Lie algebras

- named after Sofus Lie (1842–1899)
- defined over a field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$)

Definition

A Lie algebra over a field $\mathbb{F}$ is a vector space $\mathfrak{g}$ (over $\mathbb{F}$) endowed with a bilinear map, the Lie bracket

$$[\ ,\ ]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad (x,y) \mapsto [x,y],$$

satisfying

$$[x,x] = 0$$

Jacobi identity

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$

Antisymmetry:

$$0 = [x+y, x+y] \implies [x,y] = -[y,x]$$

The dimension of $\mathfrak{g}$ is defined as the dimension of $\mathfrak{g}$ as a vector space.

(*) Unless otherwise stated, all vector spaces (hence Lie algebras) are assumed finite-dim.

(*) Only need to consider Jacobi for x,y,z all distinct, since

$$[x,[x,z]] + [x,[z,x]] + \underbrace{[z,\underbrace{[x,x]}_{=0}]}_{=0} = [x,[x,z]] + [x,-[x,z]] = 0 \quad \checkmark$$

(*) The Lie bracket, a multiplication on $\mathfrak{g}$, with $[x,y]$ the Lie product of x and y.

For each $x \in \mathfrak{g}$, the adjoint mapping is given by

$$\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}, \quad y \mapsto [x,y]$$

Proposition

$$\mathrm{ad}_x \circ \mathrm{ad}_y - \mathrm{ad}_y \circ \mathrm{ad}_x = \mathrm{ad}_{[x,y]}$$

(usual composition)

Proof

$$\mathrm{ad}_x \circ \mathrm{ad}_y (z) = [x,[y,z]] \;, \quad \mathrm{ad}_y \circ \mathrm{ad}_x (z) = [y,[x,z]]$$

$$\mathrm{ad}_{[x,y]}(z) = [[x,y],z]$$

Jacobi & antisymmetry $\Longrightarrow$ result

A derivation is a linear map $\delta : \mathfrak{g} \to \mathfrak{g}$ satisfying

$$\delta([x,y]) = [\delta(x),y] + [x,\delta(y)]$$

⊛ Nilpotent if $\delta^n = 0$ for some $n \in \mathbb{N}$.

Proposition

For $x \in \mathfrak{g}$, ad_x is a derivation of $\mathfrak{g}$.

Proof

$$\mathrm{ad}_x([y,z]) = [x,[y,z]] = [[x,y],z] + [y,[x,z]]$$

$$= [\mathrm{ad}_x(y), z] + [y, \mathrm{ad}_x(z)]$$

The direct sum $\mathfrak{g}_1 \boxplus \mathfrak{g}_2$ of two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$:

Vector space $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ with

$$[x_1 + x_2, y_1 + y_2] := \underset{(\text{in } \mathfrak{g}_1)}{[x_1, y_1]} + \underset{(\text{in } \mathfrak{g}_2)}{[x_2, y_2]}$$

Let $\{x_1, \dots, x_d\}$ be a basis for $\mathfrak{g}$. Then, $[x_a, x_b] \in \mathfrak{g}$,

so $$[x_a, x_b] = \sum_{c=1}^{d} f_{ab}{}^{c} x_c \quad \text{for some } f_{ab}{}^{c} \in \mathbb{F}$$

(structure constants)

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(structure constants)

Ex Abelian: $[x, y] = 0 \quad \forall x, y \in \mathfrak{g}$

Ex One-dim Lie algebra $\mathfrak{a}$ — abelian

Ex $\mathbb{R}^3$ with $[u, v] := u \times v$ (not abelian)

Ex Three-dim Heisenberg algebra

$$[p, q] = c, \quad [c, p] = [c, q] = 0$$

— generated by p, q

$\mathfrak{g}$ is said to be generated by the subset $X \subseteq \mathfrak{g}$ if every element of $\mathfrak{g}$ can be expressed as a lin. comb. of elements of X and repeated Lie products of the elements of X:

$$\mathfrak{g} = \langle X \rangle$$

— The elements of X are then referred to as generators

Ex Given an associative algebra A with mult. $*$,

define $[x,y] := x * y - y * x$

$\hookrightarrow$ Lie(A): Lie algebra of the associative algebra A.

([x,y] often referred to as the commutator)

Ex A derivation of the alg. A is a lin. map $\delta: A \to A$ obeying the Leibniz rule

$$\delta(ab) = \delta(a)b + a\delta(b)$$

With Lie bracket $(\delta, \delta') \mapsto [\delta, \delta'] := \delta \circ \delta' - \delta' \circ \delta$, the set of all derivations of A forms a Lie alg.

Ex A_1 is a three-dim Lie alg. over $\mathbb{C}$, with basis elements e, h, f and Lie bracket

$$[h,e] := 2e, \quad [h,f] := -2f, \quad [e,f] := h$$

Let X and Y be non-empty subsets of $\mathfrak{g}$.

Product

$$[X, Y] := \operatorname{span}\{[x,y] \mid x \in X,\ y \in Y\}$$

Proposition $\quad [X, Y] = [Y, X]$

Proof - see notes

Lie subalg $\quad$ A subspace $\mathfrak{h}$ of $\mathfrak{g}$ is a Lie subalg of $\mathfrak{g}$ if $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$

$$([x,y] \in \mathfrak{h} \quad \forall x, y \in \mathfrak{h})$$

Ex
- ⊛ Zero vector space $\{0\}$
- ⊛ $\mathfrak{g}$ itself

$\Big\}$ - (trivial subalgebras)

⊛ A Lie subalg of $\mathfrak{g}$ different from $\mathfrak{g}$ is called a proper subalg of $\mathfrak{g}$.

<u>Ideal</u> A subspace i of g is an <u>ideal</u>

of g if $[g, i] \subseteq i$ $([x,y] \in i \ \forall x \in g, \forall y \in i)$

(*) i an ideal of $g \Rightarrow i$ a Lie subalg of g

(*) Converse need not be true

Ex: $\langle h \rangle$ not an ideal of A_1

<u>General notation</u>

$$\begin{cases} g: & \text{Lie alg} \\ h: & \text{Lie subalg} \\ i: & \text{ideal} \end{cases}$$

<u>Simple</u> g is <u>simple</u> if

(i) no nontrivial ideals (only $\{0\}$ and g)

(ii) not abelian

<u>Ex</u> A_1 is simple (see notes for details)

– one of our goals is to classify all simple Lie algebras over $\mathbb{C}$.

Review

<u>Def. of Lie alg.</u> g a vector space with bilinear map

$$[\ ,\ ]: g \times g \to g$$

satisfying

$$[x,x]=0 \qquad (\Leftrightarrow \text{anti-symmetry})$$

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0 \qquad (\text{Jacobi})$$

<u>Adjoint mappings</u>

$$ad_x : g \to g \ , \quad y \mapsto [x,y]$$

<u>Derivation</u> Linear map $\delta : g \to g$ satisfying

$$\delta([x,y]) = [\delta(x),y] + [x,\delta(y)]$$

$\circledast$ ad_x a derivation

<u>Ex</u> Associative alg $A \to Lie(A)$

where $[x,y] := x * y - y * x$

<u>Ex</u> A_1 , basis $\{e,h,f\}$,

$$[h,e]=2e \ , \quad [h,f]=-2f \ , \quad [e,f]=h$$

Prop For $\left\{\begin{array}{ll} \mathfrak{h}, \mathfrak{h}_1, \mathfrak{h}_2 & \text{subalgebras} \\ \mathfrak{i}, \mathfrak{i}_1, \mathfrak{i}_2 & \text{ideals} \end{array}\right\}$ of $\mathfrak{g}$,

we have

(i) $\mathfrak{h}_1 \cap \mathfrak{h}_2$ subalg

(ii) $\mathfrak{i}_1 \cap \mathfrak{i}_2$ ideal

(iii) $\mathfrak{h} + \mathfrak{i}$ subalg

(iv) $\mathfrak{i}_1 + \mathfrak{i}_2$ ideal

(v) $[\mathfrak{i}_1, \mathfrak{i}_2]$ ideal

Centre

$$Z(\mathfrak{g}) := \{x \in \mathfrak{g} \mid [x, \mathfrak{g}] = \{0\}\}$$

Prop

(i) $Z(\mathfrak{g})$ an ideal of $\mathfrak{g}$

(ii) $Z(\mathfrak{g}) = \mathfrak{g}$ iff $\mathfrak{g}$ abelian

(iii) $Z(\mathfrak{g}) = \{0\}$ for $\mathfrak{g}$ simple

Proof: Exercise!

Derived alg

$$\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$$

Prop

$\mathfrak{g}'$ an ideal of $\mathfrak{g}$

Proof: Special case of Prop ($\mathfrak{i}_1, \mathfrak{i}_2$ ideals $\Rightarrow [\mathfrak{i}_1, \mathfrak{i}_2]$ ideal), setting $\mathfrak{i}_1 = \mathfrak{i}_2 = \mathfrak{g}$

⊛ $\mathfrak{g}$ abelian: $\mathfrak{g}' = \{0\}$

⊛ $\mathfrak{g}$ simple: $\mathfrak{g}' = \mathfrak{g}$

From linear algebra

Let $W \subseteq V$ be vector spaces.

A coset of W is of the form

$$v + W := \{v + w \mid w \in W\}, \quad v \in V$$

⊛ $v + W = v' + W \iff v - v' \in W$

⊛ Unless $W = \{0\}$, the representative v of the coset $v + W$ is not unique.

Quotient space

$$V/W := \{v + W \mid v \in V\}$$

– the set of all cosets of W in V

⊛ V/W a vector space with

addition: $(v+W)+(v'+W) := (v+v')+W$

scalar multiplication: $a(v+W) := av + W$

zero vector: $0 + W$

Well-defined?

– must be indep. of representatives of the cosets

(Exercise: Verify!)

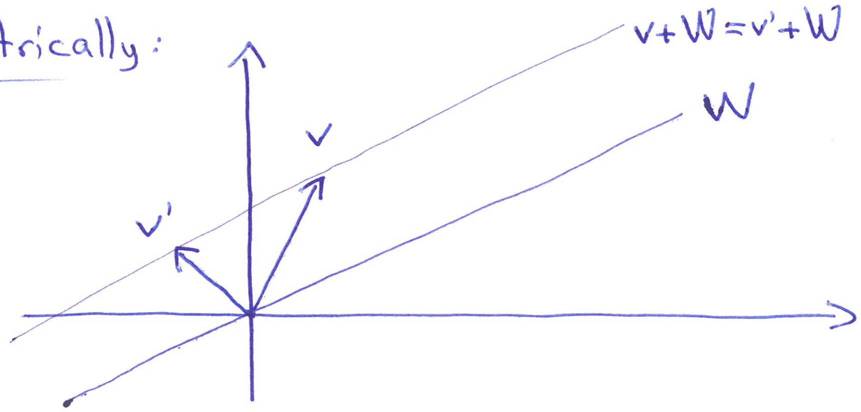
⊛ $\dim(V/W) = \dim V - \dim W$

Ex $V = \mathbb{R}^2$, $W = \operatorname{span}\{\begin{pmatrix}-2\\-1\end{pmatrix}\}$

Set $v = \begin{pmatrix}1\\2\end{pmatrix}$ and $v' = \begin{pmatrix}-1\\1\end{pmatrix}$ $\longrightarrow$ $v - v' = \begin{pmatrix}2\\1\end{pmatrix} \in W$

so $v + W = v' + W$

Geometrically:



Let i be an ideal of g. Then,

<u>cosets:</u> $x+i = \{x+z \mid z \in i\}$, $x \in g$

<u>quotient space:</u> $g/i = \{x+i \mid x \in g\}$

<u>Prop</u> g/i is a Lie alg with

$$[x+i, y+i] := [x,y] + i$$
(in g)

<u>Proof</u>

Well-defined? Yes (indep. of representatives)

Antisymmetry: $[x+i, x+i] = [x,x] + i = 0 + i$ ✓

Jacobi: see notes

<u>Prop</u> g/i abelian $\Leftrightarrow$ $g' \leq i$

<u>Proof</u> see notes

<u>Corollary</u>

g/g' is an abelian Lie alg

⊛ $\mathfrak{g}'$ measures the "non-abelianness" of $\mathfrak{g}$

– the "smaller" $\mathfrak{g}'$ is, the "larger" $\mathfrak{g}/\mathfrak{g}'$ is, and the "closer" $\mathfrak{g}$ is to being abelian.

⊛ In the extreme case: $\mathfrak{g}' = \{0\}$ and $\mathfrak{g}$ is abelian

From lin alg

Let $\phi : V \to W$ be a lin. map

kernel: $\ker(\phi) := \{ v \in V \mid \phi(v) = 0 \}$

image: $\mathrm{im}(\phi) := \{ w \in W \mid w = \phi(v) \text{ for some } v \in V \}$

⊛ $\ker(\phi)$ and $\mathrm{im}(\phi)$ are vector spaces

⊛ $\ker(\phi)$ measures the degree to which ϕ fails to be injective, with

$$\phi \text{ inj} \iff \ker(\phi) = \{0\}$$

<u>Homomorphism</u> Lin. map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$

such that $\phi(\underbrace{[x,y]}_{\text{in } \mathfrak{g}_1}) = \underbrace{[\phi(x), \phi(y)]}_{\text{in } \mathfrak{g}_2}$

– "structure preserving"

<u>Terminology</u>

<u>mono</u>-morphism : injective
<u>epi</u>- . . . . : surjective
<u>iso</u> - . . . . : bijective
<u>endo</u>- . . . . . . : $\mathfrak{g}_2 = \mathfrak{g}_1$
<u>auto</u>- . . . . . . : $\mathfrak{g}_2 = \mathfrak{g}_1$ and bijective

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(*) $\mathfrak{g}_1$ and $\mathfrak{g}_2$ <u>isomorphic</u> (written $\mathfrak{g}_1 \cong \mathfrak{g}_2$)

if $\exists$ isomorphism $\mathfrak{g}_1 \to \mathfrak{g}_2$

<u>Ex</u> $\mathfrak{a}(\mathbb{F}) \cong \mathbb{F}$

Natural homom.

$$\phi_i : g \to g/i \ , \quad x \mapsto x+i$$

Note: $\phi_i([x,y]) = [x,y]+i = [x+i, y+i] = [\phi_i(x), \phi_i(y)]$

⊛ Surjective, since any element of g/i is of the form $x+i$ for some $x \in g$, hence of the form $\phi_i(x)$.

⊛ $\ker(\phi_i) = i$

Homom. Lemma (for Lie alg)

Let $\phi : g_1 \to g_2$ be a Lie alg homom. Then,

(i) $\ker(\phi)$ is an ideal of g_1

(ii) $\mathrm{im}(\phi)$ is a Lie subalg of g_2

Proof: ϕ lin., so $\ker(\phi)$ and $\mathrm{im}(\phi)$ vector spaces.

(i) Let $x \in \ker(\phi)$ and $y \in g_1$. Then,

$$\phi([x,y]) = [\phi(x), \phi(y)] = [0, \phi(y)] = 0,$$

so $[x,y] \in \ker(\phi)$.

(ii) Let $x, y \in \mathrm{im}(\phi)$. Then, $\exists\, x', y' \in g_1$ such that $x = \phi(x')$ and $y = \phi(y')$, so

$$[x,y] = [\phi(x'), \phi(y')] = \phi([x',y']) \in \mathrm{im}(\phi)$$

Review

* $\mathfrak{h} \leq \mathfrak{g}$ a <u>Lie subalg</u> if $[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{h}$
* $\mathfrak{i} \leq \mathfrak{g}$ an <u>ideal</u> if $[\mathfrak{g},\mathfrak{i}] \leq \mathfrak{i}$
* $\mathfrak{g}$ <u>simple</u> if $\begin{cases} \text{no nontrivial ideals (only } \{0\} \text{ and } \mathfrak{g}) \\ \text{not abelian} \end{cases}$
* <u>A_1 is simple</u>
* <u>Centre</u>: $Z(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x,\mathfrak{g}] = \{0\}\}$ (an ideal)
* <u>Derived alg</u>: $\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]$ — " —
* <u>Cosets</u>: $x + \mathfrak{i} = \{x + z \mid z \in \mathfrak{i}\}$
* <u>Quotient alg</u>: $\mathfrak{g}/\mathfrak{i} = \{x + \mathfrak{i} \mid x \in \mathfrak{g}\}$
  with $[x+\mathfrak{i}, y+\mathfrak{i}] = [x,y] + \mathfrak{i}$
* $\mathfrak{g}/\mathfrak{i}$ abelian $\iff \mathfrak{g}' \leq \mathfrak{i}$
* <u>Homomorphism</u>
  Lin. map $\phi: \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\phi([x,y]) = [\phi(x),\phi(y)]$
* <u>Natural homom.</u>
  $\phi_{\mathfrak{i}}: \mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$, $x \mapsto x + \mathfrak{i}$, $\begin{cases} \phi_{\mathfrak{i}} \text{ surjective} \\ \ker(\phi_{\mathfrak{i}}) = \mathfrak{i} \end{cases}$
* <u>Homom. Lemma</u> $\phi: \mathfrak{g}_1 \to \mathfrak{g}_2$ a homom. Then,

  (i) $\ker(\phi)$ an ideal of $\mathfrak{g}_1$

  (ii) $\operatorname{im}(\phi)$ a Lie subalg of $\mathfrak{g}_2$

The isomorphism theorems (for Lie alg)

(i) Let $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a Lie alg homom. Then,

$$\mathfrak{g}_1 / \ker(\phi) \cong \operatorname{im}(\phi)$$

(ii) If $\mathfrak{h}$ is a Lie subalg and $\mathfrak{i}$ an ideal of $\mathfrak{g}$, then

$$(\mathfrak{h}+\mathfrak{i})/\mathfrak{i} \cong \mathfrak{h}/(\mathfrak{h}\cap\mathfrak{i})$$

(iii) If $\mathfrak{i} \leq \mathfrak{j}$ are ideals of $\mathfrak{g}$, then

$\mathfrak{j}/\mathfrak{i}$ is an ideal of $\mathfrak{g}/\mathfrak{i}$ and $(\mathfrak{g}/\mathfrak{i})/(\mathfrak{j}/\mathfrak{i}) \cong \mathfrak{g}/\mathfrak{j}$

<u>Proof</u> - see notes, with (ii) an exercise.

Classification

Every fin-dim simple complex Lie alg is isomorphic to one of the classical Lie alg $(r \in \mathbb{N})$

$A_r \cong sl(r+1)$, $r \geq 1$ (special)

$B_r \cong so(2r+1)$, $r \geq 2$ (orthogonal)

$C_r \cong sp(2r)$, $r \geq 3$ (symplectic)

$D_r \cong so(2r)$, $r \geq 4$ (orthogonal)

or one of the five exceptional Lie alg: E_6, E_7, E_8, F_4, G_2

Excluded duplications:

$A_1 \cong B_1 \cong C_1$, $B_2 \cong C_2$, $D_2 \cong A_1 \boxplus A_1$, $D_3 \cong A_3$

and $D_1 \cong a$ (abelian)

Note, $\mathbb{C}$ is algebraically closed ; $\mathbb{R}$ is not (f. ex. $x^2+1=0$)

Endomorphism ring For vector space V,

$$\mathrm{End}(V) := \{\phi : V \to V \mid \phi \text{ linear}\}$$

– a ring with mult. given by composition of maps

(*) $\dim(\mathrm{End}(V)) = (\dim V)^2$

General lin. alg $\mathfrak{gl}(V) = \mathrm{Lie}(\mathrm{End}(V))$

with $[x,y] := x \circ y - y \circ x$

Adjoint map

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(V), \quad x \mapsto \mathrm{ad}_x$$

– a Lie alg homom.

Prop $\ker(\mathrm{ad}) = Z(\mathfrak{g})$

Proof $\ker(\mathrm{ad})$ consists of all $x \in \mathfrak{g}$ for which $\mathrm{ad}_x = 0$, that is, for which $[x,y] = 0 \ \forall y \in \mathfrak{g}$

Corollary Any simple Lie alg is isom. to a lin. Lie alg

Proof $\mathfrak{g}$ simple $\Longrightarrow Z(\mathfrak{g}) = \{0\}$

$\Longrightarrow \ker(\mathrm{ad}) = \{0\}$

$\Longrightarrow \mathfrak{g} \cong \mathfrak{g}/\ker(\mathrm{ad})$

$\Longrightarrow \mathfrak{g} \cong \mathrm{im}(\mathrm{ad})$ (1st isom. th., using that ad is a homom)

and $\mathrm{im}(\mathrm{ad})$ is a Lie subalg of $\mathfrak{gl}(\mathfrak{g})$.

