ALGEBRAIC METHODS OF MATHEMATICAL PHYSICS

MATH3103/7133

Contents

1 Introduction

This set of notes provides a foundation for the topics to be introduced and discussed in the second half of the course. There is some overlap with content from the first half of the course. Reproducing it here allows for these notes to be reasonably self-contained.

One thing that will become apparent is that notational conventions will vary as we proceed through the notes. There are two underlying reasons for this. The first is that notational conventions are not necessarily optimal in all situations. In some instances the convention used will be one that is most beneficial for treating the problem at hand. The second, and more important reason, is that many different conventions exist out in the mathematical literature. Exposing students to this variety of conventions increases their capacity to undertake independent learning activities.

Make special note that

- Generally, Einstein summation convention will be adopted when convenient.
- The word *tensor* will be used in several different contexts.
- In some instances, tensors defined with lower indices will have their inverses expressed with upper indices (and vice versa). For example, using Einstein summation convention
$$A^{jp}A_{pk} = \delta^j{}_k.$$
On the other hand, a delta function will also commonly be represented as δ_{jk}.

Finally, these notes have not been proofread and edited to the same standard of a published textbook. If you do find any errors, please send an email to jrl@maths.uq.edu.au so a correction can be listed on the course Blackboard site.

2 Dynamics of physical systems

2.1 Classical Hamiltonian systems

For the purposes of these notes, a classical Hamiltonian H is a function of many variables where $q_j,\ p_j,\ j = 1, ..., N$ are conjugate *position* and *momentum* variables, and N is the number of *degrees of freedom*. Generally, the Hamiltonian represents the total energy of the system. The dynamical behaviour of the system is governed by Hamilton's equations

$$\frac{\mathrm{d}q_j}{\mathrm{d}t} = \frac{\partial H}{\partial p_j}, \qquad \frac{\mathrm{d}p_j}{\mathrm{d}t} = -\frac{\partial H}{\partial q_j}. \tag{1}$$

The simplest example is given by a mass m on a spring with constant k, in one dimension. The total energy is given by the sum of the kinetic and potential energies

$$\begin{aligned} H &= \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \\ &= \frac{p^2}{2m} + \frac{kx^2}{2} \end{aligned}$$

where the momentum is defined $p = mv$, $v = \mathrm{d}x/\mathrm{d}t$ is the velocity and $x = q$ is the sole co-ordinate. Hamilton's equations give

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\partial H}{\partial p} = \frac{p}{m} = v, \\ \frac{\mathrm{d}p}{\mathrm{d}t} &= -\frac{\partial H}{\partial x} = -kx. \end{aligned}$$

We can combine these two equations to obtain

$$\frac{\mathrm{d}p}{\mathrm{d}t} = m\frac{\mathrm{d}v}{\mathrm{d}t} = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx$$

which is Newton's law for this system.

In the above formulation, for any $F = F(q_j, p_j)$, $G = G(q_j, p_j)$ we define the Poisson bracket by

$$\begin{aligned} \{F,\, G\} &= \sum_{j=1}^{N} \frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j} - \sum_{j=1}^{N} \frac{\partial G}{\partial q_j}\frac{\partial F}{\partial p_j} \\ &= -\{G,\, F\}. \end{aligned} \tag{2}$$

In particular, through Hamilton's equations (1) we find

$$\begin{aligned} \{F,\, H\} &= \sum_{j=1}^{N} \frac{\partial F}{\partial q_j}\frac{\partial H}{\partial p_j} - \sum_{j=1}^{N} \frac{\partial H}{\partial q_j}\frac{\partial F}{\partial p_j} \\ &= \sum_{j=1}^{N} \frac{\partial F}{\partial q_j}\frac{\mathrm{d}q_j}{\mathrm{d}t} + \sum_{j=1}^{N} \frac{\mathrm{d}p_j}{\mathrm{d}t}\frac{\partial F}{\partial p_j} \\ &= \frac{\mathrm{d}F}{\mathrm{d}t}. \end{aligned}$$

It follows from the antisymmetry property (2) that

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0$$

which is a statement of the conservation of energy. More generally if

$$\{F,\, H\} = 0$$

then F is also conserved.

Exercise 1. *Poisson's theorem states that the Poisson bracket of two conserved quantities is also conserved. Prove Poisson's theorem.*

2.2 Two-dimensional classical oscillator

Here we examine the two-dimensional oscillator. A physical realisation of it is given by a mass on an idealised spring (natural length is zero), on a frictionless horizontal table, with the end fixed at the origin. This system has two degrees of freedom. The Hamiltonian is given by

$$\begin{aligned} H &= \frac{\vec{p}.\vec{p}}{2m} + \frac{m\omega^2 \vec{q}.\vec{q}}{2} \\ &= \frac{p_1^2 + p_2^2}{2m} + \frac{m\omega^2(q_1^2 + q_2^2)}{2}. \end{aligned}$$

We may determine the equations of motion from Hamilton's equations which yield

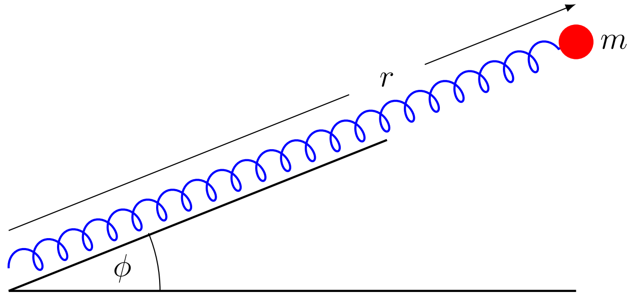
$$\frac{\mathrm{d}q_j}{\mathrm{d}t} = \frac{\partial H}{\partial p_j} = \frac{p_j}{m} \tag{3}$$

$$\frac{\mathrm{d}p_j}{\mathrm{d}t} = -\frac{\partial H}{\partial q_j} = -m\omega^2 q_j \tag{4}$$

for both $j = 1,\, 2$. In this case, the set of four differential equations has decoupled into two sets of two differential equations. It is straightforward to solve the equations, with the general solution

$$\begin{aligned} q_j &= A_j \sin(\omega t + \theta_j), \\ p_j &= m\omega A_j \cos(\omega t + \theta_j) \end{aligned}$$

and the four constants of integration are $A_j,\, \theta_j,\, j = 1,\, 2$.



It is also useful to consider the transformation to polar co-ordinates r and ϕ with conjugate momenta p_r and p_ϕ. The transformation is

$$\begin{aligned}
q_1 &= r\cos(\phi), \\
p_1 &= p_r\cos(\phi) - \frac{p_\phi}{r}\sin(\phi), \\
q_2 &= r\sin(\phi), \\
p_2 &= p_r\sin(\phi) + \frac{p_\phi}{r}\cos(\phi).
\end{aligned}$$

Next, expressing the Hamiltonian in terms of the polar co-ordinates

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) + \frac{m\omega^2 r^2}{2}$$

we can derive the equations of motion

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}, \tag{5}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2}, \tag{6}$$

$$\frac{\mathrm{d}p_r}{\mathrm{d}t} = -\frac{\partial H}{\partial r} = \frac{p_\phi^2}{mr^3} - m\omega^2 r, \tag{7}$$

$$\frac{\mathrm{d}p_\phi}{\mathrm{d}t} = -\frac{\partial H}{\partial \phi} = 0. \tag{8}$$

We immediately see from (8) that p_ϕ is a constant of the motion. Substituting into (6) yields

$$p_\phi = mr^2\frac{\mathrm{d}\phi}{\mathrm{d}t}$$

which is recognisable as the *angular momentum*. Here, we see both energy and angular momentum being conserved. Combining (5) and (7) yields the second order differential equation

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \frac{p_\phi^2}{m^2 r^3} - \omega^2 r.$$

Since p_ϕ is conserved, we again see that the equations of motion have decoupled.

2.3 Quantum Hamiltonian systems

One of the major differences between classical and quantum systems is that for quantum systems the spectra of physical observables, such as the total energy or the angular momentum, is discrete, whereas for classical systems it is continuous. To mathematically describe the quantum analogue of a classical system, we adopt the following prescription:

- The states of the system are elements of a complex Hilbert space (i.e. vector space with positive definite inner product) $\mathcal{H}$ with unit norm with respect to the inner product $\langle\Phi|\Psi\rangle \to \mathbb{C}$, for $|\Phi\rangle$, $|\Psi\rangle \in \mathcal{H}$.

- Physical observables are represented by self-adjoint operators acting on $\mathcal{H}$.

- For conjugate classical variables q and p which satisfy the Poisson bracket

$$\{q,\, p\} = 1,$$

the quantum operators must satisfy the *commutation relation*

$$[q,\, p] = i\hbar I \tag{9}$$

where the commutator is defined as

$$[A,\, B] = AB - BA = -[B,\, A].$$

Above, $\hbar$ is Planck's constant. Mostly, we will set $\hbar = 1$. The general rule in going from a classical system to a quantum analogue is to replace all Poisson brackets with commutators such that

$$\{A,\, B\} \longrightarrow -i[A,\, B]$$

where $i = \sqrt{-1}$. It looks suspicious that a complex number has appeared in the commutation relations, whereas in the Poisson brackets we deal only with real functions. The reason for this is mathematical consistency. For example, consider the operator

$$\mathcal{O} = qp - pq.$$

Taking the adjoint we find

$$\begin{aligned} \mathcal{O}^\dagger &= p^\dagger q^\dagger - q^\dagger p^\dagger \\ &= pq - qp \\ &= -\mathcal{O}, \end{aligned}$$

using the fact that q and p are assumed to be self-adjoint (to ensure that they have real eigenvalues which have physical meaning). We cannot assign $\mathcal{O}$ to be the identity operator I, since I is self-adjoint. The resolution is to assign $\mathcal{O} = iI$.

- The observable spectrum of A is given by the eigenvalues of A, and the observable states are the eigenstates.

- For an observable A let $\{|\psi_j\rangle\}$ denote normalised eigenstates (which provide an orthonormal basis for $\mathcal{H}$) and let $\{\lambda_j\}$ denote the corresponding eigenvalues. Given an arbitrary normalised state $|\Psi\rangle$ we can express it as

$$|\Psi\rangle = \sum_j c_j|\psi_j\rangle, \qquad \sum_j |c_j|^2 = 1.$$

We interpret $\mathfrak{p}_j = |c_j|^2$ as the probability that the measurement represented by A yields the result λ_j. The average, or *expectation value*, of A is given by

$$\begin{aligned}\langle A\rangle &= \sum_j \mathfrak{p}_j \lambda_j \\ &= \sum_{j,k} c_k^* \lambda_j c_j \delta_{j,k} \\ &= \sum_{j,k} c_k^* c_j \langle\psi_k| A |\psi_j\rangle \\ &= \langle\Psi| A |\Psi\rangle .\end{aligned}$$

- The time-evolution of a state, $|\Psi(t)\rangle$, is given by

  $$|\Psi(t)\rangle = \exp(-itH)|\Psi(0)\rangle.$$

  Note that if $|\Psi(0)\rangle$ is an eigenstate of the Hamiltonian H with energy E then

  $$|\Psi(t)\rangle = \exp(-itH)|\Psi(0)\rangle = \exp(-itE)|\Psi(0)\rangle$$

  so the state evolves by a scalar phase factor. It follows that

  $$H|\Psi(t)\rangle = E|\Psi(t)\rangle$$

  for all t so the energy is conserved.

- For a general observable A the time evolution of the expectation value is

  $$\begin{aligned}\langle A(t)\rangle &= \langle\Psi(t)| A |\Psi(t)\rangle \\ &= \langle\Psi| \exp(itH) A \exp(-itH) |\Psi\rangle .\end{aligned}$$

  Then assuming A has no explicit time dependence

  $$\begin{aligned}\frac{\mathrm{d}\langle A(t)\rangle}{\mathrm{d}t} &= i\langle\Psi| \exp(itH)[H,\, A] \exp(-itH) |\Psi\rangle \\ &= i\langle\Psi(t)| [H,\, A] |\Psi(t)\rangle .\end{aligned}$$

  Consequently we define

  $$\frac{\mathrm{d}A}{\mathrm{d}t} = i[H,\, A] \tag{10}$$

  such that

  $$\left\langle \frac{\mathrm{d}A}{\mathrm{d}t} \right\rangle = \frac{\mathrm{d}\langle A\rangle}{\mathrm{d}t}.$$

Throughout we assume that the Hamiltonian H has no explicit time dependence. With this definition we see that the eigenvalues of any observable A are independent of time, since for any eigenstate $|\Phi\rangle$ of A

$$\left\langle \Phi \left| \frac{\mathrm{d}A}{\mathrm{d}t} \right| \Phi \right\rangle = i\langle\Phi| [H,\, A] |\Phi\rangle = 0.$$

One of the consequences of the above formulation for quantum systems is what is known as *Heisenberg's uncertainty principle*. Suppose that we have two observables represented by operators A and B which have the same set of eigenstates Ψ_j, with eigenvalues λ_j^A and λ_j^B respectively. Then we find

$$\begin{aligned}[A,\, B]\,|\Psi_j\rangle &= (AB - BA)\,|\Psi_j\rangle \\ &= (\lambda_j^B A - \lambda_j^A B)\,|\Psi_j\rangle \\ &= (\lambda_j^B \lambda_j^A - \lambda_j^A \lambda_j^B)\,|\Psi_j\rangle \\ &= 0.\end{aligned}$$

Since this is true for all Ψ_j then we must have $[A,\, B] = 0$. Conversely if $[A,\, B] \neq 0$, then there does not exist a set of simultaneous eigenstates. In this latter case, the observables A and B cannot be determined simultaneously, which underlies the uncertainty principle.

An important property to recognise in this correspondence is that both the Poisson bracket and the commutator are *derivations*, i.e.

$$\begin{aligned}\{F,\, GK\} &= \{F,\, G\}K + G\{F,\, K\} \\ [A,\, BC] &= [A,\, B]C + B[A,\, C]\end{aligned}$$

where the terminology *derivation* stems from the similarity to the familiar product rule

$$\frac{\mathrm{d}(fg)}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}t}\, g + f\, \frac{\mathrm{d}g}{\mathrm{d}t}.$$

This result generalises, e.g.

$$[A_1 A_2 \ldots . A_L,\, B] = \sum_{j=1}^{L} A_1 A_2 \ldots A_{j-1} [A_j,\, B] A_{j+1} \ldots A_L. \tag{11}$$

As an example, consider the one-dimensional classical oscillator with the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}.$$

To analyse the quantum analogue, we first need a Hilbert space. Since the Hamiltonian describes a one-dimensional system, we choose the Hilbert space to be the space of one-variable functions f with the inner product

$$\langle f|g\rangle = \int_{-\infty}^{\infty} f^*(x) g(x) dx.$$

Exercise 2. *Show that the mapping*

$$\begin{aligned}q &\mapsto x, \\ p &\mapsto -i\hbar \frac{\mathrm{d}}{\mathrm{d}x}\end{aligned}$$

satisfies the commutation relation (9). Show that under this representation the Hamiltonian for the quantum oscillator is

$$H = -\frac{\hbar^2}{2m} \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{m\omega^2 x^2}{2}.$$

Thus the problem of determining the energy spectrum of the system is transformed to solving the eigenvalue problem

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + \frac{m\omega^2x^2}{2}\Psi = E\Psi.$$

In general, an equation of the form

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + V(x)\Psi = E\Psi$$

is known as a one-dimensional Schrödinger equation with potential $V(x)$.

Exercise 3. *Using (10), calculate expressions for* $\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}p}{\mathrm{d}t}$ *and compare them with the analogous expressions for the classical oscillator derived through Hamilton's equations.*

2.4 One-dimensional Schrödinger equation

Hereafter we set $\hbar = 1$ and $m = 1/2$, so the one-dimensional Schrödinger equation is of the form

$$-\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + V(x)\Psi = E\Psi. \tag{12}$$

Set

$$Q(u) = \prod_{j=1}^{M}(u - v_j) \tag{13}$$

and look for solutions of the form

$$\Psi = e^{v(x)}Q(u(x)). \tag{14}$$

Now

$$\begin{aligned}
\frac{\mathrm{d}\Psi}{\mathrm{d}x} &= \frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{e}^vQ + e^v\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}Q}{\mathrm{d}u}\\
\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} &= \left(\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2\right)\mathrm{e}^vQ + \left(2\frac{\mathrm{d}u}{\mathrm{d}x}.\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right)\mathrm{e}^v\frac{\mathrm{d}Q}{\mathrm{d}u}\\
&\quad + \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2\mathrm{e}^v\frac{\mathrm{d}^2Q}{\mathrm{d}u^2}.
\end{aligned}$$

Hence a solution of the following differential equation

$$-\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2\frac{\mathrm{d}^2Q}{\mathrm{d}u^2} - \left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + 2\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x}\right)\frac{\mathrm{d}Q}{\mathrm{d}u} + \left(V - \frac{\mathrm{d}^2v}{\mathrm{d}x^2} - \left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2\right)Q = EQ \tag{15}$$

where Q is a polynomial in u gives a solution of (12). Note that equation (15) must be regarded as an equation in the variable u, so expressions such as $\frac{\mathrm{d}v}{\mathrm{d}x}$ must be written in terms of u. So we manipulate (15) further to

$$-\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2\frac{\mathrm{d}^2Q}{\mathrm{d}u^2}-\left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+2\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2\frac{\mathrm{d}v}{\mathrm{d}u}\right)\frac{\mathrm{d}Q}{\mathrm{d}u}$$
$$+\left(V-\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2\frac{\mathrm{d}^2v}{\mathrm{d}u^2}-\frac{\mathrm{d}v}{\mathrm{d}u}\frac{\mathrm{d}^2u}{\mathrm{d}x^2}-\left(\frac{\mathrm{d}v}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2\right)Q=EQ \tag{16}$$

This has the general form

$$A(u)Q''(u)+B(u)Q'(u)+C(u)Q(u)=EQ(u) \tag{17}$$

where Q is given by (13) is polynomial.

