0 \leqslant j \leqslant k. \tag{3.73}$$

The induction start $j = 0$ is trivially satisfied since $ev = 0$. For the induction step, we note that

$$e(f^j v) = (h + fe)(f^{j-1} v) = (\lambda - 2(j-1)) f^{j-1} v + f(e(f^{j-1} v)). \tag{3.74}$$

The vector $(\lambda - 2(j-1)) f^{j-1} v$ is clearly in the span in (3.73). By the induction assumption, $e(f^{j-1} v) \in \operatorname{span}\{f^i v \,|\, 0 \leqslant i < j-1\}$, so $f(e(f^{j-1} v))$ is likewise in the span in (3.73).

(ii) Since the submodule W spanned by $\{v, fv, \ldots, f^k v\}$ is nonzero and V is irreducible, we have $W = V$ and $k = d$.

(iii) With respect to the ordered basis $\{v, fv, \ldots, f^d v\}$, (the endomorphism induced by) h is diagonal with trace

$$\lambda + (\lambda - 2) + \cdots + (\lambda - 2d) = (d+1)(\lambda - d). \tag{3.75}$$

Since $h = [e, f]$ implies that, in any given representation ρ, $\mathrm{tr}(R_\rho(h)) = \mathrm{tr}([R_\rho(e), R_\rho(f)]) = 0$, we conclude that $\lambda = d$.

(iv) From (ii), $\{v, fv, \ldots, f^d v\}$ is a basis for V. Written in the representation ϕ_d, a basis for V_d is given by

$$\{X^d,\ \phi_d(f)X^d,\ (\phi_d(f))^2 X^d, \ldots,\ (\phi_d(f))^d X^d\}, \tag{3.76}$$

where $(\phi_d(f))^k X^d$ is a nonzero scalar multiple of $X^{d-k}Y^k$. We now introduce the linear map

$$\varphi_d :\ V \to V_d, \qquad f^j v \mapsto (\phi_d(f))^j X^d, \tag{3.77}$$

whose action on the basis vectors for V is given explicitly. The map is seen to be invertible. It is also seen to intertwine the action of h and f, in the sense that

$$\varphi_d \circ \rho(h) = \phi_d(h) \circ \varphi_d, \qquad \varphi_d \circ \rho(f) = \phi_d(f) \circ \varphi_d, \tag{3.78}$$

where ρ is the representation corresponding to the module V. To show that it also intertwines the action of e, we use induction on j to show that

$$\varphi_d(ef^j v) = \phi_d(e)\big(\varphi_d(f^j v)\big), \qquad 0 \leqslant j \leqslant d, \tag{3.79}$$

thereby establishing that

$$\varphi_d \circ \rho(e) = \phi_d(e) \circ \varphi_d \tag{3.80}$$

and consequently that V and V_d are isomorphic as $\mathfrak{sl}(2)$-modules. The induction start $\varphi_d(ev) = \phi_d(e)(\varphi_d(v))$ is trivially satisfied as both sides of the equation are zero. For the induction step, we use the intertwining property with h and f to obtain

$$\begin{aligned}
\varphi_d(ef^j v) &= \varphi_d((h + fe)(f^{j-1}v)) = (\phi_d(h) \circ \varphi_d)(f^{j-1}v) + (\phi_d(f) \circ \varphi_d)(ef^{j-1}v) \\
&= \big(\phi_d(h) + \phi_d(f) \circ \phi_d(e)\big)\big(\varphi_d(f^{j-1}v)\big) = \big(\phi_d(e) \circ \phi_d(f)\big)\big(\varphi_d(f^{j-1}v)\big) \\
&= \phi_d(e)\big(\varphi_d(f^j v)\big),
\end{aligned} \tag{3.81}$$

where the third equality follows from the induction assumption. ■

Theorem 3.11 (Weyl's theorem for simple Lie algebras). *Every representation of a complex simple Lie algebra is completely reducible.*

Proof. This is merely a special case of the Weyl Theorem 4.16. ■

In conclusion, we have obtained the classification of (finite-dimensional) $\mathfrak{sl}(2)$-modules summarised in the following theorem.

Theorem 3.12 (CLASSIFICATION OF $\mathfrak{sl}(2)$-MODULES). *Any nonzero $\mathfrak{sl}(2)$-module V is of the form*

$$V \cong V_{d_1} \oplus \cdots \oplus V_{d_n} \tag{3.82}$$

for some $n \in \mathbb{N}$ and $d_1, \ldots, d_n \in \mathbb{N}_0$.

Proof. According to Weyl's theorem 3.11, any nonzero $\mathfrak{sl}(2)$-module is isomorphic to a direct sum of irreducible $\mathfrak{sl}(2)$-modules. By Theorem 3.10, each of these irreducible modules is isomorphic to V_d for some d. ■

REMARK

> We recall our standing assumption that, unless otherwise stated explicitly, all vector spaces are finite-dimensional in these lecture notes. All the $\mathfrak{sl}(2)$-modules considered above are thus assumed finite-dimensional. This is crucial for the validity of the main classification theorems and some of the propositions.

3.8 *Exercises*

Exercise 3.1.

(i) Show that the adjoint representation of $\mathfrak{sl}(2, \mathbb{F})$ is given by (3.7). Verify that these matrices satisfy the defining commutation relations for the matrix Lie algebra $\mathfrak{sl}(2, \mathbb{F})$.

(ii) Using (2.35), work out the adjoint representation of $\mathfrak{sl}(3, \mathbb{F})$.

Exercise 3.2.

Show that a one-dimensional $\mathfrak{g}$-module is irreducible.

Exercise 3.3.

Let $\{e, h, f\}$ be the standard basis for the Lie algebra A_1, and let ρ be a three-dimensional representation of A_1 such that

$$\rho(h) = \begin{pmatrix} 0 & 2 & 0 \\ 0 & -2 & 0 \\ 2 & -2 & 2 \end{pmatrix}, \qquad \rho(f) = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}. \tag{3.83}$$

(i) Find $\rho(e)$.

(ii) Argue that ρ is irreducible.

Exercise 3.4.

Let V and W be $\mathfrak{g}$-modules.

(i) Show that the action (3.22) turns the vector space $V \oplus W$ into a $\mathfrak{g}$-module.

(ii) Show that the action (3.29) turns the vector space $V \otimes W$ into a $\mathfrak{g}$-module.

Exercise 3.5.

Let V be a $\mathfrak{g}$-module. Show that $\{x \in \mathfrak{g} \,|\, xv = 0 \text{ for all } v \in V\}$ is an ideal of $\mathfrak{g}$.

Exercise 3.6.

Let $B = \{p, q, c\}$ be an ordered basis for the three-dimensional Heisenberg algebra, with Lie products

$$[p, q] = c, \qquad [c, p] = [c, q] = 0. \tag{3.84}$$

(i) Relative to B, work out the adjoint representation of the Heisenberg algebra.

(ii) Determine whether the adjoint representation of the Heisenberg algebra is faithful.

Exercise 3.7.

Let $\phi : V \to W$ be an intertwiner of the $\mathfrak{g}$-modules V and W. Show that $\ker(\phi)$ and $\mathrm{im}(\phi)$ are $\mathfrak{g}$-invariant subspaces of V and W, respectively.

Exercise 3.8.

Let V and W be $\mathfrak{g}$-modules. Show that the vector space $V \otimes W$ with $\mathfrak{g}$-action

$$x(v \otimes w) = (xv) \otimes (xw), \qquad x \in \mathfrak{g},\ v \in V,\ w \in W \tag{3.85}$$

is *not* in general a $\mathfrak{g}$-module.

Exercise 3.9.

Let $\mathfrak{g}_i$ and V_i, $i = 1, 2$, be Lie algebras and vector spaces, respectively. For each $i = 1, 2$, let ϕ_i be a representation of $\mathfrak{g}_i$ on V_i. The tensor product $\phi_1 \otimes \phi_2$ acts on the direct sum $\mathfrak{g}_1 \boxplus \mathfrak{g}_2$ as

$$(\phi_1 \otimes \phi_2)(x_1, x_2) := \phi_1(x_1) \otimes 1 + 1 \otimes \phi_2(x_2), \qquad \forall\, x_i \in \mathfrak{g}_i. \tag{3.86}$$

(i) Show that this is a representation of $\mathfrak{g}_1 \boxplus \mathfrak{g}_2$ on $V_1 \otimes V_2$.

(ii) Generalising Exercise 3.8, show that $(x_1, x_2) \mapsto \phi_1(x_1) \otimes \phi_2(x_2)$ does not in general yield a representation of $\mathfrak{g}_1 \boxplus \mathfrak{g}_2$ on $V_1 \otimes V_2$.

Exercise 3.10.

(i) Let $\mathfrak{U}(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$. Show that

$$[u, v \otimes w] = v \otimes [u, w] + [u, v] \otimes w, \qquad \forall\, u, v, w \in \mathfrak{U}(\mathfrak{g}). \tag{3.87}$$

(ii) Let $\{e, h, f\}$ denote the standard basis for A_1. In $\mathfrak{U}(A_1)$, let

$$f^{\otimes 0} = 1, \qquad f^{\otimes n} = \underbrace{f \otimes \cdots \otimes f}_{n \text{ factors}}, \qquad n \in \mathbb{N}. \tag{3.88}$$

Show that

$$[e, f^{\otimes n}] = n(n-1) f^{\otimes(n-1)} + n\, h \otimes f^{\otimes(n-1)}. \tag{3.89}$$

Exercise 3.11.

Verify that the multiplication rule (3.44) is well-defined. That is, verify that the product $(a + \mathfrak{i}_\mathfrak{g}) \cdot (b + \mathfrak{i}_\mathfrak{g})$ is independent of the representatives a and b.

Exercise 3.12.

Let ϕ_d be the $\mathfrak{sl}(2)$ representation described in Section 3.7. Work out the matrices $R_{\phi_d}(x)$, $x \in \{e, h, f\}$, in the ordered basis $\{X^d, X^{d-1}Y, \ldots, Y^d\}$.

Exercise 3.13.

Let $\{e, h, f\}$ be the standard basis for A_1, and let V be a finite-dimensional A_1-module. Let $v \in V$ be a highest-weight vector of weight $\lambda \in \mathbb{C}$, and assume that $f^2 v \neq 0$. Show that $ef^2 v$ is an eigenvector of h and determine the corresponding eigenvalue.

Exercise 3.14.

Let $\{e, h, f\}$ be a basis for A_1 such that

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h, \tag{3.90}$$

and let V be an irreducible A_1-module. Let $v \in V$ be an eigenvector of h such that $f^2 v \neq 0$ and $e^3 v = f^4 v = 0$. Find the possible values of $\dim(V)$.

Exercise 3.15.

Let $\{e, h, f\}$ be a basis for A_1 with Lie products

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h, \tag{3.91}$$

and let V be a finite-dimensional A_1-module. For $\lambda \in \mathbb{C}$, define

$$W_\lambda := \{v \in V \,|\, hv = \lambda v\}. \tag{3.92}$$

(i) Let $v \in W_\lambda$. Show that $e(fv) \in W_\lambda$.

(ii) Show that the number of irreducible submodules of V is given by $\dim W_0 + \dim W_1$.

(iii) Suppose $\dim V = 10$. Determine the possible values of $\dim W_2$.

Exercise 3.16.

Give a representation of $\mathfrak{gl}(2, \mathbb{C})$ that is not completely reducible.

Note that this demonstrates that Weyl's theorem 3.11 for simple Lie algebras fails to extend to the non-simple Lie algebra $\mathfrak{gl}(2, \mathbb{C})$.

Exercise 3.17.

Let $\lambda \in \mathbb{C}$. Verify that the differential operators

$$D_e = \frac{d}{dx}, \qquad D_h = \lambda - 2x\frac{d}{dx}, \qquad D_f = x\lambda - x^2\frac{d}{dx} \tag{3.93}$$

form a realisation of $\mathfrak{sl}(2)$.

Exercise 3.18.

By setting $\lambda = 0$ in Exercise 3.17, it follows that $\{\frac{d}{dx}, x\frac{d}{dx}, x^2\frac{d}{dx}\}$ is a basis for a Lie algebra. Is $S = \{\frac{d}{dx}, x\frac{d}{dx}, x^2\frac{d}{dx}, x^3\frac{d}{dx}\}$ likewise a basis for a Lie algebra? If not, can S be extended to a set generating a Lie algebra?

4 Structure theory

Any d-dimensional abelian Lie algebra $\mathfrak{g}$ is isomorphic to the direct sum of d copies of the one-dimensional abelian Lie algebra $\mathfrak{a}$:

$$\mathfrak{g} \cong \boxplus_{j=1}^{\mathrm{d}} \mathfrak{a} = \underbrace{\mathfrak{a} \boxplus \cdots \boxplus \mathfrak{a}}_{\mathrm{d\ summands}} . \tag{4.1}$$

Of course, abelian Lie algebras are very special, but it is natural to ask to what extent a given Lie algebra 'resembles' an abelian one. In an attempt to understand this and to characterise Lie algebras beyond commutativity, we will study **descending chains** of Lie subalgebras, of the form

$$\mathfrak{g} \supseteq \mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \ldots . \tag{4.2}$$

Since $\mathfrak{g}$ is assumed finite-dimensional, a descending chain will stabilise at some point, meaning that there exists a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ such that $\mathfrak{g}_n = \mathfrak{h}$ for all n larger than some finite integer. Fundamental properties of certain descending chains for which $\mathfrak{h} = \{0\}$ will give rise to the notions of *solvability* and *nilpotency* of Lie algebras, as discussed in Sections 4.1 and 4.2, respectively. Armed with these notions, we will subsequently develop a preliminary structure theory of Lie algebras.

4.1 Solvability

Define recursively the sequence of subspaces $\mathfrak{g}^{(0)}, \mathfrak{g}^{(1)}, \mathfrak{g}^{(2)}, \ldots$ of $\mathfrak{g}$ by

$$\mathfrak{g}^{(0)} := \mathfrak{g}, \qquad \mathfrak{g}^{(n+1)} := [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}]. \tag{4.3}$$

By construction, $\mathfrak{g}^{(n+1)}$ is the derived algebra of $\mathfrak{g}^{(n)}$, so according to Proposition 1.6, $\mathfrak{g}^{(n+1)}$ is an ideal of $\mathfrak{g}^{(n)}$. The sequence $\{\mathfrak{g}^{(n)}\}$ thus forms a descending chain of ideals,

$$\mathfrak{g} = \mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \supseteq \ldots , \tag{4.4}$$

called the **derived series**. The second ideal is recognised as the derived algebra of $\mathfrak{g}$ itself: $\mathfrak{g}^{(1)} = \mathfrak{g}'$. By applying Proposition 1.4 (v) iteratively, it also follows that $\mathfrak{g}^{(n)}$ is an ideal of $\mathfrak{g}$. According to Proposition 1.9, the quotient $\mathfrak{g}^{(n-1)}/\mathfrak{g}^{(n)}$ is an abelian Lie algebra for all $n \in \mathbb{N}$. The derived series thus gives rise to a sequence of abelian Lie algebras:

$$\mathfrak{g}^{(0)}/\mathfrak{g}^{(1)},\ \mathfrak{g}^{(1)}/\mathfrak{g}^{(2)}, \ldots,\ \mathfrak{g}^{(n-1)}/\mathfrak{g}^{(n)}, \ldots . \tag{4.5}$$

A Lie algebra $\mathfrak{g}$ is called **solvable** (or soluble) if its derived series terminates, meaning that $\mathfrak{g}^{(n)} = \{0\}$ for some $n \in \mathbb{N}$. A solvable Lie algebra is thus an 'almost abelian' Lie algebra in the sense that it can be constructed by successive 'extensions' of abelian Lie algebras.

The three-dimensional Heisenberg algebra introduced in Section 1.2 (see also Exercise 4.17) is an example of a solvable Lie algebra with $\mathfrak{g}^{(2)} = \{0\}$. The special linear algebra $\mathfrak{sl}(2)$, on the other hand, is not. Indeed, $\mathfrak{g} = \mathfrak{sl}(2)$ is simple and therefore equal to its derived algebra, so $\mathfrak{g}^{(n)} = \mathfrak{g}$ for all $n \in \mathbb{N}_0$.

As we have just seen, the derived series of a solvable Lie algebra is a *finite* series of ideals with abelian quotients. According to the following proposition, the 'converse' is also true.

Proposition 4.1. *Let* $\mathfrak{g}$ *admit a descending chain of ideals,*

$$\mathfrak{g} = \mathfrak{i}_0 \supseteq \mathfrak{i}_1 \supseteq \mathfrak{i}_2 \supseteq \cdots \supseteq \mathfrak{i}_n = \{0\}, \tag{4.6}$$

such that $\mathfrak{i}_{k-1}/\mathfrak{i}_k$ *is abelian for* $1 \leqslant k \leqslant n$. *Then,* $\mathfrak{g}$ *is solvable.*

Proof. By induction on k, we shall prove that $\mathfrak{g}^{(k)} \subseteq \mathfrak{i}_k$ for all $1 \leqslant k \leqslant n$. Setting $k = n$ then implies $\mathfrak{g}^{(n)} \subseteq \mathfrak{i}_n$ and consequently $\mathfrak{g}^{(n)} = \{0\}$, meaning that $\mathfrak{g}$ is solvable. Now, the induction start, $\mathfrak{g}^{(1)} \subseteq \mathfrak{i}_1$, follows from Proposition 1.9 applied to the ideal $\mathfrak{i}_1 \subseteq \mathfrak{g}$ since $\mathfrak{g}/\mathfrak{i}_1$ being abelian implies that $\mathfrak{g}' \subseteq \mathfrak{i}_1$. For the induction step, we assume that $\mathfrak{g}^{(k-1)} \subseteq \mathfrak{i}_{k-1}$, where $k \geqslant 2$, so $[\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}] \subseteq [\mathfrak{i}_{k-1}, \mathfrak{i}_{k-1}]$, meaning that $\mathfrak{g}^{(k)} \subseteq [\mathfrak{i}_{k-1}, \mathfrak{i}_{k-1}]$. By construction, the Lie algebra $\mathfrak{i}_{k-1}/\mathfrak{i}_k$ is abelian. We can therefore apply Proposition 1.9 to $\mathfrak{i}_k \subseteq \mathfrak{i}_{k-1}$, saying that $[\mathfrak{i}_{k-1}, \mathfrak{i}_{k-1}] \subseteq \mathfrak{i}_k$. Hence, $\mathfrak{g}^{(k)} \subseteq \mathfrak{i}_k$. ■

We can thus think of the derived series as the 'fastest' descending series of ideals whose successive quotients are abelian.