Square matrices

$n \in \mathbb{N}$

$$M_n(\mathbb{F}) = \{ n \times n \text{ matrix} \mid \text{entries in } \mathbb{F} \} \quad \text{- a vector space}$$

(*) $M_n(\mathbb{F})$ an alg with mult. given by the usual matrix mult.

General lin. alg

$$\mathfrak{gl}(n, \mathbb{F}) = \mathrm{Lie}(M_n(\mathbb{F}))$$

with $[x, y] = xy - yx$ (usual matrix products)

Matrix units

$$M_n(\mathbb{F}) \ni E_{ij} = \begin{array}{cc} & \begin{matrix} 1 & \cdots & j & \cdots & n \end{matrix} \\ \begin{matrix} 1 \\ \vdots \\ i \\ \vdots \\ n \end{matrix} & \begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix} \end{array}, \quad 1 \leq i, j \leq n$$

(*) $[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}$

Kronecker delta: $\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

Basis for $M_n(\mathbb{F})$: $\{ E_{ij} \mid 1 \leq i, j \leq n \}$

$\Longleftrightarrow \dim(\mathfrak{gl}(n, \mathbb{F})) = n^2$

Prop

$$Z(\mathfrak{gl}(n, \mathbb{F})) \cong \mathfrak{a}(\mathbb{F})$$

Proof - exercise!

For each $s \in M_n(\mathbb{F})$,

(matrix transpose)

$$gl_s(n,\mathbb{F}) := \{x \in M_n(\mathbb{F}) \mid x^t s = -sx\}$$

(*) $gl_{-s}(n,\mathbb{F}) = gl_s(n,\mathbb{F})$

Prop

$gl_s(n,\mathbb{F})$ is a Lie subalg of $gl(n,\mathbb{F})$

Proof

$gl_s(n,\mathbb{F})$ a vector space? ✓

Also,

$$\begin{aligned}
[x,y]^t s &= (xy-yx)^t s \\
&= y^t x^t s - x^t y^t s \\
&= y^t(-sx) - x^t(-sy) \\
&= (-y^t s)x - (-x^t s)y \\
&= syx - sxy \\
&= -s[x,y]
\end{aligned}$$

Trace

$$\mathrm{tr}: M_n(\mathbb{F}) \longrightarrow \mathbb{F}, \quad x \mapsto \sum_{k=1}^{n} x_{kk}$$

$\hookrightarrow$ a Lie alg homom. $gl(n,\mathbb{F}) \longrightarrow \mathbb{F}$, since

$$\mathrm{tr}([x,y]) = \mathrm{tr}(xy-yx) = \mathrm{tr}(xy) - \mathrm{tr}(yx) \underset{\text{cyclicity}}{=} 0 = [\mathrm{tr}(x), \mathrm{tr}(y)]$$

$$\big(\mathrm{tr}(xy) = \mathrm{tr}(yx), \quad \mathrm{tr}(abc) = \mathrm{tr}(bca) = \mathrm{tr}(cab)\big)$$

Traceless $n \times n$ matrices

$$\{x \in M_n(\mathbb{F}) \mid \mathrm{tr}(x) = 0\}$$

- a vector space
- a Lie alg with the commutator Lie bracket:

Special Linear Algebra $sl(n,\mathbb{F})$

Basis for $sl(n,\mathbb{F})$

$$\{E_{11}-E_{22}, E_{22}-E_{33}, \dots, E_{n-1,n-1}-E_{nn}\} \cup \{E_{ij} \mid i \neq j\}$$

$\hookrightarrow$ $\dim(sl(n,\mathbb{F})) = n^2 - 1$

Trace on $gl(n,\mathbb{F})$

$$tr : gl(n,\mathbb{F}) \longrightarrow \mathbb{F}$$

– a homom. with $\begin{cases} \ker(tr) = sl(n,\mathbb{F}) \\ tr \text{ surjective} \end{cases}$

$\underrightarrow{\text{(1st isom theorem)}}$ $gl(n,\mathbb{F})/sl(n,\mathbb{F}) \cong \mathbb{F}$

⊛ A coset of the form $x + sl(n,\mathbb{F})$ consists of the $n\times n$ matrices whose trace is $tr(x)$:

$$x + sl(n,\mathbb{F}) = \{ y \in M_n(\mathbb{F}) \mid tr(y) = tr(x) \}$$

$sl(2,\mathbb{F})$

An element of $sl(2,\mathbb{F})$ is of the form

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \qquad a,b,c \in \mathbb{F}$$

↪ basis: $\Big\{ \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{=e}, \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{=h}, \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{=f} \Big\}$

↪ $[h,e] = 2e$, $[h,f] = -2f$, $[e,f] = h$

so $\quad \underline{\underline{sl(2,\mathbb{C}) \cong A_1}}$

$sl(3,\mathbb{F})$ – basis:

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_\theta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad f_\theta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$sl(2,\mathbb{F})$ subalg

$$\langle e_1, h_1, f_1 \rangle \cong sl(2,\mathbb{F})$$

⊛ An ideal ?

⊛ More copies of $sl(2)$ inside $sl(3)$.

Embedding

Let $\mathfrak{h}$ be a Lie subalg of $\mathfrak{g}$.

Then, $\exists$ monomorphism (inj. homom.) $\mathfrak{h} \to \mathfrak{g}$,

called an embedding of $\mathfrak{h}$ in $\mathfrak{g}$.

Prop For $n \geq 2$, $sL(n,\mathbb{C})$ is simple.

Symplectic alg

For $n = 2r$ ($r \in \mathbb{N}$), set $s = \begin{pmatrix} 0_r & I_r \\ -I_r & 0_r \end{pmatrix}$

Then, define

$$sp(2r, \mathbb{F}) := gl_s(2r, \mathbb{F})$$
$$= \{ x \in M_{2r}(\mathbb{F}) \mid x^t s = -sx \}$$

— a Lie subalg of $gl(2r, \mathbb{F})$

Prop

$x \in sp(2r, \mathbb{F}) \iff x = \begin{pmatrix} m & p \\ q & -m^t \end{pmatrix}$ for some $m, p, q \in M_r(\mathbb{F})$ with p and q symmetric

Proof

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in M_r(\mathbb{F})$$

$\hookrightarrow$ $x^t s = \begin{pmatrix} -c^t & a^t \\ -d^t & b^t \end{pmatrix}$, $-sx = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}$

$\hookrightarrow$ $b = b^t$, $c = c^t$, $d = -a^t$

Symplectic alg

For $n = 2r$ $(r \in \mathbb{N})$, set $s = \begin{pmatrix} O_r & I_r \\ -I_r & O_r \end{pmatrix}$

Then, define

$$sp(2r, \mathbb{F}) := gl_s(2r, \mathbb{F})$$
$$= \{ X \in M_{2r}(\mathbb{F}) \mid X^t s = -sX \}$$

— a Lie subalg of $gl(2r, \mathbb{F})$

Prop

$X \in sp(2r, \mathbb{F}) \iff X = \begin{pmatrix} m & p \\ q & -m^t \end{pmatrix}$ for some $m, p, q \in M_r(\mathbb{F})$ with p and q symmetric

Proof

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in M_r(\mathbb{F})$$

$\hookrightarrow$ $X^t s = \begin{pmatrix} -c^t & a^t \\ -d^t & b^t \end{pmatrix}$, $\quad -sX = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}$

$\hookrightarrow$ $b = b^t$, $\quad c = c^t$, $\quad d = -a^t$

Cor.

$sp(2r,\mathbb{F})$ is a $(2r^2+r)$-dim Lie subalg of $sl(2r,\mathbb{F})$.

Proof

(*) $\dim = \underset{\substack{\uparrow \\ m}}{r^2} + \underset{\substack{\uparrow \\ p}}{\tfrac{1}{2}r(r+1)} + \underset{\substack{\uparrow \\ q}}{\tfrac{1}{2}r(r+1)} = 2r^2+r$

(*) $tr(x) = tr(m) + tr(-m^t) = 0$

Prop For $r \geq 2$, $sp(2r,\mathbb{C})$ is simple

Hamiltonian dynamics

- based on generalised notions of position and momentum used to describe the evolution of physical systems.

$$\left.\begin{aligned} \frac{dq}{dt} &= \frac{\partial H}{\partial p} \\ \frac{dp}{dt} &= -\frac{\partial H}{\partial q} \end{aligned}\right\} \Rightarrow \frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = s\begin{pmatrix} \partial_q H \\ \partial_p H \end{pmatrix}$$

$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- extends to higher dimensions

Antisymmetric matrix

$x \in M_n(\mathbb{F})$ such that $x^t = -x$

$\hookrightarrow$ $[x, y]^t = -[x, y]$ for x, y antisymmetric

Orthogonal alg

$$so(n, \mathbb{F}) := \{ x \in M_n(\mathbb{F}) \mid x^t = -x \}$$

with the commutator Lie bracket.

* $tr(x^t) = tr(x) \Longrightarrow tr(x) = 0 \quad \forall x \in so(n, \mathbb{F})$

  hence $so(n, \mathbb{F})$ & $so(n, \mathbb{F})$ a Lie subalg of $sl(n, \mathbb{F})$

* $so(n, \mathbb{F}) = gl_s(n, \mathbb{F})$, $s = I_n$

Basis

$$\{ E_{ij} - E_{ji} \mid 1 \leq i < j \leq n \}$$

$\hookrightarrow$ $\dim(so(n, \mathbb{F})) = \frac{1}{2} n(n-1)$

$so(3,\mathbb{F})$ – convenient basis:

$$R_1 = -(E_{23}-E_{32}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$R_2 = E_{13}-E_{31} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$R_3 = -(E_{12}-E_{21}) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\hookrightarrow [R_1,R_2]=R_3,\ [R_2,R_3]=R_1,\ [R_3,R_1]=R_2$

Prop For $n=3$ or $n\geq 5$, $so(n,\mathbb{C})$ is simple

⊛ $so(4,\mathbb{C}) \cong so(3,\mathbb{C}) \boxplus so(3,\mathbb{C})$ – not simple

⊛ $so(3,\mathbb{C}) \cong A_1$

Rotations in $\mathbb{R}^3$ – about x-axis:

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} = I_3 + \theta R_1 + O(\theta^2)$$

generator of infinitesimal rotations

⊛ Special relativity is based on Minkowski space (not Euclidean) and to describe "rotations" in Minkowski space, we generalise the orthogonal Lie algebras. As it turns out, we need $s \neq I_n$ but with eigenvalues still satisfying $\lambda^2 = 1$

$$I_n \longrightarrow I_{p,q} = \begin{pmatrix} I_p & O_{p\times q} \\ O_{q\times p} & -I_q \end{pmatrix} \quad , \quad n = p+q, \ p,q \in \mathbb{N}_0$$

Indefinite orthogonal alg

$$so(p,q,\mathbb{F}) := gl_s(p+q,\mathbb{F}) \quad , \quad s = I_{p,q}$$

$$= \{ X \in M_{p+q}(\mathbb{F}) \mid X^t I_{p,q} = -I_{p,q} X \}$$

with the commutator Lie bracket

⊛ $so(p,q,\mathbb{C}) \cong so(p+q,\mathbb{C})$

- so generalisation only worthwhile for $\mathbb{F} = \mathbb{R}$

⊛ $so(p,q,\mathbb{R})$ a Lie subalg of $sl(p+q,\mathbb{R})$

⊛ $so(p,q,\mathbb{R}) \cong so(q,p,\mathbb{R})$

⊛ $\dim so(p,q,\mathbb{R}) = \frac{1}{2}(p+q)(p+q-1)$

Conjugate transpose (hermitian transpose)

For A an $m \times n$ matrix: $A^\dagger := \overline{A^t} = (\bar{A})^t$

$$(A^\dagger)_{ij} = \overline{A_{ji}}$$

Hermitian $\quad A^\dagger = A \quad$ (requires A square)

Anti-hermitian $\quad A^\dagger = -A \quad$ (— . . —)

⊛ In quantum mechanics, physical observables are usually represented by hermitian operators, ensuring that the eigenvalues are real, allowing us to relate them to measurements of physical quantities

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Vector space?

$\{X \in M_n(\mathbb{C}) \mid X^\dagger = -X\}$ a <u>real</u> vector space

– <u>not</u> a vector space over $\mathbb{C}$, since

X anti-hermitian $\Longrightarrow$ iX hermitian

Unitary Lie alg: $\mathfrak{u}(n) := \{X \in M_n(\mathbb{C}) \mid X^\dagger = -X\}$

Special — " " — : $\mathfrak{su}(n) := \{X \in \mathfrak{u}(n) \mid \operatorname{tr}(X) = 0\}$

(both with the commutator Lie bracket)

Bases (for $n \geq 2$)

$\mathfrak{u}(n)$: $\{ i E_{ii} \mid i = 1, \dots, n\} \cup \{E_{ij} - E_{ji}, \; i(E_{ij} + E_{ji}) \mid 1 \leq i < j \leq n\}$

$\mathfrak{su}(n)$: $\{ i(E_{11} - E_{22}), i(E_{22} - E_{33}), \dots, i(E_{n-1,n-1} - E_{nn})\} \cup \{ \quad \}$

$\hookrightarrow$ $\dim \mathfrak{u}(n) = n^2$, $\dim \mathfrak{su}(n) = n^2 - 1$

$(*)$ $\mathfrak{u}(n)/\mathfrak{su}(n) \cong \mathbb{R}$

$\mathfrak{su}(2)$ – basis: $\{ u_k = -\frac{i}{2} \sigma_k \mid k = 1, 2, 3\}$

where the Pauli matrices (hermitian!) are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\hookrightarrow$ $[u_1, u_2] = u_3, \quad [u_2, u_3] = u_1, \quad [u_3, u_1] = u_2$

A representation of $\mathfrak{g}$ on V is a Lie alg homom.

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V)$$

- V called the representation space
- $\dim \rho := \dim V$
- Notation: $\rho(x)v$, $v \in V$

A (left) $\mathfrak{g}$-module is a vector space V with

$$\mathfrak{g} \times V \to V, \qquad (x,v) \mapsto xv$$

satisfying

(M1): $(ax+by)v = a(xv) + b(yv)$

(M2): $x(av+bw) = a(xv) + b(xw)$

(M3): $[x,y]v = x(yv) - y(xv)$

- Given rep $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$

  $\hookrightarrow$ V a $\mathfrak{g}$-module with $xv = \rho(x)v$

- Given $\mathfrak{g}$-module V

  $\hookrightarrow$ $\rho(x)v = xv$ defines rep $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$

Rep ρ is called <u>faithful</u> if $\ker(\rho) = \{0\}$

<u>Adjoint rep</u>

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

– a rep of $\mathfrak{g}$ on itself

$\circledast$ ad faithful $\iff$ $Z(\mathfrak{g}) = \{0\}$

<u>Ex:</u> $\mathfrak{sl}(2, \mathbb{F})$ – ad is faithful

<u>Ex:</u> $\mathfrak{gl}(2, \mathbb{F})$ – ad is <u>not</u> faithful

since $Z(\mathfrak{gl}(2, \mathbb{F})) = \mathrm{span}\{I_2\}$

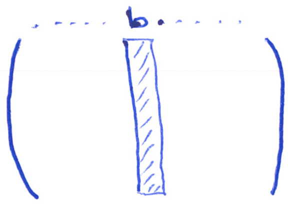
<u>Matrix rep</u> Given rep $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$

$\hookrightarrow$ matrix rep $R_\rho(X)$ of $\rho(X)$

<u>Ex</u> – adjoint rep, ordered basis $\{X_1, \dots, X_d\}$

$\hookrightarrow$ $\mathrm{ad}_{X_a}(X_b) = [X_a, X_b] = \sum_c f_{ab}{}^c X_c$

$\hookrightarrow$ $R_{\mathrm{ad}}(X_a) =$



$\hookrightarrow$ $(R_{\mathrm{ad}}(X_a))_{cb} = f_{ab}{}^c$

Ex: A_1 with ordered basis $\{e, h, f\}$

$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h$

$$R_{ad}(e) = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R_{ad}(h) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$R_{ad}(f) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

A <u>submodule</u> of the $\mathfrak{g}$-module V is a $\mathfrak{g}$-invariant subspace $U \subseteq V$. $\quad (\mathfrak{g}(U) \subseteq U)$

Thus, U is a $\mathfrak{g}$-module in its own right.

The $\mathfrak{g}$-module V is <u>irreducible</u> if its only submodules are $\{0\}$ and V.

Otherwise, V is <u>reducible</u>

<u>Submodule generated by vector</u>

For $v \in V$, $\quad \mathfrak{g}(v) = \operatorname{span}\{x_1 \cdots x_n v \mid x_1, \ldots, x_n \in \mathfrak{g}\}$

$$\text{where } \begin{cases} x_1 \cdots x_n v = x_1(x_2(\cdots(x_n v)\cdots)) \\ x_1 \cdots x_n v = v \quad \text{for } n = 0 \end{cases}$$

<u>Prop</u>

$\mathfrak{g}$-module V irred. $\iff V \subseteq \mathfrak{g}(v)$ for each nonzero $v \in V$

<u>Proof</u> – see notes

(*) Since $V \supseteq \mathfrak{g}(v)$, the condition $V \subseteq \mathfrak{g}(v)$ is equiv to $V = \mathfrak{g}(v)$

(*) Any vector in irred V can be reached from any nonzero vector in V by the action of $\mathfrak{g}$

Intertwiner

Let V_1, V_2 be $\mathfrak{g}$-modules with reps ρ_1, ρ_2.

A lin. map $\phi : V_1 \to V_2$ is an <u>intertwiner</u> if

$$\phi \circ \rho_1(x) = \rho_2(x) \circ \phi \qquad \forall x \in \mathfrak{g}$$

Commutative diagram

$$\begin{tikzcd} V_1 \arrow[r, "\rho_1(x)"] \arrow[d, "\phi"] & V_1 \arrow[d, "\phi"] \\ V_2 \arrow[r, "\rho_2(x)"] & V_2 \end{tikzcd}$$

- all paths between a given ordered pair of objects correspond to the same combined map.

Isomorphic $\mathfrak{g}$-modules

If ϕ is invertible, then $\quad \rho_1(x) = \phi^{-1} \circ \rho_2(x) \circ \phi$

$\hookrightarrow$ V_1 and V_2 are <u>isomorphic</u> $\mathfrak{g}$-modules

(equivalent/indistinguishable as $\mathfrak{g}$-modules)

Prop Let $\phi : V \to W$ be an intertwiner. Then,

(i) $\ker(\phi)$ is $\mathfrak{g}$-invariant (subspace of V)

(ii) $\operatorname{im}(\phi)$ — ... — (— ... — W)

Proof: Exercise

Schur's Lemma (for Lie alg)

Let V, W be irred. $\mathfrak{g}$-modules, and $\phi : V \to W$ an intertwiner

Then,

$$\phi = 0 \quad \text{or} \quad \phi \text{ invertible}$$

Proof

$$V \text{ irred.} \Rightarrow \ker(\phi) = \begin{cases} \{0\} & \Rightarrow \phi \text{ injective} \\ \text{or} & \\ V & \Rightarrow \phi = 0 \end{cases}$$

$$W \text{ irred.} \Rightarrow \operatorname{im}(\phi) = \begin{cases} \{0\} & \Rightarrow \phi = 0 \\ \text{or} & \\ W & \Rightarrow \phi \text{ surjective} \end{cases}$$

$$\longrightarrow \quad \phi = 0 \quad \text{or} \quad \phi \text{ invertible}$$

Corollary Let $\mathfrak{g}$ be complex and V, W irreducible $\mathfrak{g}$-modules.

(i) If $\varphi : V \to V$ an intertwiner, then

$\varphi = \lambda \, \mathrm{id}_V$ for some $\lambda \in \mathbb{C}$

(ii) If ϕ_1, ϕ_2 nonzero intertwiners $V \to W$, then

$\phi_1 = \lambda \phi_2$ for some $\lambda \in \mathbb{C}^\times$

Proof (i): Over $\mathbb{C}$, φ has at least one eigenvalue $\lambda \in \mathbb{C}$.