2.5 One-dimensional quantum oscillator

For the one-dimensional quantum oscillator we take the potential

$$V(x)=\frac{\omega^2x^2}{4}$$

and choose

$$u(x)=x, \qquad v(x)=-\frac{\omega x^2}{4}$$

in (14). This leads us to the differential equation

$$-Q''+\omega uQ'+\frac{\omega}{2}Q=EQ. \tag{18}$$

Since Q is a polynomial function of u, say of order n, to leading order we set

$$Q\sim u^n$$

and equate the terms of order n in (18). This directly gives the energy eigenvalues as

$$E_n=\omega\left(n+\frac{1}{2}\right). \tag{19}$$

Thus the energy levels are $\omega/2$, $3\omega/2$, $5\omega/2\ldots$. Note that the ground state energy for the classical model is zero, which is not the case here. This difference is a consequence of the uncertainty principle.

2.6 Lie algebras

Algebraic methods are often very powerful in the studies of quantum systems. We start by introducing the concept of a *Lie algebra*, and discuss a first example that will play a central role throughout these notes.

An algebra is a vector space V endowed with a multiplication map $m : V\times V\to V$. Below, it is assumed that all vector spaces are over the field $\mathbb{C}$. A *Lie algebra* L has a multiplication known as the bracket, denoted by $[\ ,\ ]$. It satisfies the following three properties. For all $x, y, z \in L$ and for all α, $\beta\in\mathbb{C}$:

- Antisymmetry. $[x,\, y] = -[y,\, x]$;
- Bilinearity. It is sufficient that $[x,\, \alpha y + \beta z] = \alpha[x,\, y] + \beta[x,\, z]$ holds;
- Jacobi identity. $[x,\, [y,\, z]] + [y,\, [z,\, x]] + [z,\, [x,\, y]] = 0$.

Throughout we will discuss *complex Lie algebras*, defined over the field $\mathbb{C}$. Notationally, we drop the dependence on the field so these will be denoted $gl(n)$, $o(n)$, $sp(2m)$ etc. Let L be a finite-dimensional Lie algebra, which as a vector space has basis $\{x_1, \ldots, x_n\}$. We can then write, using the Einstein summation convention:

$$[x_i, x_j] = C_{ij}^k x_k \tag{20}$$

where the $C_{ij}^k \in \mathbb{C}$ are called the *structure constants* of L. The defining properties imply constraints:

- $C_{ij}^k = -C_{ji}^k$
- $C_{jk}^l C_{il}^m + C_{ij}^l C_{kl}^m + C_{ki}^l C_{jl}^m = 0$.

Let K be a vector subspace of a Lie algebra L (viewed as a vector space). Recall the following definitions (generally for any type of algebra, we use these):

- K is a *subalgebra* of L if $[K, K] \subseteq K$, that is K is closed under $[\ ,\ ]$, viz. $\forall x \in K, [x, K] \subseteq K$.
- K is an *ideal* (or *invariant subalgebra* of L) if $[L, K] \subseteq K$, that is $\forall x \in L, [x, K] \subseteq K$.

Not all subalgebras are ideals. As $[\ ,\ ]$ is antisymmetric, ideals are two-sided. Note:

- (0) and L are ideals.
- The *commutator subalgebra* $[L, L]$ is an ideal. (This is also known as the *derived* subalgebra.) If $[L, L] = (0)$, then L is *abelian*.
- If the only ideals of a non-abelian Lie algebra are (0) and L, then L is said to be a *simple* Lie algebra (L has no proper factor algebras).

Exercise 4. *Prove that every 2-dimensional Lie algebra is not simple.*

A *Lie algebra homomorphism* between Lie algebras L and L' is a vector space mapping $\phi : L \to L'$ that preserves the bracket operation:

$$\phi([x, y]) = [\phi(x), \phi(y)]. \tag{21}$$

As for groups, we define the kernel and image of ϕ, with the comments that whilst $\ker(\phi)$ is an *ideal* of L, $\mathrm{im}(\phi)$ is only a *subalgebra* of L'.

There are several fundamentally important Lie algebras that will be examined in detail in these notes. One is the *Heisenberg algebra* (also known by other names), denoted here as as $h(1)$. This algebra has basis $\{a, b, c\}$ subject to the commutation relations

$$[a,\, b] = c, \quad [a,\, c] = [b,\, c] = 0. \tag{22}$$

This algebra is not simple. It contains a one-dimensional ideal with basis $\{c\}$. It is commonplace, for reasons which will arise later, to impose further restrictions on this algebra by identifying a as the *conjugate* of b, that is $a = b^\dagger$, and identifying $c = -I$ where I is the identity operator in the universal enveloping algebra satisfying $I^\dagger = I$. With these conditions the commutation relations are

$$[b, b^\dagger] = I, \quad [b, I] = [b^\dagger, I] = 0 \tag{23}$$

and satisfy

$$[x, y]^\dagger = [y^\dagger, x^\dagger] \tag{24}$$

for all $x, y \in h(1)$. The relation (24) shows that conjugation satisfies the defining property of an *anti-automorphism*.

2.7 One-dimensional quantum oscillator revisited

Let's re-examine the Hamiltonian for the one-dimensional quantum oscillator

$$H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}.$$

We will set $\hbar = 1$. Putting

$$\begin{aligned} b &= \sqrt{\frac{m\omega}{2}}q + i\sqrt{\frac{1}{2m\omega}}p, \\ b^\dagger &= \sqrt{\frac{m\omega}{2}}q - i\sqrt{\frac{1}{2m\omega}}p, \end{aligned}$$

then

$$\begin{aligned} H &= \omega\left(b^\dagger b + \frac{1}{2}I\right) \\ &= \omega\left(N + \frac{1}{2}I\right) \end{aligned}$$

where $N = b^\dagger b$. Note that for any non-zero state $|\Psi\rangle$, and setting $|\Phi\rangle = b\,|\Psi\rangle$, we have

$$\begin{aligned} \langle\Psi|\,H\,|\Psi\rangle &= \omega\left(\langle\Psi|\,b^\dagger b\,|\Psi\rangle + \frac{1}{2}\langle\Psi|\Psi\rangle\right) \\ &= \omega\left(\langle\Phi|\Phi\rangle + \frac{1}{2}\langle\Psi|\Psi\rangle\right) \\ &> 0. \end{aligned}$$

It follows that the spectrum of H is positive.

Exercise 5. *Using the canonical relation $[q,\, p] = iI$ show that $[b,\, b^\dagger] = I$. Also show by induction, and using the derivation property (11), that for $k \in \mathbb{N}$,*

$$[b,\, (b^\dagger)^k] = k(b^\dagger)^{k-1}.$$

We introduce the vacuum state $|0\rangle$, assumed to be normalised, satisfying

$$b\,|0\rangle = 0.$$

(**Nota Bene:** The right-hand side above denotes the zero vector. The zero vector is not $|0\rangle$.) Now construct the set of states

$$|n\rangle = \frac{1}{\sqrt{n!}}(b^\dagger)^n\,|0\rangle$$

such that

$$\begin{aligned}
N\,|n\rangle &= b^\dagger b\,|n\rangle \\
&= \frac{1}{\sqrt{n!}} b^\dagger b (b^\dagger)^n\,|0\rangle \\
&= \frac{1}{\sqrt{n!}} b^\dagger \left((b^\dagger)^n b + n(b^\dagger)^{n-1}\right)|0\rangle \\
&= \frac{1}{\sqrt{n!}} n (b^\dagger)^n\,|0\rangle \\
&= n\,|n\rangle\,.
\end{aligned}$$

The states $|n\rangle$ are normalised. To verify this, consider

$$\begin{aligned}
b^\dagger\,|n-1\rangle &= \frac{1}{\sqrt{(n-1)!}} b^\dagger (b^\dagger)^{n-1}\,|0\rangle \\
&= \frac{1}{\sqrt{(n-1)!}} (b^\dagger)^n\,|0\rangle \\
&= \sqrt{n}\,|n\rangle\,,
\end{aligned}$$

or rather

$$|n\rangle = \frac{1}{\sqrt{n}} b^\dagger\,|n-1\rangle\,.$$

Now

$$\begin{aligned}
\langle n|n\rangle &= \frac{1}{n}\left\langle n-1|bb^\dagger|n-1\right\rangle \\
&= \frac{1}{n}\langle n-1|(I+N)|n-1\rangle \\
&= \langle n-1|n-1\rangle\,.
\end{aligned}$$

Assuming $|0\rangle$ to be normalised this implies, by induction, that all states are normalised.

The above means that the action of the Hamiltonian can be determined as

$$H\,|n\rangle = \omega(n+1/2)\,|n\rangle\,,$$

in agreement with (19). In deriving the above the existence of a vacuum state was assumed. If such a state did not exist then the energies would be unbounded from below, since it can be shown that the action of the operator b on an eigenstate gives an eigenstate with lower energy eigenvalue. However, we have already established that the spectrum of H is positive, so the vacuum state must exist.

Exercise 6. *Perhaps the above construction does not account for all states. Assume that there exists a state $|\Theta\rangle$ which has properties*

$$\langle\Theta|\Theta\rangle = 1, \qquad\qquad b^\dagger|\Theta\rangle = 0.$$

Give an argument as to why the state $|\Theta\rangle$ does not exist.

The infinite-dimensional vector space

$$\mathcal{F} = \mathrm{span}\{|0\rangle,\ |1\rangle,\ |2\rangle, ...\} \tag{25}$$

is known as *Fock space*.

3 Lie algebraic methods

Another important example of a Lie algebra is the smallest, non-zero dimensional case that is simple. This is commonly referred to as $su(2)$, $so(3)$, $o(3)$, $sl(2)$, less commonly as $sp(1)$, $sp(2)$, $u(2)$, $su(1,1)$, and possibly other names. This Lie algebra has basis $\{e, f, h\}$ subject to the commutation relations

$$[h,\, e] = e, \quad [h,\, f] = -f, \quad [e,\, f] = h. \tag{26}$$

Note that a straightforward redefinition of the elements as $E = e$, $F = -f$ and $H = h$ leads to the commutation relations

$$[H,\, E] = E, \quad [H,\, F] = -F, \quad [E,\, F] = -H. \tag{27}$$

The commutation relations (26) are typically associated with $su(2)$, while (27) are typically associated with $su(1,1)$. This means that, as Lie algebras, they are isomorphic. The distinction comes with the consideration of conjugation. It is easily verified that

$$e^\dagger = f, \quad f^\dagger = e, \quad h^\dagger = h \qquad \Longleftrightarrow \qquad E^\dagger = -F, \quad F^\dagger = -E, \quad H^\dagger = H \tag{28}$$

is an anti-automorphism of (26), but not of (27). Similarly,

$$E^\dagger = F, \quad F^\dagger = E, \quad H^\dagger = H \qquad \Longleftrightarrow \qquad e^\dagger = -f, \quad f^\dagger = -e, \quad h^\dagger = h \tag{29}$$

is an anti-automorphism of (27), but not of (26). This difference does become significant in the construction of *representations* of Lie algebras, which is discussed later.

Exercise 7. *Note that in the abstract definition of a Lie algebra the commutator $[x,\, y]$ is not the same as $xy - yx$. For example, consider the vector space $\mathbb{R}^3$ which becomes an algebra under the multiplication given by the cross product. We already know the cross product is bilinear and antisymmetric. Show the cross product also satisfies the Jacobi identity; i.e.*

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0$$

establishing that $(\mathbb{R}^3, \times)$ is a Lie algebra.

Exercise 8. *Let $M_{n\times n}(\mathbb{R})$ denote the vector space of real $n \times n$. For a fixed $f = (f_{ij}) \in M_{n\times n}(\mathbb{R})$ let*

$$L_f = \{x \in M_{n\times n}(\mathbb{R}) \mid x^T f + fx = 0\}.$$

Show that L_f is a Lie algebra by establishing that it is a vector space and that it is closed under the matrix commutator $[x,\, y] = xy - yx$.

3.1 General linear and special linear Lie algebras

We define $gl(n)$ to be the n^2-dimensional complex Lie algebra with basis

$$\{a^i{}_j;\ i, j = 1, \ldots, n\} \tag{30}$$

satisfying the $gl(n, \mathbb{C})$ *commutation relations*:

$$[a^i{}_j, a^k{}_l] = \delta^k{}_j a^i{}_l - \delta^i{}_l a^k{}_j. \tag{31}$$

The generator $I_1 = \displaystyle\sum_{k=1}^{n} a^k{}_k$ commutes with all the $a^i{}_j$, viz.: $[I_1, a^i{}_j] = 0$. It is an example of a *Casimir invariant*, which generates an ideal. A convenient basis for $sl(n)$ is given by the $n^2 - 1$ linearly independent operators:

$$\left\{a^i{}_j - \frac{1}{n}\delta_{ij} I_1 :\ i, j = 1, \ldots, n,\ \text{excluding } i = j = n\right\}.$$

Another basis is:

$$\{a^i{}_j :\ i, j = 1, \ldots, n,\ i \neq j\} \cup \{a^i{}_i - a^{i+1}{}_{i+1} :\ i = 1, \ldots, n-1\}. \tag{32}$$

We have an ideal direct sum: $gl(n) = sl(n) \oplus \mathbb{C} I_1$.

3.2 Orthogonal Lie algebras

The *orthogonal Lie algebra* $o(n)$ can be viewed as a subset $o(n) \subset gl(n)$. It is spanned by the $\dfrac{1}{2}n(n-1)$ generators $\alpha^i{}_j = -\alpha^j{}_i$, defined in terms of the $gl(n)$ generators by:

$$\alpha^i{}_j = a^i{}_j - a^j{}_i,\ i, j = 1, \ldots, n. \tag{33}$$

The generators satisfy the $o(n)$ *commutation relations*:

$$[\alpha^i{}_j, \alpha^k{}_l] = (\delta^k{}_j\alpha^i{}_l - \delta^i{}_l\alpha^k{}_j) - (\delta^k{}_i\alpha^j{}_l - \delta^j{}_l\alpha^k{}_i). \tag{34}$$

These may be verified by substitution of (31) into (33), viz. (using $\alpha^i{}_j = -\alpha^j{}_i$):

$$\begin{aligned}
[\alpha^i{}_j, \alpha^k{}_l] &= [a^i{}_j - a^j{}_i, a^k{}_l - a^l{}_k] = [a^i{}_j, a^k{}_l] - [a^i{}_j, a^l{}_k] - [a^j{}_i, a^k{}_l] + [a^j{}_i, a^l{}_k] \\
&= (\delta^k{}_j a^i{}_l - \delta^i{}_l a^k{}_j) - (\delta^l{}_j a^i{}_k - \delta^i{}_k a^l{}_j) - (\delta^k{}_i a^j{}_l - \delta^j{}_l a^k{}_i) + (\delta^l{}_i a^j{}_k - \delta^j{}_k a^l{}_i) \\
&= \delta^k{}_j(a^i{}_l - a^l{}_i) - \delta^i{}_l(a^k{}_j - a^j{}_k) - \delta^k{}_i(a^j{}_l - a^l{}_j) + \delta^j{}_l(a^k{}_i - a^i{}_k) \\
&= (\delta^k{}_j\alpha^i{}_l - \delta^i{}_l\alpha^k{}_j) - (\delta^k{}_i\alpha^j{}_l - \delta^j{}_l\alpha^k{}_i).
\end{aligned}$$

3.3 Symplectic Lie algebras

For the symplectic Lie algebras $sp(n = 2m)$, a convenient basis for the $\frac{1}{2}n(n+1) = m(2m+1)$ symmetric generators $\alpha^i_j = \alpha^j_i$ is given by:

$$\{\alpha^i{}_j = f^{ik}a^k{}_j + f^{jk}a^k{}_i \mid i, j = 1, \ldots, m\}. \tag{35}$$

where f is antisymmetric, i.e. $f^{ij} = -f^{ji}$. These satisfy the $sp(m = 2p)$ *commutation relations*:

$$[\alpha^i{}_j, \alpha^k{}_l] = (f^{kj}\alpha^i{}_l - f^{il}\alpha^k{}_j) - (f^{jl}\alpha^k{}_i - f^{ki}\alpha^j{}_l). \tag{36}$$

Again, these may be verified by substitution of (31) into (35) and by implicitly summing over dummy indices m, n:

$$\begin{aligned}
[\alpha^i{}_j, \alpha^k{}_l] &= [f^{im}a^m{}_j + f^{jm}a^m{}_i, f^{kn}a^n{}_l + f^{ln}a^n{}_k] \\
&= f^{im}f^{kn}[a^m{}_j, a^n{}_l] + f^{im}f^{ln}[a^m{}_j, a^n{}_k] + f^{jm}f^{kn}[a^m{}_i, a^n{}_l] + f^{jm}f^{ln}[a^m{}_i, a^n{}_k] \\
&= f^{im}f^{kn}(\delta_{nj}a^m{}_l - \delta_{ml}a^n{}_j) + f^{im}f^{ln}(\delta_{nj}a^m{}_k - \delta_{mk}a^n{}_j) \\
&\quad + f^{jm}f^{kn}(\delta_{ni}a^m{}_l - \delta_{ml}a^n{}_i) + f^{jm}f^{ln}(\delta_{ni}a^m{}_k - \delta_{mk}a^n{}_i) \\
&= f^{im}f^{kj}a^m{}_l - f^{il}f^{kn}a^n{}_j + f^{im}f^{lj}a^m{}_k - f^{ik}f^{ln}a^n{}_j \\
&\quad + f^{jm}f^{ki}a^m{}_l - f^{jl}f^{kn}a^n{}_i + f^{jm}f^{li}a^m{}_k - f^{jk}f^{ln}a^n{}_i \\
&= f^{jk}(f^{mi}a^m{}_l + f^{nl}a^n{}_i) - f^{li}(f^{nk}a^n{}_j + f^{mj}a^m{}_k) \\
&\quad - f^{ki}(f^{mj}a^m{}_l + f^{nl}a^n{}_j) + f^{jl}(f^{nk}a^n{}_i + f^{mi}a^m{}_k) \\
&= -f^{jk}(f^{im}a^m{}_l + f^{ln}a^n{}_i) + f^{li}(f^{kn}a^n{}_j + f^{jm}a^m{}_k) \\
&\quad + f^{ki}(f^{jm}a^m{}_l + f^{ln}a^n{}_j) - f^{jl}(f^{kn}a^n{}_i + f^{im}a^m{}_k) \\
&= (f^{li}\alpha^k{}_j - f^{jk}\alpha^i{}_l) - (f^{jl}\alpha^k{}_i - f^{ki}\alpha^j{}_l) \\
&= (f^{kj}\alpha^i{}_l - f^{il}\alpha^k{}_j) - (f^{jl}\alpha^k{}_i - f^{ki}\alpha^j{}_l).
\end{aligned}$$

3.4 Unitary Lie algebras

The unitary Lie algebra $u(n)$ is spanned by the n^2 generators:

$$\{\alpha^j{}_k = a^j{}_k - a^k{}_j, \quad \beta^j{}_k = i(a^j{}_k + a^k{}_j); \qquad j, k = 1, \ldots, n\}. \tag{37}$$

3.5 Some observations

Note that over $\mathbb{C}$, $u(n)$ is isomorphic to $gl(n)$ as we can write

$$a^j{}_k = \frac{1}{2}\left(\alpha^j{}_k - i\beta^j{}_k\right).$$

However $u(n)$ and $gl(n)$ are not isomorphic over $\mathbb{R}$.