Let $\mathfrak{h}$ be a Lie subalgebra of a solvable Lie algebra $\mathfrak{g}$. Since $\mathfrak{h}^{(n)} \subseteq \mathfrak{g}^{(n)}$ for all n, $\mathfrak{h}$ is solvable. In particular, every ideal of $\mathfrak{g}$ is solvable. Likewise, any quotient algebra $\mathfrak{g}/\mathfrak{i}$ of $\mathfrak{g}$ is solvable if $\mathfrak{g}$ is. To appreciate this, one can use that

$$(\mathfrak{g}/\mathfrak{i})^{(n)} = \{x + \mathfrak{i} \mid x \in \mathfrak{g}^{(n)}\}. \tag{4.7}$$

Proposition 4.2. *Let* $\mathfrak{i}$ *be an ideal of* $\mathfrak{g}$. *If both* $\mathfrak{i}$ *and* $\mathfrak{g}/\mathfrak{i}$ *are solvable, then* $\mathfrak{g}$ *is solvable.*

Proof. Since $\mathfrak{g}/\mathfrak{i}$ is solvable, we have $(\mathfrak{g}/\mathfrak{i})^{(n)} = \{0\}$ for some n, meaning that $\mathfrak{g}^{(n)} \subseteq \mathfrak{i}$. Since $\mathfrak{i}$ is solvable, we have $\mathfrak{i}^{(m)} = \{0\}$ for some m. Observing that, by definition,

$$(\mathfrak{g}^{(n)})^{(1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}] = \mathfrak{g}^{(n+1)}, \quad (\mathfrak{g}^{(n)})^{(2)} = [(\mathfrak{g}^{(n)})^{(1)}, (\mathfrak{g}^{(n)})^{(1)}] = [\mathfrak{g}^{(n+1)}, \mathfrak{g}^{(n+1)}] = \mathfrak{g}^{(n+2)}, \tag{4.8}$$

it readily follows by induction on m that

$$\mathfrak{g}^{(n+m)} = (\mathfrak{g}^{(n)})^{(m)}. \tag{4.9}$$

Putting it together, we thus have

$$\mathfrak{g}^{(n+m)} = (\mathfrak{g}^{(n)})^{(m)} \subseteq \mathfrak{i}^{(m)} = \{0\}, \tag{4.10}$$

so $\mathfrak{g}$ is solvable. ■

Proposition 4.3. *Let* $\mathfrak{i}$ *and* $\mathfrak{j}$ *be solvable ideals of* $\mathfrak{g}$. *Then,* $\mathfrak{i} + \mathfrak{j}$ *is a solvable ideal of* $\mathfrak{g}$.

Proof. By Proposition 1.4, the sum $\mathfrak{i} + \mathfrak{j}$ is an ideal of $\mathfrak{g}$, so we just have to show it is solvable. From the same proposition, we have that $\mathfrak{i} \cap \mathfrak{j}$ is an ideal of $\mathfrak{j}$ (and of $\mathfrak{i}$). Since $\mathfrak{j}$ is solvable, so are $\mathfrak{i} \cap \mathfrak{j}$ and $\mathfrak{j}/(\mathfrak{i} \cap \mathfrak{j})$. By the second isomorphism theorem 1.12, $(\mathfrak{i} + \mathfrak{j})/\mathfrak{i} \cong \mathfrak{j}/(\mathfrak{i} \cap \mathfrak{j})$, so $(\mathfrak{i} + \mathfrak{j})/\mathfrak{i}$ is solvable. It then follows from Proposition 4.2 that $\mathfrak{i} + \mathfrak{j}$ is solvable. ■

A **maximal solvable ideal** of $\mathfrak{g}$ is one that is not contained in a larger solvable ideal of $\mathfrak{g}$.

Proposition 4.4. *Every Lie algebra has a unique maximal solvable ideal.*

Proof. By Proposition 4.3, the sum of two solvable ideals of a given Lie algebra $\mathfrak{g}$ is again a solvable ideal of $\mathfrak{g}$. Since $\mathfrak{g}$ is assumed finite, it thus has a unique maximal ideal obtained by adding up all its solvable ideals. ■

The unique maximal solvable ideal of $\mathfrak{g}$ is called the **radical** of $\mathfrak{g}$ and is denoted by $\mathrm{rad}(\mathfrak{g})$.

Proposition 4.5. *The Lie algebra* $\mathfrak{g}$ *is solvable if and only if* $\mathfrak{g} = \mathrm{rad}(\mathfrak{g})$.

Proof. This is the content of Exercise 4.5. ■

The other extreme, namely $\mathrm{rad}(\mathfrak{g}) = \{0\}$, will play a prominent role in Section 4.3 and beyond.

As indicated in the following theorem, the representation theory of solvable Lie algebras over $\mathbb{C}$ is special. In fact, the theorem holds as long as the field $\mathbb{F}$ is algebraically closed and of characteristic 0, but is not true for $\mathbb{F} = \mathbb{R}$.

Theorem 4.6 (Lie's theorem)**.** *Let* $\mathfrak{g}$ *be a solvable Lie algebra over* $\mathbb{C}$ *and* V *an irreducible* $\mathfrak{g}$*-module. Then,* $\dim V = 1$.

Proof. We refer to the literature [1, 2, 4, 7, 8] for a proof of this. ■

Corollary 4.7. *Let* $\mathfrak{g}$ *be a solvable Lie algebra over* $\mathbb{C}$ *and* V *a* $\mathfrak{g}$*-module. Then, a basis can be chosen for* V *with respect to which every element of* $\mathfrak{g}$ *is represented by an upper-triangular matrix.*

4.2 Nilpotency

Define recursively the sequence of subspaces $\mathfrak{g}^1, \mathfrak{g}^2, \dots$ of the Lie algebra $\mathfrak{g}$ by

$$\mathfrak{g}^1 := \mathfrak{g}, \qquad \mathfrak{g}^{n+1} := [\mathfrak{g}, \mathfrak{g}^n]. \tag{4.11}$$

Applying Proposition 1.4 (v) iteratively, it follows that, for $n \in \mathbb{N}$, $\mathfrak{g}^{n+1}$ is not only an ideal of $\mathfrak{g}^n$, but also of $\mathfrak{g}$. The sequence $\{\mathfrak{g}^n\}$ thus forms a descending chain of ideals,

$$\mathfrak{g} = \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \mathfrak{g}^3 \supseteq \dots, \tag{4.12}$$

called the **lower central series** (or descending central series). The second ideal is recognised as the derived algebra: $\mathfrak{g}^2 = \mathfrak{g}'$. The term "central" is justified by the content of the following proposition.

Proposition 4.8. *Let* $\{\mathfrak{g}^n \,|\, n \in \mathbb{N}\}$ *be defined as in (4.11). Then,*

$$\mathfrak{g}^n/\mathfrak{g}^{n+1} \subseteq Z(\mathfrak{g}/\mathfrak{g}^{n+1}), \qquad \forall\, n \in \mathbb{N}. \tag{4.13}$$

Proof. An element of $\mathfrak{g}^n/\mathfrak{g}^{n+1}$ is of the form $x + \mathfrak{g}^{n+1}$ where $x \in \mathfrak{g}^n$. Since $\mathfrak{g}^n \subseteq \mathfrak{g}$, this is also an element of $\mathfrak{g}/\mathfrak{g}^{n+1}$, and we need to show that it commutes with any element of $\mathfrak{g}/\mathfrak{g}^{n+1}$. Any such element is of the form $y + \mathfrak{g}^{n+1}$ for some $y \in \mathfrak{g}$. We then have

$$[x + \mathfrak{g}^{n+1}, y + \mathfrak{g}^{n+1}] = [x, y] + \mathfrak{g}^{n+1} = 0 + \mathfrak{g}^{n+1}, \tag{4.14}$$

where the last equality follows from $[x, y] \in [\mathfrak{g}, \mathfrak{g}^n] = \mathfrak{g}^{n+1}$. ■

A Lie algebra $\mathfrak{g}$ is called **nilpotent** if its lower central series terminates in the sense that $\mathfrak{g}^n = \{0\}$ for some $n \in \mathbb{N}$. It readily follows that every abelian Lie algebra is nilpotent. Indeed, the derived algebra of an abelian Lie algebra $\mathfrak{g}$ is the zero vector space so $\mathfrak{g}^2 = \{0\}$.

Proposition 4.9.

(i) $[\mathfrak{g}^m, \mathfrak{g}^n] \subseteq \mathfrak{g}^{m+n}$ *for all* $m, n \in \mathbb{N}$.

(ii) $\mathfrak{g}^{(n)} \subseteq \mathfrak{g}^{2^n}$ *for all* $n \in \mathbb{N}_0$.

Proof.

(i) We use induction on n. For $n = 1$, the assertion $[\mathfrak{g}^m, \mathfrak{g}] \subseteq \mathfrak{g}^{m+1}$ is an immediate consequence of Proposition 1.3 applied to the definition (4.11) of $\mathfrak{g}^{m+1}$. Now, suppose the assertion is true for $n = k > 1$. Using the Jacobi identity and the induction assumption, we then have

$$\begin{aligned}[\mathfrak{g}^m, \mathfrak{g}^{k+1}] &= [\mathfrak{g}^m, [\mathfrak{g}, \mathfrak{g}^k]] \subseteq [\mathfrak{g}, [\mathfrak{g}^k, \mathfrak{g}^m]] + [\mathfrak{g}^k, [\mathfrak{g}^m, \mathfrak{g}]] \\ &\subseteq [\mathfrak{g}, \mathfrak{g}^{m+k}] + [\mathfrak{g}^k, \mathfrak{g}^{m+1}] \subseteq \mathfrak{g}^{m+k+1} + \mathfrak{g}^{m+k+1} \\ &\subseteq \mathfrak{g}^{m+k+1}, \end{aligned} \tag{4.15}$$

so the result holds for all $n \in \mathbb{N}$.

(ii) We use induction on n. For $n = 0$, the assertion $[\mathfrak{g}^{(0)}, \mathfrak{g}] \subseteq \mathfrak{g}^1$ is trivially true. Now, suppose the assertion is true for $n = k > 0$. Using the result (i), we then have

$$\mathfrak{g}^{(k+1)} = [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}] \subseteq [\mathfrak{g}^{2^k}, \mathfrak{g}^{2^k}] \subseteq \mathfrak{g}^{2^k+2^k} = \mathfrak{g}^{2^{k+1}}, \tag{4.16}$$

so the result holds for all $n \in \mathbb{N}_0$. ■

Corollary 4.10. *Every nilpotent Lie algebra is solvable.*

Proof. Assuming that $\mathfrak{g}$ is nilpotent, we have $\mathfrak{g}^{2^n} = \{0\}$ for n sufficiently large. By Proposition 4.9 (ii), it then follows that $\mathfrak{g}^{(n)} = \{0\}$, so $\mathfrak{g}$ is solvable. ■

Note that the statement analogous to Proposition 4.2 for solvable Lie algebras is false for nilpotent Lie algebras. Indeed, consider the Lie algebra $\mathfrak{t}(n)$ of upper-triangular $n \times n$ matrices. The Lie algebra of strictly upper-triangular $n \times n$ matrices $\mathfrak{n}(n)$ is a nilpotent ideal of $\mathfrak{t}(n)$, and the corresponding quotient algebra $\mathfrak{d}(n)$ of diagonal $n \times n$ matrices is likewise nilpotent. However, the ambient Lie algebra $\mathfrak{t}(n)$ is *not* nilpotent (for $n > 1$).

Theorem 4.11 (Engel's theorem)**.** *A Lie algebra* $\mathfrak{g}$ *is nilpotent if and only if* ad_x *is nilpotent for each* $x \in \mathfrak{g}$.

Proof. We refer to the literature [1, 2, 4, 7, 8] for a proof of this. ■

Corollary 4.12. *A Lie algebra is nilpotent if and only if it is isomorphic to a Lie algebra of strictly upper triangular matrices.*

As an immediate application of Engel's theorem 4.11, we note that $\mathfrak{sl}(2, \mathbb{F})$ is not nilpotent. Indeed, the matrix realisation of ad_h in (3.7) is a (nonzero) diagonal matrix and is therefore not nilpotent.

Proposition 4.13. *Let* $\mathfrak{g}$ *be complex. Then,* $\mathfrak{g}$ *is solvable if and only if* $\mathfrak{g}'$ *is nilpotent.*

Proof. This is the content of Exercise 4.4. ■

4.3 Semisimplicity and reductiveness

A Lie algebra $\mathfrak{g}$ is said to be **semisimple** if it contains no nonzero solvable ideal. Since the radical of $\mathfrak{g}$ is the maximal solvable ideal of $\mathfrak{g}$,

$$\boxed{\mathfrak{g} \textit{ is semisimple} \quad \Longleftrightarrow \quad \mathrm{rad}(\mathfrak{g}) = \{0\}.}$$

In some sense, semisimple is therefore the 'opposite' of solvable. It follows readily from the definition of simple Lie algebras given in Section 1.3 that

$$\boxed{\mathfrak{g} \textit{ is simple} \quad \Longrightarrow \quad \mathfrak{g} \textit{ is semisimple.}}$$

Conversely, a semisimple Lie algebra can possess a proper ideal, in which case it is not simple, as long as the ideal is not solvable.

Remark on conventions

> It is noted that the restriction, that a simple Lie algebra may not be abelian, ensures that the one-dimensional abelian Lie algebra $\mathfrak{a}$ is not simple. Without this restriction, $\mathfrak{a}$ would have been simple but not semisimple. Despite this somewhat awkward situation, the restriction is not always imposed in the literature.

Proposition 4.14. *The Lie algebra* $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ *is semisimple.*

Proof. An ideal of $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is of the form $\mathfrak{i}/\mathrm{rad}(\mathfrak{g})$ where $\mathfrak{i}$ is an ideal of $\mathfrak{g}$. If $\mathfrak{i}/\mathrm{rad}(\mathfrak{g})$ is solvable, then, by Proposition 4.2, $\mathfrak{i}$ is solvable since $\mathrm{rad}(\mathfrak{g})$ is solvable. Since $\mathrm{rad}(\mathfrak{g}) \subseteq \mathfrak{i}$ and $\mathrm{rad}(\mathfrak{g})$ is the maximal solvable ideal of $\mathfrak{g}$, it follows that $\mathfrak{i} = \mathrm{rad}(\mathfrak{g})$. Hence, $\mathfrak{i}/\mathrm{rad}(\mathfrak{g}) = \{0\}$, so $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ does not contain any nonzero solvable ideals. ■

Theorem 4.15. *A Lie algebra is semisimple if and only if it is isomorphic to a direct sum of simple Lie algebras.*

Proof. We refer to the literature [1, 2] for a proof of this. ■

We can thus characterise a semisimple Lie algebra by

$$\mathfrak{g} \cong \mathfrak{g}_1 \boxplus \cdots \boxplus \mathfrak{g}_n, \tag{4.17}$$

where $\mathfrak{g}_i$ is simple for every $i = 1, \ldots, n$ for some positive integer $n \in \mathbb{N}$. As in the simple case, the centre of a semisimple Lie algebra $\mathfrak{g}$ is zero: $Z(\mathfrak{g}) = \{0\}$.

Theorem 4.16 (Weyl's theorem)**.** *Every representation of a complex semisimple Lie algebra is completely reducible.*

Proof. We refer to the literature [2, 7] for a proof of this. ■

A Lie algebra $\mathfrak{g}$ is said to be **reductive** if its radical equals its centre: $\mathrm{rad}(\mathfrak{g}) = Z(\mathfrak{g})$. It readily follows that

$$\boxed{\mathfrak{g} \textit{ is semisimple} \quad \Longrightarrow \quad \mathfrak{g} \textit{ is reductive.}}$$

The converse is not true, since it is possible for $\mathfrak{g}$ that $\mathrm{rad}(\mathfrak{g}) = Z(\mathfrak{g}) \neq \{0\}$. This is the case for $\mathfrak{g} = \mathfrak{gl}(2)$, for example, whose centre is generated by the 2×2 identity matrix. The extension of Theorem 4.15 for semisimple Lie algebras to the reductive ones is the content of the following result.

Theorem 4.17. *A Lie algebra is reductive if and only if it is isomorphic to a direct sum of simple or abelian Lie algebras.*

Classifying all reductive (including semisimple) Lie algebras thus boils down to classifying all simple Lie algebras – the topic of Section 6. The following table lists some of the complex Lie algebras encountered in Section 2, indicating whether they are simple, semisimple (but not simple) or reductive (but not semisimple). Some related isomorphisms are also included.