Let ρ be the rep corresponding to $\mathfrak{g}$-module V. Then,

$$(\varphi - \lambda \, \mathrm{id}_V) \circ \rho(x) = \varphi \circ \rho(x) - \lambda \rho(x)$$
$$= \rho(x) \circ \varphi - \rho(x) \lambda$$
$$= \rho(x) \circ (\varphi - \lambda \, \mathrm{id}_V)$$

so $(\varphi - \lambda \, \mathrm{id}_V)$ an intertwiner from V to V.

Let $v \in V$ be an eigenvector corresponding to λ.

Then, $(\varphi - \lambda \, \mathrm{id}_V) v = 0$, so $\varphi - \lambda \, \mathrm{id}_V$ not invertible

$\xRightarrow{\text{Schur}}$ $\varphi - \lambda \, \mathrm{id}_V = 0$, that is, $\varphi = \lambda \, \mathrm{id}_V$.

Proof of (ii) in the notes

Dual space

Given vector space V,

$$V^* := \{ f : V \to \mathbb{F} \mid f \text{ linear} \}$$

– a vector space if $(f+g)(v) = f(v) + g(v)$ and $(af)(v) = a(f(v))$

⊛ $\dim V^* = \dim V$

Dual module

Given $\mathfrak{g}$-module V,

V^* is a $\mathfrak{g}$-module (called dual module) if the action of $x \in \mathfrak{g}$ on $f \in V^*$ is given by the map $xf \in V^*$ defined by

$$xf : V \to \mathbb{F}, \quad v \mapsto -f(xv)$$

Check: (M1), (M2) ✓

(M3): For all $x, y \in \mathfrak{g}$, $f \in V^*$, $v \in V$,

$$\begin{aligned}
([x,y]f)(v) &= -f([x,y]v) \\
&= -f(x(yv) - y(xv)) \\
&= -f(x\underbrace{(yv)}_{\in V}) + f(y\underbrace{(xv)}_{\in V}) \\
&= \underbrace{(xf)}_{\in V^*}\underbrace{(yv)}_{\in V} - \underbrace{(yf)}_{\in V^*}\underbrace{(xv)}_{\in V} \\
&= -y(xf)(v) + x(yf)(v) \\
&= (x(yf) - y(xf))(v)
\end{aligned}$$

Direct sum module

$V \oplus W$ with $x(v \oplus w) = (xv) \oplus (xw)$

– simplified notation: $x(v+w) = xv + xw$

⊛ $\dim(V \oplus W) = \dim V + \dim W$

A $\mathfrak{g}$-module is completely reducible if it can be written as a direct sum of irred. $\mathfrak{g}$-modules

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A $\mathfrak{g}$-module is indecomposable if it cannot be written as a direct sum of two nonzero modules. Otherwise, it is decomposable.

Tensor product module

$V \otimes W$ with $x(v \otimes w) = (xv) \otimes w + v \otimes (xw)$

extended linearly to all of $V \otimes W$

⊛ $\dim(V \otimes W) = (\dim V)(\dim W)$

Goal: Construct associative alg whose reps are "equivalent" to the reps of $\mathfrak{g}$

$\hookrightarrow$ Universal Enveloping Algebra $\mathcal{U}(\mathfrak{g})$

First, for vector space V, define

$$V^{\otimes 0} := \mathbb{F}, \quad V^{\otimes n} := V \otimes \ldots \otimes V, \quad n \in \mathbb{N}$$

Then, $V^{\otimes n}$ a vector space with $\dim V^{\otimes n} = (\dim V)^n$

Second, form <u>infinite</u> direct sum

$$T_V := \bigoplus_{n=0}^{\infty} V^{\otimes n}$$

whose elements are given by <u>finite</u> sums of terms

Third, introduce natural mult.

$$(u_1 \otimes \ldots \otimes u_m) * (v_1 \otimes \ldots \otimes v_n) = u_1 \otimes \ldots u_m \otimes v_1 \otimes \ldots \otimes v_n$$

linearly extended to all of T_V

$\hookrightarrow$ T_V the <u>tensor alg</u> of V

$\circledast$ Identity element of T_V given by $1 \in V^{\otimes 0}$

Two-sided ideal of tensor alg $T\mathfrak{g}$

$$i_{\mathfrak{g}} := \operatorname{span}\{a \otimes (\underbrace{x \otimes y - y \otimes x}_{\in \mathfrak{g}^{\otimes 2}} - \underbrace{[x,y]}_{\in \mathfrak{g}^{\otimes 1}}) \otimes b \mid x, y \in \mathfrak{g};\ a, b \in T\mathfrak{g}\}$$

Quotient alg

$$\mathcal{U}(\mathfrak{g}) := T\mathfrak{g} / i_{\mathfrak{g}}$$

with mult. $(a + i_{\mathfrak{g}}) \cdot (b + i_{\mathfrak{g}}) := a \otimes b + i_{\mathfrak{g}}$

– known as the universal enveloping alg of $\mathfrak{g}$

In $\mathcal{U}(\mathfrak{g})$, $x \otimes y - y \otimes x - [x,y] = 0$

In $\mathcal{U}(A_1)$, $e \otimes f + f \otimes e = ([e,f] + f \otimes e) + f \otimes e$

$$= h + 2 f \otimes e$$

(sloppily: $ef + fe = h + 2fe$)

Caution: Although $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = 0$,

$$e^2 = e \otimes e \neq 0 \text{ in } \mathcal{U}(sl(2))$$

Any element of $\mathfrak{g}$ can be viewed as an element of $T\mathfrak{g}$ hence of $\mathcal{U}(\mathfrak{g})$

$\mathcal{U}(\mathfrak{g})$ instead of $\mathfrak{g}$

Advantage: - manipulate expressions associatively as $\underbrace{xyz}_{\in \mathcal{U}(\mathfrak{g})} v \in V$

- rep theories of $\mathcal{U}(\mathfrak{g})$ and $\mathfrak{g}$ equivalent

Price to pay: - work with infinite-dim alg

Poincaré-Birkhoff-Witt theorem (PBW theorem)

Let $\{x_1, \dots, x_d\}$ be an ordered basis for $\mathfrak{g}$. Then,

the elements

$$x_1^{r_1} \dots x_d^{r_d} \quad , \quad r_1, \dots, r_d \in \mathbb{N}_0,$$

form a basis for $\mathcal{U}(\mathfrak{g})$.

↑
(infinite!)

$A_1 \cong sl(2,\mathbb{C}) \equiv sl(2)$

$$[h,e] = 2e \ , \ [h,f] = -2f, \ [e,f] = h$$

Let V be an $sl(2)$-module and $v \in V$ an eigenvector of h with eigenvalue λ. Then,

$$h(ev) = [h,e]v + e(hv) = 2ev + e(\lambda v) = (\lambda+2)ev$$

$$h(fv) = \cdots\cdots\cdots\cdots = (\lambda-2)fv$$

$\hookrightarrow \begin{cases} e & \text{a } \underline{\text{raising}} \text{ operator} \\ f & \text{a } \underline{\text{lowering}} \ \text{''} \end{cases}$

– To describe irred. $sl(2)$-reps/modules, we introduce

$\underline{\mathbb{C}[X,Y]}$: space of polynomials in X,Y with complex coefficients

(*) For each $d \in \mathbb{N}_0$,

$$V_d := \text{span}_{\mathbb{C}}\{X^k Y^{d-k} \mid k = 0,1,\ldots,d\}$$

$\hookrightarrow \underline{\dim V_d = d+1}$

Differential-operator realisation

Linear map $\phi_d : sl(2) \longrightarrow gl(V_d)$, where

$$\phi_d(e) := X \frac{\partial}{\partial Y}$$

$$\phi_d(h) := X \frac{\partial}{\partial X} - Y \frac{\partial}{\partial Y}$$

$$\phi_d(f) := Y \frac{\partial}{\partial X}$$

Prop: ϕ_d a rep of $sl(2)$

Proof: Verify that $\phi_d([x,y]) = [\phi_d(x), \phi_d(y)]$ $\forall x, y \in \{e, h, f\}$ by showing that the two sides have identical actions on any given basis vector $X^a Y^b$ (details in notes)

(*) The $d+1$ basis vectors $X^k Y^{d-k}$ are ~~h~~-eigenvectors $\phi_d(h)$ with distinct eigenvalues:

$$\phi_d(h)(X^k Y^{d-k}) = (2k-d) X^k Y^{d-k} \quad , \; k = 0, 1, \dots, d$$

$\hookrightarrow$ all $\phi_d(h)$-eigenspaces are one-dim (spanned by $X^k Y^{d-k}$)

(*) $d \neq d' \implies \dim V_d \neq \dim V_{d'}$

$\implies V_d \not\cong V_{d'}$

Prop V_d is an irred. $sl(2)$-module

Proof: Show that any $u \in V_d$ can be reached from any nonzero $v \in V_d$

(i) $X^d \in g(v)$ for all nonzero $v \in V_d$

(so X^d can be reached from any $v \neq 0$)

(ii) $u \in g(X^d)$ for all $u \in V_d$

(so every u can be reached from X^d)

Prop Any $sl(2)$-module V contains an h-eigenvector v satisfying $ev = 0$

Proof Over $\mathbb{C}$, $h: V \to V$ has at least one eigenvalue λ with corresponding eigenvector w. Consider infinite sequence of vectors: $w, ew, e^2w, \ldots$

If all nonzero, then all with distinct h-eigenvalues, in contradiction with V being fin-dim.

$\hookrightarrow$ $\exists k \in \mathbb{N}_0$ such that $e^k w \neq 0$, $e^{k+1} w = 0$

Setting $v = e^k w$, then

$$hv = he^k w \underset{\uparrow}{=} (\lambda + 2k) e^k w = (\lambda + 2k) v$$

(induction on k)

Highest-weight vector (hwv)

$v \neq 0$ such that $ev = 0$ and $hv = \lambda v$ $(\lambda \in \mathbb{C})$

$\circledast$ $X^d \in V_d$ a hwv of (highest) weight d

Lowest-weight vector (lwv)

$u \neq 0$ such that $fu = 0$ and $hu = \lambda u$ $(\lambda \in \mathbb{C})$

$\circledast$ $Y^d \in V_d$ a lwv of (lowest) weight $-d$

$\circledast$ V contains a hwv. Likewise, V contains a lwv.

Classification of irred. $sl(2)$-modules

Let V be an irred. $sl(2)$-module of dim. $d+1$ $(d \in \mathbb{N}_0)$.

Then,

$$V \cong V_d$$

Proof - see notes

Structure of V_d

(lwv) (hwv)

Basis: $(f^{d+1}v=0)$ $f^d v \dots f^2 v \;\; fv \;\; v$ $(ev=0)$

$[u \dots e^{d-2}u \;\; e^{d-1}u \;\; e^d u]$

h-eigenvalues: $-d \dots d-4 \;\; d-2 \;\; d$

(lowest weight) (highest weight)

Weyl's theorem for simple Lie alg

Every (fin-dim) rep of a (fin-dim) complex simple Lie alg is completely reducible

---

Classification of sl(2)-modules

Any (fin-dim) sl(2)-module V is of the form

$$V \cong V_{d_1} \oplus \dots \oplus V_{d_n}$$

for some $d_1, \dots, d_n \in \mathbb{N}_0$

Descending chains of Lie subalg

$$\mathfrak{g} \supseteq \mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \dots$$

$\circledast$ $\mathfrak{g}$ fin-dim $\Rightarrow$ $\exists\, \mathfrak{h} \leq \mathfrak{g}$ such that $\mathfrak{g}_n = \mathfrak{h}$ for $n \gg 1$

Define recursively

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \quad \mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}]$$

(derived alg of $\mathfrak{g}^{(n)}$, so an ideal)

Derived series: $\mathfrak{g} = \mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \supseteq \dots$

$\circledast$ $\mathfrak{g}^{(n-1)}/\mathfrak{g}^{(n)}$ abelian

---

$\mathfrak{g}$ is solvable if its derived series terminates (that is, $\mathfrak{g}^{(n)} = \{0\}$ for some $n \in \mathbb{N}$)

Ex Three-dim Heisenberg alg is solvable

($[p, q] = c$, $[c, p] = [c, q] = 0$, so $\mathfrak{g}^{(2)} = \{0\}$)

Ex $sl(2)$ is not solvable

($\mathfrak{g} = sl(2)$, $\mathfrak{g}' = \mathfrak{g}$, so $\mathfrak{g}^{(n)} = \mathfrak{g}$ $\forall n \in \mathbb{N}_0$)

<u>Prop</u> $\mathfrak{i}, \mathfrak{j}$ solvable ideals of $\mathfrak{g}$ $\Rightarrow$ $\mathfrak{i}+\mathfrak{j}$ solvable ideal of $\mathfrak{g}$

<u>Proof</u> – in notes

A solvable ideal of $\mathfrak{g}$, that is not contained in a larger solvable ideal of $\mathfrak{g}$, is called a <u>maximal solvable ideal</u> of $\mathfrak{g}$

<u>Prop</u> Every (fin-dim) Lie alg has a unique maximal solvable ideal

<u>Proof</u> – obtained by "adding" all its solvable ideals

The unique maximal solvable ideal is called the <u>radical</u> and is denoted by $\mathrm{rad}(\mathfrak{g})$

<u>Prop</u> $\mathfrak{g}$ solvable $\Longleftrightarrow$ $\mathrm{rad}(\mathfrak{g}) = \mathfrak{g}$

⊛ The "other extreme", $\mathrm{rad}(\mathfrak{g}) = \{0\}$, discussed later ($\mathfrak{g}$ <u>semisimple</u>)

<u>Lie's theorem</u> Let $\mathfrak{g}$ be complex and solvable.

Then, V an irred. $\mathfrak{g}$-module $\Rightarrow \dim V = 1$

<u>Corollary</u> Let $\mathfrak{g}$ be complex and solvable, and V a $\mathfrak{g}$-module. Then, $\exists$ basis for V relative to which every element of $\mathfrak{g}$ is represented by an upper-triangular matrix.

Define recursively

$$\mathfrak{g}^1 = \mathfrak{g} \;, \quad \mathfrak{g}^{n+1} = [\mathfrak{g}, \mathfrak{g}^n]$$

$\circledast$ $\mathfrak{g}^{n+1}$ an ideal of $\mathfrak{g}^n$; also of $\mathfrak{g}$

<u>Lower central series</u> $(\mathfrak{g}^2 = \mathfrak{g}')$

$$\mathfrak{g} = \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \mathfrak{g}^3 \supseteq \dots$$

<u>Prop</u>

$$\mathfrak{g}^n / \mathfrak{g}^{n+1} \subseteq Z(\mathfrak{g}/\mathfrak{g}^{n+1})$$

<u>Proof</u> - see notes

$\mathfrak{g}$ is <u>nilpotent</u> if its lower central series terminates

(that is, $\mathfrak{g}^n = \{0\}$ for some $n \in \mathbb{N}$)

Prop

$$\mathfrak{g} \text{ nilpotent} \implies \mathfrak{g} \text{ solvable}$$

Proof (sketch): Show

(i) $[\mathfrak{g}^m, \mathfrak{g}^n] \subseteq \mathfrak{g}^{n+m}$

(ii) $\mathfrak{g}^{(n)} \subseteq \mathfrak{g}^{2^n}$ (follows from (i))

Then,

$$\begin{aligned} \mathfrak{g} \text{ nilpotent} &\implies \mathfrak{g}^{2^n} = \{0\} \text{ for } n \gg 1 \\ &\overset{(ii)}{\implies} \mathfrak{g}^{(n)} \subseteq \{0\} \text{ for } n \gg 1 \\ &\implies \mathfrak{g} \text{ solvable} \end{aligned}$$

⊗ Converse true?

No, set of upper-triangular matrices is solvable but not nilpotent

---

Engel's theorem

$$\mathfrak{g} \text{ nilpotent} \implies \mathrm{ad}_x \text{ nilpotent for each } x \in \mathfrak{g}$$

Corollary

$\mathfrak{g}$ nilpotent iff $\mathfrak{g}$ isomorphic to Lie alg of strictly upper-triangular matrices

Prop Let $\mathfrak{g}$ be complex. Then,

$$\mathfrak{g} \text{ solvable} \iff \mathfrak{g}' \text{ nilpotent}$$

Proof – exercise!

Mnemonics

$$\begin{cases} \text{solvable}: & \text{upper-triangular} & (\text{easily "solved"}) \\ \text{nilpotent}: & \text{strictly} \text{ — " —} & (\text{some power of it vanishes}) \\ \text{abelian}: & \text{diagonal} & (\text{"readily commutative"}) \end{cases}$$

$\mathfrak{g}$ is <u>semisimple</u> if it contains no nonzero solvable ideal

$\hookrightarrow$ $$\mathfrak{g} \text{ semisimple} \iff \operatorname{rad}(\mathfrak{g}) = \{0\}$$

(semisimple "opposite" of solvable)

⊛ $\mathfrak{g}$ simple $\Longrightarrow$ $\mathfrak{g}$ semisimple

---

Prop $\mathfrak{g}$ semisimple iff $\mathfrak{g}$ of the form

$$\mathfrak{g} \cong \mathfrak{g}_1 \boxplus \dots \boxplus \mathfrak{g}_n$$

where $\mathfrak{g}_1, \dots, \mathfrak{g}_n$ are simple.

Weyl's theorem Every (fin-dim) rep of a (fin-dim) complex semisimple Lie alg is completely reducible.

$\mathfrak{g}$ is reductive if $\mathrm{rad}(\mathfrak{g}) = Z(\mathfrak{g})$

$\circledast$ $\mathfrak{g}$ semisimple $\Longrightarrow$ $\mathfrak{g}$ reductive

$\circledast$ Converse true?

No, could have $\mathrm{rad}(\mathfrak{g}) = Z(\mathfrak{g}) \neq \{0\}$

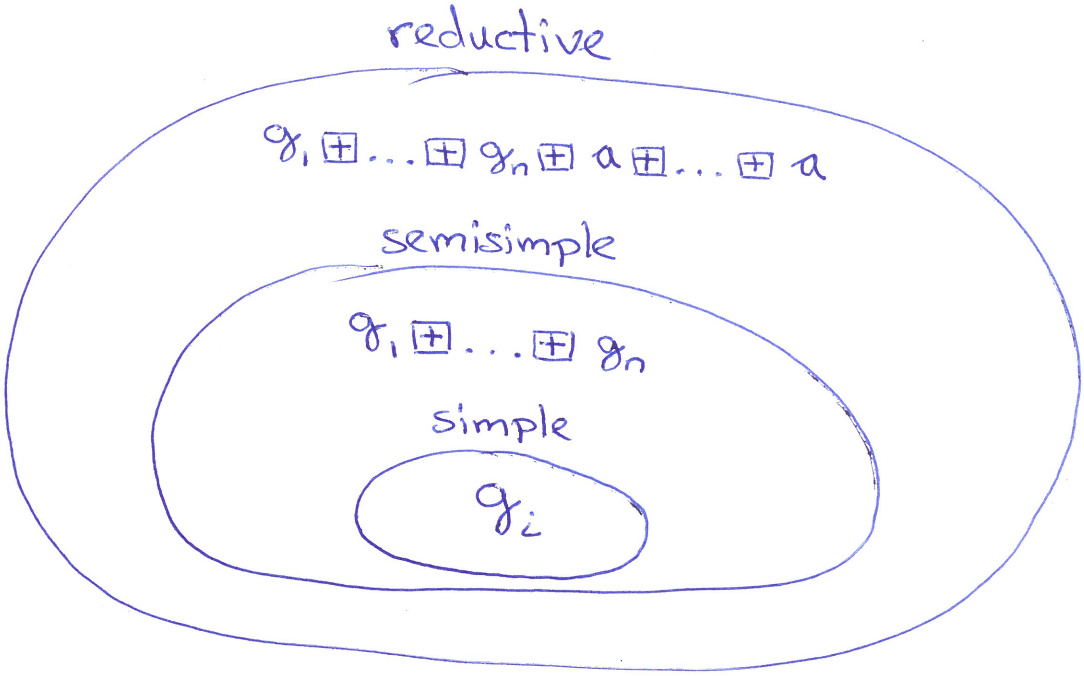
(Ex: $\mathfrak{gl}(2)$)

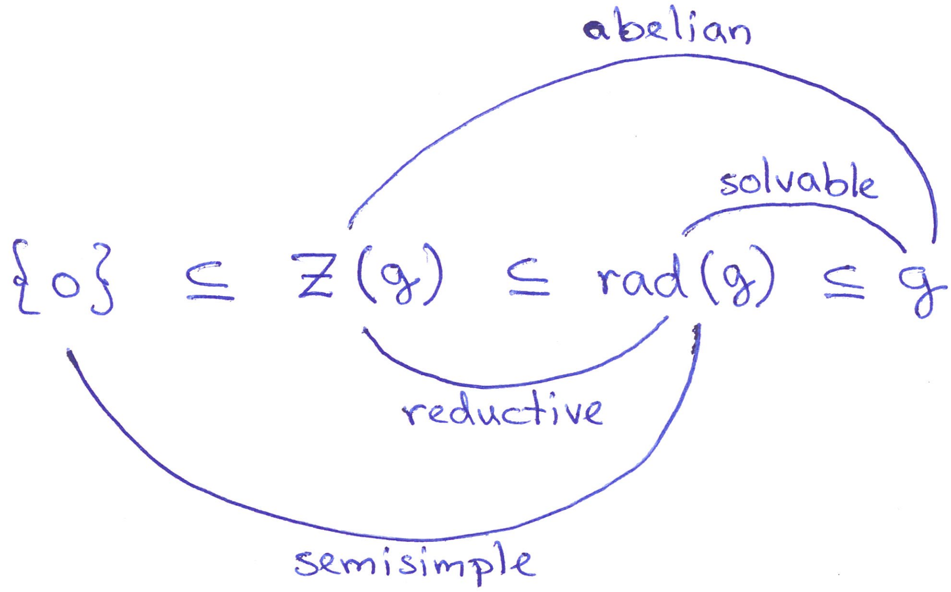
Prop $\mathfrak{g}$ reductive iff $\mathfrak{g}$ of the form

$$\mathfrak{g} \cong \mathfrak{g}_{\text{semi}} \boxplus \mathfrak{g}_{\text{abelian}}$$

($\mathfrak{g}_{\text{semi}}$ or $\mathfrak{g}_{\text{abelian}}$ could be zero).