In general, the orthogonal, symplectic and special linear Lie algebras are non-isomorphic, however examples of Lie algebra isomorphisms include $o(3) \sim su(2) \sim sp(2) \sim sl(2)$, $o(2) \sim u(1)$, and the less trivial $sp(4) \sim o(5)$.

- Where Lie algebra elements are represented by matrices, $gl(2)$ is spanned by the four 2×2 matrices $e_{11}, e_{21}, e_{12}, e_{22}$. This space is four dimensional. Observe that $I_1 = e_{11}+e_{22}$ spans a one dimensional ideal, so that $gl(2)$ is not simple. However, the subspace $sl(2)$ (traceless matrices) is simple, and it has a basis $\left\{\frac{1}{2}(e_{11} - e_{22}), e_{21}, e_{12}\right\}$.

- $sl(n)$ and $su(n)$ are simple subalgebras of $gl(n)$ and $u(n)$ respectively.

- For $o(3)$, the generators can be written as (Einstein summation convention used) $L_i = \frac{1}{2}\varepsilon_{ijk}\alpha^j{}_k$ where $i = 1, 2, 3$, satisfying $[L_i, L_j] = \varepsilon_{ijk}L_k$. For physical applications it is typical to define $L_i = \frac{i}{2}\varepsilon_{ijk}\alpha^j{}_k$ and obtain instead $[L_i, L_j] = i\varepsilon_{ijk}L_k$.

- $o(n)$ is simple for all $n \neq 4$. Indeed $o(4)$ is spanned by the six antisymmetric matrices $\{\alpha_{ij} = e_{ij} - e_{ji},\ i, j \in \{1, 2, 3, 4\},\ i < j\}$:

$$\alpha_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ etc.} \tag{38}$$

  An alternative basis is given by the six matrices $L_i^{(\pm)} = \frac{i}{2}(b_i \pm a_i)$ where we have $\{a_i = \alpha^i{}_4,\ b_i = \frac{1}{2}\varepsilon_{ijk}\alpha^j{}_k : i = 1, 2, 3\}$. These close to form two Lie algebras $L^{(+)}$ and $L^{(-)}$, each isomorphic to $o(3)$ such that $[L^{(+)}, L^{(-)}] = (0)$. Explicitly, with $\{i,\ j,\ k\}$ an even permutation of $\{1,\ 2,\ 3\}$ and $\mu,\ \nu \in \{\pm 1\}$ we have the commutator

$$\begin{aligned}
[L_i^{(\mu)},\ L_j^{(\nu)}] &= -\frac{1}{4}[\alpha^j{}_k + \mu\alpha^i{}_4,\ \alpha^k{}_i + \nu\alpha^j{}_4] \\
&= -\frac{1}{4}[e_{jk} - e_{kj} + \mu(e_{i4} - e_{4i}),\ e_{ki} - e_{ik} + \nu(e_{j4} - e_{4j})] \\
&= -\frac{1}{2}(1 + \mu\nu)\alpha^j{}_i - \frac{1}{2}(\mu + \nu)\alpha^k{}_4 \\
&= i\delta_{\mu\nu}L_k^{(\mu)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}[L_i^{(+)},\, L_i^{(-)}] &= \frac{1}{4}[\alpha^j{}_k + \alpha^i{}_4,\, \alpha^j{}_k - \alpha^i{}_4] \\ &= \frac{1}{4}[e_{jk} - e_{kj} + e_{i4} - e_{4i},\, e_{jk} - e_{kj} - e_{i4} + e_{4i}] \\ &= 0.\end{aligned}$$

Thus $L^{\pm}$ form two non-zero ideals in $o(4)$, and $o(4)$ is the direct sum of two commuting $o(3)$ Lie algebras:

$$o(4) \sim o(3) \oplus o(3). \tag{39}$$

- $sp(2m)$ is simple.
- *Lorentz Lie algebra* $o(3,1)$. The Minkowski four-dimensional space-time metric is:

$$f = \begin{pmatrix} I_3 & 0 \\ 0 & -1 \end{pmatrix} \tag{40}$$

which defines spacetime length $x^2+y^2+z^2-t^2$ (in units where $c \equiv 1$). The Lorentz Lie algebra $o(3,1)$ is spanned by the operators:

$$\{\beta^\mu{}_\nu = f^{\mu\sigma}e_{\sigma\nu} - f^{\nu\sigma}e^{\sigma\mu};\ \mu,\nu = 1,\ldots,4\}.$$

We thus have the $o(3)$ subalgebra with basis:

$$\{\alpha^i{}_j = e_{ij} - e_{ji} :\ i,j = 1,2,3\} \tag{41}$$

generating spatial rotations together with the three space time rotation generators:

$$\{\alpha^i{}_4 = e_{i4} + e_{4i} :\ i = 1,2,3\}. \tag{42}$$

Exercise 9. *So far we have only considered finite-dimensional Lie algebras. However infinite-dimensional Lie algebras also exist. Show that the differential operators*

$$\tilde{L}_n = z^{n+1}\frac{\mathrm{d}}{\mathrm{d}z}, \qquad n \in \mathbb{Z}$$

close to form a Lie algebra, which is known as the Witt algebra*. For each $n \neq 0$, show that L_0, $L_{\pm n}$ close to form a subalgebra.*

3.6 Representations of Lie algebras

Let V be an arbitrary complex vector space, which in general may be infinite-dimensional. The set of linear transformations on V is denoted $\mathrm{End}(V)$. Endowed with a bracket multiplication, specifically the usual commutator bracket for linear transformations:

$$[X,Y] = XY - YX,$$

End(V) satisfies the definition of a Lie algebra. More generally, for complex vector spaces V, W, the vector space of linear transformations from V to W is denoted Hom(V, W). Thus, in the above notation, End(V) = Hom(V, V). Equipped with an appropriate bracket, Hom(V, W) also satisfies the defining properties of a Lie algebra. If V, W are finite-dimensional, say of dimensions m and n respectively, then:

$$\dim[\mathrm{Hom}(V, W)] = nm.$$

A *representation* of a Lie algebra L on a vector space V is a Lie algebra homomorphism $\pi : L \to \mathrm{End}(V)$. A homomorphism necessarily preserves the bracket operation, viz:

$$\pi([a, b]) = [\pi(a), \pi(b)] = \pi(a)\pi(b) - \pi(b)\pi(a).$$

(The bracket on the left is the commutator in L, that on the right is the commutator in End(V).) We call V the *representation space* of π and conversely π is called the *representation afforded by* V.

In all cases, as a result of the Jacobi identity the Lie algebra L *itself* admits an *adjoint* representation, $\mathrm{ad} : L \to L$, defined for $x, y \in L$ by

$$\mathrm{ad}(x) \circ y = [x, y].$$

Explicitly, if L has a basis $\{x_1, \ldots, x_n\}$, and $[x_i, x_j] = \sum_{k=1}^{n} C_{ij}^k x_k$, then the matrix elements of $\mathrm{ad}(x_i)$ are given by $[\mathrm{ad}(x_i)]_{kj} = C_{ij}^k$. To see this

$$\sum_{k=1}^{n} C_{ij}^k x_k = \mathrm{ad}(x_i) \circ x_j = \sum_{k=1}^{n} [\mathrm{ad}(x_i)]_{kj} x_k.$$

Exercise 10. *Determine the adjoint representation for $h(1)$, with commutation relations (22), and for $su(2)$, with commutation relations (26).*

Exercise 11. *Consider n independent variables x_i, and their corresponding partial derivatives. Show that*

$$a^i{}_j = x_i \frac{\partial}{\partial x_j} \tag{43}$$

satisfy the $gl(n)$ commutation relations.

Exercise 12. *Let L be a Lie algebra with basis $\{x_1, ..., x_n\}$ and commutation relations*

$$[x_i,\, x_j] = C_{ij}^k x_k. \tag{44}$$

Show that the elements

$$X_i = \sum_{j,k} C_{ji}^k a^j{}_k$$

also satisfy the commutation relations (44) with the a_{ij} the usual basis elements for $gl(n)$ satisfying

$$[a^i{}_j,\, a^k{}_l] = \delta_j^k a^i{}_l - \delta_l^i a^k{}_j.$$

This result is known as Ado's theorem.

3.7 Unitary representations

Let L denote a Lie algebra with a conjugation $\dagger : L \to L$ satisfying the anti-automorphism property (24). A representation π of L is said to be *unitary* if, for all $x \in L$,

$$\pi(x^\dagger) = \overline{\pi^T(x)} \equiv \pi^\dagger(x)$$

where T denotes matrix transposition, and the overline denotes complex conjugation. That is, $\dagger$ on the right denotes *Hermitian conjugation.*

Sometimes a representation may not be unitary for a given basis, but is equivalent to a unitary representation by an appropriately chosen basis.

Exercise 13. *Verify that the adjoint representation for $su(2)$, with commutation relations (26) and conjugation (28), is unitary for suitable choice of basis. Determine the adjoint representation for $su(1,1)$, with commutation relations (27) and conjugation (29). Show that there does not exist a basis for which this representation is unitary.*

3.8 Lie algebra modules

Instead of considering representations, we may instead talk about modules. For a Lie algebra L, consider a complex vector space V endowed with an action $\circ : L \times V \to V$, which, for $x \in L, v \in V$ we denote by:

$$(x, v) \mapsto x \circ v.$$

Then V is called an L-module if the action has the following properties for all x, $y \in L$, v, $w \in V$ and α, $\beta \in \mathbb{C}$

- Linearity:

$$\begin{aligned} x \circ (\alpha v + \beta w) &= \alpha(x \circ v) + \beta(x \circ w), \\ (\alpha x + \beta y) \circ v &= \alpha(x \circ v) + \beta(y \circ v). \end{aligned}$$

- Homomorphism: $[x, y] \circ v = x \circ (y \circ v) - y \circ (x \circ v)$.

If $\pi : L \to \mathrm{End}(V)$ is a representation of L on a vector space V, then V becomes an L-module with action defined for $x \in L, v \in V$ by:

$$x \circ v \equiv \pi(x)v.$$

Conversely, if V is an L-module, then V determines a representation $\pi : L \to \mathrm{End}(V)$ defined by:

$$\pi(x)v \equiv x \circ v.$$

As the descriptions are equivalent, either may be chosen according to convenience. It is commonplace to drop the notations π and $\circ$ and simply write vx.

3.9 Tensor products of vector spaces

Let V, W be finite-dimensional vector spaces. The *tensor product* space $V \otimes W$ is formally defined as the vector space spanned by the vectors:

$$v \otimes w, \qquad v \in V,\ w \in W$$

where the tensor product operation $\otimes : (v, w) \mapsto v \otimes w$ satisfies the following bilinearity requirements for all $\alpha, \beta \in \mathbb{C}$, $v, v' \in V$ and $w, w' \in W$

- $(\alpha v + \beta v') \otimes w = \alpha(v \otimes w) + \beta(v' \otimes w)$;
- $v \otimes (\alpha w + \beta w') = \alpha(v \otimes w) + \beta(v \otimes w')$.

Note that these imply that $\alpha(v \otimes w) = (\alpha v) \otimes w = v \otimes (\alpha w)$. Now, let V, W have bases $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$ respectively. Then $V \otimes W$ is the mn-dimensional vector space with basis

$$\{v_i \otimes w_j \mid i = 1, \ldots, m;\ j = 1, \ldots, n\}. \tag{45}$$

Indeed, if $v = \sum_{i=1}^{m} \alpha_i v_i \in V$ and $w = \sum_{j=1}^{N} \beta_j w_j \in W$, then $v \otimes w = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_i \beta_j (v_i \otimes w_j)$.

If $A \in \mathrm{End}(V)$ and $B \in \mathrm{End}(W)$, define $A \otimes B \in \mathrm{End}(V \otimes W)$ by the action

$$(A \otimes B)(v \otimes w) = (Av) \otimes (Bw).$$

Let $[A_{ij}]$ be the matrix of A in the basis $\{v_i\}_{i=1}^m$ of V and $[B_{ij}]$ be the matrix of B in the basis $\{w_i\}_{i=1}^n$ of W, then

$$Av_i = \sum_{k=1}^{m} A_{ki} v_k, \qquad Bw_j = \sum_{q=1}^{n} B_{qj} w_q.$$

The matrix of $A \otimes B$ in the tensor product basis (45) of $V \otimes W$ is then found by inspection of

$$(A \otimes B)(v_i \otimes w_j) = \sum_{k=1}^{m} \sum_{q=1}^{n} A_{ki} B_{qj} (v_k \otimes w_q),$$

viz

$$(A \otimes B)_{kq,ij} = A_{ki} B_{qj}.$$

This is called the *tensor product of the matrices* A and B.

It can be shown that $\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\mathrm{tr}(B)$, and $\det(A \otimes B) = \det(A^n)\det(B^m)$.

3.10 Tensor products of modules

If V and W are L-modules, the tensor product $V \otimes W$ is an L-module with the definition

$$x(v \otimes w) = (xv) \otimes w + v \otimes (xw) \tag{46}$$

for $x \in L$, $v \in V$ and $w \in W$. From Exercise 14, the left side of (46) may be expressed as $\Delta(x)(v \otimes w)$.

In representation theoretic terms, where π_V and π_W are the representations afforded by V and W respectively, the *tensor product of the representations* π_V and π_W afforded by $V \otimes W$ is $\pi_{V\otimes W}$ is defined by

$$\pi_{V\otimes W}(x) = \pi_V(x) \otimes I + I \otimes \pi_W(x).$$

More generally, if $V_1, V_2, \ldots, V_k$ are L-modules we may define the vector space $V_1 \otimes V_2 \otimes \cdots \otimes V_k$, spanned by the vectors $v_1 \otimes v_2 \otimes \cdots \otimes v_k$, for $v_i \in V_i$, $i = 1, 2, \ldots, k$, which becomes an L-module with the definition

$$\begin{aligned} x \circ (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = (xv_1) \otimes v_2 \otimes \cdots \otimes v_k \,+\, v_1 \otimes (xv_2) \otimes \cdots \otimes v_k \\ +\ \ldots\ +\ v_1 \otimes v_2 \otimes \cdots \otimes (xv_k). \end{aligned}$$

In general, $V \otimes W$ is a reducible L-module. The problem of decomposing $V \otimes W$ into a direct sum of irreducible L-modules is called the *Clebsch-Gordan problem*. We will see an example of this in Sect. 3.18.

3.11 The universal enveloping algebra of a Lie algebra

Let L be a finite-dimensional Lie algebra, with basis $\{x_1, \ldots, x_n\}$. We may embed L in an infinite-dimensional associative algebra $U(L)$, called the *universal enveloping algebra* on L, which is the algebra spanned by the formal products:

$$\{1 \in \mathbb{C}, x_i, x_i x_j, x_i x_j x_k, \ldots\} \tag{47}$$

with the condition that $x_i x_j - x_j x_i = [x_i, x_j] = C_{ij}^k x_k$. Thus $U(L)$ is like the polynomial algebra $\mathbb{C}[x_1, \ldots, x_n]$ except that the variables x_i in general no longer commute. If they do, that is if L is abelian, then all $C_{ij}^k = 0$, and $U(L)$ *is* the polynomial *ring* $\mathbb{C}[x_1, \ldots, x_n]$. More formally:

$$U(L) = \bigoplus_{k=0}^{\infty} \mathbb{C}L^k = \mathbb{C} \oplus \mathbb{C}L \oplus \mathbb{C}L^2 \oplus \cdots \oplus \mathbb{C}L^k \oplus \cdots \tag{48}$$

An L-module V becomes a $U(L)$ module with the definition:

$$(x_{i_1} x_{i_2} \cdots x_{i_k})v = x_{i_1}(x_{i_2} \cdots (x_{i_k} v) \cdots), \qquad \text{for } v \in V. \tag{49}$$

If W is a subspace of V, we set $U(L)W = \{uw \;:\; u \in U(L), w \in W\}$ as the $U(L)$-*module generated by* W. When $W = \mathbb{C}v$, we write $U(L)v$ in place of $U(L)W$. Let V be an irreducible L-module, and take any non-zero $v \in V$. Now, $U(L)v \subseteq V$ is a submodule of V. Since V is irreducible, $V = U(L)v$.

Theorem 1 (Poincaré-Birkhoff-Witt (PBW) theorem)**.** *Let L be a Lie algebra with basis $\{x_1, \ldots, x_n\}$. Then a basis for $U(L)$ is given by $1 \in \mathbb{C}$, together with all* ordered *products of powers of elements of L:*

$$\{1 \in \mathbb{C},\ x_1^{m_1} \cdots x_n^{m_n},\ m_i \in \mathbb{Z}^+\}. \tag{50}$$

Exercise 14. *Show that $\Delta : L \to U(L) \otimes U(L)$ defined by*

$$\Delta(a) = I \otimes a + a \otimes I$$

for all $a \in L$ is a Lie algebra homomorphism.