Table of complex matrix Lie algebras

$\mathfrak{sl}(n)$	$n \geqslant 2$	simple	
$\mathfrak{gl}(1)$		reductive	$\mathfrak{gl}(1) \cong \mathfrak{a}$
$\mathfrak{gl}(n)$	$n \geqslant 2$	reductive	$\mathfrak{gl}(n) \cong \mathfrak{sl}(n) \boxplus \mathfrak{a}$
$\mathfrak{sp}(2r)$	$r \geqslant 1$	simple	
$\mathfrak{so}(2)$		reductive	$\mathfrak{so}(2) \cong \mathfrak{a}$
$\mathfrak{so}(3)$		simple	$\mathfrak{so}(3) \cong \mathfrak{sl}(2)$
$\mathfrak{so}(4)$		semisimple	$\mathfrak{so}(4) \cong \mathfrak{sl}(2) \boxplus \mathfrak{sl}(2)$
$\mathfrak{so}(n)$	$n \geqslant 5$	simple	

4.4 Killing form

Let V be a vector space over the field $\mathbb{F}$. A **bilinear form** on V is a bilinear map

$$\langle\ ,\ \rangle :\ V \times V \to \mathbb{F}. \tag{4.18}$$

The form is **symmetric** if $\langle v, w\rangle = \langle w, v\rangle$ for all $v, w \in V$. For example, the dot product on $\mathbb{R}^n$ is a symmetric bilinear form. More generally, for each $A \in M_n(\mathbb{F})$, we obtain a bilinear form on $\mathbb{F}^n$ by setting

$$\langle x, y\rangle = x^t A\, y, \qquad \forall\, x, y \in \mathbb{F}^n. \tag{4.19}$$

This form is symmetric if the matrix A is symmetric ($A^t = A$). The dot product on $\mathbb{R}^n$ is recovered by setting $A = I_n$ and $\mathbb{F} = \mathbb{R}$.

A symmetric bilinear form $\langle\ ,\ \rangle$ on the vector space V is said to be **non-degenerate** if for any nonzero $v \in V$ there exists $w \in V$ such that $\langle v, w\rangle \neq 0$. Equivalently, the form is non-degenerate if $\langle v, w\rangle = 0$ for all w implies that $v = 0$. A non-degenerate symmetric bilinear form on V is called an (indefinite) **inner product** on V. A practical way to test whether a symmetric bilinear form on a d-dimensional vector space V is non-degenerate is to

(i) fix an ordered basis $\{v_1, \ldots, v_{\mathrm{d}}\}$ for V;

(ii) work out the determinant of the d × d matrix with entries $\langle v_i, v_j\rangle$;

(iii) use that the form is non-degenerate if and only if the determinant is nonzero.

Since $\mathfrak{g}$ is a vector space, it makes sense to define bilinear forms on $\mathfrak{g}$. A symmetric bilinear form on $\mathfrak{g}$ is said to be **invariant** if

$$\langle [x, y], z\rangle = \langle x, [y, z]\rangle, \qquad \forall\, x, y, z \in \mathfrak{g}. \tag{4.20}$$

This invariance property can be thought of as an *associativity property.*

Let V be a vector space and $\phi : V \to V$ a linear map, and consider the trace of the matrix representation of ϕ in some basis for V. The result is, in fact, independent of the choice of basis since different bases will give rise to similar matrices and, according to (2.25), the traces of similar matrices are equal. We thus have a *basis-independent* definition of the trace $\mathrm{tr}(\phi)$ of the linear map ϕ.

The **Killing form** (or Cartan-Killing form) on $\mathfrak{g}$ is defined by

$$\kappa :\ \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}, \qquad (x, y) \mapsto \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y). \tag{4.21}$$

Since the adjoint mappings ad_x and ad_y are linear maps $\mathfrak{g} \to \mathfrak{g}$, so is their composition $\mathrm{ad}_x \circ \mathrm{ad}_y$. It follows that $\mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y) \in \mathbb{F}$, as already indicated. As discussed above, we may evaluate $\kappa(x, y)$ in any given basis for the adjoint map:

$$\kappa(x, y) = \mathrm{tr}(R_{\mathrm{ad}}(x)R_{\mathrm{ad}}(y)). \tag{4.22}$$

The Killing form on an abelian Lie algebra is readily seen to be identically zero.

Proposition 4.18. *Let κ be the Killing form on $\mathfrak{g}$. Then,*

(i) *κ is bilinear;*

(ii) *κ is symmetric;*

(iii) *κ is invariant.*

Proof. Let $x, y, z \in \mathfrak{g}$.

(i) This follows from the fact that the adjoint map ad is linear, the composition of linear maps is bilinear, and the trace operator tr is linear.

(ii) Using the cyclicity of the trace, we have

$$\kappa(x, y) = \mathrm{tr}(R_{\mathrm{ad}}(x)R_{\mathrm{ad}}(y)) = \mathrm{tr}(R_{\mathrm{ad}}(y)R_{\mathrm{ad}}(x)) = \kappa(y, x). \tag{4.23}$$

(iii) Using Proposition 1.1 and the cyclicity property and linearity of the trace, we have

$$\begin{aligned}
\kappa([x,y],z) &= \mathrm{tr}(\mathrm{ad}_{[x,y]} \circ \mathrm{ad}_z) \\
&= \mathrm{tr}((\mathrm{ad}_x \circ \mathrm{ad}_y - \mathrm{ad}_y \circ \mathrm{ad}_x) \circ \mathrm{ad}_z) \\
&= \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y \circ \mathrm{ad}_z) - \mathrm{tr}(\mathrm{ad}_y \circ \mathrm{ad}_x \circ \mathrm{ad}_z) \\
&= \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y \circ \mathrm{ad}_z) - \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_z \circ \mathrm{ad}_y) \\
&= \mathrm{tr}(\mathrm{ad}_x \circ (\mathrm{ad}_y \circ \mathrm{ad}_z - \mathrm{ad}_z \circ \mathrm{ad}_y)) \\
&= \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_{[y,z]}) \qquad (4.24) \\
&= \kappa(x,[y,z]).
\end{aligned}$$

■

With respect to the basis $\{x_a \,|\, a = 1, \ldots, \dim \mathfrak{g}\}$ for the Lie algebra $\mathfrak{g}$, we introduce

$$\kappa_{ab} := \kappa(x_a, x_b). \qquad (4.25)$$

Using (3.6), we can express this in terms of the structure constants as

$$\kappa_{ab} = \sum_{c,d} f_{ad}{}^c f_{bc}{}^d. \qquad (4.26)$$

Likewise in terms of the structure constants, the invariance property of the Killing form reads

$$\sum_d f_{ab}{}^d \kappa_{dc} = \sum_d f_{bc}{}^d \kappa_{ad}. \qquad (4.27)$$

The lefthand side of this relation suggests the introduction of structure constants with only lower indices:

$$f_{abc} := \sum_d f_{ab}{}^d \kappa_{dc}. \qquad (4.28)$$

Applying the symmetric property of the Killing form to the righthand side of (4.27), we thus have

$$f_{abc} = f_{bca}. \qquad (4.29)$$

From (1.15), we recall the antisymmetry of the structure constants in the first two indices: $f_{ab}{}^c = -f_{ba}{}^c$. As this carries over to the new structure constants as

$$f_{abc} = -f_{bac}, \qquad (4.30)$$

we see that the structure constants f_{abc} are *totally antisymmetric* in their indices.

Proposition 4.19. *The Killing form on a nilpotent Lie algebra is identically zero.*

Proof. We refer to the literature for a proof of this. ■

The converse is not true, as demonstrated in Exercise 4.14. Indeed, a non-nilpotent Lie algebra with zero Killing form can be constructed using a non-nilpotent matrix whose square has trace 0. Establishing the existence of such a 2×2 matrix is the content of Exercise 4.12.

We refer to the literature for proofs of the following two important applications of the Killing form.

Theorem 4.20 (Cartan's first criterion). *A Lie algebra $\mathfrak{g}$ is solvable if and only if $\kappa(x, y) = 0$ for all $x \in \mathfrak{g}$ and $y \in \mathfrak{g}'$.*

Theorem 4.21 (Cartan's second criterion). *A Lie algebra is semisimple if and only if its Killing form is non-degenerate.*

It follows that the Killing form provides an inner product on a semisimple Lie algebra. Cartan's second criterion is sometimes referred to as the *Cartan-Killing criterion.* It will play a crucial role in the following as our primary focus will be on semisimple Lie algebras.

4.5 *Exercises*

Exercise 4.1.

Describe the lower central series of $\mathfrak{n}^+(n, \mathbb{F})$ and use it to argue that $\mathfrak{n}^+(n, \mathbb{F})$ is nilpotent.

Exercise 4.2.

Let $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a Lie algebra epimorphism. Show that $\phi(\mathfrak{g}_1^{(n)}) = \mathfrak{g}_2^{(n)}$ for all $n \in \mathbb{N}_0$.

Exercise 4.3.

Find a two-dimensional Lie subalgebra of $\mathfrak{gl}(2, \mathbb{C})$ that is solvable but not nilpotent.

Exercise 4.4.

Let $\mathfrak{g}$ be complex. Show that $\mathfrak{g}$ is solvable if and only if $\mathfrak{g}'$ is nilpotent.

Exercise 4.5.

Show that $\mathfrak{g}$ is solvable if and only if $\mathfrak{g} = \text{rad}(\mathfrak{g})$.

Exercise 4.6.

Let $\mathfrak{i}$ be an ideal of $\mathfrak{g}$. Show that

$$\text{rad}(\mathfrak{i}) = \text{rad}(\mathfrak{g}) \cap \mathfrak{i}. \tag{4.31}$$

Exercise 4.7.

Find a basis $\{x_a \,|\, a = 1, 2, 3\}$ for $\mathfrak{sl}(2, \mathbb{C})$ such that $\kappa(x_a, x_b) = \delta_{a,b}$.

Exercise 4.8.

Let $\langle\ ,\ \rangle$ denote an invariant symmetric bilinear form on $\mathfrak{g}$. Show that the set $\{x \in \mathfrak{g} \,|\, \langle x, y\rangle = 0 \text{ for all } y \in \mathfrak{g}\}$ forms an ideal of $\mathfrak{g}$.

Exercise 4.9.

Consider the Lie algebra with ordered basis $B = \{x, y\}$ and Lie product $[x, y] = x$. Find the matrix representation of its Killing form with respect to B.

Exercise 4.10.

Work out the matrix representation R of the Killing form κ on $\mathfrak{sl}(2)$ with respect to the ordered basis $\{e, h, f\}$. Compute $\det(R)$ and use the result to argue that κ is non-degenerate.

Exercise 4.11.

Let $\mathfrak{i}$ be an ideal of $\mathfrak{g}$ and κ the Killing form on $\mathfrak{g}$, and let $\kappa_{\mathfrak{i}\times\mathfrak{i}}$ denote the Killing form restricted to $\mathfrak{i}$. Show that they agree on $\mathfrak{i}$, that is,

$$\kappa_{\mathfrak{i}\times\mathfrak{i}}(x,y) = \kappa(x,y), \qquad \forall\, x,y \in \mathfrak{i}. \tag{4.32}$$

(Note that the traces are taken over adjoint representations of dimension $\dim \mathfrak{i}$ and $\dim \mathfrak{g}$, respectively.)

Exercise 4.12.

Find a non-nilpotent 2×2 real matrix A for which $\mathrm{tr}(A) = 0$.

Exercise 4.13.

Let $\{x, y, z, u\}$ be a basis for $\mathfrak{g}$ with Lie products

$$[x,z] = y, \qquad [x,u] = x, \qquad [u,z] = z, \qquad [y,x] = [y,z] = [y,u] = 0. \tag{4.33}$$

(i) Work out the derived series for $\mathfrak{g}$ and determine whether $\mathfrak{g}$ is solvable.

(ii) Work out the lower central series for $\mathfrak{g}$ and determine whether $\mathfrak{g}$ is nilpotent.

(iii) Find the radical of $\mathfrak{g}$ and determine whether $\mathfrak{g}$ is reductive.

(iv) With respect to the ordered basis $\{x, y, z, u\}$, work out the matrix realisation of the Killing form for $\mathfrak{g}$. Apply Cartan's Second Criterion to determine whether $\mathfrak{g}$ is semisimple. Compare your conclusion with your answer to (iii).

Exercise 4.14.

For $\lambda \in \mathbb{C}$, let $\{x, y, z\}$ be a basis for the complex Lie algebra $\mathfrak{g}_{(\lambda)}$ with Lie products

$$[x,y] = y, \qquad [x,z] = \lambda z, \qquad [y,z] = 0. \tag{4.34}$$

(i) Show that $\mathfrak{g}_{(\lambda)}$ is solvable.

(ii) Show that $\mathfrak{g}_{(\lambda)}$ is not nilpotent.

(iii) Determine λ such that the Killing form is identically zero.

Note that the solution to item (iii) *gives an example of a solvable, non-nilpotent Lie algebra with vanishing Killing form.*

Exercise 4.15.

Let κ denote the Killing form on $\mathfrak{g}$. For each subset S of $\mathfrak{g}$, define

$$S^{\perp} := \{x \in \mathfrak{g} \,|\, \kappa(x,s) = 0 \text{ for all } s \in S\}.$$

(i) Show that $S^{\perp}$ is a vector space.

(ii) Show that if $\mathfrak{i}$ is an ideal of $\mathfrak{g}$, then $\mathfrak{i}^{\perp}$ is an ideal of $\mathfrak{g}$.

Exercise 4.16.

Let $\{x, y, z, u\}$ be a basis for $\mathfrak{g}$ with Lie products

$$[x, y] = [y, z] = [u, z] = y, \qquad [x, z] = [y, u] = x + z, \qquad [x, u] = 0. \tag{4.35}$$

(i) Work out the derived series for $\mathfrak{g}$ and determine whether $\mathfrak{g}$ is solvable.

(ii) Work out the lower central series for $\mathfrak{g}$ and determine whether $\mathfrak{g}$ is nilpotent.

(iii) Find the radical of $\mathfrak{g}$ and determine whether $\mathfrak{g}$ is reductive.

(iv) With respect to the ordered basis $\{x, y, z, u\}$, work out the matrix realisation of the Killing form for $\mathfrak{g}$. Apply Cartan's Second Criterion to determine whether $\mathfrak{g}$ is semisimple. Compare your conclusion with your answer to (iii).

Exercise 4.17.

In this exercise, we introduce an infinite family of **Heisenberg algebras**. For each $n \in \mathbb{N}$, we thus define the vector space

$$\mathcal{H}_n := \mathrm{span}_{\mathbb{F}}\{p_i, q_i, c \,|\, i = 1, \ldots, n\} \tag{4.36}$$

with corresponding Lie products

$$[p_i, q_j] = \delta_{ij} c, \qquad \forall\, i, j \in \{1, \ldots, n\}, \tag{4.37}$$

and

$$[p_i, p_j] = [q_i, q_j] = [c, p_i] = [c, q_i] = 0, \qquad \forall\, i, j \in \{1, \ldots, n\}. \tag{4.38}$$

(i) Verify that the Jacobi identity is satisfied.

(ii) Show that the Lie algebra $\mathcal{H}_n$ contains a one-dimensional abelian ideal $\mathfrak{i}$.

(iii) Show that $\mathcal{H}_n/\mathfrak{i}$ is abelian.

(iv) Show that $\mathcal{H}_n$ is solvable.

We have thus established that $\mathcal{H}_n$ is a $(2n+1)$-dimensional solvable Lie algebra. The Heisenberg algebras are of fundamental importance in quantum mechanics where p_i and q_i are **canonical conjugate quantities** *(such as the momentum and position generators, respectively), while $c = -\mathrm{i}\hbar\,\mathrm{id}$ where id is the identity operator in a given representation and $\hbar$ is Planck's (renormalised) constant. The reader may recognise the famous* **canonical commutation relation**

$$[p, q] = -\mathrm{i}\hbar, \tag{4.39}$$

here written in a representation in which id may be identified with its eigenvalue 1.

Exercise 4.18.

This exercise concerns the following five complex matrix Lie algebras:

$$\mathfrak{d}(3) = \{ \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \mid d_1, d_2, d_3 \in \mathbb{C}\}, \qquad \mathfrak{n}^+(3) = \{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{C}\} \tag{4.40}$$

and

$$\mathfrak{t}(2) = \{ \begin{pmatrix} d_1 & a \\ 0 & d_2 \end{pmatrix} \mid a, d_1, d_2 \in \mathbb{C}\}, \qquad \mathfrak{sl}(2) = \mathfrak{sl}(2, \mathbb{C}), \qquad \mathfrak{gl}(2) = \mathfrak{gl}(2, \mathbb{C}). \tag{4.41}$$

(i) Work out the derived series for $\mathfrak{d}(3)$, $\mathfrak{n}^+(3)$, $\mathfrak{t}(2)$, $\mathfrak{sl}(2)$ and $\mathfrak{gl}(2)$. In each case, use your result to determine whether the Lie algebra is solvable.

(ii) Work out the lower central series for $\mathfrak{d}(3)$, $\mathfrak{n}^+(3)$, $\mathfrak{t}(2)$, $\mathfrak{sl}(2)$ and $\mathfrak{gl}(2)$. In each case, use your result to determine whether the Lie algebra is nilpotent.