Classifying all reductive (including semisimple) Lie alg thus boils down to classifying all simple Lie alg


reductive
$\mathfrak{g}_1 \boxplus \dots \boxplus \mathfrak{g}_n \boxplus \mathfrak{a} \boxplus \dots \boxplus \mathfrak{a}$
semisimple
$\mathfrak{g}_1 \boxplus \dots \boxplus \mathfrak{g}_n$
simple
$\mathfrak{g}_i$


abelian
solvable
$\{0\} \subseteq Z(\mathfrak{g}) \subseteq \mathrm{rad}(\mathfrak{g}) \subseteq \mathfrak{g}$
reductive
semisimple

A bilinear form $\langle\ ,\ \rangle : V \times V \to \mathbb{F}$ is <u>symmetric</u> if $\langle v, w \rangle = \langle w, v \rangle$

A symmetric bil. form is <u>non-degenerate</u> if for any nonzero $v \in V$, $\exists\, w \in V$ such that $\langle v, w \rangle \neq 0$

$$(\langle v, w \rangle = 0 \ \forall w \Rightarrow v = 0)$$

Let $\{v_1, \dots, v_d\}$ be an ordered basis for V. Then,

$$\langle\ ,\ \rangle \text{ non-deg} \iff \det(\langle v_i, v_j \rangle) \neq 0$$

A symmetric bil. form $\langle\ ,\ \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ is <u>invariant</u> if $\langle [x, y], z \rangle = \langle x, [y, z] \rangle$

<u>Killing form</u>

$$\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}, \quad (x, y) \mapsto \operatorname{tr}(\operatorname{ad}_x \circ \operatorname{ad}_y)$$

⊗ tr is basis indep., so we may evaluate $\kappa(x,y)$ in any given basis for the adjoint map:

$$\kappa(x, y) = \operatorname{tr}(R_{ad}(x) R_{ad}(y))$$

Prop

(i) κ is bilinear (since ad is linear)

(ii) κ is symmetric (since tr is cyclic)

(iii) κ is invariant

Proof (iii)

$$\begin{aligned}
\kappa([x,y],z) &= tr(ad_{[x,y]} \circ ad_z) \\
&= tr((ad_x \circ ad_y - ad_y \circ ad_x) \circ ad_z) \\
&= tr(ad_x \circ ad_y \circ ad_z) - tr(ad_y \circ ad_x \circ ad_z) \\
&= tr(ad_x \circ ad_y \circ ad_z) - tr(ad_x \circ ad_z \circ ad_y) \\
&= tr(ad_x \circ (ad_y \circ ad_z - ad_z \circ ad_y)) \\
&= tr(ad_x \circ ad_{[y,z]}) \\
&= \kappa(x,[y,z])
\end{aligned}$$

With respect to the basis $\{X_a \mid a = 1, \ldots, \dim \mathfrak{g}\}$,

$$\kappa_{ab} := \kappa(X_a, X_b)$$

<u>Prop</u> The Killing form of a nilpotent Lie alg is identically zero.

Cartan's first criterion

$\mathfrak{g}$ solvable $\iff \kappa(x,y)=0 \quad \forall x\in\mathfrak{g},\, y\in\mathfrak{g}'$

Cartan's second criterion

$\mathfrak{g}$ semisimple $\iff \kappa$ non-degenerate

$y \in \mathfrak{g}$ is <u>ad-diagonalisable</u> if ad_y is diagonalisable, that is, if $\exists$ basis $\{X_a \mid a = 1, \dots, \dim \mathfrak{g}\}$ for $\mathfrak{g}$ such that

$$[y, X_a] = \lambda_a X_a \quad \text{for some } \lambda_a \in \mathbb{C}$$

The <u>normaliser</u> of a Lie subalg $\mathfrak{h}$ of $\mathfrak{g}$ is given by

$$N_{\mathfrak{g}}(\mathfrak{h}) := \{x \in \mathfrak{g} \mid [x, \mathfrak{h}] \subseteq \mathfrak{h}\}$$

$\mathfrak{h} \leq \mathfrak{g}$ is a <u>Cartan subalg</u> if

(i) $\mathfrak{h}$ nilpotent

(ii) $\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h})$

<u>Prop:</u> For $\mathfrak{g}$ semisimple complex,

(i) $\mathfrak{g}$ has a Cartan subalg

(ii) any two Cartan subalg of $\mathfrak{g}$ are related by an automorphism of $\mathfrak{g}$

⊛ All Cartan subalg of $\mathfrak{g}$ have the same dimension, known as the <u>rank</u>:

$$\mathrm{rank}(\mathfrak{g}) = \dim \mathfrak{h} \qquad (\mathfrak{h} \text{ a Cartan subalg})$$

⊛ Elements of $\mathfrak{h}$ known as Cartan generators

⊛ Two Lie alg can only be isomorphic if they have the same rank

---

⊛ If Cartan subalg $\hat{\mathfrak{h}}$ contains Cartan subalg $\mathfrak{h}$, then $\hat{\mathfrak{h}} = \mathfrak{h}$

Prop Let $\mathfrak{h}$ be a Cartan subalg of semisimple complex $\mathfrak{g}$. Then,

(i) $\mathfrak{h}$ abelian (commutativity)

(ii) for every $h \in \mathfrak{h}$, ad_h is diagonalisable (ad-diagon...)

(iii) if $x \in \mathfrak{g}$ satisfies $[x, \mathfrak{h}] = \{0\}$, then $x \in \mathfrak{h}$ (maximality)

⊛ A Cartan subalg of $\mathfrak{g}$ is not contained in any larger abelian Lie subalg of $\mathfrak{g}$

From now on,

$\mathfrak{g}$ assumed fin-dim, complex and semisimple

⊛ Notation for simple $\mathfrak{g}$: X_r

with $X \in \{A, \ldots, G\}$ and $\text{rank}(X_r) = r$

Let V be a complex vector space and C a family of lin. operators $V \to V$.

The elements of C are <u>simultaneously diagonalisable</u> if $\exists$ basis $\{v_1, \dots, v_n\}$ for V such that each v_k is an eigenvector of every element of C

---

* A commuting family of diagonalisable matrices are simultaneously diagonalisable

---

* The adjoint operators ad_h, $h \in \mathfrak{h}$, are sim. diagon.

<u>Remark on applications:</u> Let a quantum mechanical system have $\mathfrak{g}$ as its symmetry alg. Then, $\mathrm{rank}(\mathfrak{g})$ provides the maximal number of quantum numbers which can be used to label the states in the Hilbert space. The corresponding eigenvalues of the Cartan generators are related to quantities that can be measured simultaneously.

Let $\{h_1, \dots, h_r\}$ be a basis for $\mathfrak{h}$.

$$\hookrightarrow \quad \mathrm{ad}_{h_i}(h_j) = 0 \cdot h_j \,, \; \forall\, i,j = 1, \dots, r$$

(eigenvector) (eigenvalue)

To have a full basis for $\mathfrak{g}$ consisting ~~of~~ exclusively of eigenvectors common to all ad_{h_i}, we still have to identify an additional $\dim \mathfrak{g} - r$ lin. indep. such vectors. Denote them by e_α:

$$\mathrm{ad}_{h_i}(e_\alpha) = [h_i, e_\alpha] = \alpha_{(i)} e_\alpha$$

for some eigenvalues $\alpha_{(i)} \in \mathbb{C}$.

⊛ For fixed e_α, the eigenvalues $\alpha_{(i)}$ cannot be zero for all i

---

<u>Roots</u>: The eigenvalues $\alpha_{(i)}$ associated with e_α form an r-dim vector α called a <u>root</u>

⊛ All roots are nonzero: $\alpha \neq 0$

<u>Root system</u>: the set of roots, denoted by Φ

⊛ Φ is a finite set

For $h = \sum_{i=1}^{r} \gamma^i h_i$,

$$\mathrm{ad}_h(e_\alpha) = \alpha(h)\, e_\alpha \quad , \quad \alpha(h) = \sum_{i=1}^{r} \gamma^i \alpha_{(i)}$$

In particular, $\alpha(h_i) = \alpha_{(i)}$

Lin. map $\alpha : \mathfrak{h} \to \mathbb{C}$

$\hookrightarrow$ $\alpha \in \mathfrak{h}^*$ (dual space to $\mathfrak{h}$)

For $h = \sum_{i=1}^{r} \gamma^i h_i$,

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In particular, $\alpha(h_i) = \alpha_{(i)}$

Lin. map $\alpha : \mathfrak{h} \to \mathbb{C}$

$\hookrightarrow$ $\alpha \in \mathfrak{h}^*$ (dual space to $\mathfrak{h}$)

For $\mu \in \mathfrak{h}^*$,

$$\mathfrak{g}_\mu := \{ x \in \mathfrak{g} \mid [h, x] = \mu(h) x \ , \ \forall h \in \mathfrak{h} \}$$

⊛ $\Phi = \{ \alpha \in \mathfrak{h}^* \mid \alpha \neq 0, \ \mathfrak{g}_\alpha \neq \{0\} \}$

Root space For $\alpha \in \Phi$, $\mathfrak{g}_\alpha$ a root space

---

Root-space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

Cartan-Weyl basis

$$\{h_i \mid i=1,\dots,r\} \cup \{e_\alpha \mid \alpha \in \Phi\}$$

(convenient shorthand: $\{h_i; e_\alpha\}$)

Note, a great deal of arbitrariness:

(i) a Cartan subalg $\mathfrak{h}$

(ii) a basis for $\mathfrak{h}$

(iii) nonzero elements $e_\alpha \in \mathfrak{g}_\alpha$

Prop

$$[\mathfrak{g}_\mu, \mathfrak{g}_\nu] \subseteq \mathfrak{g}_{\mu+\nu}$$

Proof: Let $x \in \mathfrak{g}_\mu$, $y \in \mathfrak{g}_\nu$ and $h \in \mathfrak{h}$. Then,

$$\begin{aligned}
[h,[x,y]] &= [[h,x],y] + [x,[h,y]] \\
&= \mu(h)[x,y] + \nu(h)[x,y] \\
&= (\mu+\nu)(h)[x,y]
\end{aligned}$$

so $[x,y] \in \mathfrak{g}_{\mu+\nu}$

Prop

(i) If $\alpha, \beta \in \Phi$ such that $\alpha+\beta \neq 0$, then $\kappa(\ell_\alpha, \ell_\beta) = 0$

(ii) $\kappa(h_i, \ell_\alpha) = 0$

(iii) The restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{h}$ is non-degenerate

---

Proof

(i) Since $\alpha+\beta \neq 0$, $\exists h \in \mathfrak{h}$ such that $(\alpha+\beta)(h) \neq 0$.

Also,
$$\alpha(h)\kappa(\ell_\alpha, \ell_\beta) = \kappa([h,\ell_\alpha], \ell_\beta)$$
$$= -\kappa(\ell_\alpha, [h,\ell_\beta])$$
(invariance)
$$= -\beta(h)\kappa(\ell_\alpha, \ell_\beta)$$

so $(\alpha+\beta)(h)\kappa(\ell_\alpha, \ell_\beta) = 0$, hence $\kappa(\ell_\alpha, \ell_\beta) = 0$

(ii) similar to (i)...

(iii) Proof by contradiction, using Cartan's second criterion

Prop

(i) $\mathfrak{h}^* = \mathrm{span}_{\mathbb{C}}(\Phi)$

(ii) For each $\alpha \in \Phi$, the only roots prop. to α are $\pm\alpha$.

(iii) Each root space $\mathfrak{g}_\alpha$ is one-dim

$$(\text{so } |\Phi| = \dim \mathfrak{g} - \mathrm{rank}(\mathfrak{g}))$$

(iv) $\kappa(e_\alpha, e_{-\alpha}) \neq 0$

Define $\phi : \mathfrak{h}^* \to \mathfrak{h}$, $\mu \mapsto t_\mu$,

where $\kappa(t_\mu, h) = \mu(h) \quad \forall h \in \mathfrak{h}$

⊛ ϕ linear

⊛ ϕ injective (κ non-deg, $\ker(\phi) = \{0\}$)

⊛ ϕ surjective ($\dim \mathfrak{h}^* = \dim \mathfrak{h}$)

Prop

(i) $\mathfrak{h}^* = \mathrm{span}_{\mathbb{C}}(\Phi)$

(ii) For each $\alpha \in \Phi$, the only roots prop. to α are $\pm\alpha$.

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(so $|\Phi| = \dim \mathfrak{g} - \mathrm{rank}(\mathfrak{g})$)

(iv) $\kappa(e_\alpha, e_{-\alpha}) \neq 0$

Define $\phi : \mathfrak{h}^* \to \mathfrak{h}$, $\mu \mapsto t_\mu$,

where $\kappa(t_\mu, h) = \mu(h) \quad \forall h \in \mathfrak{h}$

- ϕ linear
- ϕ injective (κ non-deg, $\ker(\phi) = \{0\}$)
- ϕ surjective ($\dim \mathfrak{h}^* = \dim \mathfrak{h}$)

The map ϕ provides a pairing of the elements $\mu \in \mathfrak{h}^*$ and $t_\mu \in \mathfrak{h}$.

Prop: The vectors t_α, $\alpha \in \Phi$, are all nonzero and span $\mathfrak{h}$.

Proof: ϕ bijective and Φ spans $\mathfrak{h}^*$.

Prop For all $\alpha, \beta \in \Phi$,

$$\kappa(t_\alpha, t_\alpha) \neq 0, \quad \kappa(t_\alpha, t_\beta) \in \mathbb{Q}, \quad \frac{2\kappa(t_\alpha, t_\beta)}{\kappa(t_\alpha, t_\alpha)} \in \mathbb{Z}$$

Prop $[e_\alpha, e_{-\alpha}] = \kappa(e_\alpha, e_{-\alpha}) t_\alpha \neq 0$

Proof: Let $h \in \mathfrak{h}$ and $\alpha \in \Phi$. Then,

$$\begin{aligned} [h, [e_\alpha, e_{-\alpha}]] &= [e_\alpha, [h, e_{-\alpha}]] + [[h, e_\alpha], e_{-\alpha}] \\ &= -\alpha(h)[e_\alpha, e_{-\alpha}] + \alpha(h)[e_\alpha, e_{-\alpha}] \\ &= 0 \end{aligned}$$

$\hookrightarrow [e_\alpha, e_{-\alpha}] \in \mathfrak{h}$

so

$$\begin{aligned} \kappa(h, [e_\alpha, e_{-\alpha}]) &= \kappa([h, e_\alpha], e_{-\alpha}) \\ &= \alpha(h) \kappa(e_\alpha, e_{-\alpha}) \\ &= \kappa(h, t_\alpha) \kappa(e_\alpha, e_{-\alpha}) \\ &= \kappa(h, \kappa(e_\alpha, e_{-\alpha}) t_\alpha) \end{aligned}$$

$\hookrightarrow [e_\alpha, e_{-\alpha}] = \kappa(e_\alpha, e_{-\alpha}) t_\alpha$ (since κ restricted to $\mathfrak{h}$ is non-deg)

$\neq 0$ (since $\kappa(e_\alpha, e_{-\alpha}) \neq 0$, $t_\alpha \neq 0$)

The bijection $\phi : \mathfrak{h}^* \to \mathfrak{h}$ can be used to define an (indefinite) inner product on $\mathfrak{h}^*$:

$$\langle \mu, \nu \rangle := \kappa(t_\mu, t_\nu)$$

(*) In particular, $\langle \alpha, \alpha \rangle = \kappa(t_\alpha, t_\alpha) \neq 0$

Coroots

$$\alpha^\vee := \frac{2}{\langle \alpha, \alpha \rangle} \alpha \quad , \quad \alpha \in \Phi$$

Relative normalisations

Since κ is non-deg, for each $\alpha \in \Phi$, we can (and will!) choose $e_\alpha \in \mathfrak{g}_\alpha$ and $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that

$$\kappa(e_\alpha, e_{-\alpha}) = \frac{2}{\langle \alpha, \alpha \rangle}$$

Dual coroots

$$h_\alpha := \frac{2}{\langle \alpha, \alpha \rangle} t_\alpha \quad , \quad \alpha \in \Phi$$

Cartan-Weyl basis rev.

$$[h_i, h_j] = 0 \quad , \quad [h_i, e_\alpha] = \alpha(h_i) e_\alpha$$

$$[e_\alpha, e_\beta] = \begin{cases} N_{\alpha,\beta}\, e_{\alpha+\beta} & \text{if } \alpha+\beta \in \Phi \\ h_\alpha & \text{if } \alpha+\beta = 0 \\ 0 & \text{otherwise} \end{cases}$$

(nonzero structure constant)

<u>sL(2) subalg</u> For each $\alpha \in \Phi$,

$$\alpha(h_\alpha) = \kappa(t_\alpha, h_\alpha) = \frac{2}{\langle\alpha,\alpha\rangle}\kappa(t_\alpha, t_\alpha) = 2$$

so

$$\begin{cases} [h_\alpha, \ell_\alpha] = \alpha(h_\alpha)\ell_\alpha = 2\ell_\alpha \\ [h_\alpha, \ell_{-\alpha}] = -\alpha(h_\alpha)\ell_{-\alpha} = -2\ell_{-\alpha} \\ [\ell_\alpha, \ell_{-\alpha}] = h_\alpha \end{cases}$$

$\Rightarrow$ <u>$\{\ell_\alpha, h_\alpha, \ell_{-\alpha}\}$ a basis for A_1</u>

Recall: $\mathfrak{h} = \text{span}\{t_\alpha \mid \alpha \in \Phi\}$

<u>Prop</u> Let $\alpha^1, \dots, \alpha^r \in \Phi$ such that $\{t_{\alpha^1}, \dots, t_{\alpha^r}\}$ is a basis for $\mathfrak{h}$. Then, for $\alpha \in \Phi$, $\exists\, q_1, \dots, q_r \in \mathbb{Q}$ such that

$$t_\alpha = \sum_{i=1}^{r} q_i t_{\alpha^i}$$

Define $\mathfrak{h}_{\mathbb{R}} := \text{span}_{\mathbb{R}}\{t_\alpha \mid \alpha \in \Phi\}$

<u>Prop</u> Let $h, h' \in \mathfrak{h}_{\mathbb{R}}$. Then, $\kappa(h, h') \in \mathbb{R}$ and $\kappa(h,h) \geq 0$. Moreover, if $\kappa(h,h) = 0$, then $h = 0$.