Exercise 15. *Show that in the universal enveloping algebra*

$$[AB,\, C] = A\,[B,\, C] + [A,\, C]\,B.$$

for all $A,\ B,\ B \in U(L)$.

Exercise 16. *A bilinear form $(\ ,\ ) : V \times V \to \mathbb{F}$ is invariant if*

$$(ax, y) + (x, ay) = 0 \qquad \forall a \in L,\ x, y \in V. \tag{51}$$

For the adjoint representation show (51) is equivalent to

$$([x, y], z) = (x, [y, z]) \qquad x, y, z \in L. \tag{52}$$

We thus call a bilinear form $(\ ,\ ) : L \times L \to \mathbb{F}$ invariant if it satisfies (52). Moreover we say that $(\ ,\ )$ is symmetric if $(x, y) = (y, x)$. Show that the kernel

$$K = \{x \in L | (x, y) = 0, \qquad \forall y \in L\}$$

of an invariant, symmetric bilinear form on a Lie algebra L is an ideal of L. Hence deduce that any non-zero invariant, symmetric bilinear form on a simple Lie algebra is non-degenerate. Show that a bilinear form defined by

$$(x, y) = \mathrm{tr}\,(\pi(x)\pi(y))$$

for an arbitrary representation π is both symmetric and invariant.

Let $(\ ,\ )$ be a non-degenerate, invariant, symmetric bilinear form on a complex Lie algebra L and let $\{x_i\}_{i=1}^N$ be a basis for L satisfying

$$[x_i,\, x_j] = C_{ij}^k x_k$$

where throughout summation on repeated indices is assumed. Setting $g_{ij} = (x_i,\, x_j)$ we define

$$x^i = (g^{-1})_{ik} x_k.$$

Show that $\{x^i\}_{i=1}^N$ forms a dual basis for L; i.e.

$$(x^i,\, x_j) = \delta_j^i.$$

Using (52), show that

$$[x^k,\, x_i] = C^k_{ij} x^j.$$

Now defining $C = x_i x^i$ *show that*

$$[C,\, x_j] = 0, \qquad \forall x_j \in L,$$

i.e. C *is a Casimir operator.*

Let $\{L_i\}_{i=1}^3$ *be a basis for* $o(3)$ *with relations*

$$[L_i,\, L_j] = \varepsilon_{ijk} L_k.$$

Show that a bilinear form defined by

$$(L_i,\, L_j) = \delta_{ij}$$

is invariant and symmetric and determine the associated Casimir element.

3.12 Universal enveloping algebra of $o(3)$

Recall (3.2) that $o(3)$ is the three-dimensional complex Lie algebra spanned by generators $\{x_1, x_2, x_3\}$ satisfying $[x_i, x_j] = \varepsilon_{ijk} x_k$. A basis for the enveloping algebra $U \equiv U(o(3))$ is given by the products $\{x_1^{m_1} x_2^{m_2} x_3^{m_3} : m_i \in \mathbb{Z}^+\}$, e.g.:

$$\begin{aligned} x_3 x_2 x_1 &= x_3[x_2, x_1] + x_3 x_1 x_2 \\ &= x_3[x_2, x_1] + [x_3, x_1]x_2 + x_1 x_3 x_2 \\ &= x_3[x_2, x_1] + [x_3, x_1]x_2 + x_1[x_3, x_2] + x_1 x_2 x_3 \\ &= -x_3^2 + x_2^2 - x_1^2 + x_1 x_2 x_3. \end{aligned}$$

Now introduce generators $L_\alpha = i x_\alpha$ for $\alpha = 1, 2, 3$. The L_α satisfy $[L_j, L_k] = i\varepsilon_{jkl} L_l$. It is convenient to work with the alternative (weight) basis:

$$L_\pm = L_1 \pm i L_2, \qquad L_0 = L_3 \tag{53}$$

which satisfies:

$$[L_0, L_\pm] = \pm L_\pm, \qquad [L_+, L_-] = 2L_0. \tag{54}$$

We can regard L_0 and $L_\pm$ as spanning one-dimensional abelian subalgebras of $o(3)$ with universal enveloping algebras U_0 and $U_\pm$ respectively:

$$\begin{aligned} U_0 &= \bigoplus_{k=0}^{\infty} \mathbb{C}(L_0)^k \equiv \mathbb{C}[L_0] \\ &= \mathbb{C} \oplus \mathbb{C}L_0 \oplus \mathbb{C}L_0^2 \oplus \mathbb{C}L_0^3 \oplus \cdots \\ U_\pm &= \bigoplus_{k=0}^{\infty} \mathbb{C}(L_\pm)^k \equiv \mathbb{C}[L_\pm] \\ &= \mathbb{C} \oplus \mathbb{C}L_\pm \oplus \mathbb{C}L_\pm^2 \oplus \mathbb{C}L_\pm^3 \oplus \cdots \end{aligned}$$

Using the PBW Theorem, U has a basis of generator products: $(L_-)^{k_-}(L_0)^{k_0}(L_+)^{k_+}$, where $k_0, k_\pm \in \mathbb{N} \cup \{0\}$. Thus:

$$U = U_- U_0 U_+.$$

An important element of U is the *quadratic Casimir invariant*

$$L^2 = L_1^2 + L_2^2 + L_3^2.$$

Exercise 17. *Show that L^2 may alternatively be expressed*

$$L^2 = L_0^2 + L_0 + L_- L_+ = L_0^2 - L_0 + L_+ L_-.$$

and that

$$[L^2,\, L_\alpha] = 0, \qquad \alpha = 1, 2, 3.$$

Since L^2 commutes with the action of $o(3)$, Schur's lemma shows that it reduces to a scalar multiple of the identity on an irreducible $o(3)$-module.

Exercise 18. *Consider the polynomial algebra $\mathbb{C}[x_1,\, x_2,\, x_3]$ with $x_1,\, x_2,\, x_3$ three independent real variables. Recall from Exercise 11 that the differential operators*

$$a^i{}_j = x_i \frac{\partial}{\partial x_j} \tag{55}$$

satisfy the $gl(3)$ commutation relations. Use Ado's theorem to find expressions for the $o(3)$ generators $L_1,\, L_2,\, L_3$ in terms of (55). Show that

$$L^2 = \mathbf{x}.(\mathbf{x}.\nabla + 2)\nabla - \mathbf{x}.\mathbf{x}\nabla^2$$

where

$$\begin{aligned} \mathbf{x} &= (x_1,\, x_2,\, x_3), \\ \nabla &= \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right), \\ \nabla^2 &= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}. \end{aligned}$$

You will find it helpful to use the result

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}.$$

3.13 Irreducible $o(3)$-modules

Let V be a finite-dimensional *irreducible* $o(3)$-module. Then L_0 has at least one non-zero eigenvector $v \in V$, such that:

$$L_0 v = \lambda v. \tag{56}$$

Now consider the vectors $L_\pm v$:

$$L_0(L_\pm v) = [L_0, L_\pm]v + L_\pm L_0 v = \pm L_\pm v + \lambda L_\pm v = (\lambda \pm 1)(L_\pm v). \tag{57}$$

That is, $\lambda \pm 1$ is an eigenvalue of L_0, for eigenvector $L_\pm v$, i.e. $L_\pm$ increases (respectively decreases) the eigenvalue. We refer to L_+ (respectively L_-) as a *raising* (respectively *lowering*) generator.

Consider the following sequence of eigenvectors of L_0:

$$v, L_+v, L_+^2v, L_+^3v, \ldots, L_+^{k'}v, \ldots \tag{58}$$

with eigenvalues $\lambda, \lambda+1, \lambda+2, \ldots, \lambda+k', \ldots$ respectively. Then, since V is finite-dimensional, for some integer k' we have $L_+^{k'}v \neq 0$ and $L_+^{k'+1}v = 0$. Therefore, there exists a (maximal) vector $v_+ = L_+^{k'}v$ such that:

- v_+ is an eigenvector of L_0, with, say, eigenvalue l, viz. $L_0 v_+ = l v_+$.
- $L_+ v_+ = 0$.

Since V is irreducible, $V = Uv_+$. But $U = U_- U_0 U_+$, hence $V = Uv_+ = U_-U_0U_+v_+$, and since $L_+v_+ = 0$, we have $U_+v_+ = (\mathbb{C} \oplus \mathbb{C}L \oplus \mathbb{C}L^2 \oplus \cdots)v_+ = \mathbb{C}v_+$, therefore $V = U_-U_0v_+$. Since $L_0v_+ = lv_+$, then $U_0v_+ = \mathbb{C}v_+$, so $V = U_-v_+$, thus:

$$V = \mathbb{C}v_+ \oplus \mathbb{C}L_-v_+ \oplus \mathbb{C}L_-^2v_+ \oplus \cdots \tag{59}$$

Thus V has a basis of eigenvectors $\{v_+, L_-v_+, L_-^2v_+, \ldots\}$ of L_0 with eigenvalues $l, l-1, l-2, \ldots,$ Again, since V is finite-dimensional, this sequence terminates, that is, there exists an index k such that $L_-^k v_+ \neq 0$ yet $L_-^{k+1}v_+ = 0$, so we have a vector $v_- = L_-^k v_+$ which satisfies $L_0 v_- = (l-k)v_-$ and $L_-v_- = 0$.

Now L^2 reduces to a scalar multiple of the identity on V (by Schur's Lemma, as V is irreducible), so:

$$L^2 v_+ = (L_0^2 + L_0 + L_-L_+)v_+ = (l^2 + l)v_+ = l(l+1)v_+. \tag{60}$$

That is, L^2 takes constant value $l(l+1)$ on V. Also

$$l(l+1)v_- = L^2v_- = (L_0^2 - L_0 + L_+L_-)v_- = [(l-k)^2 - (l-k)]v_-. \tag{61}$$

Hence $l(l+1) = (l-k)^2 - (l-k) = (k-l)^2 + (k-l)$, therefore $l^2 + l - (k-l)^2 - (k-l) = 0$, which implies $2l - k^2 + 2kl - k = 0$ or $(2l-k)(k+1) = 0$. As $k > 0$, we thus require $2l = k \in \mathbb{Z}^+$. As $k \in \mathbb{Z}^+$, l is an integer or half-integer, non-negative, and the corresponding module, herein denoted V_l, has dimension $2l+1$. We have thus proved:

Theorem 2. *The finite-dimensional irreducible $o(3)$-modules V are characterised by a* quantum number $l \in \frac{1}{2}\mathbb{Z}^+$*, and are $(2l+1)$-dimensional. Each V_l has a basis of eigenvectors of L_0 with eigenvalues $-l, (-l+1), \ldots, (l-1), l$ (symmetric about 0).*

3.14 Explicit construction of representations for $o(3)$

Let the $o(3)$-module V_l have an orthonormal basis of $(2l+1)$ eigenvectors $\{\psi^l_m \mid m = -l, \ldots, l\}$, for $l \in \frac{1}{2}\mathbb{Z}^+$. By demanding orthonormality, we have implicitly *defined* an inner product on V_l, in which L_0 is self-adjoint (Hermitian): $(\psi^l_m, \psi^l_{m'}) = \delta_{mm'}$. Then:

$$L_0\psi^l_m = m\psi^l_m, \qquad L^2\psi^l_m = l(l+1)\psi^l_m. \tag{62}$$

We require constants $\alpha^\pm_m$ which correspond to the raising and lowering operators $L_\pm$ as $L_\pm\psi^l_m = \alpha^\pm_m\psi^l_{m\pm1}$. Now examine:

$$\alpha^-_{m+1}\alpha^+_m\psi^l_m = L_-L_+\psi^l_m = [L^2 - L_0^2 - L_0]\psi^l_m = [l(l+1) - m(m+1)]\psi^l_m, \tag{63}$$

thus:

$$\alpha^-_{m+1}\alpha^+_m = (l-m)(l+m+1). \tag{64}$$

One solution of (64) is to require $\alpha^-_{m+1} = \alpha^+_m = \sqrt{(l-m)(l+m+1)}$, viz.:

$$L_\pm\psi^l_m = \sqrt{(l \mp m)(l \pm m + 1)}\psi^l_{m\pm1} \tag{65}$$

It is simple to check directly that this definition satisfies the $o(3)$ commutation relations, and that $(L_+)^\dagger = L_-$, or equivalently, that $L^\dagger_\alpha = L_\alpha$, $\alpha = 1, 2, 3$.

We may now obtain the corresponding representation in the basis ψ^l_m. In terms of $(2l+1)\times(2l+1)$ matrices, we have the representation (recall that $l \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$, hence $2l+1 \in \{1, 2, 3, 4, \ldots\}$):

$$\begin{aligned}
\pi(L_0) &= \mathrm{diag}(l, l-1, \ldots, -l) \\
\pi(L_+) &= \mathrm{supdiag}(\sqrt{2l}, \sqrt{2(2l-1)}, \ldots, \sqrt{2l}) \\
\pi(L_-) &= \mathrm{subdiag}(\sqrt{2l}, \sqrt{2(2l-1)}, \ldots, \sqrt{2l}) = \pi(L_+)^\dagger.
\end{aligned}$$

(All matrix elements here are real, so $\pi(L_+)^\dagger = \pi(L_+)^T$.)

Note that $L_0\psi^l_l = l\psi^l_l$, $L_+\psi^l_l = 0$. In representation theory, ψ^l_l is called a *maximal/highest weight state vector*, with l the *maximal/highest weight of the representation.* We will denote the representation corresponding to V_l by π_l. If $l \in \mathbb{Z}$, π_l is called a *tensor* representation, and if l is a half-odd integer, then π_l is a *spinor* representation.

We represent the ith j-dimensional unit vector by $e^{(j)}_i$. In the basis $\{\psi^l_{-l}, \ldots, \psi^l_l\}$, the vector ψ^l_m is represented by the unit vector $e^{(2l+1)}_{l-m+1}$. Consider:

- $l = \dfrac{1}{2}$, the *fundamental spinor representation.* $V_{\frac{1}{2}}$ has dimension $2l+1 = 2$, and has an orthonormal basis $\{\psi^{\frac{1}{2}}_{-\frac{1}{2}}, \psi^{\frac{1}{2}}_{\frac{1}{2}}\}$, such that:

$$L_0\psi^{\frac{1}{2}}_{\pm\frac{1}{2}} = \pm\tfrac{1}{2}\psi^{\frac{1}{2}}_{\pm\frac{1}{2}}, \qquad L_\pm\psi^{\frac{1}{2}}_{\mp\frac{1}{2}} = \psi^{\frac{1}{2}}_{\pm\frac{1}{2}}, \qquad L_\pm\psi^{\frac{1}{2}}_{\pm\frac{1}{2}} = 0.$$

Where the basis vectors are represented by $\psi^{\frac{1}{2}}_{-\frac{1}{2}} = e_2^{(2)}$, and $\psi^{\frac{1}{2}}_{\frac{1}{2}} = e_1^{(2)}$, we therefore have the matrix representation:

$$\pi_{\frac{1}{2}}(L_0) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \pi_{\frac{1}{2}}(L_+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \pi_{\frac{1}{2}}(L_-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (66)$$

Recall that $L_1 = \frac{1}{2}(L_- + L_+)$, $L_2 = \frac{i}{2}(L_- - L_+)$ and $L_3 = L_0$. Then $\pi_{\frac{1}{2}}(L_i) = \frac{1}{2}\sigma_i$, for $i = 1, 2, 3$, where σ_i are the Pauli sigma matrices, viz.

$$\sigma_1 \equiv \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 \equiv \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 \equiv \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- $l = 1$, the *fundamental vector/tensor representation.* (In general, a tensor representation is a tensor product of vector representations.) The module V_1 has dimension $2l + 1 = 3$, and has an orthonormal basis $\{\psi^1_{-1}, \psi^1_0, \psi^1_1\}$ such that:

$$\begin{aligned} L_\pm \psi^1_{\pm 1} &= 0, & L_\pm \psi^1_{\mp 1} &= \sqrt{2}\psi^1_0, \\ L_\pm \psi^1_0 &= \sqrt{2}\psi^1_{\pm 1}, & L_0 \psi^1_m &= m\psi^1_m, \qquad m = -1, 0, 1. \end{aligned}$$

Where the basis vectors are represented by $\psi^1_{-1} = e_3^{(3)}, \psi^1_0 = e_2^{(3)}$, and $\psi^1_1 = e_1^{(3)}$, we therefore have the matrix representation:

$$\pi_1(L_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \pi_1(L_+) = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\pi_1(L_-) = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \pi_1(L_+)^\dagger.$$

In terms of the original operators, we have:

$$\pi_1(L_1) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \pi_1(L_2) = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\pi_1(L_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \pi_1(L_0). \quad (67)$$

It is also instructive to write out this entire example in Dirac's bra-ket notation. We use $l \in \frac{1}{2}\mathbb{Z}^+$, and again illustrate with the cases $l = \frac{1}{2}$ and $l = 1$.