(iii) Find the radical of $\mathfrak{d}(3)$, $\mathfrak{n}^+(3)$, $\mathfrak{t}(2)$, $\mathfrak{sl}(2)$ and $\mathfrak{gl}(2)$. In each case, use your result to determine whether the Lie algebra is semisimple, reductive or neither.

(iv) In a basis of your choice, work out the Killing form for $\mathfrak{d}(3)$, $\mathfrak{n}^+(3)$, $\mathfrak{t}(2)$, $\mathfrak{sl}(2)$ and $\mathfrak{gl}(2)$. In each case, apply Cartan's Second Criterion to determine whether the Lie algebra is semisimple. Compare your findings with your answers to (iii).

5 Cartan-Weyl basis

The representation theory of $A_1 \cong \mathfrak{sl}(2)$ was studied in some detail in Section 3.7. With reference to the usual basis $\{e, h, f\}$ for A_1, we found that h is diagonalisable in any (finite-dimensional) representation ρ, with integer eigenvalues. To reach this conclusion, we first identified an eigenvector of $\rho(h)$ and then used $\rho(e)$ and $\rho(f)$ to construct new eigenvectors with eigenvalues raised or lowered in increments of 2. Since the representation is finite-dimensional, the ensuing chain of eigenvalues must terminate in both directions, and this can only happen if the highest (and lowest) eigenvalue in the chain is a nonnegative (respectively nonpositive) integer. Because of the complete reducibility of the representation, the full set of eigenvalues is merely the collection of such chains of eigenvalues.

Admittedly, A_1 is very special, but it is nevertheless natural to try to identify and explore the key properties that allowed us to classify its finite-dimensional representations. We shall thus apply the following initial strategy in our study of other (semi)simple complex Lie algebras. Let $\mathfrak{g}$ be such a Lie algebra. First, we will look for an abelian Lie subalgebra (generalising the one generated by h in the case of A_1) whose generators h_j should be diagonalisable in any given (finite-dimensional) representation. In particular, they must be diagonalisable in the *adjoint* representation. An element $y \in \mathfrak{g}$ is thus said to be **ad-diagonalisable** if ad_y is diagonalisable, that is, if there exists a basis $\{x_a \,|\, a = 1, \ldots, \dim \mathfrak{g}\}$ for $\mathfrak{g}$ such that

$$[y, x_a] = \lambda_a x_a \tag{5.1}$$

for some $\lambda_1, \ldots, \lambda_{\dim \mathfrak{g}} \in \mathbb{C}$. To exploit the resemblance with A_1 to its fullest, we will look for a *maximal* set of ad-diagonalisable generators h_j and then study their eigenvalues and eigenvectors in the adjoint representation. These eigenvectors are elements of the Lie algebra and their properties can be examined with the help of the Killing form, in particular.

5.1 Cartan subalgebras

Let $\mathfrak{h}$ be a Lie subalgebra of the Lie algebra $\mathfrak{g}$. It is recalled from Exercise 1.21 that the normaliser $N_{\mathfrak{g}}(\mathfrak{h})$ of $\mathfrak{h}$ in $\mathfrak{g}$ is given by

$$N_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} \,|\, [x, \mathfrak{h}] \subseteq \mathfrak{h}\}. \tag{5.2}$$

The Lie subalgebra $\mathfrak{h}$ is a **Cartan subalgebra** if

(i) $\mathfrak{h}$ is nilpotent;

(ii) $\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h})$.

Any nilpotent Lie algebra is clearly a Cartan subalgebra of itself. The notion of a Cartan subalgebra is otherwise rather abstract — even the *existence* of Cartan subalgebras is a subtle issue in general. Fortunately, it becomes much more tangible for $\mathfrak{g}$ complex and semisimple.

Theorem 5.1. *Let $\mathfrak{g}$ be a semisimple complex Lie algebra. Then,*

(i) *$\mathfrak{g}$ has a Cartan subalgebra;*

(ii) *any two Cartan subalgebras of* $\mathfrak{g}$ *are related by an automorphism of* $\mathfrak{g}$.

Proof. We refer to the literature for a proof of this. ■

As a consequence, even though $\mathfrak{g}$ can have many distinct Cartan subalgebras, they all have the same dimension. This common dimension is known as the **rank** of the Lie algebra and is denoted by

$$\mathrm{rank}(\mathfrak{g}) := \dim \mathfrak{h}, \tag{5.3}$$

where $\mathfrak{h}$ is any Cartan subalgebra of $\mathfrak{g}$. It is stressed that the rank is a property of the Lie algebra $\mathfrak{g}$. Consequently, two Lie algebras can only be isomorphic if they have the same rank, and if a Cartan subalgebra $\hat{\mathfrak{h}}$ contains the Cartan subalgebra $\mathfrak{h}$, then $\hat{\mathfrak{h}} = \mathfrak{h}$. Given a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, we refer to the elements of $\mathfrak{h}$ as the corresponding **Cartan generators** of $\mathfrak{g}$.

Theorem 5.2. *Let* $\mathfrak{h}$ *be a Cartan subalgebra of the semisimple complex Lie algebra* $\mathfrak{g}$. *Then,*

(i) $\mathfrak{h}$ *is abelian;*

(ii) *for every* $h \in \mathfrak{h}$, ad_h *is diagonalisable;*

(iii) *if* $x \in \mathfrak{g}$ *satisfies* $[x, \mathfrak{h}] = \{0\}$, *then* $x \in \mathfrak{h}$.

Proof. We refer to the literature for a proof of this. ■

It follows from the *maximality property* (iii) that a Cartan subalgebra is a maximal set of 'commuting' elements in the sense that it cannot be extended by any element $x \in \mathfrak{g}\backslash\mathfrak{h}$ and remain abelian. In other words, a Cartan subalgebra of $\mathfrak{g}$ is not contained in any larger abelian Lie subalgebra of $\mathfrak{g}$.

In the remainder of these lecture notes,
all Lie algebras are assumed finite-dimensional, complex and semisimple.

As the reader is encouraged to verify, the set of diagonal $n \times n$ matrices with zero trace generates a Cartan subalgebra of $\mathfrak{sl}(n, \mathbb{C})$. A basis for this subalgebra is given by the diagonal matrices in the basis (2.27) for $\mathfrak{sl}(n, \mathbb{C})$,

$$\{E_{11} - E_{22}, E_{22} - E_{33}, \ldots, E_{n-1,n-1} - E_{nn}\}, \tag{5.4}$$

so

$$\mathrm{rank}(\mathfrak{sl}(n, \mathbb{C})) = n - 1. \tag{5.5}$$

More generally, referring to the simple Lie algebras discussed at the beginning of Section 2,

$$\mathrm{rank}(X_r) = r, \tag{5.6}$$

where X denotes any of the classical Lie algebras A, B, C, D, or any of the exceptional Lie algebras E, F, G. In particular, the rank of $A_r \cong \mathfrak{sl}(r+1)$ is r, as already indicated in (5.5).

Although a Cartan subalgebra of $\mathfrak{g}$ is abelian and maximal in the sense indicated in Theorem 5.2, its dimension $\mathrm{rank}(\mathfrak{g})$ need not be the maximal dimension of an abelian Lie subalgebra of $\mathfrak{g}$. To illustrate this, let us consider the set of $2n \times 2n$ matrices of the form

$$\begin{pmatrix} 0_n & x \\ 0_n & 0_n \end{pmatrix}, \qquad x \in M_n(\mathbb{C}). \tag{5.7}$$

These matrices generate an n^2-dimensional abelian Lie subalgebra of $\mathfrak{sl}(2n)$, where $n^2 \geqslant 2n-1 = \mathrm{rank}(\mathfrak{sl}(2n))$. However, as shown in Exercise 5.2, this Lie subalgebra is *not* a Cartan subalgebra.

5.2 Roots

Let V be an n-dimensional vector space and C a family of linear operators $V \to V$. The elements of C are said to be **simultaneously diagonalisable** if there exists a basis $\{v_1, \ldots, v_n\}$ for V where each v_k is an eigenvector of every element of C.

Theorem 5.3 (Simultaneous diagonalisability theorem)**.** *Let V be a vector space and C a family of mutually commuting linear operators on V. If each of these operators is diagonalisable, then they are simultaneously diagonalisable.*

Proof. We refer to the literature for a proof of this. ∎

Thus,

a commuting family of diagonalisable matrices are simultaneously diagonalisable.

Remark on applications

> A well-known rule of central importance in quantum mechanics is that *a pair of commuting matrices can be diagonalised simultaneously*. However, as indicated in Theorem 5.3, this is only true in general if both matrices are diagonalisable. This obvious and rather innocent looking assumption is occasionally ignored in the physics literature, but has tremendous ramifications. In fact, there are still many unanswered questions in linear algebra associated with non-diagonalisable matrices.

Now, let $\mathfrak{h}$ be any given Cartan subalgebra of the Lie algebra $\mathfrak{g}$, and let $\{h_i \,|\, i = 1, \ldots, r\}$, $r = \mathrm{rank}(\mathfrak{g})$, denote a basis for $\mathfrak{h}$. According to Theorem 5.2, $\mathfrak{h}$ is abelian, so, by Proposition 1.1, the adjoint operators $\mathrm{ad}_h \in \mathfrak{gl}(\mathfrak{g})$, $h \in \mathfrak{h}$, all commute:

$$[\mathrm{ad}_h, \mathrm{ad}_{h'}] = \mathrm{ad}_{[h,h']} = 0, \qquad \forall\, h, h' \in \mathfrak{h}. \tag{5.8}$$

Since every $h \in \mathfrak{h}$ is ad-diagonalisable, it then follows from Theorem 5.3 that the adjoint operators ad_h are simultaneously diagonalisable.

Remark on applications

> Let a quantum mechanical system have $\mathfrak{g}$ as its symmetry algebra. Then, the rank of $\mathfrak{g}$ provides the maximal number of quantum numbers which can be used to label the states in the Hilbert space. The corresponding eigenvalues of the Cartan generators are related to quantities that can be measured simultaneously.

Let h_i be an element of the basis $\{h_1, \ldots, h_r\}$ for $\mathfrak{h}$. We already know r linearly independent eigenvectors of ad_{h_i}, namely the basis vectors $h_1, \ldots, h_r$, and that the corresponding eigenvalues are all 0:

$$\mathrm{ad}_{h_i}(h_j) = 0, \qquad \forall\, i, j = 1, \ldots, r. \tag{5.9}$$

To have a full basis for $\mathfrak{g}$ consisting exclusively of eigenvectors common to all ad_{h_i}, we still have to identify an additional $\dim \mathfrak{g} - r$ linearly independent such vectors. Denoting them by e_α (their indices are discussed below), we thus have

$$\mathrm{ad}_{h_i}(e_\alpha) = [h_i, e_\alpha] = \alpha_{(i)} e_\alpha \tag{5.10}$$

for some eigenvalues $\alpha_{(i)} \in \mathbb{C}$. For fixed e_α, the eigenvalues $\alpha_{(i)}$ cannot be zero for all i. Indeed, if they were, then $\mathfrak{h}$ could be extended by e_α and thus not be a *maximal* abelian Lie subalgebra.

Taken together, the eigenvalues $\alpha_{(i)}$ associated with e_α form an r-dimensional vector α with i labelling its components. Such vectors are called **roots**, and by the maximality argument above, all roots are nonzero: $\alpha \neq 0$. The set of roots is denoted by Φ and is called the **root system** of $\mathfrak{g}$ relative to $\mathfrak{h}$. As stressed, the definition of roots is *relative* to the given Cartan subalgebra $\mathfrak{h}$. Since $\mathfrak{g}$ is finite-dimensional, Φ is a finite set. It is also noted that the roots are independent of the normalisation of the eigenvectors e_α.

By the linearity of the adjoint map ad, for a general element $h \in \mathfrak{h}$ written as $h = \sum_{i=1}^r \gamma^i h_i$, $\gamma^i \in \mathbb{C}$, we have

$$\mathrm{ad}_h(e_\alpha) = \alpha(h) e_\alpha, \qquad \alpha(h) := \sum_{i=1}^r \gamma^i \alpha_{(i)}. \tag{5.11}$$

In particular, $\alpha(h_i) = \alpha_{(i)}$. Since the eigenvalue $\alpha(h)$ is an element of $\mathbb{C}$, we see that α defines a linear map

$$\alpha: \ \mathfrak{h} \to \mathbb{C}. \tag{5.12}$$

With reference to the definition of dual spaces in Section 3.3, this means that

$$\alpha \in \mathfrak{h}^*. \tag{5.13}$$

Remark on terminology

The complex numbers $\alpha(h)$ are eigenvalues of ad_h and are thus solutions to the characteristic equation $\det(\lambda I - \mathrm{ad}_h) = 0$. Such solutions are often referred to as *roots*, hence the terminology.

Remark on notation

Other common notations for the set Φ of roots are Δ and R.

The generators e_α are ocassionally called **ladder operators** (or step operators) due to their action on $\mathfrak{g}$-modules. To appreciate this, let V be a $\mathfrak{g}$-module and $v \in V$ an eigenvector of h, so $hv = \lambda v$ for some $\lambda \in \mathbb{C}$. Then,

$$h(e_\alpha v) = e_\alpha(hv) + [h, e_\alpha]v = (\lambda + \alpha(h)) e_\alpha v, \tag{5.14}$$

so $e_\alpha v$ is zero or an eigenvector of h with eigenvalue shifted by $\alpha(h)$ compared to the eigenvalue associated with v. The matching terminology and the resemblance of this result with the content of Proposition 3.6 pertaining to $\mathfrak{sl}(2)$ is, of course, no coincidence. However, we have not yet developed any notion of positivity or negativity for the action of e_α, so it does not make sense at this point to refer to e_α as raising or lowering the eigenvalue.

5.3 Root-space decomposition

We continue to let $\mathfrak{h}$ denote a fixed Cartan subalgebra of the finite-dimensional, semisimple complex Lie algebra $\mathfrak{g}$. Given an element $\mu \in \mathfrak{h}^*$ of the space dual to $\mathfrak{h}$, we now define

$$\mathfrak{g}_\mu := \{x \in \mathfrak{g} \,|\, [h, x] = \mu(h)x, \ \forall\, h \in \mathfrak{h}\}. \tag{5.15}$$

We can then view the root system of $\mathfrak{g}$ (relative to $\mathfrak{h}$) as

$$\Phi = \{\alpha \in \mathfrak{h}^* \,|\, \alpha \neq 0, \ \mathfrak{g}_\alpha \neq \{0\}\}. \tag{5.16}$$

For $\alpha \in \Phi$, $\mathfrak{g}_\alpha$ is known as a **root space**. Moreover, the vector space $\mathfrak{g}$ decomposes into common eigenspaces of ad_h for all $h \in \mathfrak{h}$ as

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha. \tag{5.17}$$

This is known as the **root-space decomposition** (or Cartan decomposition) of $\mathfrak{g}$ relative to $\mathfrak{h}$. It is stressed that this is only a direct sum of vector spaces, not of Lie algebras. Since the root system Φ is finite-dimensional, it follows that the root-space decomposition is finite.

It is noted that $\mathfrak{g}_0$ is nothing but the centraliser of $\mathfrak{h}$ in $\mathfrak{g}$, see Exercise 1.22. Since $\mathfrak{h}$ is a *maximal* abelian Lie subalgebra, we conclude that

$$\mathfrak{g}_0 = \mathfrak{h}, \tag{5.18}$$

allowing us to write the root-space decomposition as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha. \tag{5.19}$$

As confirmed below in Theorem 5.7, this corresponds to working in a **Cartan-Weyl basis**

$$\{h_i \,|\, i = 1, \ldots, r\} \sqcup \{e_\alpha \,|\, \alpha \in \Phi\} \tag{5.20}$$

for $\mathfrak{g}$. In fact, this involves a great deal of arbitrariness as it relies on the choice of

(i) a Cartan subalgebra;

(ii) a basis for the Cartan subalgebra;

(iii) specific nonzero elements $e_\alpha \in \mathfrak{g}_\alpha$.

For reasons to become clear, it is nevertheless a much celebrated type of basis. A convenient shorthand for the basis (5.20) is $\{h_i; e_\alpha\}$.

Proposition 5.4. *Let $\mu, \nu \in \mathfrak{h}^*$. Then,*

$$[\mathfrak{g}_\mu, \mathfrak{g}_\nu] \subseteq \mathfrak{g}_{\mu+\nu}. \tag{5.21}$$

Proof. Let $x \in \mathfrak{g}_\mu$, $y \in \mathfrak{g}_\nu$, and $h \in \mathfrak{h}$. Then,

$$[h, [x, y]] = [[h, x], y] + [x, [h, y]] = \mu(h)[x, y] + \nu(h)[x, y] = (\mu + \nu)(h)[x, y], \tag{5.22}$$

so $[x, y] \in \mathfrak{g}_{\mu+\nu}$. ∎

Proposition 5.5. *Let $\{h_i; e_\alpha\}$ be a Cartan-Weyl basis for $\mathfrak{g}$.*

(i) *If $\alpha, \beta \in \Phi$ such that $\alpha + \beta \neq 0$, then $\kappa(e_\alpha, e_\beta) = 0$.*

(ii) $\kappa(h_i, e_\alpha) = 0$.

(iii) *The restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{h}$ is non-degenerate.*

Proof. Let $e_\alpha \in \mathfrak{g}_\alpha$ and $e_\beta \in \mathfrak{g}_\beta$.