<u>Proof</u> – in lecture notes

<u>Corollary</u> κ restricted to $\mathfrak{h}_{\mathbb{R}}$ is a positive-definite inner product $\mathfrak{h}_{\mathbb{R}} \times \mathfrak{h}_{\mathbb{R}} \to \mathbb{R}$

By applying the bijection $\phi : \mathfrak{h}^* \to \mathfrak{h}$ to the subset $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{h}$ we obtain $\mathfrak{h}^*_{\mathbb{R}} = \phi^{-1}(\mathfrak{h}_{\mathbb{R}})$

– a Euclidean space with $\langle\, , \rangle : \mathfrak{h}^*_{\mathbb{R}} \times \mathfrak{h}^*_{\mathbb{R}} \to \mathbb{R}$

– called the <u>Euclidean root space</u>

(*) Φ spans an r-dim real Euclidean vector space

<u>Angle</u> $(0 \leq \theta \leq \pi)$ between $\alpha, \beta \in \Phi$:

$$\langle \alpha, \beta \rangle = |\alpha| |\beta| \cos\theta$$

<u>Length</u>

$$|\alpha| := \sqrt{\langle \alpha, \alpha \rangle}$$

Φ (and $\mathfrak{g}$ itself) is <u>simply-laced</u> if $|\alpha| = |\beta| \quad \forall \alpha, \beta \in \Phi$

Since $\quad 4(\cos\theta)^2 = \underbrace{\langle \alpha^\vee, \beta \rangle}_{\in \mathbb{Z}} \underbrace{\langle \alpha, \beta^\vee \rangle}_{\in \mathbb{Z}} \in \mathbb{Z}$

and

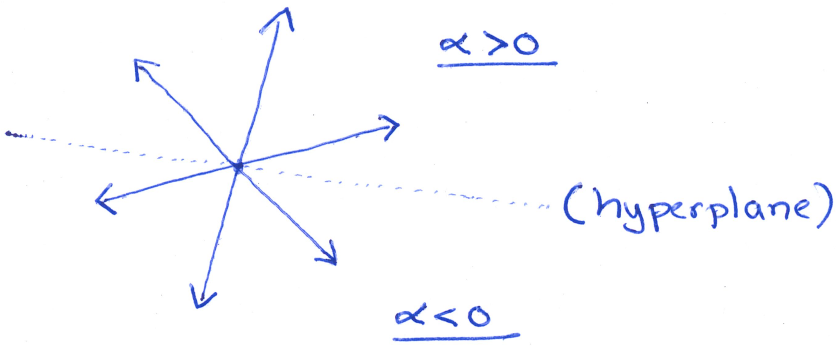
$$0 \leq (\cos\theta)^2 \leq 1$$

we have $\quad 4(\cos\theta)^2 \in \{0, 1, 2, 3, \cancel{4}\}$

for $\beta \neq \pm\alpha$

$$\hookrightarrow \theta \in \left\{ \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6} \right\}$$

$\Phi \subset \mathfrak{h}_{\mathbb{R}}^*$



$$\hookrightarrow \Phi_+ = \{\alpha \in \Phi \mid \alpha > 0\}, \quad \Phi_- = \{\alpha \in \Phi \mid \alpha < 0\}$$

Raising/Lowering operators

$$f_\alpha := e_{-\alpha}, \quad \alpha > 0$$

$$\hookrightarrow \{e_\alpha \mid \alpha \in \Phi\} = \underbrace{\{e_\alpha \mid \alpha > 0\}}_{\text{(raising)}} \cup \underbrace{\{f_\alpha \mid \alpha > 0\}}_{\text{(lowering)}}$$

Triangular decomposition

$$\mathfrak{g}_+ := \mathrm{span}_{\mathbb{C}}\{e_\alpha \mid \alpha > 0\}, \quad \mathfrak{g}_- := \mathrm{span}_{\mathbb{C}}\{f_\alpha \mid \alpha > 0\}$$

$$\hookrightarrow \quad \mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$$

Since Φ is finite, one cannot keep on adding positive (negative) roots and still get a root, so $\mathfrak{g}_+$ ($\mathfrak{g}_-$) is nilpotent

$\alpha \in \Phi_+$ is a <u>simple root</u> if it cannot be written as the sum of two positive roots.

The set of simple roots is denoted by Φ_s.

<u>Prop</u> Φ_s is a basis for $\mathfrak{h}_{\mathbb{R}}^*$

<u>Proof outline</u> Show

(i) Every $\alpha \in \Phi_+$ is a sum of simple roots

(ii) $\mathrm{span}_{\mathbb{R}}(\Phi_s) = \mathfrak{h}_{\mathbb{R}}^*$

(iii) Φ_s is lin. indep.

<u>Note on (i):</u> Let $\alpha \in \Phi_+$. Then, either $\alpha \in \Phi_s$ or $\exists \beta_1, \beta_2 \in \Phi_+$ such that $\alpha = \beta_1 + \beta_2$. In the latter case, repeat argument for β_1 and β_2. Continue process. Since Φ_+ is finite, the process will eventually terminate.

<u>Note on (ii):</u> We know $\mathrm{span}_{\mathbb{R}}(\Phi) = \mathfrak{h}_{\mathbb{R}}^*$. Since $\alpha \in \Phi$ iff $-\alpha \in \Phi$, we have $\mathrm{span}_{\mathbb{R}}(\Phi) = \mathrm{span}_{\mathbb{R}}(\Phi_+)$. Hence, using (i), $\mathrm{span}_{\mathbb{R}}(\Phi_+) = \mathrm{span}_{\mathbb{R}}(\Phi_s)$.

<u>Note on (iii):</u> Form general lin. comb. of the simple roots. Assume expression is zero. Show all coefficients are zero.

⊛ $|\Phi_s| = \dim \mathfrak{h}_{\mathbb{R}}^* = r \longrightarrow \Phi_s = \{\alpha_i \mid i = 1, \dots, r\}$

<u>Exercise:</u> Show that $\alpha_i - \alpha_j \notin \Phi$.

⊛ $\langle \alpha_i, \alpha_j \rangle \leq 0 \quad , \quad i \neq j$

$\hookrightarrow$ angle between two simple roots cannot be acute.

⊛ Let $\alpha \in \Phi$. Then,

$$\alpha = \begin{cases} \sum_{i=1}^{r} a^i \alpha_i & , \quad \alpha > 0 \\ -\sum_{i=1}^{r} a^i \alpha_i & , \quad \alpha < 0 \end{cases}$$

with $a^1, \dots, a^r$ uniquely given nonneg. integers.

⊛ No root is a lin. comb. of simple roots with coefficients of <u>both</u> signs.

$\langle \alpha_i, \alpha_j \rangle$ measures the non-orthogonality of α_i and α_j

Renormalise to get integers ; then form

<u>Cartan matrix</u> $\quad A_{ij} := \langle \alpha_i^\vee, \alpha_j \rangle$

(relative to the ordering $\alpha_1, \dots, \alpha_r$)

⊛ A need not be symmetric

⊛ A is <u>symmetrisable</u>: There exist diagonal D and symmetric S such that

$$A = DS$$

– explicitly, $D_{ii} = \dfrac{2}{\langle \alpha_i, \alpha_i \rangle}$, $S_{ij} = \langle \alpha_i, \alpha_j \rangle$

$\hookrightarrow \quad \det D > 0 \;,\; \det S > 0$ (Φ_s basis for real Euclidean vectorspace)

$\hookrightarrow \quad \det A = \det D \cdot \det S > 0$

<u>Prop</u>

(i) $A_{ii} = 2 \quad \forall i$

(ii) $A_{ij} \in \{0, -1, -2, -3\}$ if $i \neq j$

(iii) $A_{ij} \in \{-2, -3\} \Longrightarrow A_{ji} = -1$

(iv) $A_{ij} = 0 \Longleftrightarrow A_{ji} = 0$

(v) $\det A > 0$

<u>Proof</u> – in the lecture notes

<u>Chevalley generators</u> For each $i = 1, \dots, r$,

$$e_i := e_{\alpha_i} \quad , \quad h_i := h_{\alpha_i} \quad , \quad f_i := f_{\alpha_i}$$

$$\longrightarrow \begin{cases} [h_i, h_j] = 0 \quad , \quad [e_i, f_j] = \delta_{ij} h_j \\ [h_i, e_j] = A_{ij} e_j \ , \ [h_i, f_j] = -A_{ij} f_j \end{cases}$$

<u>Ex</u> For the Lie alg A_1,

$$A = (2) \qquad \text{(a } 1 \times 1 \text{ matrix)}$$

<u>Serre relations</u> For $i \neq j$,

$$(\mathrm{ad}_{e_i})^{1-A_{ij}}(e_j) = 0 = (\mathrm{ad}_{f_i})^{1-A_{ij}}(f_j)$$

Dynkin diagram

Each simple root is represented by a node •

The nodes are joined pairwise according to

$\theta = \frac{\pi}{2}$: 

$\theta = \frac{2\pi}{3}$:  (same length)

$\theta = \frac{3\pi}{4}$:  ("Inequality sign"

$\theta = \frac{5\pi}{6}$:  on root lengths)

⊛ $\mathfrak{g}$ simple iff its Dynkin diagram is connected

⊛ A Dynkin diagram is indep. of

- the division $\Phi = \Phi_+ \cup \Phi_-$
- the ordering of the simple roots

⊛ The Cartan matrix <u>does</u> depend on the ordering

⊛ Two permutation equivalent Cartan matrices describe isomorphic Lie alg

(A, A' perm. equiv. if $A_{ij} = A'_{\sigma(i)\sigma(j)}$ for some perm. σ)

Classification procedure

Any two semisimple Lie alg with the same Cartan matrix are isomorphic. Moreover,

$\mathfrak{g}$ simple $\longleftrightarrow$ Dynkin diagram connected

$\longleftrightarrow$ A indecomposable

($M \in M_n(\mathbb{C})$ is indec. if it is not perm. equiv. to any $\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$ with M_1 and M_2 ~~nonzero~~ square matrices)

Rank $r=1$

Cartan matrix: (2)

Dynkin diagram: •

Root system:  α_1

$(\mathfrak{g} \cong A_1)$

Rank $r=2$

$$A = \begin{pmatrix} 2 & -n \\ -m & 2 \end{pmatrix}, \quad m,n \in \mathbb{N}_0$$

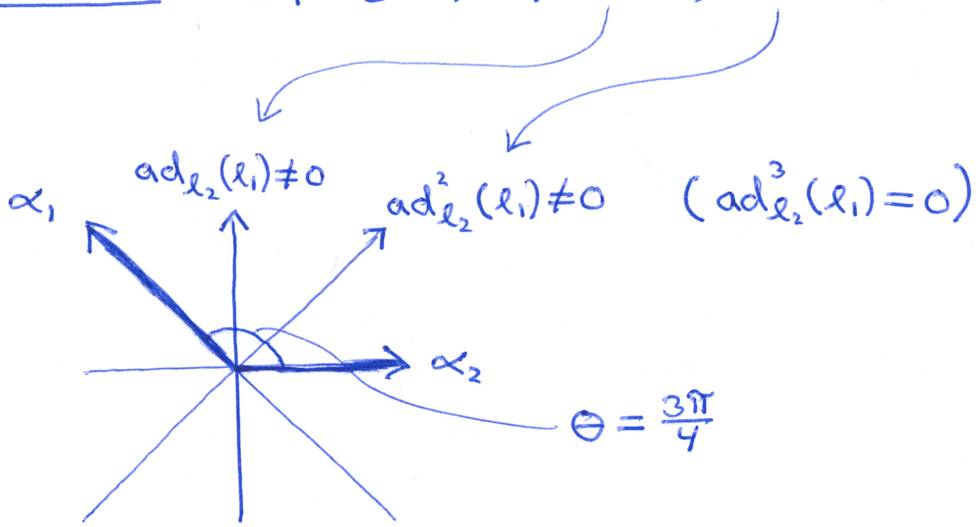
with $0 < \det A = 4 - mn$ (may assume $m \geq n$)

D_2: $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ - decomposable, not simple

Indeed, $D_2 \cong A_1 \boxplus A_1$

A_2: $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, B_2: $\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$, G_2: $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

Root system for B_2: $\Phi_+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, \alpha_1+2\alpha_2\}$



Summary of path

Fin-dim semisimple complex $\mathfrak{g}$

$\downarrow$

Cartan subalg $\mathfrak{h}$

$\downarrow$

Root system $\Phi \subset \mathfrak{h}^*$

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

$\downarrow$

Simple roots : $\Phi_s = \{\alpha_1, \dots, \alpha_r\}$

$\downarrow$

Cartan matrix and Dynkin diagram

Serre's theorem

Serre's theorem

Let A be an $r \times r$ matrix with Cartan-matrix properties, and let

$$\mathfrak{g} = \langle E_i, H_i, F_i \mid i = 1, \dots, r \rangle$$

be a complex Lie alg subject to

(S1) $[H_i, H_j] = 0$

(S2) $[H_i, E_j] = A_{ij} E_j \quad , \quad [H_i, F_j] = -A_{ij} F_j$

(S3) $[E_i, F_j] = \delta_{ij} H_j$

(S4) $(\mathrm{ad}_{E_i})^{1-A_{ij}}(E_j) = 0 = (\mathrm{ad}_{F_i})^{1-A_{ij}}(F_j) \quad \forall\, i \neq j$

Then, $\mathfrak{g}$ is finite-dim and semisimple,

$\{H_1, \dots, H_r\}$ spans a Cartan subalg,

and the corresponding root system has Cartan matrix A.

Classification

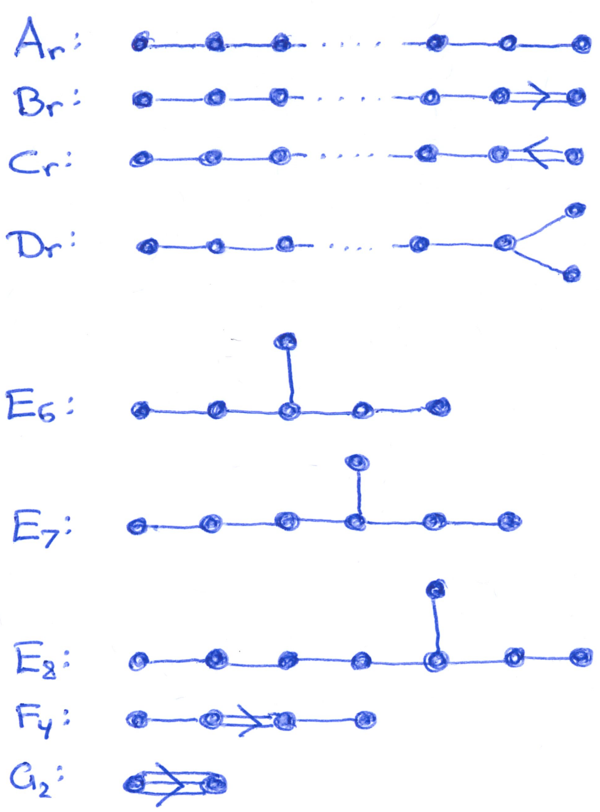
A finite-dim simple complex Lie alg is isomorphic to one of the classical Lie alg

$$A_r \ (r \geq 1), \quad B_r \ (r \geq 2), \quad C_r \ (r \geq 3), \quad D_r \ (r \geq 4)$$

or one of the exceptional Lie alg

$$E_6, \quad E_7, \quad E_8, \quad F_4, \quad G_2$$

Dynkin diagrams



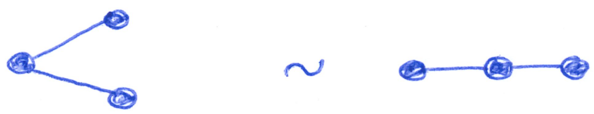
$A_1 \cong B_1 \cong C_1$ rev.

A_1, B_1, C_1 all represented by a single node, hence isomorphic

$B_2 \cong C_2$ rev.

$D_2 \cong A_1 \boxplus A_1$ rev.

$D_3 \cong A_3$ rev.



From Dynkin to Cartan

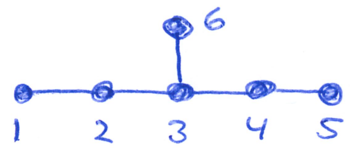
To associate a Cartan matrix to a given Dynkin diagram, we label the nodes by $i = 1, \dots, r$.

Canonical labelling:

(i) Label the nodes from left to right

(ii) In the case of D_r or E_r, the r'th node is the elevated one.

Ex



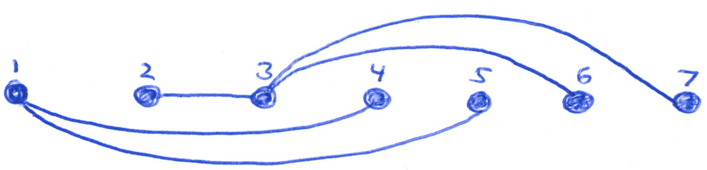
$(\mathfrak{g} \cong E_6)$

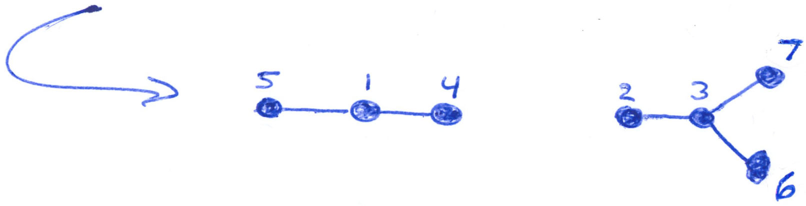
$$A = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix} \end{array}$$

Ex Given the Cartan matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

we read off the Dynkin diagram





so $\quad \mathfrak{g} \cong A_3 \boxplus D_4$

ALGEBRAIC METHODS OF MATHEMATICAL PHYSICS

MATH3103/7133

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1 Introduction

This set of notes provides a foundation for the topics to be introduced and discussed in the second half of the course. There is some overlap with content from the first half of the course. Reproducing it here allows for these notes to be reasonably self-contained.

One thing that will become apparent is that notational conventions will vary as we proceed through the notes. There are two underlying reasons for this. The first is that notational conventions are not necessarily optimal in all situations. In some instances the convention used will be one that is most beneficial for treating the problem at hand. The second, and more important reason, is that many different conventions exist out in the mathematical literature. Exposing students to this variety of conventions increases their capacity to undertake independent learning activities.

Make special note that

- Generally, Einstein summation convention will be adopted when convenient.
- The word *tensor* will be used in several different contexts.
- In some instances, tensors defined with lower indices will have their inverses expressed with upper indices (and vice versa). For example, using Einstein summation convention
$$A^{jp}A_{pk} = \delta^j{}_k.$$
On the other hand, a delta function will also commonly be represented as δ_{jk}.

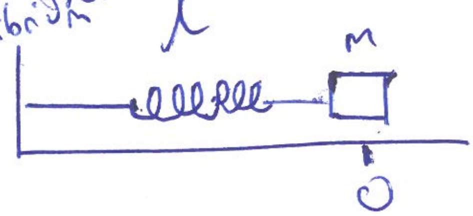
Finally, these notes have not been proofread and edited to the same standard of a published textbook. If you do find any errors, please send an email to jrl@maths.uq.edu.au so a correction can be listed on the course Blackboard site.