- $l = \frac{1}{2}$. A basis for the 2-dimensional space $V_{\frac{1}{2}}$ is written as $\{|1\rangle, |2\rangle\}$, that is, we replace $\psi^{\frac{1}{2}}_{\frac{1}{2}}$ with $|1\rangle$ (the highest-weight vector of the representation) and $\psi^{\frac{1}{2}}_{-\frac{1}{2}}$ with

$|2\rangle$ (the lowest weight vector of the representation). These vectors satisfy:

$$
\begin{aligned}
L_0 |1\rangle &= \frac{1}{2} |1\rangle, & L_0 |2\rangle &= -\frac{1}{2} |2\rangle, \\
L_+ |1\rangle &= 0, & L_- |2\rangle &= 0, \\
L_+ |2\rangle &= |1\rangle, & L_- |1\rangle &= |2\rangle,
\end{aligned}
$$

that is the representation has weights $(l, -l) = \left(\frac{1}{2}, -\frac{1}{2}\right)$. We obtain:

$$
\begin{aligned}
\pi_{\frac{1}{2}}(L_0) &= \frac{1}{2}(|1\rangle\langle 1| - |2\rangle\langle 2|), \\
\pi_{\frac{1}{2}}(L_+) &= |1\rangle\langle 2|, \\
\pi_{\frac{1}{2}}(L_-) &= |2\rangle\langle 1| = \pi_{\frac{1}{2}}(L_+)^\dagger.
\end{aligned}
$$

Reverting to the original operators:

$$
\begin{aligned}
\pi_{\frac{1}{2}}(L_1) &= \frac{1}{2}\pi_{\frac{1}{2}}(L_- + L_+) = \frac{1}{2}(|1\rangle\langle 2| + |2\rangle\langle 1|) = \frac{1}{2}\sigma_1 \\
\pi_{\frac{1}{2}}(L_2) &= \frac{i}{2}\pi_{\frac{1}{2}}(L_- - L_+) = \frac{i}{2}(|2\rangle\langle 1| - |1\rangle\langle 2|) = \frac{1}{2}\sigma_2 \\
\pi_{\frac{1}{2}}(L_3) &= \pi_{\frac{1}{2}}(L_0) = \frac{1}{2}(|1\rangle\langle 1| - |2\rangle\langle 2|) = \frac{1}{2}\sigma_3.
\end{aligned} \tag{68}
$$

- $l = 1$. A basis for the 3-dimensional space V_1 is written as $\{|1\rangle, |2\rangle, |3\rangle\}$, that is, we replace ψ^1_1 with $|1\rangle$ (the highest-weight vector of the representation), ψ^1_{-1} with $|3\rangle$ (the lowest weight vector of the representation) and ψ^1_0 with $|2\rangle$. These vectors satisfy:

$$
\begin{aligned}
L_0 |1\rangle &= |1\rangle, & L_0 |2\rangle &= 0\,|2\rangle = 0, & L_0 |3\rangle &= -|3\rangle, \\
L_+ |1\rangle &= 0, & L_+ |2\rangle &= \sqrt{2}\,|1\rangle, & L_+ |3\rangle &= \sqrt{2}\,|2\rangle, \\
L_- |1\rangle &= \sqrt{2}\,|2\rangle, & L_- |2\rangle &= \sqrt{2}\,|3\rangle, & L_- |3\rangle &= 0,
\end{aligned}
$$

that is the representation has weights $(l, l-1, -l) = (1, 0, -1)$. The factors of $\sqrt{2}$ are required to normalise $|2\rangle$ and $|3\rangle$. (Due to the simplicity of the $l = \frac{1}{2}$ case, the normalisation constant of $|2\rangle$ was unity.) In the $l = 1$ case, say that $|2\rangle = xL_- |1\rangle$, for some (scalar) normalisation constant x. Then $\langle 2| = x^* \langle 1| L_+$ as $L_-{}^\dagger = L_+$. We define $|1\rangle$ as normalised, viz. $\langle 1|1\rangle = 1$. To normalise $|2\rangle$, we require: $\langle 2|2\rangle = 1$, viz.: $x^* x \langle 1| L_+ L_- |1\rangle = 1$. As $L_+ L_- = L^2 - L_0^2 + L_0$, and L^2 and L_0 have eigenvalues $l^2 + l$ and l on $|1\rangle$ respectively, then:

$$
\begin{aligned}
\langle 1| L_+ L_- |1\rangle &= \langle 1| [L^2 - L_0^2 + L_0] |1\rangle = \langle 1| L^2 |1\rangle - \langle 1| L_0^2 |1\rangle + \langle 1| L_0 |1\rangle \\
&= \langle 1| [l^2 + l] |1\rangle - \langle 1| l^2 |1\rangle + \langle 1| l |1\rangle = [l^2 + l] \langle 1|1\rangle - l^2 \langle 1|1\rangle + l \langle 1|1\rangle \\
&= l^2 + l - l^2 + l = 2l = 2,
\end{aligned}
$$

hence the normalisation condition is $x^*x = \frac{1}{2}$. Up to an (arbitrary) phase factor, then $x = \frac{1}{\sqrt{2}}$, and hence $L_- |1\rangle = \sqrt{2}\,|2\rangle$. Similarly $L_- |2\rangle = \sqrt{2}\,|3\rangle$.

We obtain:

$$\pi_1(L_0) = |1\rangle\langle 1| + 0\,|2\rangle\langle 2| - |3\rangle\langle 3| = |1\rangle\langle 1| - |3\rangle\langle 3|\,,$$
$$\pi_1(L_-) = \sqrt{2}(|2\rangle\langle 1| + |3\rangle\langle 2|), \qquad \pi_1(L_+) = \sqrt{2}(|1\rangle\langle 2| + |2\rangle\langle 3|) = \pi_1(L_-)^\dagger.$$

Reverting to the original operators:

$$\begin{aligned}
\pi_1(L_1) &= \frac{1}{2}\pi_1(L_- + L_+) = \frac{1}{\sqrt{2}}(|2\rangle\langle 1| + |1\rangle\langle 2| + |3\rangle\langle 2| + |2\rangle\langle 3|) \\
\pi_1(L_2) &= \frac{i}{2}\pi_1(L_- - L_+) = \frac{i}{\sqrt{2}}(|2\rangle\langle 1| - |1\rangle\langle 2| + |3\rangle\langle 2| - |2\rangle\langle 3|) \\
\pi_1(L_3) &= \pi_1(L_0) = |1\rangle\langle 1| - |3\rangle\langle 3|\,. \qquad (69)
\end{aligned}$$

Inspection reveals that the expressions in (68) and (69) may be expanded to those of (66) and (67) respectively, through the identification of $|i\rangle$ with the ith unit (column) vector in the space V_l, for $i = 1, \ldots, \dim(V_l)$, where $\langle i|$ is then the dual (conjugate transpose) of the ith column vector, viz. the ith row vector.

Exercise 19. *The $o(3)$* Verma *module is an infinite-dimensional highest-weight module that is constructed as follows. Start with highest-weight vector $|\Psi_\lambda(0)\rangle$, $\lambda \in \mathbb{C}$, defined to have the properties (using the $\{e,\, f,\, h\}$ basis)*

$$\begin{aligned}
e|\Psi_\lambda(0)\rangle &= 0, \\
h|\Psi_\lambda(0)\rangle &= \lambda|\Psi_\lambda(0)\rangle.
\end{aligned}$$

Then construct the infinite set of basis vectors

$$B = \{|\Psi_\lambda(k)\rangle = f^k|\Psi_\lambda(0)\rangle : k = 0, ..., \infty\}.$$

(i) Derive an expression for $[e,\, f^k]$.

(ii) Determine all values of k and λ such that $|\Psi_\lambda(k)\rangle$ is a highest-weight vector.

3.15 The Clebsch-Gordan problem for $o(3)$

Let V_{l_1}, V_{l_2} be irreducible $o(3)$-modules with highest weight l_1, l_2 respectively, and let $|l_1, m_1\rangle\,, |l_2, m_2\rangle$ (for $m_j = -l_j, \ldots, l_j$) respectively be the usual orthonormal bases of L_0 eigenstates. (Here we adopt the Dirac notation in a way such that $|l, m\rangle \equiv \psi^l_m$.) The tensor product states:

$$|l_1, m_1\rangle \otimes |l_2, m_2\rangle \qquad (70)$$

form an orthonormal basis for the tensor product module $V_{l_1} \otimes V_{l_2}$. We have:

$$\begin{aligned}
L_0(|l_1, m_1\rangle \otimes |l_2, m_2\rangle) &= (L_0\,|l_1, m_1\rangle) \otimes |l_2, m_2\rangle + |l_1, m_1\rangle \otimes (L_0\,|l_2, m_2\rangle) \\
&= (m_1 + m_2)(|l_1, m_1\rangle \otimes |l_2, m_2\rangle),
\end{aligned}$$

thus the product states (70) are eigenstates of L_0 with eigenvalues $(m_1 + m_2)$. Our aim is to obtain the decomposition of $V_{l_1} \otimes V_{l_2}$ into irreducible $o(3)$-modules:

$$V_{l_1} \otimes V_{l_2} = \bigoplus_{l} V_l.$$

Firstly, the state with maximum L_0 eigenvalue $(l_1 + l_2)$ is $\psi_0 = |l_1, l_1\rangle \otimes |l_2, l_2\rangle$, which is an $o(3)$ *highest-weight state* since:

$$L_+\psi_0 = (L_+ |l_1, l_1\rangle) \otimes |l_2, l_2\rangle + |l_1, l_1\rangle \otimes (L_+ |l_2, l_2\rangle) = 0.$$

Thus the $o(3)$-module $V_{l_1+l_2}$ occurs in $V_{l_1} \otimes V_{l_2}$.

Secondly, two eigenstates of L_0 have eigenvalue $l_1 + l_2 - 1$, namely $|l_1, l_1 - 1\rangle \otimes |l_2, l_2\rangle$ and $|l_1, l_1\rangle \otimes |l_2, l_2 - 1\rangle$. A linear combination of these must occur in $V_{l_1+l_2}$, and another, denoted ψ_1, must belong to the orthocomplement $V_{l_1+l_2}^{\perp}$, of $V_{l_1+l_2}$. Then $V_{l_1+l_2}^{\perp}$ is an $o(3)$-module and $l_1 + l_2 - 1$ is the highest weight in this space, in other words

$$L_0\psi_1 = (l_1 + l_2 - 1)\psi_1, \qquad L_+\psi_1 = 0,$$

so the $o(3)$-module $V_{l_1+l_2-1}$ must also occur in the space $V_{l_1} \otimes V_{l_2}$.

Continuing in this manner, we eventually arrive at the decomposition:

$$V_{l_1} \otimes V_{l_2} = \bigoplus_{l=|l_1-l_2|}^{l_1+l_2} V_l. \tag{71}$$

This result is also known as the addition rule for angular momenta.

- For $l \geqslant \dfrac{1}{2}$, we have: $V_l \otimes V_{1/2} = V_{l+1/2} \oplus V_{l-1/2}$, for example $V_{1/2} \otimes V_{1/2} = V_1 \oplus V_0$.
- For $l \geqslant 1$, we have: $V_l \otimes V_1 = V_{l+1} \oplus V_l \oplus V_{l-1}$, for example $V_1 \otimes V_1 = V_2 \oplus V_1 \oplus V_0$.

3.16 Two-dimensional quantum oscillator

The Hamiltonian reads

$$\begin{aligned} H &= \frac{\vec{p}.\vec{p}}{2m} + \frac{m\omega^2 \vec{q}.\vec{q}}{2} \\ &= \frac{p_1^2 + p_2^2}{2m} + \frac{m\omega^2(q_1^2 + q_2^2)}{2}. \end{aligned}$$

For $j = 1$, 2 set

$$\begin{aligned} b_j &= \sqrt{\frac{m\omega}{2}} q_j + i\sqrt{\frac{1}{2m\omega}} p_j, \\ b_j^\dagger &= \sqrt{\frac{m\omega}{2}} q_j - i\sqrt{\frac{1}{2m\omega}} p_j, \end{aligned}$$

which satisfy

$$[b_1,\, b_2] = [b_1^\dagger,\, b_2^\dagger] = 0,$$
$$[b_j,\, b_k^\dagger] = \delta_{jk} I.$$

The Hamiltonian can now be expressed as

$$H = \omega(b_1^\dagger b_1 + b_2^\dagger b_2 + I).$$

Defining $a^j{}_k = b_j^\dagger b_k$, these operators satisfy the $gl(2)$ commutation relations

$$[a^j{}_k,\, a^l{}_p] = \delta^l_k a^j{}_p - \delta^j_p a^l{}_k.$$

We make a transformation to a new set of operators

$$\begin{aligned} N &= a^1{}_1 + a^2{}_2, \\ e &= \frac{1}{\sqrt{2}} a^1{}_2, \\ f &= \frac{1}{\sqrt{2}} a^2{}_1, \\ h &= \frac{1}{2}\left(a^1{}_1 - a^2{}_2\right). \end{aligned}$$

The set $\{e,\, f,\, h\}$ satisfy the commutation relations (26) of the $o(3)$ Lie algebra and N is central; it commutes with all other elements. (Note that N does not equal the operator L^2 discussed earlier.)

We can now determine that the following identification is in fact an $o(3)$-module isomorphism

$$|l,\, m\rangle = C_{lm}(b_1^\dagger)^{(m+l)}(b_2^\dagger)^{(l-m)}\,|0\rangle \tag{72}$$

where

$$C_{lm} = \frac{1}{\sqrt{(l+m)!(l-m)!}}.$$

In this case $|0\rangle$ is the vacuum state satisfying

$$b_1\,|0\rangle = b_2\,|0\rangle = 0.$$

To see that the isomorphism holds, we begin by noting that for all j and k,

$$a^j{}_k\,|0\rangle = b_j^\dagger b_k\,|0\rangle = 0.$$

By induction it can be shown

$$[a^j{}_k,\, (b_m^\dagger)^n] = n\delta_{km} b_j^\dagger (b_m^\dagger)^{(n-1)},$$

and specifically

$$\begin{aligned}\left[a^1{}_1,\,(b_m^\dagger)^n\right] &= n\delta_{1m}(b_1^\dagger)^n,\\ \left[a^2{}_2,\,(b_m^\dagger)^n\right] &= n\delta_{2m}(b_2^\dagger)^n.\end{aligned}$$

Next, we can show that

$$\begin{aligned}a^1{}_1\left|l,\,m\right\rangle &= C_{lm}a^1{}_1(b_1^\dagger)^{(l+m)}(b_2^\dagger)^{(l-m)}\left|0\right\rangle\\ &= C_{lm}\left((b_1^\dagger)^{(l+m)}a^1{}_1 + (l+m)(b_1^\dagger)^{(l+m)}\right)(b_2^\dagger)^{(l-m)}\left|0\right\rangle\\ &= C_{lm}(b_1^\dagger)^{(l+m)}(b_2^\dagger)^{(l-m)}a^1{}_1\left|0\right\rangle + (l+m)C_{lm}(b_1^\dagger)^{(l+m)}(b_2^\dagger)^{(l-m)}\left|0\right\rangle\\ &= (l+m)\left|l,\,m\right\rangle.\end{aligned}$$

Similarly we can establish $a^2{}_2\left|l,\,m\right\rangle = (l-m)\left|l,\,m\right\rangle$. It then follows that

$$\begin{aligned}h\left|l,\,m\right\rangle &= \frac{1}{2}(a^1{}_1 - a^2{}_2)\left|l,\,m\right\rangle\\ &= m\left|l,\,m\right\rangle,\\ N\left|l,\,m\right\rangle &= (a^1{}_1 + a^2{}_2)\left|l,\,m\right\rangle\\ &= 2l\left|l,\,m\right\rangle.\end{aligned}$$

Next we find

$$\begin{aligned}a^1{}_2\left|l,\,m\right\rangle &= C_{lm}a^1{}_2(b_1^\dagger)^{(l+m)}(b_2^\dagger)^{(l-m)}\left|0\right\rangle\\ &= C_{lm}(b_1^\dagger)^{(l+m)}\left((b_2^\dagger)^{(l-m)}a_2^1 + (l-m)b_1^\dagger(b_2^\dagger)^{(l-m-1)}\right)\left|0\right\rangle\\ &= (l-m)C_{lm}(b_1^\dagger)^{(l+m+1)}(b_2^\dagger)^{(l-m-1)}\left|0\right\rangle\\ &= \frac{(l-m)C_{lm}}{C_{l(m+1)}}\left|l,\,m+1\right\rangle\end{aligned}$$

and similarly $a^2{}_1\left|l,\,m\right\rangle = \dfrac{(l+m)C_{lm}}{C_{l(m-1)}}\left|l,\,m-1\right\rangle$. Evaluating

$$\begin{aligned}e\left|l,\,l\right\rangle &= \frac{1}{\sqrt{2}}a^1{}_2\frac{1}{\sqrt{(2l)!}}(b_1^\dagger)^{2l}\left|0\right\rangle\\ &= \frac{1}{\sqrt{2(2l)!}}(b_1^\dagger)^{(2l+1)}b_2\left|0\right\rangle\\ &= 0\end{aligned}$$

shows that $\left|l,\,l\right\rangle$ is a highest-weight state. Likewise one can show that $\left|l,\,-l\right\rangle$ is a lowest weight state.

Finally, $H\left|l,\,m\right\rangle = \omega(N+I)\left|l,\,m\right\rangle = \omega(2l+1)\left|l,\,m\right\rangle$ so the energy levels of the system are ω, 2ω, 3ω..... Moreover, we can determine the degeneracies of these levels. For each l, there are $2l+1$ states $\left|l,\,m\right\rangle$ which have energy $E = (2l+1)\omega$. Or more concisely, each energy level $n\omega$ with $n = 1, 2, 3, \ldots$ has degeneracy n.

3.17 Quantising angular momentum

Let

$$\mathbf{r} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$

denote the position vector for some particle with momentum

$$\mathbf{p} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}.$$

Recall that we have the commutation relations ($[x, y] = xy - yx$)

$$[q_1, p_1] = [q_2, p_2] = [q_3, p_3] = iI,$$

with all other commutators vanishing; e.g.

$$[q_1, p_2] = 0.$$

In analogy with the classical definition of angular momentum we set

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}.$$

In component form this reads

$$L_a = \sum_{b,c=1}^{3} \varepsilon_{abc} q_b p_c \tag{73}$$

where ε_{abc} is the Levi-Civita tensor; viz.

$$\varepsilon_{123} = 1, \quad \varepsilon_{abc} = -\varepsilon_{bac} = -\varepsilon_{acb} = -\varepsilon_{cba}.$$

Exercise 20. *Show that the components of the angular momentum operator as given by (73) satisfy the $o(3)$ commutation relations*

$$[L_a, L_b] = i\sum_{c=1}^{3} \varepsilon_{abc} L_c. \tag{74}$$

In principle, the half-odd integer values of angular momentum do not arise in the form

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}.$$

Rather, they correspond to an internal structure of particles which we refer to as *spin*. In such a case it is more common to use the symbol $\mathbf{S}$ instead of $\mathbf{L}$, and label coordinates with x, y, and z instead of 1, 2, and 3. Particles with half-odd integer spin are called *fermions*, while those of integer spin are called *bosons*. In these cases the weight of a state $|\Phi\rangle$ simply measures the spin of the state of the particle.