(i) Since $\alpha + \beta \neq 0$, there exists $h \in \mathfrak{h}$ such that $(\alpha + \beta)(h) \neq 0$. Using the invariance (4.20) of the Killing form, we have

$$\alpha(h)\kappa(e_\alpha, e_\beta) = \kappa([h, e_\alpha], e_\beta) = -\kappa(e_\alpha, [h, e_\beta]) = -\beta(h)\kappa(e_\alpha, e_\beta). \tag{5.23}$$

It follows that $(\alpha + \beta)(h)\kappa(e_\alpha, e_\beta) = 0$. Hence, $\kappa(e_\alpha, e_\beta) = 0$.

(ii) Since α is a root, $\alpha \neq 0$, so there exists $h \in \mathfrak{h}$ such that $\alpha(h) \neq 0$. Again using the invariance of the Killing form, we have

$$\alpha(h)\kappa(h_i, e_\alpha) = \kappa(h_i, [h, e_\alpha]) = \kappa([h_i, h], e_\alpha) = 0, \tag{5.24}$$

so $\kappa(h_i, e_\alpha) = 0$.

(iii) We prove this by contradiction. Assume that the restricted Killing form is degenerate. Then, there exists nonzero $h \in \mathfrak{h}$ such that $\kappa(h, h') = 0$ for all $h' \in \mathfrak{h}$. By (ii) and the bilinearity of the Killing form, we thus have $\kappa(h, x) = 0$ for all $x \in \mathfrak{g}$. Since $\mathfrak{g}$ is complex and semisimple, this contradicts Cartan's second criterion, Theorem 4.21, so the restricted Killing form is non-degenerate. ■

Corollary 5.6. *Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$.*

(i) *If $\alpha \in \mathfrak{h}^*$ is a root of $\mathfrak{g}$, then so is $-\alpha$.*

(ii) *The roots of $\mathfrak{g}$ span $\mathfrak{h}^*$.*

Proof. This is the content of Exercise 5.12. ■

Rephrased, we thus have

$$\textbf{(i)}: \quad \alpha \in \Phi \quad \Leftrightarrow \quad -\alpha \in \Phi, \qquad \text{and} \qquad \textbf{(ii)}: \quad \mathfrak{h}^* = \mathrm{span}_{\mathbb{C}}(\Phi). \tag{5.25}$$

Although the roots span $\mathfrak{h}^*$, it is emphasised that the root system is *not* a vector space.

Theorem 5.7. *Let $\alpha \in \Phi$.*

(i) *The only roots proportional to α are $\pm\alpha$.*

(ii) *The root space $\mathfrak{g}_\alpha$ is one-dimensional.*

Proof. We refer to the literature for a proof of this. ■

It follows that the number of distinct roots is given by

$$|\Phi| = \dim \mathfrak{g} - \mathrm{rank}(\mathfrak{g}). \tag{5.26}$$

Corollary 5.8. *Let $\{h_i; e_\alpha\}$ be a Cartan-Weyl basis for $\mathfrak{g}$. Then, $\kappa(e_\alpha, e_{-\alpha}) \neq 0$.*

5.4 Inner product on $\mathfrak{h}^*$

Proposition 5.5 implies that the Killing form is block-diagonal in the Cartan-Weyl basis. In a somewhat sloppy but self-explanatory notation, we thus have

$$(\kappa_{ab}) = \begin{pmatrix} (\kappa_{ij}) & 0 \\ 0 & (\kappa_{\alpha\beta}) \end{pmatrix}, \qquad \det(\kappa_{ab}) = \det(\kappa_{ij})\det(\kappa_{\alpha\beta}), \tag{5.27}$$

where the zeros indicate zero matrices of the appropriate sizes. Since the Killing form is non-degenerate, we can define the **inverse Killing form** κ^{-1} with matrix entries κ^{ab}. Using the matrix inverse of (5.27), we then have

$$(\kappa^{ab}) = \begin{pmatrix} (\kappa^{ij}) & 0 \\ 0 & (\kappa^{\alpha\beta}) \end{pmatrix} = \begin{pmatrix} (\kappa_{ij}) & 0 \\ 0 & (\kappa_{\alpha\beta}) \end{pmatrix}^{-1}. \tag{5.28}$$

As argued just below (5.30), the non-degeneracy of the Killing form on the Cartan subalgebra induces a bijection between $\mathfrak{h}^*$ and $\mathfrak{h}$,

$$\phi: \ \mathfrak{h}^* \to \mathfrak{h}, \qquad \mu \mapsto t_\mu, \tag{5.29}$$

by setting

$$\kappa(t_\mu, h) = \mu(h), \qquad \forall\, h \in \mathfrak{h}. \tag{5.30}$$

Indeed, because of the non-degeneracy of κ, the map is not only linear, but also injective (its kernel is $\{0\}$). Furthermore, because $\dim \mathfrak{h}^* = \dim \mathfrak{h}$, its image must be the whole of $\mathfrak{h}$, so the map is surjective and therefore bijective. The map thus provides a *pairing* of the elements $\mu \in \mathfrak{h}^*$ and $t_\mu \in \mathfrak{h}$. The unique element $t_\mu = \sum_i \mu^i h_i$ associated with μ can be obtained by solving, using the inverse Killing form, the system of linear equations (5.30) for the coefficients μ^i. The bijective property of ϕ readily implies the following result.

Proposition 5.9. *The vectors t_α, $\alpha \in \Phi$, are all nonzero and span $\mathfrak{h}$.*

Moreover, according to the following result, a Cartan generator of the form t_α is 'non-orthogonal' to itself with respect to the Killing form.

Theorem 5.10. *Let $\alpha, \beta \in \Phi$. Then,*

$$\kappa(t_\alpha, t_\alpha) \neq 0, \qquad \kappa(t_\alpha, t_\beta) \in \mathbb{Q}, \qquad \frac{2\,\kappa(t_\alpha, t_\beta)}{\kappa(t_\alpha, t_\alpha)} \in \mathbb{Z}. \tag{5.31}$$

Proof. We refer to the literature for a proof of this. ∎

Lemma 5.11. *Let $\alpha \in \Phi$. Then,*

$$[e_\alpha, e_{-\alpha}] = \kappa(e_\alpha, e_{-\alpha}) t_\alpha \neq 0. \tag{5.32}$$

Proof. Let $h \in \mathfrak{h}$. Then,

$$[h, [e_\alpha, e_{-\alpha}]] = [e_\alpha, [h, e_{-\alpha}]] + [[h, e_\alpha], e_{-\alpha}] = -\alpha(h)[e_\alpha, e_{-\alpha}] + \alpha(h)[e_\alpha, e_{-\alpha}] = 0, \tag{5.33}$$

so $[e_\alpha, e_{-\alpha}]$ is 0 or an eigenvector of ad_h with eigenvalue 0 for all $h \in \mathfrak{h}$. In either case, $[e_\alpha, e_{-\alpha}] \in \mathfrak{h}$. To determine which element of $\mathfrak{h}$ it is, we let $h \in \mathfrak{h}$ and consider

$$\kappa(h, [e_\alpha, e_{-\alpha}]) = \kappa([h, e_\alpha], e_{-\alpha}) = \alpha(h)\kappa(e_\alpha, e_{-\alpha}) = \kappa(t_\alpha, h)\kappa(e_\alpha, e_{-\alpha}) = \kappa(h, \kappa(e_\alpha, e_{-\alpha})t_\alpha). \tag{5.34}$$

Since κ restricted to $\mathfrak{h}$ is non-degenerate, we conclude that $[e_\alpha, e_{-\alpha}] = \kappa(e_\alpha, e_{-\alpha})t_\alpha$. By Corollary 5.8 and Proportion 5.9, $\kappa(e_\alpha, e_{-\alpha}) \neq 0$ and $t_\alpha \neq 0$, so $[e_\alpha, e_{-\alpha}] \neq 0$. ■

In summary, the Lie bracket relations satisfied by the elements of the Cartan-Weyl basis $\{h_i; e_\alpha\}$ are of the form

$$[h_i, h_j] = 0, \qquad [h_i, e_\alpha] = \alpha(h_i)e_\alpha, \qquad [e_\alpha, e_\beta] = \begin{cases} N_{\alpha,\beta}\, e_{\alpha+\beta}, & \text{if } \alpha + \beta \in \Phi, \\ \kappa(e_\alpha, e_{-\alpha})\, t_\alpha, & \text{if } \alpha + \beta = 0, \\ 0, & \text{otherwise,} \end{cases} \tag{5.35}$$

where $N_{\alpha,\beta}$ is part of the structure constant

$$f_{\alpha\beta}{}^{c} = N_{\alpha,\beta}\, \delta^{c}_{\alpha+\beta}, \qquad \alpha + \beta \in \Phi. \tag{5.36}$$

Proposition 5.12. *Let $\alpha, \beta \in \Phi$ such that $\alpha + \beta \in \Phi$. Then, $N_{\alpha,\beta} \neq 0$.*

Proof. We refer to the literature for a proof of this. ■

It follows, in particular, that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$, not just $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ as in Proposition 5.4.

The bijection $\phi : \mathfrak{h}^* \to \mathfrak{h}$ in (5.29) can be used to define an (indefinite) inner product on $\mathfrak{h}^*$ as

$$\langle \mu, \nu \rangle := \kappa(t_\mu, t_\nu). \tag{5.37}$$

Another common notation for this is $\mu \cdot \nu \equiv \langle \mu, \nu \rangle$. To examine this inner product, let $h, h' \in \mathfrak{h}$. Then, ad_h and $\mathrm{ad}_{h'}$ are diagonal matrices with the diagonal entries corresponding to the Cartan generators all zero. Since $\mathfrak{g}_\alpha$ is one-dimensional for each $\alpha \in \Phi$, $\alpha(h)$ appears exactly once on the diagonal of ad_h. The form of $\mathrm{ad}_{h'}$ is of course similar. It follows that

$$\kappa(h, h') = \mathrm{tr}(\mathrm{ad}_h\, \mathrm{ad}_{h'}) = \sum_{\alpha \in \Phi} \alpha(h)\alpha(h'). \tag{5.38}$$

In particular,

$$\langle \mu, \mu \rangle = \kappa(t_\mu, t_\mu) = \sum_{\alpha \in \Phi} \alpha(t_\mu)\alpha(t_\mu) = \sum_{\alpha \in \Phi} (\kappa(t_\alpha, t_\mu))^2 = \sum_{\alpha \in \Phi} \langle \alpha, \mu \rangle^2, \tag{5.39}$$

where the third equality follows from (5.30) with $\mu = \alpha$ and $h = t_\mu$.

5.5 $\mathfrak{sl}(2)$ subalgebras

Since

$$\langle\alpha,\alpha\rangle = \kappa(t_\alpha, t_\alpha) \neq 0, \tag{5.40}$$

we can use expressions like $\langle\alpha,\alpha\rangle$ to normalise the various roots and generators. In particular, we define **coroots** by

$$\alpha^\vee := \frac{2}{\langle\alpha,\alpha\rangle}\,\alpha, \qquad \alpha \in \Phi, \tag{5.41}$$

the set of which is denoted by $\Phi^\vee$.

So far, we have not normalised the ladder generators e_α. However, since the Killing form is non-degenerate, for each $\alpha \in \Phi$, we can choose $e_\alpha \in \mathfrak{g}_\alpha$ and $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that

$$\kappa(e_\alpha, e_{-\alpha}) = \frac{2}{\langle\alpha,\alpha\rangle}. \tag{5.42}$$

This fixes the relative normalisations of e_α and $e_{-\alpha}$. Accompanying this, we also introduce the **dual coroots**

$$h_\alpha := \frac{2}{\langle\alpha,\alpha\rangle}\,t_\alpha, \qquad \alpha \in \Phi. \tag{5.43}$$

Remark on terminology

There is no general consensus on the use of the terms *coroot* and *dual root*. Here, we follow the school referring to the renormalised roots $\alpha^\vee$ as coroots. Since

$$h_\alpha = \frac{2}{\langle\alpha,\alpha\rangle}\,t_\alpha = t_{\alpha^\vee}, \tag{5.44}$$

h_α is the unique element of $\mathfrak{h}^*$ paired with $\alpha^\vee$ under the bijection $\phi : \mathfrak{h}^* \to \mathfrak{h}$, mapping $\alpha^\vee$ to $t_{\alpha^\vee}$. It thus seems natural to refer to h_α as the corresponding *dual* coroot.

Imposing the normalisation (5.42) allows us to write the Lie bracket relations (5.35) of the chosen Cartan-Weyl basis as

$$[h_i, h_j] = 0, \qquad [h_i, e_\alpha] = \alpha(h_i)e_\alpha, \qquad [e_\alpha, e_\beta] = \begin{cases} N_{\alpha,\beta}\, e_{\alpha+\beta}, & \text{if } \alpha+\beta \in \Phi, \\ h_\alpha, & \text{if } \alpha+\beta = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{5.45}$$

Henceforth, we shall assume that (5.42) has been imposed.

Lemma 5.13. *For each $\alpha \in \Phi$,*

$$\alpha(h_\alpha) = 2. \tag{5.46}$$

Proof. Let $\alpha \in \Phi$. Then,

$$\alpha(h_\alpha) = \kappa(t_\alpha, h_\alpha) = \frac{2}{\langle\alpha,\alpha\rangle}\,\kappa(t_\alpha, t_\alpha) = 2. \tag{5.47}$$

■

Proposition 5.14. *Let $\alpha \in \Phi$. Choose elements $e_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ such that $\kappa(e_\alpha, e_{-\alpha}) = \frac{2}{\langle\alpha,\alpha\rangle}$. Then,*

$$[h_\alpha, e_\alpha] = 2\, e_\alpha, \qquad [h_\alpha, e_{-\alpha}] = -2\, e_{-\alpha}, \qquad [e_\alpha, e_{-\alpha}] = h_\alpha. \tag{5.48}$$

Proof. Using Lemma 5.13, we readily obtain

$$[h_\alpha, e_\alpha] = \alpha(h_\alpha)e_\alpha = 2\, e_\alpha, \qquad [h_\alpha, e_{-\alpha}] = -\alpha(h_\alpha)e_{-\alpha} = -2\, e_{-\alpha}. \tag{5.49}$$

The relation $[e_\alpha, e_{-\alpha}] = h_\alpha$ already appeared in (5.45). ■

For each $\alpha \in \Phi$, we recognise $\{e_\alpha, h_\alpha, e_{-\alpha}\}$ as a basis for $A_1 \cong \mathfrak{sl}(2)$. We have thus identified a whole family of such Lie subalgebras of $\mathfrak{g}$ and can therefore use results for $\mathfrak{sl}(2)$ to gain insight into the structure of $\mathfrak{g}$. In fact, one can use that the h-eigenvalues are integers in any finite-dimensional representation of $\mathfrak{sl}(2)$ (knowledge obtained in Section 3.7) to deduce that $\langle\alpha^\vee, \beta\rangle \in \mathbb{Z}$ for all roots $\alpha, \beta \in \Phi$ of $\mathfrak{g}$, see Theorem 5.10.

5.6 *Exercises*

Exercise 5.1.

(i) Show that every nonzero abelian subalgebra of $\mathfrak{sl}(2)$ is isomorphic to $\mathfrak{a}$.

(ii) Give an example of a nonzero abelian subalgebra of $\mathfrak{sl}(2)$ that is not a Cartan subalgebra of $\mathfrak{sl}(2)$.

Exercise 5.2.

(i) Show that the matrices in (5.7) generate a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(2n)$.

(ii) Show that $\mathfrak{h}$ in (i) is not a Cartan subalgebra of $\mathfrak{sl}(2n)$.

Exercise 5.3.

Identify a Cartan subalgebra of $\mathfrak{g}$ if $\mathfrak{g} = \mathfrak{sp}(2r)$ and if $\mathfrak{g} = \mathfrak{so}(n)$.

Exercise 5.4.

Let $\{e, h, f\}$ denote the standard basis for the Lie algebra A_1, and let $\mathfrak{h}$ denote the Cartan subalgebra generated by h. Find $x \in A_1 \backslash \mathfrak{h}$ such that ad_x is diagonalisable with real eigenvalues.

Exercise 5.5.

Let $\{e, h, f\}$ denote the standard basis for the Lie algebra A_1.

(i) Find the eigenvalues and corresponding eigenvectors of ad_{e-f}.

(ii) Show that $\mathrm{span}\{e - f\}$ is a Cartan subalgebra of A_1.

Exercise 5.6.

Find all Cartan subalgebras of $\mathfrak{sl}(2)$.

Exercise 5.7.

Show that a pair of commuting and diagonalisable matrices are simultaneously diagonalisable.

Exercise 5.8.

Let Φ be the root system of a finite-dimensional semisimple complex Lie algebra $\mathfrak{g}$. For $\alpha \in \Phi$, consider $x \in \mathfrak{g}_\alpha$. Show that ad_x is nilpotent.

Exercise 5.9.

Let $\alpha \in \Phi$. Show that

$$\kappa(t_\alpha, t_\alpha)\, \kappa(h_\alpha, h_\alpha) = 4. \tag{5.50}$$

Exercise 5.10.

Let $\{h_i; e_\alpha\}$ be a Cartan-Weyl basis for a Lie algebra, Φ the corresponding root system and κ the Killing form. Show that

$$\kappa_{ij} = \sum_{\alpha \in \Phi} \alpha(h_i)\, \alpha(h_j). \tag{5.51}$$

Exercise 5.11.

Let $\{h_i; e_\alpha\}$ be a Cartan-Weyl basis for a Lie algebra. Verify that setting $N_{-\alpha,-\beta} = -N_{\alpha,\beta}$ does not cause any problems.