Let x denote the displacement from equilibrium

in equilibrium

spring constant k

frictionless surface



Extend the spring and release

$m\ddot{x} = F = -kx \quad (*)$

2 Dynamics of physical systems

2.1 Classical Hamiltonian systems

For the purposes of these notes, a classical Hamiltonian H is a function of many variables where q_j, p_j, $j = 1, ..., N$ are conjugate *position* and *momentum* variables, and N is the number of *degrees of freedom*. Generally, the Hamiltonian represents the total energy of the system. The dynamical behaviour of the system is governed by Hamilton's equations

$$\frac{\mathrm{d}q_j}{\mathrm{d}t} = \frac{\partial H}{\partial p_j}, \qquad \frac{\mathrm{d}p_j}{\mathrm{d}t} = -\frac{\partial H}{\partial q_j}. \tag{1}$$

The simplest example is given by a mass m on a spring with constant k, in one dimension. The total energy is given by the sum of the kinetic and potential energies

Potential $V = \frac{1}{2}x^2 k$

$F = -\frac{dV}{dx}$

$$\begin{aligned} H &= \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \\ &= \frac{p^2}{2m} + \frac{kx^2}{2} \end{aligned}$$

$= \frac{1}{2}mA^2\omega^2\cos^2(\omega t+\phi) + \frac{A^2}{2}k\sin^2(\omega t+\phi) = mkA^2$

where the momentum is defined $p = mv$, $v = \mathrm{d}x/\mathrm{d}t$ is the velocity and $x = q$ is the sole co-ordinate. Hamilton's equations give

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\partial H}{\partial p} = \frac{p}{m} = v, \\ \frac{\mathrm{d}p}{\mathrm{d}t} &= -\frac{\partial H}{\partial x} = -kx. \end{aligned}$$

General solution of $(*)$

$x(t) = A\sin(\omega t + \phi)$

where $\omega = \sqrt{\frac{k}{m}}$. Then

$v = \frac{dx}{dt} = A\omega\cos(\omega t + \phi)$

We can combine these two equations to obtain

$$\frac{\mathrm{d}p}{\mathrm{d}t} = m\frac{\mathrm{d}v}{\mathrm{d}t} = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx$$

which is Newton's law for this system.

In the above formulation, for any $F = F(q_j, p_j)$, $G = G(q_j, p_j)$ we define the Poisson bracket by

$$\begin{aligned} \{F, G\} &= \sum_{j=1}^{N} \frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j} - \sum_{j=1}^{N} \frac{\partial G}{\partial q_j}\frac{\partial F}{\partial p_j} \\ &= -\{G, F\}. \end{aligned} \tag{2}$$

In particular, through Hamilton's equations (1) we find

$$\begin{aligned} \{F, H\} &= \sum_{j=1}^{N} \frac{\partial F}{\partial q_j}\frac{\partial H}{\partial p_j} - \sum_{j=1}^{N} \frac{\partial H}{\partial q_j}\frac{\partial F}{\partial p_j} \\ &= \sum_{j=1}^{N} \frac{\partial F}{\partial q_j}\frac{\mathrm{d}q_j}{\mathrm{d}t} + \sum_{j=1}^{N} \frac{\mathrm{d}p_j}{\mathrm{d}t}\frac{\partial F}{\partial p_j} \\ &= \frac{\mathrm{d}F}{\mathrm{d}t}. \end{aligned}$$

It follows from the antisymmetry property (2) that

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0$$

which is a statement of the conservation of energy. More generally if

$$\{F,\, H\} = 0$$

then F is also conserved.

Exercise 1. *Poisson's theorem states that the Poisson bracket of two conserved quantities is also conserved. Prove Poisson's theorem.*

2.2 Two-dimensional classical oscillator

Here we examine the two-dimensional oscillator. A physical realisation of it is given by a mass on an idealised spring (natural length is zero), on a frictionless horizontal table, with the end fixed at the origin. This system has two degrees of freedom. The Hamiltonian is given by

$\vec{p} = p_x \underline{i} + p_y \underline{j}$

$\vec{p} \cdot \vec{p} = p_x^2 + p_y^2$

$$\begin{aligned} H &= \frac{\vec{p}.\vec{p}}{2m} + \frac{m\omega^2 \vec{q}.\vec{q}}{2} \\ &= \frac{p_1^2 + p_2^2}{2m} + \frac{m\omega^2 (q_1^2 + q_2^2)}{2}. \end{aligned}$$

$\vec{q} = x\underline{i} + y\underline{j}$

$\vec{q} \cdot \vec{q} = x^2 + y^2.$

We may determine the equations of motion from Hamilton's equations which yield

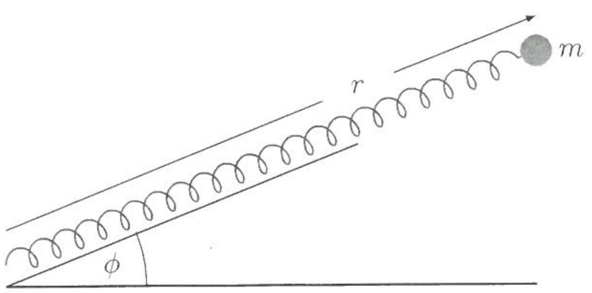
$$\frac{\mathrm{d}q_j}{\mathrm{d}t} = \frac{\partial H}{\partial p_j} = \frac{p_j}{m} \tag{3}$$

$$\frac{\mathrm{d}p_j}{\mathrm{d}t} = -\frac{\partial H}{\partial q_j} = -m\omega^2 q_j \tag{4}$$

for both $j = 1,\, 2$. In this case, the set of four differential equations has decoupled into two sets of two differential equations. It is straightforward to solve the equations, with the general solution

$$\begin{aligned} q_j &= A_j \sin(\omega t + \theta_j), \\ p_j &= m\omega A_j \cos(\omega t + \theta_j) \end{aligned}$$

and the four constants of integration are A_j, θ_j, $j = 1,\, 2$.



It is also useful to consider the transformation to polar co-ordinates r and ϕ with conjugate momenta p_r and p_ϕ. The transformation is

$$
\begin{aligned}
q_x \equiv \quad q_1 &= r\cos(\phi), \\
p_x \equiv \quad p_1 &= p_r\cos(\phi) - \frac{p_\phi}{r}\sin(\phi), && \text{— not obvious} \\
q_y \equiv \quad q_2 &= r\sin(\phi), \\
p_y \equiv \quad p_2 &= p_r\sin(\phi) + \frac{p_\phi}{r}\cos(\phi). && \text{— not obvious}
\end{aligned}
$$

Next, expressing the Hamiltonian in terms of the polar co-ordinates

$$
H = \frac{1}{2m}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) + \frac{m\omega^2 r^2}{2}
$$

we can derive the equations of motion

$$
\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}, \tag{5}
$$

$$
\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2}, \tag{6}
$$

$$
\frac{\mathrm{d}p_r}{\mathrm{d}t} = -\frac{\partial H}{\partial r} = \frac{p_\phi^2}{mr^3} - m\omega^2 r, \tag{7}
$$

$$
\frac{\mathrm{d}p_\phi}{\mathrm{d}t} = -\frac{\partial H}{\partial \phi} = 0. \tag{8}
$$

We immediately see from (8) that p_ϕ is a constant of the motion. Substituting into (6) yields

$$
p_\phi = mr^2\frac{\mathrm{d}\phi}{\mathrm{d}t}
$$

which is recognisable as the *angular momentum*. Here, we see both energy and angular momentum being conserved. Combining (5) and (7) yields the second order differential equation

$$
\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \frac{p_\phi^2}{m^2 r^3} - \omega^2 r.
$$

Since p_ϕ is conserved, we again see that the equations of motion have decoupled.

2.3 Quantum Hamiltonian systems

One of the major differences between classical and quantum systems is that for quantum systems the spectra of physical observables, such as the total energy or the angular momentum, is discrete, whereas for classical systems it is continuous. To mathematically describe the quantum analogue of a classical system, we adopt the following prescription:

- The states of the system are elements of a complex Hilbert space (i.e. vector space with positive definite inner product) $\mathcal{H}$ with unit norm with respect to the inner product $\langle\Phi|\Psi\rangle \to \mathbb{C}$, for $|\Phi\rangle$, $|\Psi\rangle \in \mathcal{H}$.

- Physical observables are represented by self-adjoint operators acting on $\mathcal{H}$.

- For conjugate classical variables q and p which satisfy the Poisson bracket

$$\{q, p\} = 1,$$

the quantum operators must satisfy the *commutation relation*

$$[q, p] = i\hbar I \tag{9}$$

where the commutator is defined as

$$[A, B] = AB - BA = -[B, A].$$

Above, $\hbar$ is Planck's constant. Mostly, we will set $\hbar = 1$. The general rule in going from a classical system to a quantum analogue is to replace all Poisson brackets with commutators such that

$$\{A, B\} \longrightarrow -i[A, B]$$

where $i = \sqrt{-1}$. It looks suspicious that a complex number has appeared in the commutation relations, whereas in the Poisson brackets we deal only with real functions. The reason for this is mathematical consistency. For example, consider the operator

$$\mathcal{O} = qp - pq.$$

$(AB)^\dagger = B^\dagger A^\dagger$

Taking the adjoint we find

$$\begin{aligned} \mathcal{O}^\dagger &= p^\dagger q^\dagger - q^\dagger p^\dagger \\ &= pq - qp \\ &= -\mathcal{O}, \end{aligned}$$

using the fact that q and p are assumed to be self-adjoint (to ensure that they have real eigenvalues which have physical meaning). We cannot assign $\mathcal{O}$ to be the identity operator I, since I is self-adjoint. The resolution is to assign $\mathcal{O} = iI$.

- The observable spectrum of A is given by the eigenvalues of A, and the observable states are the eigenstates.

- For an observable A let $\{|\psi_j\rangle\}$ denote normalised eigenstates (which provide an orthonormal basis for $\mathcal{H}$) and let $\{\lambda_j\}$ denote the corresponding eigenvalues. Given an arbitrary normalised state $|\Psi\rangle$ we can express it as

$$|\Psi\rangle = \sum_j c_j|\psi_j\rangle, \qquad \sum_j |c_j|^2 = 1.$$

We interpret $\mathfrak{p}_j = |c_j|^2$ as the probability that the measurement represented by A yields the result λ_j. The average, or *expectation value*, of A is given by

$$\begin{aligned}\langle A\rangle &= \sum_j \mathfrak{p}_j\lambda_j \\ &= \sum_{j,k} c_k^*\lambda_j c_j\delta_{j,k} \\ &= \sum_{j,k} c_k^* c_j \langle\psi_k|\, A\,|\psi_j\rangle \\ &= \langle\Psi|\, A\,|\Psi\rangle\,.\end{aligned}$$

$|\psi\rangle$ - column vector

$\langle\psi|$ - row vector

$\exp(A) = \sum_{n=0}^{\infty}\frac{1}{n!}A^n$

- The time-evolution of a state, $|\Psi(t)\rangle$, is given by

$$|\Psi(t)\rangle = \exp(-itH)|\Psi(0)\rangle.$$

Note that if $|\Psi(0)\rangle$ is an eigenstate of the Hamiltonian H with energy E then

$$|\Psi(t)\rangle = \exp(-itH)|\Psi(0)\rangle = \exp(-itE)|\Psi(0)\rangle$$

so the state evolves by a scalar phase factor. It follows that

$$H|\Psi(t)\rangle = E|\Psi(t)\rangle$$

for all t so the energy is conserved.

- For a general observable A the time evolution of the expectation value is

$$\begin{aligned}\langle A(t)\rangle &= \langle\Psi(t)|\, A\,|\Psi(t)\rangle \\ &= \langle\Psi|\exp(itH)A\exp(-itH)\,|\Psi\rangle\,.\end{aligned}$$

Then assuming A has no explicit time dependence

$$\begin{aligned}\frac{\mathrm{d}\,\langle A(t)\rangle}{\mathrm{d}t} &= i\,\langle\Psi|\exp(itH)[H,\,A]\exp(-itH)\,|\Psi\rangle \\ &= i\,\langle\Psi(t)|\,[H,\,A]\,|\Psi(t)\rangle\,.\end{aligned}$$

Consequently we define

$$\frac{\mathrm{d}A}{\mathrm{d}t} = i[H,\,A] \tag{10}$$

such that

$$\left\langle\frac{\mathrm{d}A}{\mathrm{d}t}\right\rangle = \frac{\mathrm{d}\,\langle A\rangle}{\mathrm{d}t}.$$

Throughout we assume that the Hamiltonian H has no explicit time dependence. With this definition we see that the eigenvalues of any observable A are independent of time, since for any eigenstate $|\Phi\rangle$ of A

$$\left\langle\Phi\left|\frac{\mathrm{d}A}{\mathrm{d}t}\right|\Phi\right\rangle = i\,\langle\Phi|\,[H,\,A]\,|\Phi\rangle = 0.$$

We interpret $\mathfrak{p}_j = |c_j|^2$ as the probability that the measurement represented by A yields the result λ_j. The average, or *expectation value*, of A is given by

$$
\begin{aligned}
\langle A \rangle &= \sum_j \mathfrak{p}_j \lambda_j \\
&= \sum_{j,k} c_k^* \lambda_j c_j \delta_{j,k} \\
&= \sum_{j,k} c_k^* c_j \langle \psi_k | A | \psi_j \rangle \\
&= \langle \Psi | A | \Psi \rangle .
\end{aligned}
$$

$|\psi\rangle$ - column vector

$\langle \psi|$ - row vector

$\exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$

- The time-evolution of a state, $|\Psi(t)\rangle$, is given by

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so the state evolves by a scalar phase factor. It follows that

$$H|\Psi(t)\rangle = E|\Psi(t)\rangle$$

for all t so the energy is conserved.

- For a general observable A the time evolution of the expectation value is

$$
\begin{aligned}
\langle A(t) \rangle &= \langle \Psi(t) | A | \Psi(t) \rangle \\
&= \langle \Psi | \exp(itH) A \exp(-itH) | \Psi \rangle .
\end{aligned}
$$

Then assuming A has no explicit time dependence

$$
\begin{aligned}
\frac{\mathrm{d}\langle A(t) \rangle}{\mathrm{d}t} &= i \langle \Psi | \exp(itH)[H, A] \exp(-itH) | \Psi \rangle \\
&= i \langle \Psi(t) | [H, A] | \Psi(t) \rangle . = i \langle \psi(t) | (HA - AH) | \psi(t) \rangle
\end{aligned}
$$

Consequently we define

$$\frac{\mathrm{d}A}{\mathrm{d}t} = i[H, A] \tag{10}$$

such that

$$\left\langle \frac{\mathrm{d}A}{\mathrm{d}t} \right\rangle = \frac{\mathrm{d}\langle A \rangle}{\mathrm{d}t}.$$

Throughout we assume that the Hamiltonian H has no explicit time dependence. With this definition we see that the eigenvalues of any observable A are independent of time, since for any eigenstate $|\Phi\rangle$ of A

$$\left\langle \Phi \middle| \frac{\mathrm{d}A}{\mathrm{d}t} \middle| \Phi \right\rangle = i \langle \Phi | [H, A] | \Phi \rangle = 0.$$

One of the consequences of the above formulation for quantum systems is what is known as *Heisenberg's uncertainty principle.* Suppose that we have two observables represented by operators A and B which have the same set of eigenstates Ψ_j, with eigenvalues λ_j^A and λ_j^B respectively. Then we find

$$\begin{aligned}
[A,\, B]\,|\Psi_j\rangle &= (AB - BA)\,|\Psi_j\rangle \\
&= (\lambda_j^B A - \lambda_j^A B)\,|\Psi_j\rangle \\
&= (\lambda_j^B \lambda_j^A - \lambda_j^A \lambda_j^B)\,|\Psi_j\rangle \\
&= 0.
\end{aligned}$$

Since this is true for all Ψ_j then we must have $[A,\, B] = 0$. Conversely if $[A,\, B] \neq 0$, then there does not exist a set of simultaneous eigenstates. In this latter case, the observables A and B cannot be determined simultaneously, which underlies the uncertainty principle.

An important property to recognise in this correspondence is that both the Poisson bracket and the commutator are *derivations*, i.e.

$$\begin{aligned}
\{F,\, GK\} &= \{F,\, G\}K + G\{F,\, K\} \\
[A,\, BC] &= [A,\, B]C + B[A,\, C]
\end{aligned}$$

where the terminology *derivation* stems from the similarity to the familiar product rule

$$\frac{\mathrm{d}(fg)}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}t}g + f\frac{\mathrm{d}g}{\mathrm{d}t}.$$

This result generalises, e.g.

$$[A_1 A_2 \ldots . A_L,\, B] = \sum_{j=1}^{L} A_1 A_2 \ldots A_{j-1}[A_j,\, B]A_{j+1}\ldots A_L. \tag{11}$$

As an example, consider the one-dimensional classical oscillator with the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}.$$

To analyse the quantum analogue, we first need a Hilbert space. Since the Hamiltonian describes a one-dimensional system, we choose the Hilbert space to be the space of one-variable functions f with the inner product

$$\langle f|g\rangle = \int_{-\infty}^{\infty} f^*(x)g(x)dx.$$

Exercise 2. *Show that the mapping*

$$\begin{aligned}
q &\mapsto x, \\
p &\mapsto -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}
\end{aligned}$$

satisfies the commutation relation (9). Show that under this representation the Hamiltonian for the quantum oscillator is

$$H = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{m\omega^2 x^2}{2}.$$

Thus the problem of determining the energy spectrum of the system is transformed to solving the eigenvalue problem

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + \frac{m\omega^2 x^2}{2}\Psi = E\Psi.$$

$\omega = \sqrt{\frac{k}{m}}$

In general, an equation of the form

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + V(x)\Psi = E\Psi$$

is known as a one-dimensional Schrödinger equation with potential $V(x)$.

Exercise 3. *Using (10), calculate expressions for* $\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}p}{\mathrm{d}t}$ *and compare them with the analogous expressions for the classical oscillator derived through Hamilton's equations.*

2.4 One-dimensional Schrödinger equation

Hereafter we set $\hbar = 1$ and $m = 1/2$, so the one-dimensional Schrödinger equation is of the form

$$-\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + V(x)\Psi = E\Psi. \tag{12}$$

Set

$$Q(u) = \prod_{j=1}^{M}(u - v_j) \tag{13}$$

and look for solutions of the form

$$\Psi = e^{v(x)}Q(u(x)). \tag{14}$$

Now

$$\begin{aligned}
\frac{\mathrm{d}\Psi}{\mathrm{d}x} &= \frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{e}^v Q + e^v\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}Q}{\mathrm{d}u}\\
\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} &= \left(\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2\right)\mathrm{e}^v Q + \left(2\frac{\mathrm{d}u}{\mathrm{d}x}.\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right)\mathrm{e}^v\frac{\mathrm{d}Q}{\mathrm{d}u}\\
&\quad + \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2\mathrm{e}^v\frac{\mathrm{d}^2Q}{\mathrm{d}u^2}.
\end{aligned}$$

Hence a solution of the following differential equation

$$-\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2\frac{\mathrm{d}^2Q}{\mathrm{d}u^2} - \left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + 2\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x}\right)\frac{\mathrm{d}Q}{\mathrm{d}u} + \left(V - \frac{\mathrm{d}^2v}{\mathrm{d}x^2} - \left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2\right)Q = EQ \tag{15}$$

where Q is a polynomial in u gives a solution of (12). Note that equation (15) must be regarded as an equation in the variable u, so expressions such as $\frac{\mathrm{d}v}{\mathrm{d}x}$ must be written in terms of u. So we manipulate (15) further to

$$-\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \frac{\mathrm{d}^2Q}{\mathrm{d}u^2} - \left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + 2\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \frac{\mathrm{d}v}{\mathrm{d}u}\right)\frac{\mathrm{d}Q}{\mathrm{d}u}$$
$$+\left(V - \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \frac{\mathrm{d}^2v}{\mathrm{d}u^2} - \frac{\mathrm{d}v}{\mathrm{d}u}\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - \left(\frac{\mathrm{d}v}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2\right)Q = EQ \tag{16}$$

This has the general form

$$A(u)Q''(u) + B(u)Q'(u) + C(u)Q(u) = EQ(u) \tag{17}$$

where Q is given by (13) is polynomial.

2.5 One-dimensional quantum oscillator (set $m = 1/2$)

For the one-dimensional quantum oscillator we take the potential

$$V(x) = \frac{\omega^2 x^2}{4}$$

and choose

$$u(x) = x, \qquad v(x) = -\frac{\omega x^2}{4}$$

in (14). This leads us to the differential equation

$$-Q'' + \omega u Q' + \frac{\omega}{2}Q = EQ. \tag{18}$$

Since Q is a polynomial function of u, say of order n, to leading order we set

$$Q \sim u^n \qquad Q' \sim n u^{n-1}, \quad Q'' \sim n(n-1)u^{n-2}$$

and equate the terms of order n in (18). This directly gives the energy eigenvalues as

$$E_n = \omega\left(n + \frac{1}{2}\right). \tag{19}$$

Thus the energy levels are $\omega/2$, $3\omega/2$, $5\omega/2\ldots$. Note that the ground state energy for the classical model is zero, which is not the case here. This difference is a consequence of the uncertainty principle.