In order to illustrate this difference, consider the case of a spin-1/2 particle. For a fixed choice of co-ordinates there are two basis states

$$|\uparrow\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The spin operators are represented via the Pauli matrices, up to a scalar factor,

$$\pi(S^x) = \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \pi(S^y) = \frac{1}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \pi(S^z) = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

satisfying the relations (74). Now suppose that we rotate the $\mathbf{k} - \mathbf{i}$ plane by an angle θ to define a new co-ordinate system. Now we have

$$\tilde{S}^z = S^z \cos\theta + S^x \sin\theta$$

and using the Pauli matrices

$$\begin{aligned}\tilde{S}^z &= \frac{1}{2}\cos\theta\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \frac{1}{2}\sin\theta\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\\ &= \frac{1}{2}\begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}.\end{aligned} \tag{75}$$

Diagonalising this matrix we obtain the eigenvectors

$$|\tilde{\uparrow}\rangle = \begin{pmatrix}\cos(\theta/2)\\ \sin(\theta/2)\end{pmatrix} = \cos\frac{\theta}{2}\,|\uparrow\rangle + \sin\frac{\theta}{2}\,|\downarrow\rangle \tag{76}$$

with eigenvalue $+1/2$ and

$$|\tilde{\downarrow}\rangle = \begin{pmatrix}-\sin(\theta/2)\\ \cos(\theta/2)\end{pmatrix} = -\sin\frac{\theta}{2}\,|\uparrow\rangle + \cos\frac{\theta}{2}\,|\downarrow\rangle$$

with eigenvalue $-1/2$.

A curious situation occurs when we put $\theta = 2\pi$. We find that

$$\begin{aligned}|\tilde{\uparrow}\rangle &= -\,|\uparrow\rangle\,,\\ |\tilde{\downarrow}\rangle &= -\,|\downarrow\rangle\,.\end{aligned}$$

In other words, rotating the universe by 2π causes the states of all spin-1/2 particles to change by the phase factor -1. This is an example of a *Berry phase*, and is a distinguishing feature between intrinsic spin and our usual notion of angular momentum.

For later use we set $S^{\pm} = S^x \pm iS^y$ for which the following commutation relations hold

$$[S^z,\, S^{\pm}] = \pm S^{\pm}, \qquad [S^+,\, S^-] = 2S^z. \tag{77}$$

Note that not all representations of $o(3)$ fit into the above description.

Exercise 21. *Given the commutation relation* $[b,\, b^\dagger] = I$*, show that*

$$\begin{aligned}S^+ &= -\frac{1}{2}(b^\dagger)^2,\\ S^- &= \frac{1}{2}b^2,\\ S^z &= \frac{1}{4}(2N + I)\end{aligned}$$

provides an $o(3)$ representation acting on Fock space $\mathcal{F}$ given by (25), by showing that the commutation relations (77) are preserved. What are the highest-weight and-lowest weight states?

Exercise 22. *For any constant N, show that*

$$S^z = x\frac{\mathrm{d}}{\mathrm{d}x} - \frac{N}{2}$$
$$S^+ = Nx - x^2\frac{\mathrm{d}}{\mathrm{d}x},$$
$$S^- = \frac{\mathrm{d}}{\mathrm{d}x},$$

provides an $o(3)$ representation by showing that the commutation relations (77) are preserved. Assuming that the representation acts on the space of polynomial functions of x, what are the highest-weight and lowest-weight states?

3.18 Coupling angular momentum states

Suppose that we have N identical particles of spin l. Mathematically we represent the total space of states by a tensor product

$$V_l \otimes V_l \otimes .... \otimes V_l \qquad - N \text{ copies}$$

which is the vector space with basis

$$|l,\, \mathbf{m}\rangle = |l,\, m_1\rangle \otimes |l,\, m_2\rangle \otimes .... \otimes |l,\, m_N\rangle\,. \tag{78}$$

The action of the spin operators S^a, $a = x,\, y,\, z$ is given by

$$\Delta^{(N)}(S^a) = \sum_{j=1}^{N} I \otimes I \otimes \cdots \otimes \underbrace{S^a}_{jth} \otimes \cdots \otimes\ I$$
$$= \sum_{j=1}^{N} S_j^a$$

and it can be shown that $\Delta^{(N)}$ determines a Lie algebra homomorphism; i.e.

$$\left[\Delta^{(N)}(S^a),\, \Delta^{(N)}(S^b)\right] = i\varepsilon_{abc}\Delta^{(N)}(S^c).$$

For example, for the state (78) above

$$\Delta^{(N)}(S^z)\,|l,\, \mathbf{m}\rangle = \sum_{i=1}^{N} m_i\,|l,\, \mathbf{m}\rangle$$

so the total $\mathbf{k}$ component of spin is simply the sum of the individual $\mathbf{k}$ components of spin. For the square of the spin, the situation is more complicated. For example, for just

2 particles

$$\begin{aligned}\Delta(S^2) &= \Delta(\mathbf{S}.\mathbf{S}) \\ &= \sum_{a=x,y,z} (S^a \otimes I + I \otimes S^a)(S^a \otimes I + I \otimes S^a) \\ &= S^2 \otimes I + I \otimes S^2 + 2 \sum_{a=x,y,z} S^a \otimes S^a\end{aligned}$$

or equivalently

$$S^2 \equiv S_1^2 + S_2^2 + 2\mathbf{S}_1.\mathbf{S}_2. \tag{79}$$

The allowed values for the square of the spin of two coupled particles are given from the general result

$$V_l \otimes V_k = \bigoplus_{j=|l-k|}^{l+k} V_j.$$

In other words, if we have two particles with the square of the spins given by $l(l+1)$ and $k(k+1)$ respectively, these particles may be coupled so that the allowed values of the square of the spin for the two particle system lie in the set

$$\{j(j+1): \quad j = |l-k|, |l-k|+1, ..., l+k-1, l+k\}.$$

We will not go into the proof of this general result but for later use it is instructive to consider the simplest case of the coupling of two spin-1/2 particles; viz.

$$V_{1/2} \otimes V_{1/2} = V_1 \oplus V_0. \tag{80}$$

The explicit matrix representatives of the spin operators are

$$S^x = \frac{1}{2}\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right),$$

$$S^y = \frac{1}{2}\left(\begin{array}{cc|cc} 0 & -i & -i & 0 \\ i & 0 & 0 & -i \\ \hline i & 0 & 0 & -i \\ 0 & i & i & 0 \end{array}\right),$$

$$S^z = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

where the correspondence between ket vectors and column vectors is

$$|\uparrow\rangle \otimes |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |\uparrow\rangle \otimes |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle \otimes |\uparrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle \otimes |\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

A basis for the spin-1 space of states is given by

$$\begin{gathered} |\uparrow\rangle \otimes |\uparrow\rangle \\ \tfrac{1}{\sqrt{2}} \left(|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle \right) \\ |\downarrow\rangle \otimes |\downarrow\rangle \end{gathered} \tag{81}$$

while for the spin-0 space there is the single vector

$$\frac{1}{\sqrt{2}} \left(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle \right). \tag{82}$$

Qualitatively, if we have two spin-1/2 particles they may be coupled together so the resultant system has either spin-1 or spin-0. The spin-0 system can only occupy one state which is that represented by (82), while a spin-1 system can occupy one of three eigenstates of S^z (spin 1, 0 -1) which are represented by (81).

4 Representation theory of $gl(n)$

Recall that $gl(n)$ is spanned by the n^2 abstract operators $\{a^i{}_j\}$ satisfying the $gl(n)$ commutation relations

$$[a^i{}_j, a^k{}_l] = \delta^k{}_j a^i{}_l - \delta^i{}_l a^k{}_j.$$

In particular, the subspace of *diagonal generators* $a^k{}_k$ form a maximal commutative subalgebra

$$[a^j{}_j,\, a^k{}_k] = 0 \qquad \forall j,\, k$$

and thus span an n-dimensional abelian subalgebra $\mathcal{L}_0$, which will play a role analogous to L_0 in the $o(3)$ case. For a reductive Lie algebra a maximal commutative subalgebra is called a *Cartan subalgebra*, usually represented as H.

4.1 Weights and roots for $gl(n)$

Let V be a $gl(n)$-module. A vector $v \in V$ is called a *weight vector* if it is an eigenvector of all elements of the Cartan subalgebra, viz. $\exists\{\lambda_k\} \in \mathbb{C}$ such that:

$$a^k{}_k v = \lambda_k v, \qquad k = 1, \ldots, n.$$

We call $\lambda = (\lambda_1, \ldots, \lambda_n)$ the *weight* of v, which can be expanded in terms of the basis $\{\epsilon_k : k = 1, ..., n\}$ as

$$\lambda = \sum_{k=1}^{n} \lambda_k \epsilon_k.$$

We impose a partial ordering on the weights given by the *lexical ordering*, viz. $\lambda \geq \mu$ if and only if the first non-zero component of the weight $\lambda - \mu$ is positive, e.g.:

$$(2,-1,0) \geq (1,3,-2) \geq (1,0,0) \geq (0,2,0) \geq (0,1,-5) \geq (0,0,-1) \geq (-2,0,-1).$$

In order to define this ordering, we have assumed $\lambda_k \in \mathbb{R}$, which will later be shown to be true.

Now, the $gl(n)$ generators $a^i{}_j$ are *themselves* weight vectors under the adjoint representation:

$$\mathrm{ad}(a^k{}_k) \circ a^i{}_j = [a^k{}_k, a^i{}_j] = (\delta^i{}_k - \delta^j{}_k) a^i{}_j.$$

In terms of the fundamental weights ϵ_i, $(i = 1, \ldots, n)$, the eigenvector $a^i{}_j$ has weight $\epsilon_i - \epsilon_j$. The non-zero weights of the adjoint representation are called *roots*, and the set of them is called Φ:

$$\Phi = \{\epsilon_i - \epsilon_j \mid i, j = 1, \ldots, n,\ i \neq j\}.$$

We call Φ the *root system* for $gl(n)$. It has $n(n-1)$ elements. A root $\alpha \in \Phi$ is called *positive* (respectively *negative*) if it is greater than (respectively less than) 0 under the lexical ordering. The set of the roots is thus $\Phi = \Phi^+ \cup \Phi^-$, where $\Phi^- = -\Phi^+$ (each set has half the roots).

$$\Phi^+ = \{\epsilon_i - \epsilon_j \mid i, j = 1, \ldots, n,\ i < j\}, \qquad \Phi^- = \{\epsilon_i - \epsilon_j \mid i, j = 1, \ldots, n,\ i > j\}.$$

Now consider the set:

$$\Delta = \{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid i = 1, \ldots, n-1\} \subseteq \Phi^+.$$

Every root in Φ^+ (respectively Φ^-) is a positive (respectively negative) $\mathbb{Z}$-linear combination of roots in Δ, that is for $i < j$ we have:

$$\begin{aligned}
\epsilon_i - \epsilon_j &= (\epsilon_i - \epsilon_{i+1}) + (\epsilon_{i+1} - \epsilon_{i+2}) + \cdots + (\epsilon_{j-2} - \epsilon_{j-1}) + (\epsilon_{j-1} - \epsilon_j) \\
&= \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} \in \Phi^+.
\end{aligned}$$

The roots in Δ are called *simple roots* - these are the positive roots that cannot be expressed as a sum of two positive roots.

4.2 The Cartan-Weyl decomposition for $gl(n)$

If v is a weight vector of a $gl(n)$-module V, of weight $\lambda = (\lambda_1, \ldots, \lambda_n)$, then the vector $a^i{}_j v$ is also a weight vector, of weight $\lambda + (\epsilon_i - \epsilon_j)$:

$$\begin{aligned}
a^k{}_k(a^i{}_j v) &= [a^k{}_k, a^i{}_j] v + a^i{}_j a^k{}_k v = (\delta^i{}_k a^k{}_j - \delta^k{}_j a^i{}_k) v + a^i{}_j \lambda_k v \\
&= (\delta^i{}_k - \delta^k{}_j)(a^i{}_j v) + \lambda_k (a^i{}_j v) \\
&= [\lambda_k + \delta^k{}_i - \delta^k{}_j](a^i{}_j v)
\end{aligned}$$

Hence, if $i < j$, it is useful to visualise this as a vector equation:

$$\begin{pmatrix} a^1_1 \\ \vdots \\ a^i{}_i \\ \vdots \\ a^j{}_j \\ \vdots \\ a^n{}_n \end{pmatrix} (a^i{}_j v) = \begin{pmatrix} \lambda_1 + 0 - 0 \\ \vdots \\ \lambda_i + 1 - 0 \\ \vdots \\ \lambda_j + 0 - 1 \\ \vdots \\ \lambda_n + 0 - 0 \end{pmatrix} (a^i{}_j v) = (\lambda + \epsilon_i - \epsilon_j)(a^i{}_j v).$$

A generator of the form $a^i{}_j$ with $i < j$ is called a *raising* generator, since it increases the weight of a weight vector, and corresponds to a positive root. Similarly, a generator of the form $a^i{}_j$ with $i > j$ is called a *lowering* generator, since it decreases the weight of a weight vector, and corresponds to a negative root.

The vector space spanned by the raising (resp. lowering) generators is denoted by $\mathcal{L}_+$ (resp. $\mathcal{L}_-$). The $\mathcal{L}_\pm$ are in fact subalgebras of $gl(n)$:

$$[\mathcal{L}_+, \mathcal{L}_+] \subseteq \mathcal{L}_+, \qquad [\mathcal{L}_-, \mathcal{L}_-] \subseteq \mathcal{L}_-.$$

(e.g. $[a^1{}_2, a^2{}_3] = a^1{}_3$, etc.) Also, $\mathcal{L}_0$ is a subalgebra of $gl(n)$. We thus obtain the following direct sum decomposition into Lie subalgebras (the *Cartan-Weyl decomposition*):

$$gl(n) = \mathcal{L}_- \oplus \mathcal{L}_0 \oplus \mathcal{L}_+.$$

Observe that $\mathcal{L}_0, \mathcal{L}_\pm$ are analogous to $L_0, L_\pm$ in the representation theory of $o(3)$. If $U_0, U_\pm$ denote the universal enveloping algebras of $\mathcal{L}_0, \mathcal{L}_\pm$ respectively, then the PBW theorem implies that the universal enveloping algebra U of $gl(n)$ may be written:

$$U = U_- U_0 U_+.$$

We choose $\{a^i{}_j : 1 \leq i, j \leq n\}$ as a basis for $gl(n)$. Then a basis for U is given by all products of basis elements of $gl(n)$, ordered with lowering generators on the left, raising generators on the right, and Cartan generators in the middle.

4.3 Highest weights and irreducible modules for $gl(n)$

Given a $gl(n)$-module V, we say that a weight ν occurs in V if there exists a non-zero $v \in V$ of weight ν. The corresponding *weight space* of V is the vector space V_ν of all such vectors:

$$V_\nu = \{v \in V : a^k{}_k v = \nu_k v,\ k = 1, \ldots, n\}.$$

We denote the *multiplicity* of the weight ν in V by $\dim(V_\nu)$. We have:

Theorem 3 (Cartan)**.** *Every finite-dimensional irreducible $gl(n)$ module V admits a basis of weight vectors. Moreover, V admits a* unique *vector of highest weight λ. The weight λ occurs with unit multiplicity in V, and all other weights in V have weight strictly less than λ.*

Thus we can *classify* and uniquely label *all* the finite-dimensional irreducible $gl(n)$-modules V in terms of their *highest weights* λ, and usually refer to the module as $V(\lambda)$. We prove the theorem in a similar fashion to Theorem 2.

Proof. Let V be a finite-dimensional irreducible $gl(n)$ module. From linear algebra, the $a^i{}_i$ must have at least one non-zero eigenvector $v \in V$, say of weight ν. Since V is irreducible, $V = Uv$, and as U is spanned by all generator products:

$$a^{i_1}{}_{j_1} a^{i_2}{}_{j_2} \cdots a^{i_k}{}_{j_k}$$

hence V is spanned by all vectors $a^{i_1}{}_{j_1} a^{i_2}{}_{j_2} \cdots a^{i_k}{}_{j_k} v$. This latter vector is a weight vector, of weight:

$$\nu + (\epsilon_{i_1} - \epsilon_{j_1}) + (\epsilon_{i_2} - \epsilon_{j_2}) + \cdots + (\epsilon_{i_k} - \epsilon_{j_k}).$$

(Since $a^k{}_l v$ has weight $\nu + (\epsilon_k - \epsilon_l)$, then $a^p{}_q a^k{}_l v$ has weight $\nu + (\epsilon_k - \epsilon_l) + (\epsilon_p - \epsilon_q)$.) Therefore, V has a basis of weight vectors. Since V is finite-dimensional, there must exist a weight vector $v_+ \in V$ with weight λ which is maximal under the lexical ordering. Then:

$$a^k{}_k v_+ = \lambda_k v_+, \qquad \mathcal{L}_+ v_+ = (0)$$

Thus $\mathcal{U}_+ v_+ = \mathbb{C} v_+$, so $\mathcal{U}_0 v_+ = \mathbb{C} v_+$, so in fact $V = \mathcal{U} v_+ = \mathcal{U}_- \mathcal{U}_0 \mathcal{U}_+ v_+ = \mathcal{U}_- v_+$. Now observe:

$$\begin{aligned} \mathcal{U}_- &= \mathbb{C} \oplus \mathcal{L}_- \oplus \mathcal{L}_-^2 \oplus \mathcal{L}_-^3 \oplus \cdots = \mathbb{C} \oplus \{\mathcal{L}_- \oplus \mathcal{L}_-^2 \oplus \mathcal{L}_-^3 \oplus \cdots\} \\ &= \mathbb{C} \oplus \{\mathbb{C} \oplus \mathcal{L}_- \oplus \mathcal{L}_-^2 \oplus \cdots\}\mathcal{L}_- = \mathbb{C} \oplus \mathcal{U}_- \mathcal{L}_- \end{aligned}$$

Then $V = \mathcal{U}_- v_+ = \mathbb{C} v_+ \oplus \mathcal{U}_- \mathcal{L}_- v_+$, where $\mathcal{U}_- \mathcal{L}_- v_+$ is spanned by weight vectors of weight strictly less than λ. Thus the maximal weight λ is unique. Then λ occurs with unit multiplicity in V, and all other weights $\mu \in V$ satisfy $\mu < \lambda$. □

Theorem 4. *Let V be a finite-dimensional $gl(n)$-module.*

1. *If V is cyclically generated by a highest-weight vector v_+, i.e. $V = \mathcal{U} v_+$ (V is* standard cyclic*), then V is irreducible.*
2. *V is irreducible if and only if it admits a unique (up to a scalar multiple) maximal weight vector.*

In fact, this theorem also applies to any finite-dimensional semi-simple Lie algebra.