Exercise 5.12.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let $\alpha \in \mathfrak{h}^*$ be a root of $\mathfrak{g}$.

(i) Show that $-\alpha$ is a root of $\mathfrak{g}$.

(ii) Show that the roots of $\mathfrak{g}$ span $\mathfrak{h}^*$.

6 Classification of simple Lie algebras

In this section,

> *all Lie algebras are assumed finite-dimensional, complex and semisimple.*

6.1 Euclidean root space

It is recalled from Proposition 5.9 that the elements of the form t_α, $\alpha \in \Phi$, span $\mathfrak{h}$. A basis for $\mathfrak{h}$ can thus be formed by such elements.

Theorem 6.1. *Let $\alpha \in \Phi$ and let $\alpha^1, \ldots, \alpha^r \in \Phi$ such that $\{t_{\alpha^1}, \ldots, t_{\alpha^r}\}$ is a basis for $\mathfrak{h}$. Then, there exist $q_1, \ldots, q_r \in \mathbb{Q}$ such that*

$$t_\alpha = \sum_{i=1}^{r} q_i\, t_{\alpha^i}. \tag{6.1}$$

Proof. We refer to the literature for a proof of this. ∎

Accordingly, we denote by $\mathfrak{h}_{\mathbb{Q}}$ the set of all elements of the form $\sum_{i=1}^{r} q_i\, t_{\alpha^i}$ with $q_i \in \mathbb{Q}$ and by $\mathfrak{h}_{\mathbb{R}}$ the set of all such elements with $q_i \in \mathbb{R}$. Theorem 6.1 shows that $\mathfrak{h}_{\mathbb{Q}}$ and $\mathfrak{h}_{\mathbb{R}}$ are *independent* of the choice of basis $\{t_{\alpha^1}, \ldots, t_{\alpha^r}\}$. Furthermore, $\mathfrak{h}_{\mathbb{R}}$ is the real vector space spanned by the elements of the form t_α, $\alpha \in \Phi$.

Proposition 6.2. *Let $h, h' \in \mathfrak{h}_{\mathbb{R}}$. Then, $\kappa(h, h') \in \mathbb{R}$ and $\kappa(h, h) \geqslant 0$. Moreover, if $\kappa(h, h) = 0$, then $h = 0$.*

Proof. Let

$$h = \sum_{i=1}^{r} q_i t_{\alpha^i}, \qquad h' = \sum_{j=1}^{r} q'_j t_{\alpha^j}, \qquad q_i, q'_j \in \mathbb{R}. \tag{6.2}$$

Then,

$$\kappa(h, h') = \sum_{i,j=1}^{r} q_i q'_j \kappa(t_{\alpha^i}, t_{\alpha^j}). \tag{6.3}$$

By Theorem 5.10, $\kappa(t_{\alpha^i}, t_{\alpha^j}) \in \mathbb{Q}$, so $\kappa(h, h') \in \mathbb{R}$. Using (5.38), we have

$$\kappa(h, h) = \sum_{i,j=1}^{r} q_i q_j \sum_{\alpha \in \Phi} \alpha(t_{\alpha^i})\alpha(t_{\alpha^j}) = \sum_{\alpha \in \Phi} \Big(\sum_{i=1}^{r} q_i \alpha(t_{\alpha^i})\Big)\Big(\sum_{j=1}^{r} q_j \alpha(t_{\alpha^j})\Big) = \sum_{\alpha \in \Phi} \Big(\sum_{i=1}^{r} q_i \alpha(t_{\alpha^i})\Big)^2. \tag{6.4}$$

This is a sum of squares of real numbers, so $\kappa(h, h) \geqslant 0$. If $\kappa(h, h) = 0$, then (6.4) implies that

$$\sum_{i=1}^{r} q_i \alpha(t_{\alpha^i}) = 0, \qquad \forall\, \alpha \in \Phi. \tag{6.5}$$

For $\alpha = \alpha^j$, we thus have

$$0 = \sum_{i=1}^{r} q_i \alpha^j(t_{\alpha^i}) = \sum_{i=1}^{r} q_i \kappa(t_{\alpha^j}, t_{\alpha^i}). \tag{6.6}$$

Since κ restricted to $\mathfrak{h}$ is invertible, we conclude that $q_i = 0$ for all i, that is, $h = 0$. ∎

It follows that the Killing form restricted to $\mathfrak{h}_{\mathbb{R}}$ is a positive-definite inner product $\mathfrak{h}_{\mathbb{R}} \times \mathfrak{h}_{\mathbb{R}} \to \mathbb{R}$. The vector space $\mathfrak{h}_{\mathbb{R}}$ endowed with κ as inner product is thus a Euclidean space.

By applying the inverse of the bijection ϕ (5.29) to the subset $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{h}$, we obtain $\mathfrak{h}_{\mathbb{R}}^* := \phi^{-1}(\mathfrak{h}_{\mathbb{R}})$. This subset of $\mathfrak{h}^*$ is a real vector space and is spanned by the set of roots. The reality of the Killing form on $\mathfrak{h}_{\mathbb{R}}$ carries over to $\mathfrak{h}_{\mathbb{R}}^*$:

$$\langle\ ,\ \rangle:\ \mathfrak{h}_{\mathbb{R}}^* \times \mathfrak{h}_{\mathbb{R}}^* \to \mathbb{R}. \tag{6.7}$$

This is a symmetric positive-definite bilinear form, turning $\mathfrak{h}_{\mathbb{R}}^*$ into a Euclidean space. We refer to $\mathfrak{h}_{\mathbb{R}}^*$ endowed with this form as the **Euclidean root space**. It follows that

the root system spans a real Euclidean vector space of dimension r.

Remark on terminology

The term *root space* is used to denote two different things, namely spaces of the form $\mathfrak{g}_\alpha$ and the dual space $\mathfrak{h}_{\mathbb{R}}^*$ to $\mathfrak{h}_{\mathbb{R}}$. Albeit common practice and not likely to cause confusion, this is unfortunate, so we have added the qualifier *Euclidean* to distinguish the Euclidean root space $\mathfrak{h}_{\mathbb{R}}^*$ from the root spaces $\mathfrak{g}_\alpha$.

With respect to the Euclidean form on $\mathfrak{h}_{\mathbb{R}}^*$, we introduce the angle $0 \leqslant \theta \leqslant \pi$ between $\alpha, \beta \in \Phi$ by

$$\langle \alpha, \beta \rangle = |\alpha|\,|\beta| \cos\theta, \tag{6.8}$$

where

$$|\alpha| := \sqrt{\langle \alpha, \alpha \rangle}, \qquad \alpha \in \Phi. \tag{6.9}$$

It follows that

$$4(\cos\theta)^2 = \frac{4\,\langle \alpha, \beta \rangle^2}{\langle \alpha, \alpha \rangle \langle \beta, \beta \rangle} = \langle \alpha^\vee, \beta \rangle \langle \alpha, \beta^\vee \rangle. \tag{6.10}$$

From Theorem 5.10, we know that $\langle \alpha^\vee, \beta \rangle$ and $\langle \alpha, \beta^\vee \rangle$ are integers, so

$$4(\cos\theta)^2 \in \{0, 1, 2, 3, 4\}. \tag{6.11}$$

However, if $(\cos\theta)^2 = 1$, then α and β are linearly dependent, so

$$4(\cos\theta)^2 \in \{0, 1, 2, 3\} \quad \text{for} \quad \beta \neq \pm\alpha. \tag{6.12}$$

Without loss of generality, we may choose the two roots $\beta \neq \pm\alpha$ such that $|\alpha| \leqslant |\beta|$. The solutions to the relations above are then classified as follows:

θ	$\langle \alpha^\vee, \beta \rangle$	$\langle \alpha, \beta^\vee \rangle$	$\lvert\beta\rvert/\lvert\alpha\rvert$
$\pi/6$	3	1	$\sqrt{3}$
$\pi/4$	2	1	$\sqrt{2}$
$\pi/3$	1	1	1
$\pi/2$	0	0	undetermined
$2\pi/3$	-1	-1	1
$3\pi/4$	-2	-1	$\sqrt{2}$
$5\pi/6$	-3	-1	$\sqrt{3}$

A root system where all roots are of the same length is said to be **simply-laced**. Correspondingly, a Lie algebra is said to be simply-laced if its root system is.

Proposition 6.3. *If $\alpha, \beta \in \Phi$ are such that $\alpha \neq \pm\beta$, then*

$$\begin{aligned} \langle \alpha, \beta \rangle > 0 \quad &\Rightarrow \quad \alpha - \beta \in \Phi; \\ \langle \alpha, \beta \rangle < 0 \quad &\Rightarrow \quad \alpha + \beta \in \Phi. \end{aligned} \tag{6.13}$$

Proof. We refer to the literature for a proof of this. ■

6.2 Triangular decomposition

The root system of a finite-dimensional Lie algebra is finite, so it is possible to find a hyperplane in $\mathfrak{h}_{\mathbb{R}}^*$ that does not contain any roots. This hyperplane divides $\mathfrak{h}_{\mathbb{R}}^*$ into two half-spaces. Relative to this division, we declare the roots ($\alpha \in \Phi \subset \mathfrak{h}_{\mathbb{R}}^*$) in one of these half-spaces to be positive (indicated by $\alpha > 0$) and the ones in the other half-space to be negative (indicated by $\alpha < 0$). The sets of positive and negative roots are thus given by

$$\Phi_+ = \{\alpha \in \Phi \,|\, \alpha > 0\}, \qquad \Phi_- = \{\alpha \in \Phi \,|\, \alpha < 0\}. \tag{6.14}$$

By construction, $\Phi_- = \Phi \backslash \Phi_+$. The set of ladder operators is correspondingly divided into two disjoint sets:

$$\{e_\alpha \,|\, \alpha \in \Phi\} = \{e_\alpha \,|\, \alpha > 0\} \sqcup \{e_{-\alpha} \,|\, \alpha > 0\}. \tag{6.15}$$

Mimicking the notation $\{e, h, f\}$ used to describe A_1, we introduce

$$f_\alpha := e_{-\alpha}, \qquad \alpha > 0, \tag{6.16}$$

whereby

$$\{e_\alpha \,|\, \alpha \in \Phi\} = \{e_\alpha, f_\alpha \,|\, \alpha > 0\}. \tag{6.17}$$

A ladder operator of the form e_α or f_α, $\alpha > 0$, is referred to as a **raising operator** or **lowering operator**, respectively.

Let us also introduce the linear spans

$$\mathfrak{g}_+ := \text{span}_{\mathbb{C}}\{e_\alpha \,|\, \alpha > 0\}, \qquad \mathfrak{g}_- := \text{span}_{\mathbb{C}}\{f_\alpha \,|\, \alpha > 0\}. \tag{6.18}$$

These vector spaces can be used to form the **triangular decomposition**

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+ \tag{6.19}$$

of $\mathfrak{g}$. Since $\mathfrak{g}_0 = \mathfrak{h}$, this is the same as

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+. \tag{6.20}$$

It is stressed that this is only a direct sum of vector spaces, not of Lie algebras. However, several Lie subalgebras are readily formed out of the three direct summands in the triangular decomposition, see Exercise 6.1. In particular, both $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are seen to be nilpotent Lie subalgebras of $\mathfrak{g}$. The triangular decomposition plays an important role in the representation theory of $\mathfrak{g}$, where the generators of the three direct summands $\mathfrak{g}_+$, $\mathfrak{h}$ and $\mathfrak{g}_-$ mimic the roles played by e, h and f in the case of $\mathfrak{sl}(2)$. However, the study of this is beyond the scope of these lecture notes.

6.3 Simple roots

With respect to the disjoint union decomposition $\Phi = \Phi_+ \sqcup \Phi_-$, a positive root is said to be a **simple root** if it cannot be written as the sum of two positive roots. We denote the set of simple roots by Φ_s.

REMARK ON NOTATION AND TERMINOLOGY

> In the literature, Φ_s is occasionally denoted by Π and referred to as the *fundamental system* relative to the decomposition $\Phi = \Phi_+ \sqcup \Phi_-$.

Proposition 6.4. *Φ_s forms a basis for $\mathfrak{h}^*_{\mathbb{R}}$.*

Proof. This is the content of Exercise 6.5. ■

It follows that $|\Phi_s| = \dim \mathfrak{h}^*_{\mathbb{R}} = r$, meaning that the number of simple roots is given by the rank of $\mathfrak{g}$. This allows us to denote the simple roots by $\alpha_1, \ldots, \alpha_r$, that is,

$$\Phi_s = \{\alpha_i \,|\, i = 1, \ldots, r\}. \tag{6.21}$$

As shown in Exercise 6.4, the difference of two simple roots is not a root. Proposition 6.3 thus implies that

$$\langle \alpha_i, \alpha_j \rangle \leqslant 0, \qquad i \neq j, \tag{6.22}$$

so the angle between two distinct simple roots cannot be acute.

Corollary 6.5. *Let $\alpha \in \Phi$. Then, there exist unique nonnegative integers $a^1, \ldots, a^r$ such that*

$$\alpha = \sum_{i=1}^{r} a^i \alpha_i \;\; (\textit{if } \alpha > 0), \qquad \alpha = -\sum_{i=1}^{r} a^i \alpha_i \;\; (\textit{if } \alpha < 0). \tag{6.23}$$

Proof. This is the content of Exercise 6.6. ■

It follows that *no* root is a linear combination of simple roots with coefficients of *both* signs.

6.4 Cartan matrix

Let $\alpha_1, \ldots, \alpha_r$ fix an ordering of the simple roots. With respect to this, the *non-orthogonality* of the simple roots is encoded in the **Cartan matrix** A whose entries are given by the **Cartan integers** defined by

$$A_{ij} := \langle \alpha_i^\vee, \alpha_j \rangle. \tag{6.24}$$

That these are indeed integers follows from Theorem 5.10. Note that A need not be symmetric. However, it is an example of a **symmetrisable matrix** as it can be written as the product of a diagonal matrix D and a symmetric matrix S:

$$A = DS. \tag{6.25}$$

In the case of the Cartan matrix,

$$D_{ii} = \frac{2}{\langle \alpha_i, \alpha_i \rangle}, \qquad S_{ij} = \langle \alpha_i, \alpha_j \rangle. \tag{6.26}$$

Since the simple roots form a basis for the real Euclidean vector space $\mathfrak{h}_{\mathbb{R}}^*$, it follows that

$$\det(S) > 0. \tag{6.27}$$

Remark on convention

> In the literature, the Cartan matrix is occasionally defined as the transpose of the matrix defined by (6.24).

Proposition 6.6. *The Cartan matrix A has the following properties:*

(i) $A_{ii} = 2$ *for all* i*;*

(ii) $A_{ij} \in \{0, -1, -2, -3\}$ *if* $i \neq j$*;*

(iii) $A_{ij} \in \{-2, -3\} \;\Rightarrow\; A_{ji} = -1$*;*

(iv) $A_{ij} = 0 \;\Leftrightarrow\; A_{ji} = 0$*;*

(v) $\det A > 0$*.*

Proof.

(i) We have

$$A_{ii} = \langle \alpha_i^\vee, \alpha_i \rangle = \frac{2\,\langle \alpha_i, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = 2. \tag{6.28}$$

(ii) Together with (6.22), the analysis in Section 6.1 implies the result.

(iii) This is an immediate consequence of the analysis in Section 6.1.

(iv) We have

$$A_{ij} = 0 \iff \langle \alpha_i^\vee, \alpha_j \rangle = 0 \iff \langle \alpha_i, \alpha_j^\vee \rangle = 0 \iff A_{ji} = 0. \tag{6.29}$$

(v) Write $A = DS$ as in (6.25). The entries (6.26) of the diagonal matrix D are all positive, so $\det(D) > 0$. Since $\det(A) = \det(D)\det(S)$ and $\det(S) > 0$, we thus have $\det(A) > 0$.

∎

6.5 Chevalley generators

Let $\{\alpha_i \,|\, i = 1, \ldots, r\}$ denote a set of simple roots. Associated with this, we introduce the $3r$ **Chevalley generators**

$$e_i := e_{\alpha_i}, \qquad h_i := h_{\alpha_i}, \qquad f_i := f_{\alpha_i}, \qquad i = 1, \ldots, r. \tag{6.30}$$

It is straightforward to verify that they satisfy the Lie bracket relations

$$[h_i, h_j] = 0, \qquad [h_i, e_j] = A_{ij} e_j, \qquad [h_i, f_j] = -A_{ij} f_j, \qquad [e_i, f_j] = \delta_{ij} h_j, \tag{6.31}$$

whose nontrivial structure constants are given by the Cartan integers. The Chevalley generators are seen to be elements of a Cartan-Weyl basis for $\mathfrak{g}$. In particular, the Chevalley generators h_i form a basis for a Cartan subalgebra.

Since $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are nilpotent Lie algebras, Engel's theorem 4.11 implies that there exist integers $p_{ij} > 0$ and $q_{ij} > 0$ such that $(\mathrm{ad}_{e_i})^{p_{ij}}(e_j) = 0$ and $(\mathrm{ad}_{f_i})^{q_{ij}}(f_j) = 0$. It turns out that these 'degrees of nilpotency' are encoded in the Cartan matrix as

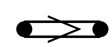
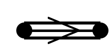
$$(\mathrm{ad}_{e_i})^{1-A_{ij}}(e_j) = 0, \qquad (\mathrm{ad}_{f_i})^{1-A_{ij}}(f_j) = 0, \qquad i \neq j. \tag{6.32}$$

These relations are known as **Serre relations**. The remarkable simplicity of the relations (6.31) and (6.32) satisfied by the Chevalley generators demonstrate the fundamental nature of the data stored in the Cartan matrix. We will get back to this when we discuss Serre's theorem 6.7 in Section 6.7.