2.6 Lie algebras

Algebraic methods are often very powerful in the studies of quantum systems. We start by introducing the concept of a *Lie algebra*, and discuss a first example that will play a central role throughout these notes.

An algebra is a vector space V endowed with a multiplication map $m : V \times V \to V$. Below, it is assumed that all vector spaces are over the field $\mathbb{C}$. A *Lie algebra* L has a multiplication known as the bracket, denoted by $[\ ,\ ]$. It satisfies the following three properties. For all $x, y, z \in L$ and for all α, $\beta \in \mathbb{C}$:

- Antisymmetry. $[x,\, y] = -[y,\, x]$;
- Bilinearity. It is sufficient that $[x,\, \alpha y + \beta z] = \alpha[x,\, y] + \beta[x,\, z]$ holds;
- Jacobi identity. $[x,\, [y,\, z]] + [y,\, [z,\, x]] + [z,\, [x,\, y]] = 0$.

Throughout we will discuss *complex Lie algebras*, defined over the field $\mathbb{C}$. Notationally, we drop the dependence on the field so these will be denoted $gl(n)$, $o(n)$, $sp(2m)$ etc. Let L be a finite-dimensional Lie algebra, which as a vector space has basis $\{x_1, \ldots, x_n\}$. We can then write, using the Einstein summation convention:

$$[x_i, x_j] = C_{ij}^k x_k \equiv \sum_{k=1}^{n} C_{ij}^{k} x_k \tag{20}$$

where the $C_{ij}^k \in \mathbb{C}$ are called the *structure constants* of L. The defining properties imply constraints:

- $C_{ij}^k = -C_{ji}^k$
- $C_{jk}^l C_{il}^m + C_{ij}^l C_{kl}^m + C_{ki}^l C_{jl}^m = 0$.

Let K be a vector subspace of a Lie algebra L (viewed as a vector space). Recall the following definitions (generally for any type of algebra, we use these):

- K is a *subalgebra* of L if $[K, K] \subseteq K$, that is K is closed under $[\ ,\ ]$, viz. $\forall x \in K, [x, K] \subseteq K$.
- K is an *ideal* (or *invariant subalgebra* of L) if $[L, K] \subseteq K$, that is $\forall x \in L, [x, K] \subseteq K$.

Not all subalgebras are ideals. As $[\ ,\ ]$ is antisymmetric, ideals are two-sided. Note:

- (0) and L are ideals.
- The *commutator subalgebra* $[L, L]$ is an ideal. (This is also known as the *derived* subalgebra.) If $[L, L] = (0)$, then L is *abelian*.
- If the only ideals of a non-abelian Lie algebra are (0) and L, then L is said to be a *simple* Lie algebra (L has no proper factor algebras).

Exercise 4. *Prove that every 2-dimensional Lie algebra is not simple.*

A *Lie algebra homomorphism* between Lie algebras L and L' is a vector space mapping $\phi : L \to L'$ that preserves the bracket operation:

$$\phi([x, y]) = [\phi(x), \phi(y)]. \tag{21}$$

As for groups, we define the kernel and image of ϕ, with the comments that whilst $\ker(\phi)$ is an *ideal* of L, $\mathrm{im}(\phi)$ is only a *subalgebra* of L'.

There are several fundamentally important Lie algebras that will be examined in detail in these notes. One is the *Heisenberg algebra* (also known by other names), denoted here as as $h(1)$. This algebra has basis $\{a, b, c\}$ subject to the commutation relations

$$[a,\, b] = c, \quad [a,\, c] = [b,\, c] = 0. \tag{22}$$

$\tilde{a} = \phi(a) = ka$

$\tilde{b} = \phi(b) = \ell b$

$[\tilde{a}, \tilde{b}] = k\ell[a, b] = k\ell c = \tilde{c}$

Note: Since $[b, b^+] = I$, the algebra does not have finite-dimensional reps. since $tr(\pi[X, Y]) = tr(\pi(X)\pi(Y) - \pi(Y)\pi(X)) = 0$.

This algebra is not simple. It contains a one-dimensional ideal with basis $\{c\}$. It is commonplace, for reasons which will arise later, to impose further restrictions on this algebra by identifying a as the *conjugate* of b, that is $a = b^\dagger$, and identifying $c = -I$ where I is the identity operator in the universal enveloping algebra satisfying $I^\dagger = I$. With these conditions the commutation relations are

$$[b,\, b^\dagger] = I, \quad [b,\, I] = [b^\dagger,\, I] = 0 \tag{23}$$

and satisfy

$$[x,\, y]^\dagger = [y^\dagger,\, x^\dagger] \tag{24}$$

for all $x,\, y \in h(1)$. The relation (24) shows that conjugation satisfies the defining property of an *anti-automorphism.*

2.7 One-dimensional quantum oscillator revisited

$q^\dagger = q, \quad p^\dagger = p$

Let's re-examine the Hamiltonian for the one-dimensional quantum oscillator

$$H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}.$$

From page 6

We will set $\hbar = 1$. Putting

$[q, p] = i\hbar I \equiv iI$.

$$b = \sqrt{\frac{m\omega}{2}}q + i\sqrt{\frac{1}{2m\omega}}p,$$
$$b^\dagger = \sqrt{\frac{m\omega}{2}}q - i\sqrt{\frac{1}{2m\omega}}p,$$

$\Rightarrow [b, b^+] = I$.

then

$$H = \omega\left(b^\dagger b + \frac{1}{2}I\right)$$
$$= \omega\left(N + \frac{1}{2}I\right)$$

where $N = b^\dagger b$. Note that for any non-zero state $|\Psi\rangle$, and setting $|\Phi\rangle = b\,|\Psi\rangle$, we have

$$\frac{1}{\omega}\langle\Psi|\, H\,|\Psi\rangle = \langle\Psi|\, b^\dagger b\,|\Psi\rangle + \frac{1}{2}\langle\Psi|\Psi\rangle$$
$$= \langle\Phi|\Phi\rangle + \frac{1}{2}\langle\Psi|\Psi\rangle$$
$$> 0.$$

It follows that the spectrum of H is positive.

Exercise 5. *Using the canonical relation $[q,\, p] = iI$ show that $[b,\, b^\dagger] = I$. Also show by induction, and using the derivation property (11), that for $k \in \mathbb{N}$,*

$$[b,\, (b^\dagger)^k] = k(b^\dagger)^{k-1}.$$

$bb^+ - b^+ b = I$. Set $N = b^+ b \Rightarrow bb^+ = N + I$.

$[N, b^+] = [b^+ b, b^+] = b^+ [b, b^+] + \cancel{[b^+, b^+]} b = b^+$

$[N, b] = [b^+ b, b] = b^+ \cancel{[b, b]} + [b^+, b] b = -b$

We introduce the vacuum state $|0\rangle$, assumed to be normalised, satisfying

$$b\,|0\rangle = 0.$$

(**Nota Bene:** The right-hand side above denotes the zero vector. The zero vector is not $|0\rangle$.) Now construct the set of states

$$|n\rangle = \frac{1}{\sqrt{n!}}(b^\dagger)^n\,|0\rangle$$

such that

$$\begin{aligned}
N\,|n\rangle &= b^\dagger b\,|n\rangle \\
&= \frac{1}{\sqrt{n!}} b^\dagger b (b^\dagger)^n\,|0\rangle \\
&= \frac{1}{\sqrt{n!}} b^\dagger \left((b^\dagger)^n b + n(b^\dagger)^{n-1}\right)|0\rangle \\
&= \frac{1}{\sqrt{n!}} n (b^\dagger)^n\,|0\rangle \\
&= n\,|n\rangle\,.
\end{aligned}$$

If $N|\alpha\rangle = \alpha|\alpha\rangle$, set $|\beta\rangle = b|\alpha\rangle$, so $\langle\beta| = \langle\alpha|b^+$. Then

$\langle\beta|\beta\rangle = \langle\alpha|b^+ b|\alpha\rangle$
$= \alpha\langle\alpha|\alpha\rangle$

$\Rightarrow \alpha$ is non-negative

Moreover,

$N|\beta\rangle$
$= N b|\alpha\rangle$
$= [N,b]|\alpha\rangle + bN|\alpha\rangle$
$= -b|\alpha\rangle + \alpha b|\alpha\rangle$
$= (\alpha-1)|\beta\rangle$

The states $|n\rangle$ are normalised. To verify this, consider

$$\begin{aligned}
b^\dagger\,|n-1\rangle &= \frac{1}{\sqrt{(n-1)!}} b^\dagger (b^\dagger)^{n-1}\,|0\rangle \\
&= \frac{1}{\sqrt{(n-1)!}} (b^\dagger)^n\,|0\rangle \\
&= \sqrt{n}\,|n\rangle\,,
\end{aligned}$$

or rather

$$|n\rangle = \frac{1}{\sqrt{n}} b^\dagger\,|n-1\rangle\,.$$

Now

$$\begin{aligned}
\langle n|n\rangle &= \frac{1}{n}\langle n-1|bb^\dagger|n-1\rangle \\
&= \frac{1}{n}\langle n-1|(I+N)|n-1\rangle \\
&= \langle n-1|n-1\rangle\,.
\end{aligned}$$

Assuming $|0\rangle$ to be normalised this implies, by induction, that all states are normalised.

The above means that the action of the Hamiltonian can be determined as

$$H\,|n\rangle = \omega(n+1/2)\,|n\rangle\,,$$

in agreement with (19). In deriving the above the existence of a vacuum state was assumed. If such a state did not exist then the energies would be unbounded from below, since it can be shown that the action of the operator b on an eigenstate gives an eigenstate with lower energy eigenvalue. However, we have already established that the spectrum of H is positive, so the vacuum state must exist.

Exercise 6. *Perhaps the above construction does not account for all states. Assume that there exists a state* $|\Theta\rangle$ *which has properties*

$$\langle\Theta|\Theta\rangle = 1, \qquad\qquad b^\dagger|\Theta\rangle = 0.$$

Give an argument as to why the state $|\Theta\rangle$ *does not exist.*

The infinite-dimensional vector space

$$\mathcal{F} = \mathrm{span}\{|0\rangle,\, |1\rangle,\, |2\rangle, \ldots\} \tag{25}$$

is known as *Fock space.*

3 Lie algebraic methods

$[h,f] = hf - fh = -f$

Another important example of a Lie algebra is the smallest, non-zero dimensional case that is simple. This is commonly referred to as $su(2)$, $so(3)$, $o(3)$, $sl(2)$, less commonly as $sp(1)$, $sp(2)$, $u(2)$, $su(1,1)$, and possibly other names. This Lie algebra has basis $\{e, f, h\}$ subject to the commutation relations

Note: $[A,B]^\dagger = [B^\dagger, A^\dagger]$ must be satisfied.

$$[h,\, e] = e, \quad [h,\, f] = -f, \quad [e,\, f] = h. \tag{26}$$

Note that a straightforward redefinition of the elements as $E = e$, $F = -f$ and $H = h$ leads to the commutation relations

$$[H,\, E] = E, \quad [H,\, F] = -F, \quad [E,\, F] = -H. \tag{27}$$

The commutation relations (26) are typically associated with $su(2)$, while (27) are typically associated with $su(1,1)$. This means that, as Lie algebras, they are isomorphic. The distinction comes with the consideration of conjugation. It is easily verified that

$$e^\dagger = f, \quad f^\dagger = e, \quad h^\dagger = h \qquad \Longleftrightarrow \qquad E^\dagger = -F, \quad F^\dagger = -E, \quad H^\dagger = H \tag{28}$$

is an anti-automorphism of (26), but not of (27). Similarly,

$$E^\dagger = F, \quad F^\dagger = E, \quad H^\dagger = H \qquad \Longleftrightarrow \qquad e^\dagger = -f, \quad f^\dagger = -e, \quad h^\dagger = h \tag{29}$$

is an anti-automorphism of (27), but not of (26). This difference does become significant in the construction of *representations* of Lie algebras, which is discussed later.

$e^\dagger = [h,e]^\dagger = [e^\dagger, h^\dagger] = [f,h] = f$

Exercise 7. *Note that in the abstract definition of a Lie algebra the commutator* $[x, y]$ *is not the same as* $xy - yx$. *For example, consider the vector space* $\mathbb{R}^3$ *which becomes an algebra under the multiplication given by the cross product. We already know the cross product is bilinear and antisymmetric. Show the cross product also satisfies the Jacobi identity; i.e.*

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0$$

establishing that $(\mathbb{R}^3, \times)$ *is a Lie algebra.*

Exercise 8. *Let* $M_{n\times n}(\mathbb{R})$ *denote the vector space of real* $n \times n$. *For a fixed* $f = (f_{ij}) \in M_{n\times n}(\mathbb{R})$ *let*

$$L_f = \{x \in M_{n\times n}(\mathbb{R}) \mid x^T f + fx = 0\}.$$

Show that L_f *is a Lie algebra by establishing that it is a vector space and that it is closed under the matrix commutator* $[x, y] = xy - yx$.

3.1 General linear and special linear Lie algebras

We define $gl(n)$ to be the n^2-dimensional complex Lie algebra with basis

$$\{a^i{}_j;\ i, j = 1, \ldots, n\} \tag{30}$$

satisfying the $gl(n, \mathbb{C})$ *commutation relations*:

$$[a^i{}_j, a^k{}_l] = \delta^k{}_j a^i{}_l - \delta^i{}_l a^k{}_j. \tag{31}$$

The generator $I_1 = \sum_{k=1}^{n} a^k{}_k$ commutes with all the $a^i{}_j$, viz.: $[I_1, a^i{}_j] = 0$. It is an example of a *Casimir invariant*, which generates an ideal. A convenient basis for $sl(n)$ is given by the $n^2 - 1$ linearly independent operators:

$$\left\{a^i{}_j - \frac{1}{n}\delta_{ij} I_1 : \ i, j = 1, \ldots, n, \text{ excluding } i = j = n\right\}.$$

Another basis is:

$$\{a^i{}_j :\ i, j = 1, \ldots, n,\ i \neq j\} \cup \{a^i{}_i - a^{i+1}{}_{i+1} :\ i = 1, \ldots, n-1\}. \tag{32}$$

We have an ideal direct sum: $gl(n) = sl(n) \oplus \mathbb{C} I_1$.

3.2 Orthogonal Lie algebras

The *orthogonal Lie algebra* $o(n)$ can be viewed as a subset $o(n) \subset gl(n)$. It is spanned by the $\frac{1}{2}n(n-1)$ generators $\alpha^i{}_j = -\alpha^j{}_i$, defined in terms of the $gl(n)$ generators by:

$$\alpha^i{}_j = a^i{}_j - a^j{}_i,\ i, j = 1, \ldots, n. \tag{33}$$

For $gl(n)$, customary to consider $(a^i{}_j)^\dagger = a^j{}_i$
$(i)^\dagger = -i$

The generators satisfy the $o(n)$ *commutation relations*:

$$[\alpha^i{}_j, \alpha^k{}_l] = (\delta^k{}_j\alpha^i{}_l - \delta^i{}_l\alpha^k{}_j) - (\delta^k{}_i\alpha^j{}_l - \delta^j{}_l\alpha^k{}_i). \tag{34}$$

These may be verified by substitution of (31) into (33), viz. (using $\alpha^i{}_j = -\alpha^j{}_i$):

$$\begin{aligned}
[\alpha^i{}_j, \alpha^k{}_l] &= [a^i{}_j - a^j{}_i, a^k{}_l - a^l{}_k] = [a^i{}_j, a^k{}_l] - [a^i{}_j, a^l{}_k] - [a^j{}_i, a^k{}_l] + [a^j{}_i, a^l{}_k] \\
&= (\delta^k{}_j a^i{}_l - \delta^i{}_l a^k{}_j) - (\delta^l{}_j a^i{}_k - \delta^i{}_k a^l{}_j) - (\delta^k{}_i a^j{}_l - \delta^j{}_l a^k{}_i) + (\delta^l{}_i a^j{}_k - \delta^j{}_k a^l{}_i) \\
&= \delta^k{}_j(a^i{}_l - a^l{}_i) - \delta^i{}_l(a^k{}_j - a^j{}_k) - \delta^k{}_i(a^j{}_l - a^l{}_j) + \delta^j{}_l(a^k{}_i - a^i{}_k) \\
&= (\delta^k{}_j\alpha^i{}_l - \delta^i{}_l\alpha^k{}_j) - (\delta^k{}_i\alpha^j{}_l - \delta^j{}_l\alpha^k{}_i).
\end{aligned}$$

3.3 Symplectic Lie algebras

For the symplectic Lie algebras $sp(n = 2m)$, a convenient basis for the $\frac{1}{2}n(n+1) = m(2m+1)$ symmetric generators $\alpha^i_j = \alpha^j_i$ is given by:

$$\{\alpha^i{}_j = f^{ik}a^k{}_j + f^{jk}a^k{}_i \mid i, j = 1, \ldots, m\}. \tag{35}$$

where f is antisymmetric, i.e. $f^{ij} = -f^{ji}$. These satisfy the $sp(m = 2p)$ *commutation relations*:

$$[\alpha^i{}_j, \alpha^k{}_l] = (f^{kj}\alpha^i{}_l - f^{il}\alpha^k{}_j) - (f^{jl}\alpha^k{}_i - f^{ki}\alpha^j{}_l). \tag{36}$$

Again, these may be verified by substitution of (31) into (35) and by implicitly summing over dummy indices m, n:

$$\begin{aligned}
[\alpha^i{}_j, \alpha^k{}_l] &= [f^{im}a^m{}_j + f^{jm}a^m{}_i, f^{kn}a^n{}_l + f^{ln}a^n{}_k] \\
&= f^{im}f^{kn}[a^m{}_j, a^n{}_l] + f^{im}f^{ln}[a^m{}_j, a^n{}_k] + f^{jm}f^{kn}[a^m{}_i, a^n{}_l] + f^{jm}f^{ln}[a^m{}_i, a^n{}_k] \\
&= f^{im}f^{kn}(\delta_{nj}a^m{}_l - \delta_{ml}a^n{}_j) + f^{im}f^{ln}(\delta_{nj}a^m{}_k - \delta_{mk}a^n{}_j) \\
&\quad + f^{jm}f^{kn}(\delta_{ni}a^m{}_l - \delta_{ml}a^n{}_i) + f^{jm}f^{ln}(\delta_{ni}a^m{}_k - \delta_{mk}a^n{}_i) \\
&= f^{im}f^{kj}a^m{}_l - f^{il}f^{kn}a^n{}_j + f^{im}f^{lj}a^m{}_k - f^{ik}f^{ln}a^n{}_j \\
&\quad + f^{jm}f^{ki}a^m{}_l - f^{jl}f^{kn}a^n{}_i + f^{jm}f^{li}a^m{}_k - f^{jk}f^{ln}a^n{}_i \\
&= f^{jk}(f^{mi}a^m{}_l + f^{nl}a^n{}_i) - f^{li}(f^{nk}a^n{}_j + f^{mj}a^m{}_k) \\
&\quad - f^{ki}(f^{mj}a^m{}_l + f^{nl}a^n{}_j) + f^{jl}(f^{nk}a^n{}_i + f^{mi}a^m{}_k) \\
&= -f^{jk}(f^{im}a^m{}_l + f^{ln}a^n{}_i) + f^{li}(f^{kn}a^n{}_j + f^{jm}a^m{}_k) \\
&\quad + f^{ki}(f^{jm}a^m{}_l + f^{ln}a^n{}_j) - f^{jl}(f^{kn}a^n{}_i + f^{im}a^m{}_k) \\
&= (f^{li}\alpha^k{}_j - f^{jk}\alpha^i{}_l) - (f^{jl}\alpha^k{}_i - f^{ki}\alpha^j{}_l) \\
&= (f^{kj}\alpha^i{}_l - f^{il}\alpha^k{}_j) - (f^{jl}\alpha^k{}_i - f^{ki}\alpha^j{}_l).
\end{aligned}$$

3.4 Unitary Lie algebras

The unitary Lie algebra $u(n)$ is spanned by the n^2 generators:

$$\{\alpha^j{}_k = a^j{}_k - a^k{}_j, \quad \beta^j{}_k = i(a^j{}_k + a^k{}_j); \qquad j, k = 1, \ldots, n\}. \tag{37}$$

ε_{ijk} – Levi-Civita symbol. Defined by $\varepsilon_{123} = 1$, $\varepsilon_{ijk} = -\varepsilon_{jik} = \varepsilon_{jki} = -\varepsilon_{kji} =$ etc. In particular $\varepsilon_{112} = -\varepsilon_{112} = 0$

3.5 Some observations

Note that over $\mathbb{C}$, $u(n)$ is isomorphic to $gl(n)$ as we can write

$$a^j{}_k = \frac{1}{2}\left(\alpha^j{}_k - i\beta^j{}_k\right).$$

However $u(n)$ and $gl(n)$ are not isomorphic over $\mathbb{R}$.