4.4 Limitations on possible highest weights

We have seen that the finite-dimensional $o(3)$-modules have highest weights $l \in \frac{1}{2}\mathbb{Z}^+$. Here, we obtain analogous conditions on the highest weights λ of the finite-dimensional irreducible $gl(n)$-modules. To this end, corresponding to each of the $\frac{n}{2}(n-1)$ pairs of distinct integers $1 \le i < j \le n$, we have an $o(3)$-subalgebra of $gl(n)$, with generators:

$$L_+ = a^i{}_j, \qquad L_0 = \frac{1}{2}(a^i{}_i - a^j{}_j), \qquad L_- = a^j{}_i.$$

Now let v_+ be the maximal weight vector of the finite dimensional irreducible $gl(n)$-module $V(\lambda)$. Then:

$$L_+v_+ = a^i{}_j v_+ = 0, \qquad L_0 v_+ = \frac{1}{2}(\lambda_i - \lambda_j)v_+.$$

(Compare with $L_+\psi^l_l = 0$; and $L_0\psi^l_l = l\psi^l_l$.) Therefore v_+ must be the highest-weight vector for an irreducible $o(3)$-module contained in $V(\lambda)$, so that: $\frac{1}{2}(\lambda_i - \lambda_j) \in \frac{1}{2}\mathbb{Z}^+$, hence $\lambda_i - \lambda_j \in \mathbb{Z}^+$ for $i < j$. Thus the highest weights λ must have components satisfying:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n, \qquad \lambda_i - \lambda_j \in \mathbb{Z}. \tag{83}$$

A weight λ satisfying (83) is called a *dominant weight*, written $\lambda \in D^+$.

For example, let V be the n-dimensional $gl(n)$ module with basis $\{v^k\}$ satisfying $a^i{}_j v^k = \delta^k{}_j v^i$. Now $a^i{}_i v^k = \delta^k{}_i v^i = \delta^i{}_k v_k$ hence v_k is an eigenvector, indeed it is a weight vector, with weight ϵ_k. Clearly v_1 is the unique highest-weight vector, with weight ϵ_1. In fact ϵ_1 is the unique dominant weight in $V(\epsilon_1)$, and hence $V(\epsilon_1)$ must be irreducible.

It can be shown that corresponding to *each* dominant weight λ, there exists a finite-dimensional irreducible $gl(n)$-module, labelled $V(\lambda)$. The problem of the *explicit* construction (cf. the $o(3)$ case) of $gl(n)$ representations is in general non-trivial. The construction is aided by using a (*symmetry adapted*) basis (e.g. the *Gel'fand-Tsetlin basis*).

Let ρ be the half-sum of the positive roots of $gl(n)$, viz.:

$$\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha.$$

With this, we have *Weyl's dimension formula*

$$\dim[V(\lambda)] = \prod_{\alpha\in\Phi^+} \frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)}.$$

which will not be proved, but is useful to know.

Exercise 23. *Explicitly determine ρ for $gl(n)$ and then use Weyl's formula to calculate* $\dim[V(\lambda)]$ *for the following choices:*

(i) $\lambda = \epsilon_1$

(ii) $\lambda = 2\epsilon_1$

(iii) $\lambda = \epsilon_1 + \epsilon_2$

(iv) $\lambda = \epsilon_1 - \epsilon_n$

Exercise 24. *Define elements A^m_{ij} of the universal enveloping algebra $U(gl(n))$ recursively by*

$$A^m_{ij} = \sum_{k=1}^{n} a^i{}_k A^{m-1}_{kj}$$
$$A^1_{ij} = a^i{}_j.$$

Show that these elements satisfy

$$[a^i{}_j,\, A^m_{kl}] = \delta^k_j A^m_{il} - \delta^i_l A^m_{kj}.$$

Show that

$$I_m = \sum_{k=1}^{n} A^m_{kk}$$

are Casimir invariants; i.e.

$$[I_m,\, a^i{}_j] = 0 \qquad\qquad 1 \leq i, j \leq n.$$

Determine the eigenvalue of the Casimir invariant I_2 on any irreducible module with highest weight

$$\lambda = (\lambda_1,\, \lambda_2,\, ...,\, \lambda_n).$$

- Considering $gl(n)$, let e_i, $i = 1, \ldots, n$ be the usual elementary basis of $V = \mathbb{C}^n$. Then V constitutes an irreducible $gl(n)$-module with the definition:

  $$a^i{}_j e_k = \delta_{jk} e_i.$$

  Observe that e_i is a weight vector of weight ϵ_i, and $\epsilon_1 = (1, \dot{0}_{n-1})$, corresponding to the weight of e_1, is the unique highest-weight vector of V: we call V the fundamental *vector representation* of $gl(n)$.

  Now let $V \otimes V$ be the tensor product space with basis $e_i \otimes e_j$ (the *rank two tensor representation*), which becomes a $gl(n)$-module with the action:

  $$a^i{}_j(e_k \otimes e_q) = (a^i{}_j e_k) \otimes e_q + e_k \otimes (a^i{}_j e_q) = \delta_{jk}(e_i \otimes e_q) + \delta_{jq}(e_k \otimes e_i).$$

  We write the Clebsch-Gordan decomposition: $V = V^+ \oplus V^-$, where:

  $$\begin{aligned} V^+ &= \text{span}\{e_i \otimes e_j + e_j \otimes e_i \mid i \leq j,\ i, j = 1, \ldots, n\}, \\ V^- &= \text{span}\{e_i \otimes e_j - e_j \otimes e_i \mid i < j,\ i, j = 1, \ldots, n\}. \end{aligned}$$

  Here, V^+ is irreducible, with highest weight $(2, \dot{0}_{n-1})$ and highest-weight vector $e_1 \otimes e_1$; and V^- is irreducible, with highest weight $(1, 1, \dot{0}_{n-2})$, and highest-weight vector $\frac{1}{\sqrt{2}}(e_1 \otimes e_2 - e_2 \otimes e_1)$.

- More generally, consider the rank k tensor product module:

  $$V^k = \underbrace{V \otimes \cdots \otimes V.}_{k \text{ fold product}}$$

  Let V^+ be the subspace spanned by the symmetrised vectors $(i_1, i_2, \ldots, i_k = 1, \ldots, n)$:

  $$\frac{1}{\sqrt{k!}} \sum_{\pi \in S_k} e_{i_{\pi(1)}} \otimes e_{i_{\pi(2)}} \otimes \cdots \otimes e_{i_{\pi(k)}}, \qquad i_1 \leq i_2 \leq \cdots \leq i_k.$$

Similarly, let V^- be the subspace spanned by the antisymmetrised vectors ($i_1, i_2, \ldots, i_k = 1, \ldots, n$):

$$\frac{1}{\sqrt{k!}} \sum_{\pi \in S_k} \mathrm{sn}(\pi) e_{i_{\pi(1)}} \otimes e_{i_{\pi(2)}} \otimes \cdots \otimes e_{i_{\pi(k)}}, \qquad i_1 < i_2 < \cdots < i_k.$$

Then $V^\pm$ are irreducible $gl(n)$-modules with highest weights $(k, \dot{0}_{n-1})$ and $(\dot{1}_k, \dot{0}_{n-k})$ respectively. The representations afforded by $V^\pm$ are respectively called the *symmetric* and *antisymmetric rank k tensor representations* of $gl(n)$. In general the complete reduction of V^k into irreducible tensors of various symmetries can be achieved through the use of *Young diagrams*.

Exercise 25. *The Lie algebra $gl(n)$ with basis $\{a^i{}_j\}_{i,j=1}^n$ has a $gl(n-1)$ subalgebra with basis $\{a^i{}_j\}_{i,j=1}^{n-1}$. Viewing $gl(n)$ as a $gl(n-1)$-module under the commutator, determine the decomposition into irreducible submodules and give the highest weight for each submodule.*

4.5 The boson and fermion calculi

In many areas of quantum systems, especially many-body systems of identical particles, it is convenient to formulate problems in terms of canonical boson and/or fermion operators. For bosonic systems a set of n canonical boson operators $\{b_j,\, b_j^\dagger : 1 \leq j \leq n\}$ satisfy the commutation relations

$$[b_j,\, b_k] = [b_j^\dagger,\, b_k^\dagger] = 0, \qquad [b_j,\, b_k^\dagger] = \delta_{jk} I \tag{84}$$

It is usual to refer to the $\{b_j^\dagger\}$ as *creation operators* and the $\{b_j\}$ as *annihilation operators*.

Exercise 26. *Show that the canonical boson operators realise the $gl(n)$ algebra by setting*

$$a^j{}_k = b_j^\dagger b_k$$

The operator:

$$N = \sum_{j=1}^n a^j{}_j = \sum_{i=1}^n b_j^\dagger b_j,$$

is referred to as the boson *number operator*; it is the first order Casimir invariant of $gl(n)$.

For canonical fermion operators $\{c_j,\, c_j^\dagger : 1 \leq j \leq n\}$ the situation is similar except now the following *anti-commutation* relations are satisfied

$$\{c_j,\, c_k\} = \{c_j^\dagger,\, c_k^\dagger\} = 0, \qquad \{c_j,\, c_k^\dagger\} = \delta_{jk} I \tag{85}$$

where the anti-commutator $\{\,,\,\}$ is realised as

$$\{A,\, B\} = AB + BA.$$

Exercise 27. *Show that for arbitrary operators A, B, C, the derivation rule extends to incorporate anti-commutators as*

$$\begin{aligned} [AB,\, C] &= A\{B,\, C\} - \{A,\, C\}B \\ [A,\, BC] &= \{A,\, B\}C - B\{A,\, C\} \end{aligned}$$

Using the above result, show that the canonical fermion operators realise the $gl(n)$ algebra by setting

$$a^j{}_k = c_j^\dagger c_k$$

Analogously to the case of bosons, the operator

$$N = \sum_{j=1}^{n} a^j{}_j = \sum_{i=1}^{n} c_j^\dagger c_j,$$

is referred to as the fermion number operator.

4.6 Fermionic Fock space

Using the realisation of $gl(n)$ in terms of canonical boson or fermion operators, we can systematically construct representations. Below we describe this in terms of fermion operators. We introduce a *vacuum state* $|0\rangle$, defined by:

$$c_i \,|0\rangle = 0, \qquad i = 1, \ldots, n.$$

The *fermionic Fock space* $\mathcal{F}$ is spanned by states:

$$c_{i_1}^\dagger c_{i_2}^\dagger \cdots c_{i_k}^\dagger \,|0\rangle\,, \qquad i_1, i_2, \ldots, i_k = 1, \ldots, n. \tag{86}$$

Since the $c_i^\dagger$ anticommute, we have $\left(c_i^\dagger\right)^2 = 0$, from which it can be seen that no two subscripts $i_1, i_2, \ldots, i_k$ in (86) can be equal. Observe that:

$$c_i^\dagger c_j^\dagger \,|0\rangle = -c_j^\dagger c_i^\dagger \,|0\rangle\,,$$

and more generally:

$$c_{i_1}^\dagger c_{i_2}^\dagger \cdots c_{i_k}^\dagger \,|0\rangle = \mathrm{sn}(\pi) c_{i_{\pi(1)}}^\dagger c_{i_{\pi(2)}}^\dagger \cdots c_{i_{\pi(k)}}^\dagger \,|0\rangle\,,$$

where π is any permutation of the numbers $1, 2, \ldots, k$, and $\mathrm{sn}(\pi)$ is the sign of the permutation π (which is 1 if π is an even permutation, that is, it can be represented by an even number of index swaps, and -1 if π is odd). It follows that a full set of states is given by:

$$c_{i_1}^\dagger c_{i_2}^\dagger \cdots c_{i_k}^\dagger \,|0\rangle\,, \qquad i_1 < i_2 < \cdots < i_k; \quad i_1, i_2, \ldots, i_k = 1, \ldots, n, \tag{87}$$

where $k \leq n$ (i.e. we cannot have states with greater than n fermions). Observe:

$$a^i{}_i \,|0\rangle = c_i^\dagger c_i \,|0\rangle = 0, \qquad a^i{}_i c_j^\dagger \,|0\rangle = c_i^\dagger c_i c_j^\dagger \,|0\rangle = c_i^\dagger \{c_i, c_j^\dagger\} \,|0\rangle - c_i^\dagger c_j^\dagger c_i \,|0\rangle = \delta^j{}_i c_j^\dagger \,|0\rangle\,,$$

and, more generally:

$$a^i{}_i c^\dagger_{j_1} c^\dagger_{j_2} \cdots c^\dagger_{j_k} |0\rangle = (\delta^{j_1}{}_i + \delta^{j_2}{}_i + \cdots + \delta^{j_k}{}_i) c^\dagger_{j_1} c^\dagger_{j_2} \cdots c^\dagger_{j_k} |0\rangle \, .$$

It follows that the state (87) is an eigenstate of the number operator N with eigenvalue k, and is moreover a $gl(n)$ weight state, of weight:

$$\epsilon_{j_1} + \epsilon_{j_2} + \cdots + \epsilon_{j_k}.$$

We conclude that the states (87) are linearly independent, since their $gl(n)$ weights are distinct, and thus constitute a basis for $\mathcal{F}$. We will denote by $\mathcal{F}_k$ (for $k = 0, \ldots, l$) the space of eigenstates of N with eigenvalue k, referred to as the *space of k fermion states*. We then have a vector space decomposition:

$$\mathcal{F} = \bigoplus_{k=0}^{n} \mathcal{F}_k,$$

where $\mathcal{F}_0$ is the one-dimensional space spanned by $|0\rangle$. Clearly the dimension of $\mathcal{F}_k$ is given by the number of k-tuples $(j_1, j_2, \ldots, j_k)$ such that $j_1 < j_2 < \cdots < j_k$ and $j_1, j_2, \ldots, j_k = 1, \ldots, n$, viz.: $\dim(\mathcal{F}_k) = \left(\begin{array}{c} n \\ k \end{array} \right)$. In particular, $\mathcal{F}_n$ is one-dimensional, and spanned by the state:

$$\psi_0 = c^\dagger_1 c^\dagger_2 \cdots c^\dagger_n |0\rangle \, , \tag{88}$$

and $\dim(\mathcal{F}) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) = (1+1)^n = 2^n$.

Exercise 28. *For $n = 4$, there is a representation of the $o(4)$ algebra (satisfying commutation relations (34) given by fermion operators as a result of Ado's theorem, viz.*

$$\alpha^i{}_j = c^\dagger_i c_j + c_i c^\dagger_j$$

In the decomposition $\mathcal{F} = \bigoplus_{k=0}^{4} \mathcal{F}_k$, identify those $\mathcal{F}_k$ that are irreducible $o(4)$-modules.

5 Lie algebras in quantum mechanics

In quantum mechanics, the state of a physical system is specified by a vector (often called *wavefunction*) $\psi \in \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space. Physical observables are associated with self-adjoint operators acting on $\mathcal{H}$. In particular, the *Hamiltonian H* of the system, the quantum analogue of the energy in classical mechanics, constitutes a self-adjoint operator on $\mathcal{H}$. So too do the coordinates q_i and momenta p_i of particles of the system. These are no longer commutative operators, but instead satisfy the *canonical commutation relations*:

$$[q_k, p_j] = i\hbar\delta^k_j, \qquad [q_k, q_j] = [p_k, p_j] = 0.$$

Here $2\pi\hbar$ is Planck's constant – we commonly choose units that set $\hbar$ to unity. The state ψ of a system is determined by *Schrödinger's time-dependent wave equation*:

$$i\hbar\frac{\mathrm{d}\psi}{\mathrm{d}t} = H\psi. \tag{89}$$

A physical observable A evolves in time according to:

$$i\hbar\frac{\mathrm{d}A}{\mathrm{d}t} = [A, H]. \tag{90}$$

Thus, any self-adjoint operator A on $\mathcal{H}$ which commutes with H will be a constant of the motion. According to the *correspondence principle*, a quantum Hamiltonian may be constructed from a classical one by making the substitution: q_i remains the same, except that now it is an *operator*, and p_j is replaced with the operator $-i\hbar\partial/\partial q_j$. This way, (89) may be viewed as a partial differential equation. Setting

$$\psi(\vec{q}, t) = \psi(\vec{q})e^{-iEt/\hbar}$$

then ψ solves (89) provided that $\psi(\vec{q})$ satisfies *Schrödinger's time-independent wave equation*:

$$H\psi = E\psi. \tag{91}$$

It is a trademark of quantum mechanics that (in bound-state problems), the E are always given by a discrete set (*energy levels*):

$$E_n, \qquad n = 0, 1, 2, \ldots$$

which are eigenvalues of H. The corresponding eigenspaces:

$$\mathcal{E}_n = \{\psi \in \mathcal{H} \mid H\psi = E_n\psi\}$$

are referred to as the nth *energy level* of the system (the states in which the system has energy E_n).

We define the *degeneracy* of the nth energy state as $\dim(\mathcal{E}_n)$. Lie algebras are useful in understanding degeneracies. They are also used to provide quantum numbers for labelling physical states, and to simplify calculations for numerical approximations of complex systems (e.g. molecules).

5.1 Symmetries and Lie algebras

Associated with a Hamiltonian H of a quantum system is a set $\mathcal{D}$ of symmetry operators satisfying

$$[A,\, H] = 0, \qquad A \in \mathcal{D}.$$

In view of (90), such operators are constants of the motion. These operators span a complex vector space which we denote L_H.