6.6 Dynkin diagrams

To each (finite-dimensional semisimple complex) Lie algebra $\mathfrak{g}$, we can associate a so-called *Dynkin diagram*. As we will see, this diagram provides an incredibly compact way of conveying the structure of $\mathfrak{g}$. Along with the Cartan matrices, the Dynkin diagrams also provide a framework for the classification of simple Lie algebras.

The **Dynkin diagram** of $\mathfrak{g}$ can be constructed from the root system or be extracted from the Cartan matrix. Each simple root is represented by a node $\bullet$. The nodes are joined pairwise by a number of line segments depending on the angle θ between the corresponding simple roots. According to (6.22) and our discussion in Section 6.1, this angle can take on the four possible values $\frac{\pi}{2}$, $\frac{2\pi}{3}$, $\frac{3\pi}{4}$ and $\frac{5\pi}{6}$. The rule for the number of line segments between a pair of nodes is given as follows:

$\theta = \frac{\pi}{2}$:	
$\theta = \frac{2\pi}{3}$:	
$\theta = \frac{3\pi}{4}$:	
$\theta = \frac{5\pi}{6}$:	

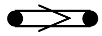
A *single* line thus indicates that the two roots have the *same* length, while two or three lines indicates that the two roots have *different* lengths. An arrow is included in the latter case, pointing from the *longer* to the *shorter* root. The arrow may thus be regarded as an *inequality sign* on the lengths of the two simple roots. A Dynkin diagram need not be connected. Indeed,

> $\mathfrak{g}$ *is simple if and only if its Dynkin diagram is connected.*

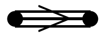
The number of links between the nodes of the two distinct simple roots α_i and α_j is encoded in the Cartan matrix as

$$n_{ij} = A_{ij}A_{ji}, \qquad i \neq j. \tag{6.33}$$

In accordance with the geometric interpretation of n_{ij} as a 'measure' of the angle between the two simple roots, the expression (6.33) is indeed symmetric in i and j. From the analysis in Section 6.1, we see that arrows between nodes in a Dynkin diagram only appear in the following situations:



$$|\alpha_i| = \sqrt{2}\,|\alpha_j| \qquad A_{ij} = -1,\ A_{ji} = -2,$$



$$|\alpha_i| = \sqrt{3}\,|\alpha_j| \qquad A_{ij} = -1,\ A_{ji} = -3.$$

In case of a single link, we simply have

$$|\alpha_i| = |\alpha_j| \qquad A_{ij} = A_{ji} = -1.$$

A Dynkin diagram is independent of the division $\Phi = \Phi_+ \sqcup \Phi_-$ of the root system into positive and negative roots. It is also independent of the ordering of the corresponding simple roots. The Cartan matrix, on the other hand, depends on this ordering. However, modulo permutations of the simple roots, the information stored in the Dynkin diagram and that in the Cartan matrix is the same. The permutation ambiguity in the relation between Dynkin diagrams and Cartan matrices can be resolved by grouping the Cartan matrices using the following notion of *permutation equivalence*: Two matrices are **permutation equivalent** if one can be obtained from the other by a simultaneous permutation of the rows and columns. The two Cartan matrices A and A' are thus equivalent if both are $r \times r$ matrices (for some $r \in \mathbb{N}$) and there exists a permutation σ of $1, \ldots, r$ such that

$$A'_{ij} = A_{\sigma(i)\sigma(j)}. \tag{6.34}$$

To indicate that the two matrices are permutation equivalent, we may write

$$A' \sim A. \tag{6.35}$$

It is stressed that a simultaneous permutation of the rows and columns of a Cartan matrix merely amounts to a relabelling of the set of simple roots. It follows that

two permutation equivalent Cartan matrices describe isomorphic Lie algebras.

We say that a matrix M is **indecomposable** if it is *not* permutation equivalent to a block-diagonal matrix

$$\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, \tag{6.36}$$

where M_1 and M_2 are square matrices, while the zeros are zero matrices of the appropriate sizes. Indecomposability will play a role when specialising from semisimple to simple Lie algebras.

6.7 Classification procedure

A key to the classification of finite-dimensional simple complex Lie algebras is that

any two semisimple Lie algebras with the same Cartan matrix are isomorphic.

As one can show, the classification of finite-dimensional semisimple complex Lie algebras thus amounts to a classification of permutation inequivalent square matrices A with the following properties:

(C1) $A_{ii} = 2$;

(C2) $A_{ij} \in -\mathbb{N}_0$ for $i \neq j$;

(C3) $A_{ij} = 0 \;\Leftrightarrow\; A_{ji} = 0$;

(C4) $\det A > 0$.

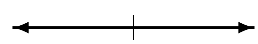
Restricting to *simple* Lie algebras amounts to imposing the additional condition

(C5) A is indecomposable.

Rank $r = 1$**:** A Cartan matrix of rank 1 is a 1×1 matrix. It is thus a diagonal matrix, whose single entry is 2. This is the Cartan matrix associated with $\mathfrak{sl}(2)$, also known as

$$A_1 : \ (2). \tag{6.37}$$

Associated with this are

the Dynkin diagram: $\bullet$, and the root system:  α_1 ,

where we have indicated the $|\Phi| = 2$ roots in the $(r = 1)$-dimensional Euclidean root space, drawn horizontally. The only positive root is the simple root α_1. The only other root is $-\alpha_1$.

Rank $r = 2$**:** A Cartan matrix of rank 2 is of the form

$$A = \begin{pmatrix} 2 & -n \\ -m & 2 \end{pmatrix}, \qquad n, m \in \mathbb{N}_0, \tag{6.38}$$

where

$$0 < \det A = 4 - nm \tag{6.39}$$

with $n = m = 0$ or $n, m > 0$. Since A is a 2×2 matrix, a simultaneous permutation of its rows and columns corresponds to taking the transpose of A. It thus suffices to consider $m \geqslant n$, in which case the complete set of solutions to (6.39) is

$$(n, m) \in \{(0,0), (1,1), (1,2), (1,3)\}, \tag{6.40}$$

corresponding to the Cartan matrices

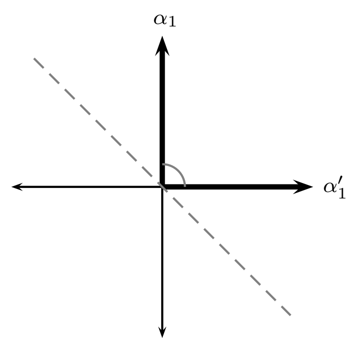
$$D_2 : \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad A_2 : \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \qquad B_2 : \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \qquad G_2 : \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}. \tag{6.41}$$

The first of these matrices violates condition (C5), reflecting that the semisimple Lie algebra D_2 is not simple. The form of the Cartan matrix is in accordance with the direct sum decomposition

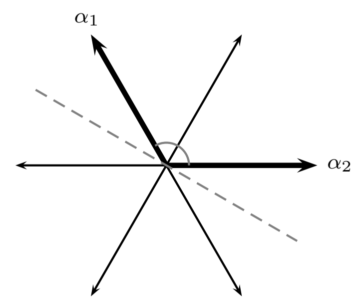
$$D_2 \cong A_1 \boxplus A_1. \tag{6.42}$$

The remaining three Cartan matrices all correspond to simple Lie algebras. The corresponding Dynkin diagrams are

while the corresponding root systems are given by

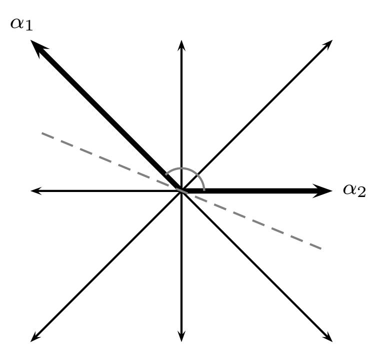


D_2 ($\theta = \frac{\pi}{2}$)

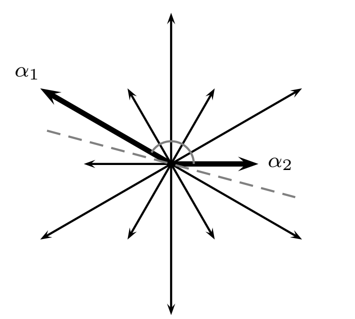


A_2 ($\theta = \frac{2\pi}{3}$)

and



B_2 ($\theta = \frac{3\pi}{4}$)



G_2 ($\theta = \frac{5\pi}{6}$)

In each root system, the chosen hyperplane dividing the plane into two is indicated by a dashed line. Since $r = 2$, the Euclidean root space is two-dimensional, so the hyperplane is merely a line. The angle between the two simple roots is also indicated.

Rank $r = 3$**:** A similar analysis demonstrates that there are only three permutation inequivalent Cartan matrices associated with simple Lie algebras, namely

$$A_3 : \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \qquad B_3 : \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}, \qquad C_3 : \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}. \tag{6.43}$$

The reader is encouraged to verify this.

Rank $r \geqslant 4$**:** Completing the classification of permutation inequivalent Cartan matrices for general r yields the celebrated classification of simple Lie algebras alluded to at the beginning of Section 2 and summarised in Section 6.9.

6.8 Serre's theorem

As the following theorem shows, one can characterise a semisimple complex Lie algebra in terms of generators and relations depending only on data from a Cartan matrix. In fact, the result also provides a proof of existence of all the simple Lie algebras, including the exceptional ones.

Theorem 6.7 (Serre's theorem). *Let A be an $r \times r$ matrix with the Cartan-matrix properties* (C1)-(C4)*, and let*

$$\mathfrak{g} = \langle E_i, H_i, F_i \,|\, i = 1, \ldots, r \rangle \tag{6.44}$$

be the complex Lie algebra generated by the elements E_i, H_i, F_i, $i = 1, \ldots, r$, subject to the relations

(S1) $[H_i, H_j] = 0$,

(S2) $[H_i, E_j] = A_{ij} E_j$, $[H_i, F_j] = -A_{ij} F_j$,

(S3) $[E_i, F_j] = \delta_{ij} H_j$,

(S4) $(\mathrm{ad}_{E_i})^{1-A_{ij}}(E_j) = (\mathrm{ad}_{F_i})^{1-A_{ij}}(F_j) = 0$ *for all* $i \neq j$.

Then, $\mathfrak{g}$ is finite-dimensional and semisimple, $\{H_1, \ldots, H_r\}$ spans a Cartan subalgebra, and the corresponding root system has Cartan matrix A.

Proof. We refer to the literature for a proof of this. ∎

As the notation suggests, the generators E_i, H_i, F_i are exactly the Chevalley generators e_i, h_i, f_i of the ensuing Lie algebra.

Remark on construction

> To be precise, the construction in Serre's theorem is based on (i) the tensor algebra with basis $\{E_i, H_i, F_i \,|\, i = 1, \ldots, r\}$, (ii) the quotient algebra of this tensor algebra by the relations $[A, B] + [B, A] = 0$ and $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$, and (iii) the quotient algebra of the resulting free Lie algebra by the relations (S1)-(S4).

It is emphasised that, using the notation of Chevalley generators, the Serre relations (S4) provide the *smallest* number of occurrences of e_i and f_i in nested Lie brackets like

$$[e_i, [e_i, \ldots, [e_i, e_j] \ldots]] = 0, \qquad [f_i, [f_i, \ldots, [f_i, f_j] \ldots]] = 0, \qquad i \neq j. \tag{6.45}$$

In particular,

$$(\mathrm{ad}_{e_i})^{-A_{ij}}(e_j) \neq 0, \qquad (\mathrm{ad}_{f_i})^{-A_{ij}}(f_j) \neq 0, \qquad i \neq j. \tag{6.46}$$

Let us illustrate this by an example. For $\mathfrak{g} = A_1$, there are no Serre relations, so the simplest nontrivial example is $\mathfrak{g} = A_2$. In this case, the Serre relations are given by

$$[e_1, [e_1, e_2]] = [e_2, [e_2, e_1]] = 0, \qquad [f_1, [f_1, f_2]] = [f_2, [f_2, f_1]] = 0 \tag{6.47}$$

and tacitly imply that

$$[e_1, e_2] \neq 0, \qquad [f_1, f_2] \neq 0. \tag{6.48}$$

From (6.48), we learn that

$$\pm(\alpha_1 + \alpha_2) \in \Phi, \tag{6.49}$$

while the Serre relations (6.47) imply that, for all $n_1, n_2 \in \mathbb{N}$,

$$\pm(n_1\alpha_1 + n_2\alpha_2) \notin \Phi \qquad \text{if} \quad n_1 \geqslant 2 \quad \text{or} \quad n_2 \geqslant 2. \tag{6.50}$$

6.9 Summary

This section summarises some of the key ingredients and results in the classification of finite-dimensional semisimple complex Lie algebras. The following flow chart provides a rough outline of the path we have taken through the theory of these Lie algebras.

$$\begin{tikzcd}
\boxed{\text{Finite-dimensional semisimple complex Lie algebra } \mathfrak{g}} \arrow[d, "\text{we choose}"] \\
\boxed{\text{Cartan subalgebra } \mathfrak{h}} \arrow[d, "\text{this yields}"] \\
\boxed{\begin{array}{c} \text{Root system } \Phi \subset \mathfrak{h}^* \\ \text{and root-space decomposition } \mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \end{array}} \arrow[d, "\text{we choose}"] \\
\boxed{\text{Set of simple roots } \Phi_s = \{\alpha_1, \ldots, \alpha_r\}} \arrow[d, "\text{this yields}"] \\
\boxed{\text{Cartan matrix and Dynkin diagram}} \arrow[d, dashed, "\text{Serre's theorem}"] \\
\boxed{\text{Finite-dimensional semisimple complex Lie algebra } \mathfrak{g}}
\end{tikzcd}$$

As a semisimple Lie algebra is a direct sum of simple Lie algebras, it suffices to classify the simple ones. Thus, let $\mathfrak{g}$ be a finite-dimensional simple complex Lie algebra. The classification then states that it is isomorphic to one of the classical Lie algebras

$$A_r \ (r \geqslant 1), \quad B_r \ (r \geqslant 2), \quad C_r \ (r \geqslant 3), \quad D_r \ (r \geqslant 4), \tag{6.51}$$

or one of the exceptional Lie algebras

$$E_6, \quad E_7, \quad E_8, \quad F_4, \quad G_2. \tag{6.52}$$

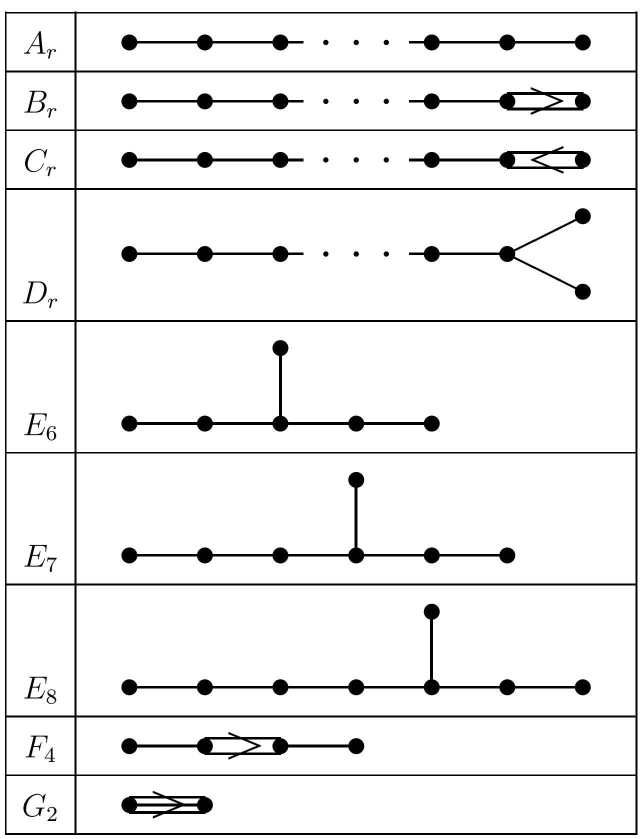
Some basic data of these Lie algebras are summarised as follows:

$\mathfrak{g}$	A_r	B_r	C_r	D_r	E_6	E_7	E_8	F_4	G_2
$\dim(\mathfrak{g})$	r^2+2r	$2r^2+r$	$2r^2+r$	$2r^2-r$	78	133	248	52	14
$\mathrm{rank}(\mathfrak{g})$	r	r	r	r	6	7	8	4	2
$\lvert\Phi_+\rvert$	$\frac{1}{2}(r^2+r)$	r^2	r^2	r^2-r	36	63	120	24	6
simply-laced	✓	✗	✗	✓	✓	✓	✓	✗	✗

Here, we recall our notation $r = \mathrm{rank}(\mathfrak{g})$ and that

$$\dim(\mathfrak{g}) = r + |\Phi| = r + 2|\Phi_+|, \qquad |\Phi_s| = r. \tag{6.53}$$

The following table lists the Dynkin diagrams of the simple complex Lie algebras.



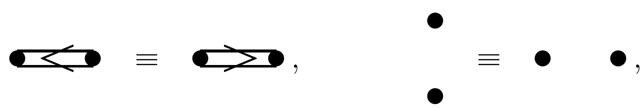
Remark on applications

> Although not discussed in these lecture notes, the Dynkin diagrams also appear in other contexts. The simply-laced ones (of A, D or E type), in particular, play a prominent role in quite a few classification schemes in pure mathematics and mathematical physics.

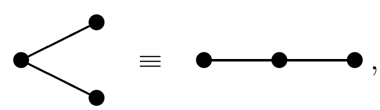
Let us readdress the isomorphisms

$$C_1 \cong B_1 \cong A_1, \qquad C_2 \cong B_2, \qquad D_2 \cong A_1 \boxplus A_1, \qquad D_3 \cong A_3, \tag{6.54}$$

but this time from the perspective of the associated Dynkin diagrams. For $r = 1$, the three Lie algebras C_1, B_1 and A_1 are all represented by a single node and are thus isomorphic. For completeness, we also recall that $D_1 \cong \mathfrak{a}$, but this is abelian and therefore not semisimple. For $r = 2$, the isomorphisms $C_2 \cong B_2$ and $D_2 \cong A_1 \boxplus A_1$ reflect the identifications



justified by simple diagram rotations. For $r = 3$, the isomorphism $D_3 \cong A_3$ corresponds to



obtained by 'straightening' and rotating the diagram on the left.

As already discussed, the specific form of a Cartan matrix depends on the ordering of the simple roots. To associate a Cartan matrix to a given Dynkin diagram, we thus need to label the nodes by $i = 1, \ldots, r$ to fix the ordered basis in which the Cartan matrix will be expressed. With reference to the table above, our convention is as follows:

(i) Label the nodes from left to right.

This rule suffices in most cases, although an ambiguity remains for D_r and E_r.

(ii) In the case of D_r or E_r, the r'th node is the elevated one.

The corresponding Cartan matrix is then uniquely determined. In the following list of Cartan matrices, all nonzero entries are indicated. If a zero entry is indicated explicitly, it appears in smaller font for the sake of readability. With these conventions, the Cartan matrices of the classical series are given by

$$
A_r : \begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & 2 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{pmatrix}, \qquad B_r : \begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & 2 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -2 & 2 \end{pmatrix},
$$

$$
C_r : \begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & 2 & -1 & \\ & & & & -1 & 2 & -2 \\ & & & & & -1 & 2 \end{pmatrix}, \qquad D_r : \begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & 2 & -1 & -1 \\ & & & & -1 & 2 & 0 \\ & & & & -1 & 0 & 2 \end{pmatrix}, \tag{6.55}
$$

while the Cartan matrices of the exceptional Lie algebras are given by

$$
E_6 : \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}, \qquad E_7 : \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}, \tag{6.56}
$$

$$
E_8 : \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}, \tag{6.57}
$$

and

$$
F_4 : \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \qquad G_2 : \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}. \tag{6.58}
$$

6.10 *Exercises*

Exercise 6.1.

Let $\mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$ be a triangular decomposition of a semisimple Lie algebra. Relative to this, identify as many abelian, solvable and nilpotent Lie subalgebras as possible.

Exercise 6.2.

Let $\alpha, \beta \in \Phi$ such that $\alpha + \beta \in \Phi$. Show that

$$h_{\alpha+\beta} = \frac{\langle \alpha, \alpha \rangle}{\langle \alpha+\beta, \alpha+\beta \rangle} h_\alpha + \frac{\langle \beta, \beta \rangle}{\langle \alpha+\beta, \alpha+\beta \rangle} h_\beta. \tag{6.59}$$

Exercise 6.3.

Let $\alpha, \beta, \gamma \in \Phi$ such that $\alpha + \beta + \gamma = 0$. Show that

$$\frac{N_{\alpha,\beta}}{\langle \gamma, \gamma \rangle} = \frac{N_{\beta,\gamma}}{\langle \alpha, \alpha \rangle} = \frac{N_{\gamma,\alpha}}{\langle \beta, \beta \rangle}. \tag{6.60}$$

Exercise 6.4.

Show that the difference of two simple roots is not a root.

Exercise 6.5.

(i) Show that every $\alpha \in \Phi_+$ is a sum of simple roots.

(ii) Show that $\mathrm{span}_{\mathbb{R}}(\Phi_s) = \mathfrak{h}^*_{\mathbb{R}}$.

(iii) Show that Φ_s is linearly independent.

Exercise 6.6.

Let $\alpha \in \Phi$ and let $\Phi_s = \{\alpha_1, \ldots, \alpha_r\}$ be a set of distinct simple roots. Show that there exist nonnegative integers $a^1, \ldots, a^r$ such that

$$\alpha = \sum_{i=1}^r a^i \alpha_i \ \ (\text{if } \alpha > 0), \qquad \alpha = -\sum_{i=1}^r a^i \alpha_i \ \ (\text{if } \alpha < 0). \tag{6.61}$$

Exercise 6.7.

Following our conventions, the Cartan matrices of A_3, B_3 and F_4 are given by

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}, \qquad F = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}. \tag{6.62}$$

In each case, determine all the permutation equivalent matrices. That is, find all matrices A', B' and F' such that $A' \sim A$, $B' \sim B$ and $F' \sim F$.

Exercise 6.8.

Show that the determinant of the Cartan matrix of A_r is given by $r + 1$.

Exercise 6.9.

Given the Cartan matrix of B_2 in (6.41), use Serre's theorem to recover the root system and Lie bracket relations.

Exercise 6.10.

For G_2, determine how many roots have length $|\alpha_1|$ and how many have length $|\alpha_2|$.

Exercise 6.11.

Let Φ be the root system of a finite-dimensional semisimple complex Lie algebra, and let $\alpha, \beta \in \Phi$. Determine the possible values of the difference $\beta(h_\alpha) - \alpha(h_\beta)$.

Exercise 6.12.

Show that

$$\begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -2 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix} \tag{6.63}$$

is a Cartan matrix of a simple complex Lie algebra.

Exercise 6.13.

Let $\alpha_1 \neq \alpha_2$ be simple roots of the Lie algebra G_2, and suppose that $|\alpha_1 + \alpha_2| = \sqrt{5}$. Determine $|\alpha_1 - \alpha_2|$.

Exercise 6.14.

Let $\{\alpha_1, \alpha_2, \alpha_3\}$ be a set of distinct simple roots of the Lie algebra B_3, and suppose that

$$|\alpha_1| = |\alpha_3|, \qquad \langle \alpha_2, \alpha_3 \rangle = 0.$$

Determine the corresponding Cartan matrix.

Exercise 6.15.

Let

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

be the E_6 Cartan matrix corresponding to the simple roots $\alpha_1, \ldots, \alpha_6$, and suppose that $|\alpha_4| = \sqrt{3}$. Determine $\langle \alpha_3 + \alpha_4, \alpha_6 \rangle$.

Exercise 6.16.

Draw the Dynkin diagrams of all simple complex Lie algebras for which the root system contains 36 positive roots.

Exercise 6.17.

Label the nodes of the Dynkin diagram for E_6 such that the corresponding Cartan matrix is given by

$$\begin{pmatrix} 2 & 0 & 0 & -1 & 0 & -1 \\ 0 & 2 & -1 & 0 & 0 & -1 \\ 0 & -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ -1 & -1 & 0 & 0 & -1 & 2 \end{pmatrix} \tag{6.64}$$

Exercise 6.18.

Determine the complex Lie algebra whose Cartan matrix is given by

$$\begin{pmatrix} 2 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}. \tag{6.65}$$

A Algebraic notions

The notes in this appendix are very sketchy and only meant as a rough guide.

A.1 Some basic algebraic notions

Cartesian product

For sets S_1 and S_2, the **Cartesian product** $S_1 \times S_2$ is the set of all ordered pairs (s_1, s_2) where $s_1 \in S_1$ and $s_2 \in S_2$.

Binary operation

A **binary operation** $*$ on a set S to the set S' is a map

$$* : \; S \times S \to S'. \tag{A.1}$$

Binary composition

A binary operation $*$ on a set S is a **binary composition** in S if the image is in S,

$$* : \; S \times S \to S. \tag{A.2}$$

A binary composition is thus a **closed** binary operation. Although we are distinguishing between the two types of operations (binary operation and binary composition), this is not always done in the literature where $S' = S$ may simply be required in the definition of a binary operation.

Commutativity

A binary operation $*$ on a set S is **commutative** if

$$a * b = b * a, \qquad \forall\, a, b \in S. \tag{A.3}$$

In functional notation,

$$f(a,b) = f(b,a). \tag{A.4}$$

Associativity

A binary composition $*$ on a set S is **associative** if

$$(a * b) * c = a * (b * c), \qquad \forall\, a, b, c \in S. \tag{A.5}$$

In functional notation,

$$f(f(a,b),c) = f(a, f(b,c)). \tag{A.6}$$

Semigroup

A **semigroup** is a set endowed with an associative binary composition.

Unit element

An element 1 of a semigroup S is called a **unit element** if

$$1 * s = s * 1 = s, \qquad \forall\, s \in S. \tag{A.7}$$

If $1'$ is likewise a unit element, then $1' = 1' * 1 = 1$. Hence, if a unit element exists, it is unique.

Monoid

A **monoid** is a semigroup containing a unit element.

Inverse

An element m in a monoid M is said to be **invertible** if there exists $m^{-1} \in M$ such that

$$m * m^{-1} = 1 = m^{-1} * m. \tag{A.8}$$

If n likewise satisfies $m * n = 1 = n * m$, then $n = (m^{-1} * m) * n = m^{-1} * (m * n) = m^{-1}$. Hence, if m is invertible, its **inverse** m^{-1} is unique. Furthermore, m is the inverse of m^{-1}, that is, $(m^{-1})^{-1} = m$.

Group

A **group** is a monoid whose elements are all invertible. A commutative group is often said to be **abelian**.

Ring

A **ring** is a set R endowed with two binary compositions, **addition** $+$ and **multiplication** $\cdot$, and containing a distinguished element, $0 \in R$, such that

(1) $(R, +, 0)$ is an abelian group;

(2) $(R, \cdot)$ is a semigroup;

(3) The distributive laws,

$$a \cdot (b + c) = a \cdot b + a \cdot c, \qquad (a + b) \cdot c = a \cdot c + b \cdot c, \tag{A.9}$$

hold for all $a, b, c \in R$.

The element $0 \in R$ is often called the **zero element** of R. A ring R is said to be a **unital ring** if the semigroup $(R, \cdot)$ has a unit element 1, that is, if the semigroup is, in fact, a monoid $(R, \cdot, 1)$. A ring is said to be commutative if the semigroup $(R, \cdot)$ is commutative.

Field

A **field** is a commutative ring in which every nonzero element has a multiplicative inverse. As a ring, a field is thus unital. Important examples of fields are $\mathbb{R}$ and $\mathbb{C}$, the sets of real and complex numbers. In fact, in this course, we will only consider these two fields. It is customary to write $\mathbb{F}^{\times} = \mathbb{F} \backslash \{0\}$ for the set of nonzero elements of the field $\mathbb{F}$.

Vector space

A **vector space** V over the field $\mathbb{F}$ is an abelian group endowed with a (scalar) multiplication,

$$\mathbb{F} \times V \to V, \qquad (a, v) \mapsto av, \tag{A.10}$$

satisfying

$$a(u + v) = au + av, \quad (a + b)v = av + bv, \quad (ab)v = a(bv), \quad a0 = 0, \tag{A.11}$$

for all $a, b \in \mathbb{F}$ and all $u, v \in V$. The group structure of the vector space is with respect to addition, so the zero vector $0 \in V$ is the corresponding unit element $(0 + v = v + 0 = v$ and

$v + (-v) = 0)$.

External direct sum of vector spaces

Let V and W be vector spaces over $\mathbb{F}$. Their **external direct sum** $V \oplus W$ is likewise a vector space over $\mathbb{F}$. The elements of the direct sum are written $v \oplus w$, where $v \in V$ and $w \in W$, and satisfy

$$(v_1 + v_2) \oplus (w_1 + w_2) = (v_1 \oplus w_1) + (v_2 \oplus w_2), \qquad a(v \oplus w) = (av) \oplus (aw), \tag{A.12}$$

for all $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$ and $a \in \mathbb{F}$. Let B_V be a basis for V and B_W for W. A basis for the direct sum $V \oplus W$ is then given by

$$\{v \oplus 0 \,|\, v \in B_V\} \sqcup \{0 \oplus w \,|\, w \in B_W\}. \tag{A.13}$$

For V and W finite-dimensional, it follows that

$$\dim(V \oplus W) = \dim V + \dim W. \tag{A.14}$$

If confusion is unlikely to arise, it is common to simplify the notation and write $v + w$ instead of $v \oplus w$.

Internal direct sum of vector spaces

Let U_1 and U_2 be subspaces of the vector space V. The (vector-space) **sum** of U_1 and U_2 is defined as

$$U_1 + U_2 := \{u_1 + u_2 \,|\, u_1 \in U_1,\ u_2 \in U_2\}, \tag{A.15}$$

where the addition of u_1 and u_2 is carried out in V. Thus, with its operations inherited from V, $U_1 + U_2$ is a subspace of V. If each element $w \in U_1 + U_2$ can be written in a unique way as

$$w = u_1 + u_2, \qquad u_1 \in U_1,\ u_2 \in U_2, \tag{A.16}$$

then the sum $U_1 + U_2$ is said to be **direct**. This is often referred to as an **internal direct sum** and is denoted by $U_1 \oplus U_2$. Note that the sum $U_1 + U_2$ is direct if and only if $U_1 \cap U_2 = \{0\}$, and that (for U_1 and U_2 finite-dimensional)

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2). \tag{A.17}$$

Despite the use of $\oplus$ to indicate both external and internal direct sums, the two notions should not be confused. Notably, one can form the external direct sum of any two vector spaces over the same field, whereas an internal direct sum can only be formed of two subspaces of the same ambient vector space. Both types of sum can thus be formed of a pair of subspaces of the same vector space. In that case, albeit different, the two constructions are isomorphic and often not distinguished.

Tensor product of vector spaces

Let V and W be vector spaces over $\mathbb{F}$. Their **tensor product** $V \otimes W$ is likewise a vector space over $\mathbb{F}$, with elements given by linear combinations of terms of the form $v \otimes w$, where $v \in V$ and $w \in W$, satisfying

$$\begin{aligned} &(v_1 + v_2) \otimes (w_1 + w_2) = v_1 \otimes w_1 + v_1 \otimes w_2 + v_2 \otimes w_1 + v_2 \otimes w_2, \\ &a(v \otimes w) = (av) \otimes w = v \otimes (aw), \end{aligned} \tag{A.18}$$

for all $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$ and $a \in \mathbb{F}$. Let $\{v^1, \ldots, v^{\dim V}\}$ be a basis for V and $\{w^1, \ldots, w^{\dim W}\}$ for W. A basis for the tensor product $V \otimes W$ is then given by

$$\{v^i \otimes w^j \mid i = 1, \ldots, \dim V,\ j = 1, \ldots, \dim W\}. \tag{A.19}$$

It readily follows that the dimension of the tensor product is

$$\dim(V \otimes W) = (\dim V)(\dim W). \tag{A.20}$$

Linear map

Let V and W be vector spaces over the same field $\mathbb{F}$. A function $f : V \to W$ is a **linear map** if

$$f(x + y) = f(x) + f(y), \qquad f(\alpha x) = \alpha f(x), \tag{A.21}$$

for all $x, y \in V$ and all $\alpha \in \mathbb{F}$. In particular, $f(0) = 0$, where the argument is the zero element $0 = 0_V$ of V, while the righthand side is the zero element $0 = 0_W$ of W. The **identity map** defined as

$$\mathrm{id}_V :\ V \to V, \qquad v \mapsto v, \tag{A.22}$$

is an example of a linear map.

Bilinear map

Let V_1, V_2 and W be vector spaces over the same field $\mathbb{F}$. A function $f : V_1 \times V_2 \to W$ is a **bilinear map** if it is linear in each of its two arguments.

Algebra

An **algebra** A is a vector space endowed with a binary composition $*$ which is bilinear and satisfies

$$(ax) * (by) = (ab)(x * y) \tag{A.23}$$

for all $a, b \in \mathbb{F}$ and all $x, y \in A$. We may thus think of an algebra as a ring on which the field acts in a way that is 'compatible' with the multiplication and addition rules of the ring. If the multiplication operation $*$ is associative (or commutative), the algebra is said to be associative (or commutative). If the algebra possesses a (multiplicative) unit element, then the algebra is called unital.

A.2 Some advanced algebraic notions

Characteristic of a ring

The **characteristic of a ring** R is the smallest positive integer n such that

$$\underbrace{a + \cdots + a}_{n \text{ summands}} = 0 \tag{A.24}$$

for some nonzero $a \in R$. If no such integer exists, the ring is said to be of characteristic 0. In a unital ring (with multiplicative unit element 1), the characteristic may equivalently be taken to be the smallest positive integer n such that

$$\underbrace{1 + \cdots + 1}_{n \text{ summands}} = 0. \tag{A.25}$$

Again, if no such integer exists, the ring is said to be of characteristic 0.

CHARACTERISTIC OF A FIELD

As a field is a unital ring, the **characteristic of a field** is the smallest positive integer n such that

$$\underbrace{1 + \cdots + 1}_{n \text{ summands}} = 0, \tag{A.26}$$

where 1 is the multiplicative unit element. If no such integer exists, the field is said to be of characteristic 0. The characteristic of a field is either 0 or a prime number. The fields $\mathbb{R}$ and $\mathbb{C}$ are both of characteristic 0. A field of nonzero characteristic is said to be of **finite characteristic** (or positive characteristic).

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