Exercise 29. *Show that if A, $B \in L_H$ then $[A, B] \in L_H$.*

In view of the above, the elements of L_H close to form a Lie algebra. Now we define the *action* of one operator on another in terms of the commutator

$$A \circ B = [A,\, B]$$

which in particular satsifies the derivation property

$$A \circ (BC) = (A \circ B)C + B(A \circ C).$$

Powers of the action are defined recursively

$$A^m \circ B = A \circ (A^{m-1} \circ B).$$

In terms of this action we can define a transformation on B through

$$U_A(\lambda) \circ B = \exp(i\lambda A) \circ B, \qquad A^0 \circ B = B$$

where λ is a real parameter.

Exercise 30. *Show that*

$$U_A(\lambda) \circ B = \exp(i\lambda A) B \exp(-i\lambda A).$$

Hint: *Differentiate both sides of the expression.*

5.2 Single particle in a central potential

Let

$$r^2 = q_1^2 + q_2^2 + q_3^2, \qquad p^2 = p_1^2 + p_2^2 + p_3^2.$$

A Hamiltonian in a central potential has the form

$$H = \frac{p^2}{2m} + V(r). \tag{92}$$

Choosing units such that $\hbar = 1$ we have canonical commutation relations:

$$[q_j, p_k] = i\delta^j_k, \qquad [q_j, q_k] = [p_j, p_k] = 0. \tag{93}$$

Classically, the system admits spherical (or rotational) symmetry, as $V = V(r)$. We seek the quantum analogue of this result. The quantum angular momentum vector operator is $\vec{L} = \vec{r} \times \vec{p}$, alternatively written:

$$L_i = \varepsilon_{ijk} q_j p_k, \qquad \text{(implicit sum on } j \text{ and } k\text{)},$$

that is $L_1 = q_2 p_3 - q_3 p_2$, and cyclic permutations. Using (93), the L_i satisfy the $o(3)$ commutation relations:

$$[L_1, L_2] = iL_3 \qquad \text{(and cyclic permutations)}.$$

Exercise 31. *We say that $\vec{A}$ is an $o(3)$ tensor operator if*

$$[L_j,\ A_k] = i\varepsilon_{jkl}A_l$$

where ε_{jkl} is the alternating tensor. Throughout we adopt the Einstein summation convention over repeated indices and note that

$$\varepsilon_{jkl}\varepsilon_{jmn} = \delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm}.$$

1. *Show that*

$$\varepsilon_{jkl}\varepsilon_{jkn} = 2\delta_{ln}.$$

2. *Given that $\vec{A}$ and $\vec{B}$ are tensor operators, show that*

   (i) $\vec{r}$, $\vec{p}$, $\vec{A}\times\vec{B}$ *are all tensor operators*

   (ii) $[\vec{L},\ \vec{A}\cdot\vec{B}] = 0$

   (iii) $[\vec{L}\cdot\vec{L},\ \vec{A}] = 2\vec{A} + 2i\vec{A}\times\vec{L}$

   (iv) $\vec{L}\cdot\vec{r} = \vec{r}\cdot\vec{L} = \vec{L}\cdot\vec{p} = \vec{p}\cdot\vec{L} = 0$

   (v) $\vec{L}\times\vec{A} + \vec{A}\times\vec{L} = 2i\vec{A}$

   (vi) $(\vec{A}\times\vec{L})\cdot\vec{B} + (\vec{A}\times\vec{B})\cdot\vec{L} = 2i(\vec{A}\cdot\vec{B})$

   (vii) $\vec{A}\cdot(\vec{L}\times\vec{B}) + (\vec{A}\times\vec{B})\cdot\vec{L} = 2i(\vec{A}\cdot\vec{B})$

   (viii) $(\vec{A}\times\vec{L})\times\vec{B} = \vec{L}(\vec{A}\cdot\vec{B}) - \vec{A}(\vec{L}\cdot\vec{B}) + i(\vec{A}\times\vec{B})$.

3. *Assume that*

$$[r,\ q_j] = 0,$$
$$[r,\ p_j] = \frac{iq_j}{r}.$$

   For H of the form (92), deduce that

$$[L_j, H] = 0, \qquad j = 1,2,3.$$

The angular momentum vector $\vec{L}$ is a constant of the motion as it commutes with the Hamiltonian. It follows that the energy levels (eigenspaces) $\mathcal{E}_n$ of the system

$$\mathcal{E}_n = \{\psi \in \mathcal{H} \mid H\psi = E_n\psi\},$$

give rise to $o(3)$-modules. Indeed since $\psi \in \mathcal{E}_n$, we have $L_i\psi \in \mathcal{E}_n$ as:

$$H(L_i\psi) = L_iH\psi = L_iE_n\psi = E_n(L_i\psi).$$

In the absence of any further symmetries, we would expect $\mathcal{E}_n$ to give rise to an irreducible $o(3)$ module. Therefore, on the basis of $o(3)$ representation theory, we would expect the energy levels to be $(2l+1)$-fold degenerate, for $l \in \frac{1}{2}\mathbb{Z}^+$. Setting $n = 2l$, we label energy levels by $n = 0,1,2,3,\ldots$, and obtain $\dim(\mathcal{E}_n) = n+1$.)

This is generally true for spherical symmetry, but for two very important quantum mechanical systems, there is actually even greater degeneracy - the simple harmonic oscillator in three dimensions and the hydrogen atom. In following subsections, we will explore these.

5.3 Three-dimensional isotropic simple harmonic oscillator

Again, we have a central field problem, this time:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2$$

where ω is the frequency of the oscillator. The energy levels are labelled by a principal quantum number $n \in \mathbb{Z}^+$, and are $\frac{1}{2}(n+1)(n+2)$ fold degenerate, again greater than that predicted by $o(3)$ symmetry. As for the hydrogen atom, this occurs due to extra symmetries. We introduce operators:

$$b_j = \sqrt{\frac{m\omega}{2}}q_j + i\sqrt{\frac{1}{2m\omega}}p_j, \qquad b_j^\dagger = \sqrt{\frac{m\omega}{2}}q_j - i\sqrt{\frac{1}{2m\omega}}p_j.$$

These satisfy the canonical boson commutation relations:

$$[b_i, b_j^\dagger] = \delta^i_j, \qquad [b_i, b_j] = [b_i^\dagger, b_j^\dagger] = 0.$$

The operators $b^i_j = b_i^\dagger b_j$ satisfy the $gl(3)$ commutation relations:

$$[b^i_j, b^k_l] = \delta^k_j b^i_l - \delta^i_l b^k_j.$$

As before, the first order invariant $N = \sum_{i=1}^{n} b^i_i$ commutes with the $gl(3)$ generators. Now in terms of the b^i_j we have:

$$H = \omega\left(N + \frac{3}{2}I\right).$$

In particular, $[H, b^i_j] = 0$, for $i, j = 1, 2, 3$. Therefore, we expect energy levels to give rise to irreducible $gl(3)$-modules. To construct energy eigenvectors of H, we follow the approach of Section 4.6 and introduce the (normalised) vacuum state $|0\rangle$, defined such that:

$$b_i |0\rangle = 0, \qquad i = 1, 2, 3, \qquad \langle 0|0\rangle = 1.$$

The energy eigenstates are then constructed within Fock space (see Section 4.6), $\mathcal{F}_m$, which is spanned by the states:

$$\left\{(b_1^\dagger)^{m_1}(b_2^\dagger)^{m_2}(b_3^\dagger)^{m_3}|0\rangle \,|m_i \in \mathbb{Z}^+\right\}. \tag{94}$$

Note that:

$$b^i_j |0\rangle = b_i^\dagger b_j |0\rangle = 0, \qquad [b^i_j, b_k^\dagger] = \delta^k_j b_i^\dagger.$$

Exercise 32. *Show by induction that*

$$[b^i_j, (b_k^\dagger)^n] = n\delta^k_j b_i^\dagger (b_k^\dagger)^{n-1}.$$

From the above, it follows that states in (94) are eigenstates of the number operator $N = \sum_{i=1}^{3} b_i^\dagger b_i$, with eigenvalues $m = m_1 + m_2 + m_3$. The states in (94) are called *weight m boson states*; they form a (weight) basis for $\mathcal{F}_m$ with weight (m_1, m_2, m_3), and give rise to $gl(3)$ modules.[1]

Only the state $\left(b_1^\dagger\right)^m |0\rangle$ is maximal, as it's the only state on which *all* the raising operators will vanish. Therefore, $\mathcal{F}_m$ is an irreducible $gl(3)$ module with highest weight $(m, 0, 0)$. Observe that $\dim(\mathcal{F}_m) = \frac{1}{2}(m+1)(m+2)$ (cf. Exercise 23), and on $\mathcal{F}_m$, H has eigenvalue:

$$E_m = \omega \left(m + \frac{3}{2} \right).$$

In summary, the energy spectrum and degeneracies of the simple harmonic oscillator can be derived by exploiting the $gl(3)$ symmetry.

5.4 Non-relativistic hydrogen atom

This was one of the first systems to be studied quantum mechanically. Specifically we are interested in the Hamiltonian associated with the central field problem of a nucleus of charge Z (in units of e, the electron charge) and a single electron of mass m. The quantum analogue of the classical Hamiltonian is:

$$H = \frac{p^2}{2m} - \frac{Ze^2}{r}. \tag{95}$$

From spectroscopy, the energy levels are found to not be $(n+1)$-fold degenerate as predicted by $o(3)$ symmetry).[2] The different symmetry arises from the fact that this Hamiltonian admits extra constants of the motion in addition to L. These extra symmetries are often called *hidden symmetries*.

Firstly, we consider the *classical* hydrogen atom. For bound-state problems, the electron is 'captured' by the field, and the orbit is an ellipse. From Hamilton's equations:

$$\frac{\mathrm{d}q_j}{\mathrm{d}t} = \frac{\partial H}{\partial p_j} = \frac{p_j}{m}, \qquad \frac{\mathrm{d}p_j}{\mathrm{d}t} = -\frac{\partial H}{\partial q_j} = -\frac{Ze^2}{r^3} q_j,$$

thus $\vec{L} = \vec{r} \times \vec{p}$ is a constant in time, as:

$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = \left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \times \vec{p} \right) + \left(\vec{r} \times \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \right) = \frac{1}{m}(\vec{p} \times \vec{p}) - \frac{Ze^2}{r^3}(\vec{r} \times \vec{r}) = \vec{0}.$$

Clearly, $\vec{L} \perp \vec{r}$ and $\vec{p}$. The inverse square law is the *only* central force giving rise to closed orbits – in this case ellipses in time, in the plane perpendicular to $\vec{L}$. There is another

[1] The 3 is the number of degrees of freedom of the system in the physical space. In general, for an oscillator with n degrees of freedom, we would work with $gl(n)$ modules.

[2] Here, n is called the *Bohr principal quantum number*.

constant of the motion, the *Rungé vector* $\vec{A}$, peculiar to the inverse square law of force, pointing along the major axis of the ellipse. As $\vec{A}$ is in the plane of the ellipse, it is a linear combination of the vectors $\vec{r}$ and $\vec{L} \times \vec{p}$. Set

$$\vec{A} = \frac{1}{Ze^2m}(\vec{L} \times \vec{p}) + \frac{\vec{r}}{r}.$$

Exercise 33. *Show by direct differentiation that*

$$\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} = \vec{0}.$$

In the light of the classical case, we consider the quantum situation. The quantum analogue of the *Rungé vector* is the *Pauli-Rungé vector*:

$$\vec{A} = \frac{1}{2Ze^2m}(\vec{L} \times \vec{p} - \vec{p} \times \vec{L}) + \frac{\vec{r}}{r}.$$

Exercise 34. *Show that*

$$[A_j,\, H] = 0$$

with H given by (95). Note from Exercise 31 that $\vec{A}$ is an $o(3)$ tensor operator, so that

$$[\vec{A} \cdot \vec{A},\, L_j] = 0.$$

Show that

$$\vec{A} \cdot \vec{A} = I + \frac{2}{Z^2e^4m}(I + \vec{L} \cdot \vec{L})H$$

Introducing the operators

$$a_i = \sqrt{-\frac{Z^2e^4m}{2H}}A_i,$$

we have

$$I = -\frac{2}{Z^2e^4m}(I + \vec{a} \cdot \vec{a} + \vec{L} \cdot \vec{L})H \tag{96}$$

Exercise 35. *Show that*

$$[a_1, a_2] = iL_3, \qquad [L_1, L_2] = iL_3, \qquad [L_1, a_2] = ia_3 \qquad \textit{(and cyclic permutations)}$$

The above are seen to be the defining relations of $o(4)$, so we expect the energy levels of the hydrogen atom to give rise to irreducible $o(4)$-modules, not irreducible $o(3)$-modules. Now we consider representations of $o(4)$. The operators:

$$\vec{M}^{\pm} = \frac{1}{2}(\vec{L} \pm \vec{a})$$

give rise to two new $o(3)$ algebras which commute:

$$[M^+_\alpha, M^-_\beta] = 0$$

in agreement with our previous observation from Eq. 39:

$$o(4) \sim o(3) \oplus o(3). \tag{97}$$

To construct irreducible $o(4)$-modules, consider the tensor product space:

$$V(l_+, l_-) \equiv V^+_{l_+} \otimes V^-_{l_-}$$

where $V^\pm_{l_j}$, for $l_j \in \frac{1}{2}\mathbb{Z}^+$, denote irreducible $(2l_j+1)$-dimensional $o(3)$-modules for the Lie algebras $\vec{M}^\pm$ respectively, and under the action of $\vec{M}^\mp$ the space $V^\pm_{l_+}$ is a sum of one-dimensional modules. This space has dimension $(2l_+ + 1)(2l_- + 1)$, and admits a basis:

$$\{\psi^{l_+}_{m_+} \otimes \psi^{l_-}_{m_-};\ -l_j \leq m_j \leq l_j,\ j = \pm\}.$$

The action of the $o(4)$ algebra is

$$\begin{aligned}
M^+_\alpha(\psi^{l_+}_{m_+} \otimes \psi^{l_-}_{m_-}) &= M^+_\alpha \psi^{l_+}_{m_+} \otimes \psi^{l_-}_{m_-} + \psi^{l_+}_{m_+} \otimes M^+_\alpha \psi^{l_-}_{m_-} \\
&= (L_\alpha \psi^{l_+}_{m_+}) \otimes \psi^{l_-}_{m_-}, \\
M^-_\alpha(\psi^{l_+}_{m_+} \otimes \psi^{l_-}_{m_-}) &= M^-_\alpha \psi^{l_+}_{m_+} \otimes \psi^{l_-}_{m_-} + \psi^{l_+}_{m_+} \otimes M^-_\alpha \psi^{l_-}_{m_-} \\
&= \psi^{l_+}_{m_+} \otimes (L_\alpha \psi^{l_-}_{m_-}).
\end{aligned}$$

It can be shown that this defines an irreducible $o(4)$-module, and moreover, *all* irreducible $o(4)$-modules are of this form.

Exercise 36. *Show that*

$$\vec{A} \cdot \vec{L} = \vec{L} \cdot \vec{A} = 0,$$

and hence $\vec{M}^+ \cdot \vec{M}^+ = \vec{M}^- \cdot \vec{M}^-$.

Since $l_+(l_+ + 1) = l_-(l_- + 1)$, we set $l_+ = l_- = l_0$. These $(2l_0+1)^2$-dimensional modules $V(l_0, l_0)$ are called *symmetric* modules. We identify the integer $n = 2l_0 + 1$ with the Bohr principal quantum number, which explains the observed n^2-fold degeneracy as due to $o(4)$ symmetry.

Moreover,

$$\vec{M}^+ \cdot \vec{M}^+ = \vec{M}^- \cdot \vec{M}^- = \frac{1}{4}(\vec{L} + \vec{a})^2 = l_0(l_0+1) = \frac{1}{4}(n^2 - 1).$$

Now from (96) we have

$$1 = -\frac{2}{Z^2 e^4 m}(1 + (n^2 - 1))E_n,$$

which rearranges to give the *Bohr-Heisenberg formula* for the nth energy eigenvalue:

$$E_n = -\frac{Z^2 e^4 m}{2n^2}$$

with degeneracy n^2.

We have seen that the energy levels of the hydrogen atom constitute irreps of the Lie algebra $o(4)$. In the light of the above, we rewrite (97) as:

$$o(4) = o_+(3) \oplus o_-(3).$$

The $o(4)$-modules are given by the tensor products of $o(3)$-modules:

$$V(l_+, l_-) = V_{l_+} \otimes V_{l_-}.$$

For the hydrogen atom, $l_+ = l_- = l_0$, and the nth energy level is given by:

$$V_n = V_{l_0} \otimes V_{l_0}, \qquad n = 2l_0 + 1.$$

The total angular momentum $\vec{L} = \vec{M}^+ + \vec{M}^-$ coincides with the orbital angular momentum (recall that $\vec{M}^\pm = \frac{1}{2}(\vec{L} \pm \vec{a})$), so that the allowed orbital angular momenta l (always *integers*) are given by the vector addition rule (71):

$$V_n = \bigoplus_{l=0}^{n-1} V_l.$$

Thus, the energy eigenstates for the hydrogen atom may be labelled

$$|n,\ l,\ m\rangle\,, \qquad n > l \geq 0, \qquad -l \leq m \leq l, \tag{98}$$

where the quantum numbers n, l and m are $o(4)$, $o(3)$, and $o(2)$ *labels* respectively. We say that the states (98) are *symmetry adapted* to the subalgebra chain:

$$o(4) \supset o(3) \supset o(2).$$

This method of labelling states via subalgebra chains is commonly employed in quantum mechanics.

If we incorporate electron spin, the energy levels of the hydrogen atom are not n^2-degenerate, but $2n^2$-fold degenerate. Using a more accurate analysis using Dirac's relativistic theory in which the electron spin arises naturally, we find that the levels are *not actually* degenerate, but rather made up of finely separated levels. In the relativistic theory, the Lorentz Lie algebra plays a fundamental role and the true symmetry group for the relativistic hydrogen atom is actually $o(3)$ as in a typical spherically symmetric system.