In general, the orthogonal, symplectic and special linear Lie algebras are non-isomorphic, however examples of Lie algebra isomorphisms include $o(3) \sim su(2) \sim sp(2) \sim sl(2)$, $o(2) \sim u(1)$, and the less trivial $sp(4) \sim o(5)$.

- Where Lie algebra elements are represented by matrices, $gl(2)$ is spanned by the four 2×2 matrices $e_{11}, e_{21}, e_{12}, e_{22}$. This space is four dimensional. Observe that $I_1 = e_{11} + e_{22}$ spans a one dimensional ideal, so that $gl(2)$ is not simple. However, the subspace $sl(2)$ (traceless matrices) is simple, and it has a basis $\left\{\frac{1}{2}(e_{11} - e_{22}), e_{21}, e_{12}\right\}$.

- $sl(n)$ and $su(n)$ are simple subalgebras of $gl(n)$ and $u(n)$ respectively.

- For $o(3)$, the generators can be written as (Einstein summation convention used) $L_i = \frac{1}{2}\varepsilon_{ijk}\alpha^j{}_k$ where $i = 1, 2, 3$, satisfying $[L_i, L_j] = \varepsilon_{ijk}L_k$. For physical applications it is typical to define $L_i = \frac{i}{2}\varepsilon_{ijk}\alpha^j{}_k$ and obtain instead $[L_i, L_j] = i\varepsilon_{ijk}L_k$.

- $o(n)$ is simple for all $n \neq 4$. Indeed $o(4)$ is spanned by the six antisymmetric matrices $\{\alpha_{ij} = e_{ij} - e_{ji},\ i, j \in \{1, 2, 3, 4\},\ i < j\}$:

$$\alpha_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ etc.} \tag{38}$$

An alternative basis is given by the six matrices $L_i^{(\pm)} = \frac{i}{2}(b_i \pm a_i)$ where we have $\{a_i = \alpha^i{}_4,\ b_i = \frac{1}{2}\varepsilon_{ijk}\alpha^j{}_k : i = 1, 2, 3\}$. These close to form two Lie algebras $L^{(+)}$ and $L^{(-)}$, each isomorphic to $o(3)$ such that $[L^{(+)}, L^{(-)}] = (0)$. Explicitly, with $\{i,\ j,\ k\}$ an even permutation of $\{1,\ 2,\ 3\}$ and $\mu,\ \nu \in \{\pm 1\}$ we have the commutator

$$\begin{aligned}
[L_i^{(\mu)},\ L_j^{(\nu)}] &= -\frac{1}{4}[\alpha^j{}_k + \mu\alpha^i{}_4,\ \alpha^k{}_i + \nu\alpha^j{}_4] \\
&= -\frac{1}{4}[e_{jk} - e_{kj} + \mu(e_{i4} - e_{4i}),\ e_{ki} - e_{ik} + \nu(e_{j4} - e_{4j})] \\
&= -\frac{1}{2}(1 + \mu\nu)\alpha^j{}_i - \frac{1}{2}(\mu + \nu)\alpha^k{}_4 \\
&= i\delta_{\mu\nu}L_k^{(\mu)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}[L_i^{(+)},\, L_i^{(-)}] &= \frac{1}{4}[\alpha^j{}_k + \alpha^i{}_4,\, \alpha^j{}_k - \alpha^i{}_4] \\ &= \frac{1}{4}[e_{jk} - e_{kj} + e_{i4} - e_{4i},\, e_{jk} - e_{kj} - e_{i4} + e_{4i}] \\ &= 0.\end{aligned}$$

Thus $L^{\pm}$ form two non-zero ideals in $o(4)$, and $o(4)$ is the direct sum of two commuting $o(3)$ Lie algebras:

$$o(4) \sim o(3) \oplus o(3). \tag{39}$$

- $sp(2m)$ is simple.
- *Lorentz Lie algebra* $o(3,1)$. The Minkowski four-dimensional space-time metric is:

$$f = \begin{pmatrix} I_3 & 0 \\ 0 & -1 \end{pmatrix} \tag{40}$$

which defines spacetime length $x^2 + y^2 + z^2 - t^2$ (in units where $c \equiv 1$). The Lorentz Lie algebra $o(3,1)$ is spanned by the operators:

$$\{\beta^\mu{}_\nu = f^{\mu\sigma} e_{\sigma\nu} - f^{\nu\sigma} e^{\sigma\mu};\ \mu, \nu = 1, \ldots, 4\}.$$

We thus have the $o(3)$ subalgebra with basis:

$$\{\alpha^i{}_j = e_{ij} - e_{ji} :\ i, j = 1, 2, 3\} \tag{41}$$

generating spatial rotations together with the three space time rotation generators:

$$\{\alpha^i{}_4 = e_{i4} + e_{4i} :\ i = 1, 2, 3\}. \tag{42}$$

Exercise 9. *So far we have only considered finite-dimensional Lie algebras. However infinite-dimensional Lie algebras also exist. Show that the differential operators*

$$\tilde{L}_n = z^{n+1} \frac{\mathrm{d}}{\mathrm{d}z}, \qquad n \in \mathbb{Z}$$

close to form a Lie algebra, which is known as the Witt algebra. *For each $n \neq 0$, show that L_0, $L_{\pm n}$ close to form a subalgebra.*

3.6 Representations of Lie algebras

Let V be an arbitrary complex vector space, which in general may be infinite-dimensional. The set of linear transformations on V is denoted $\mathrm{End}(V)$. Endowed with a bracket multiplication, specifically the usual commutator bracket for linear transformations:

$$[X, Y] = XY - YX,$$

End(V) satisfies the definition of a Lie algebra. More generally, for complex vector spaces V, W, the vector space of linear transformations from V to W is denoted Hom(V, W). Thus, in the above notation, End(V) = Hom(V, V). Equipped with an appropriate bracket, Hom(V, W) also satisfies the defining properties of a Lie algebra. If V, W are finite-dimensional, say of dimensions m and n respectively, then:

$$\dim[\mathrm{Hom}(V, W)] = nm.$$

A *representation* of a Lie algebra L on a vector space V is a Lie algebra homomorphism $\pi : L \to \mathrm{End}(V)$. A homomorphism necessarily preserves the bracket operation, viz:

$$\pi([a, b]) = [\pi(a), \pi(b)] = \pi(a)\pi(b) - \pi(b)\pi(a).$$

(The bracket on the left is the commutator in L, that on the right is the commutator in End(V).) We call V the *representation space* of π and conversely π is called the *representation afforded by* V.

In all cases, as a result of the Jacobi identity the Lie algebra L *itself* admits an *adjoint* representation, ad : $L \to L$, defined for $x, y \in L$ by

$$\mathrm{ad}(x) \circ y = [x, y].$$

Explicitly, if L has a basis $\{x_1, \ldots, x_n\}$, and $[x_i, x_j] = \sum_{k=1}^{n} C_{ij}^k x_k$, then the matrix elements of ad(x_i) are given by $[\mathrm{ad}(x_i)]_{kj} = C_{ij}^k$. To see this

$$\sum_{k=1}^{n} C_{ij}^k x_k = \mathrm{ad}(x_i) \circ x_j = \sum_{k=1}^{n} [\mathrm{ad}(x_i)]_{kj} x_k.$$

Exercise 10. *Determine the adjoint representation for $h(1)$, with commutation relations (22), and for $su(2)$, with commutation relations (26).*

Exercise 11. *Consider n independent variables x_i, and their corresponding partial derivatives. Show that*

$$a^i{}_j = x_i \frac{\partial}{\partial x_j} \tag{43}$$

satisfy the $gl(n)$ commutation relations.

Exercise 12. *Let L be a Lie algebra with basis $\{x_1, ..., x_n\}$ and commutation relations*

$$[x_i,\, x_j] = C_{ij}^k x_k. \tag{44}$$

Show that the elements

$$X_i = \sum_{j,k} C_{ji}^k a^j{}_k$$

also satisfy the commutation relations (44) with the a_{ij} the usual basis elements for $gl(n)$ satisfying

$$[a^i{}_j,\, a^k{}_l] = \delta_j^k a^i{}_l - \delta_l^i a^k{}_j.$$

This result is known as Ado's theorem.

3.7 Unitary representations

Let L denote a Lie algebra with a conjugation $\dagger : L \to L$ satisfying the anti-automorphism property (24). A representation π of L is said to be *unitary* if, for all $x \in L$,

$$\pi(x^\dagger) = \overline{\pi^T(x)} \equiv \pi^\dagger(x)$$

where T denotes matrix transposition, and the overline denotes complex conjugation. That is, $\dagger$ on the right denotes *Hermitian conjugation.*

Sometimes a representation may not be unitary for a given basis, but is equivalent to a unitary representation by an appropriately chosen basis.

Exercise 13. *Verify that the adjoint representation for $su(2)$, with commutation relations (26) and conjugation (28), is unitary for suitable choice of basis. Determine the adjoint representation for $su(1,1)$, with commutation relations (27) and conjugation (29). Show that there does not exist a basis for which this representation is unitary.*

3.8 Lie algebra modules

Instead of considering representations, we may instead talk about modules. For a Lie algebra L, consider a complex vector space V endowed with an action $\circ : L \times V \to V$, which, for $x \in L, v \in V$ we denote by:

$$(x, v) \mapsto x \circ v.$$

Then V is called an L-module if the action has the following properties for all x, $y \in L$, v, $w \in V$ and α, $\beta \in \mathbb{C}$

- Linearity:

$$x \circ (\alpha v + \beta w) = \alpha(x \circ v) + \beta(x \circ w),$$
$$(\alpha x + \beta y) \circ v = \alpha(x \circ v) + \beta(y \circ v).$$

- Homomorphism: $[x, y] \circ v = x \circ (y \circ v) - y \circ (x \circ v)$.

If $\pi : L \to \mathrm{End}(V)$ is a representation of L on a vector space V, then V becomes an L-module with action defined for $x \in L, v \in V$ by:

$$x \circ v \equiv \pi(x)v.$$

Conversely, if V is an L-module, then V determines a representation $\pi : L \to \mathrm{End}(V)$ defined by:

$$\pi(x)v \equiv x \circ v.$$

As the descriptions are equivalent, either may be chosen according to convenience. It is commonplace to drop the notations π and $\circ$ and simply write vx.

3.9 Tensor products of vector spaces

Let V, W be finite-dimensional vector spaces. The *tensor product* space $V \otimes W$ is formally defined as the vector space spanned by the vectors:

$$v \otimes w, \qquad v \in V,\ w \in W$$

where the tensor product operation $\otimes : (v, w) \mapsto v \otimes w$ satisfies the following bilinearity requirements for all $\alpha, \beta \in \mathbb{C}$, $v, v' \in V$ and $w, w' \in W$

- $(\alpha v + \beta v') \otimes w = \alpha(v \otimes w) + \beta(v' \otimes w)$;
- $v \otimes (\alpha w + \beta w') = \alpha(v \otimes w) + \beta(v \otimes w')$.

Note that these imply that $\alpha(v \otimes w) = (\alpha v) \otimes w = v \otimes (\alpha w)$. Now, let V, W have bases $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$ respectively. Then $V \otimes W$ is the mn-dimensional vector space with basis

$$\{v_i \otimes w_j \mid i = 1, \ldots, m;\ j = 1, \ldots, n\}. \tag{45}$$

Indeed, if $v = \sum_{i=1}^{m} \alpha_i v_i \in V$ and $w = \sum_{j=1}^{N} \beta_j w_j \in W$, then $v \otimes w = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_i \beta_j (v_i \otimes w_j)$.

If $A \in \mathrm{End}(V)$ and $B \in \mathrm{End}(W)$, define $A \otimes B \in \mathrm{End}(V \otimes W)$ by the action

$$(A \otimes B)(v \otimes w) = (Av) \otimes (Bw).$$

Let $[A_{ij}]$ be the matrix of A in the basis $\{v_i\}_{i=1}^{m}$ of V and $[B_{ij}]$ be the matrix of B in the basis $\{w_i\}_{i=1}^{n}$ of W, then

$$Av_i = \sum_{k=1}^{m} A_{ki} v_k, \qquad Bw_j = \sum_{q=1}^{n} B_{qj} w_q.$$

The matrix of $A \otimes B$ in the tensor product basis (45) of $V \otimes W$ is then found by inspection of

$$(A \otimes B)(v_i \otimes w_j) = \sum_{k=1}^{m} \sum_{q=1}^{n} A_{ki} B_{qj} (v_k \otimes w_q),$$

viz

$$(A \otimes B)_{kq,ij} = A_{ki} B_{qj}.$$

This is called the *tensor product of the matrices* A and B.

It can be shown that $\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\mathrm{tr}(B)$, and $\det(A \otimes B) = \det(A^n)\det(B^m)$.

3.10 Tensor products of modules

If V and W are L-modules, the tensor product $V \otimes W$ is an L-module with the definition

$$x(v \otimes w) = (xv) \otimes w + v \otimes (xw) \tag{46}$$

for $x \in L$, $v \in V$ and $w \in W$. From the co-product (??), the left side of (46) may be expressed as $\Delta(x)(v \otimes w)$.

In representation theoretic terms, where π_V and π_W are the representations afforded by V and W respectively, the *tensor product of the representations* π_V and π_W afforded by $V \otimes W$ is $\pi_{V\otimes W}$ is defined by

$$\pi_{V\otimes W}(x) = \pi_V(x) \otimes I + I \otimes \pi_W(x).$$

More generally, if $V_1, V_2, \ldots, V_k$ are L-modules we may define the vector space $V_1 \otimes V_2 \otimes \cdots \otimes V_k$, spanned by the vectors $v_1 \otimes v_2 \otimes \cdots \otimes v_k$, for $v_i \in V_i$, $i = 1, 2, \ldots, k$, which becomes an L-module with the definition

$$\begin{aligned} x \circ (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = (xv_1) \otimes v_2 \otimes \cdots \otimes v_k + v_1 \otimes (xv_2) \otimes \cdots \otimes v_k \\ + \ldots + v_1 \otimes v_2 \otimes \cdots \otimes (xv_k). \end{aligned}$$

In general, $V \otimes W$ is a reducible L-module. The problem of decomposing $V \otimes W$ into a direct sum of irreducible L-modules is called the *Clebsch-Gordan problem.* We will see an example of this in Sect. 3.18.

3.11 The universal enveloping algebra of a Lie algebra

Let L be a finite-dimensional Lie algebra, with basis $\{x_1, \ldots, x_n\}$. We may embed L in an infinite-dimensional associative algebra $U(L)$, called the *universal enveloping algebra* on L, which is the algebra spanned by the formal products:

$$\{1 \in \mathbb{C}, x_i, x_i x_j, x_i x_j x_k, \ldots\} \tag{47}$$

with the condition that $x_i x_j - x_j x_i = [x_i, x_j] = C_{ij}^k x_k$. Thus $U(L)$ is like the polynomial algebra $\mathbb{C}[x_1, \ldots, x_n]$ except that the variables x_i in general no longer commute. If they do, that is if L is abelian, then all $C_{ij}^k = 0$, and $U(L)$ *is* the polynomial *ring* $\mathbb{C}[x_1, \ldots, x_n]$. More formally:

$$U(L) = \bigoplus_{k=0}^{\infty} \mathbb{C}L^k = \mathbb{C} \oplus \mathbb{C}L \oplus \mathbb{C}L^2 \oplus \cdots \oplus \mathbb{C}L^k \oplus \cdots \tag{48}$$

An L-module V becomes a $U(L)$ module with the definition:

$$(x_{i_1} x_{i_2} \cdots x_{i_k})v = x_{i_1}(x_{i_2} \cdots (x_{i_k} v) \cdots), \qquad \text{for } v \in V. \tag{49}$$

If W is a subspace of V, we set $U(L)W = \{uw \,:\, u \in U(L), w \in W\}$ as the *$U(L)$-module generated by W*. When $W = \mathbb{C}v$, we write $U(L)v$ in place of $U(L)W$. Let V be an irreducible L-module, and take any non-zero $v \in V$. Now, $U(L)v \subseteq V$ is a submodule of V. Since V is irreducible, $V = U(L)v$.

$L = o(3)$ Lie algebra $\{e, f, h\}$

$$[h, e] = e \ , \quad [h, f] = -f \ , \quad [e, f] = h$$

Conjugation operation: $\dagger : L \to L$ such that
$[A, B]^\dagger = [B^\dagger, A^\dagger]$, and $(\alpha A + \beta B)^\dagger = \bar{\alpha} A^\dagger + \bar{\beta} B^\dagger$

For $o(3)$ $e^\dagger = f$, $f^\dagger = e$, $h^\dagger = h$.

Consider ~~…~~ for $\alpha > 0$ $\quad \pi_\alpha(h) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$,

$\pi_\alpha(e) = \begin{pmatrix} 0 & \frac{\alpha}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}$, $\quad \pi_\alpha(f) = \begin{pmatrix} 0 & 0 \\ \frac{\alpha^{-1}}{\sqrt{2}} & 0 \end{pmatrix}$.

This provides a representation e.g.

$$\pi_\alpha(e)\pi_\alpha(f) - \pi_\alpha(f)\pi_\alpha(e)$$

$$= \begin{pmatrix} 0 & \frac{\alpha}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \frac{\alpha^{-1}}{\sqrt{2}} & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \frac{\alpha^{-1}}{\sqrt{2}} & 0 \end{pmatrix}\begin{pmatrix} 0 & \frac{\alpha}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} = \pi_\alpha(h)$$

This representation is unitary (see page 20)
for $\alpha = 1$, and non-unitary otherwise.

$$\searrow \quad \pi_1^\dagger(a) = \pi_1(a^\dagger) \qquad a = e, f, h.$$

However Π_2 is equivalent to a unitary representation. Set

$$X = \begin{pmatrix} \alpha^{-\frac{1}{2}} & 0 \\ 0 & \alpha^{\frac{1}{2}} \end{pmatrix}, \quad X^{-1} = \begin{pmatrix} \alpha^{\frac{1}{2}} & 0 \\ 0 & \alpha^{-\frac{1}{2}} \end{pmatrix}$$

Then $X \, \Pi_\alpha(a) X^{-1} = \Pi_1(a) \qquad a = e, f, h$

From page 14, $su(1,1)$ relations

$$[H, E] = E, \quad [H, F] = -F, \quad [E, F] = -H.$$

We then have the representation family

$$\Pi_\alpha(H) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad \Pi_\alpha(E) = \begin{pmatrix} 0 & \frac{\alpha}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}$$

$$\Pi_\alpha(F) = \begin{pmatrix} 0 & 0 \\ -\frac{\alpha^{-1}}{\sqrt{2}} & 0 \end{pmatrix}$$

This representation cannot be made unitary by a basis transformation.

Let V have basis $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$. If $W = V \otimes V$,

then W has basis

$$\left\{ w_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ w_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ w_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ w_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

Notationally, we can write

$$w_1 \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\quad w_2 \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},\quad w_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\quad w_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\qquad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$A \otimes B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array} \right)$$

Recall, (setting $\alpha = 1$)

$$\pi(e) = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}, \quad \pi(f) = \begin{pmatrix} 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \quad \pi(h) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

We now construct the tensor product representation

$$(\pi \otimes \pi)(e) = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \end{pmatrix} = (\pi \otimes \pi)^{\dagger}(f)$$

$$(\pi \otimes \pi)(h) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & & & 0 \\ & \frac{1}{2} & & \\ & & -\frac{1}{2} & \\ 0 & & & -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & & & 0 \\ & -\frac{1}{2} & & \\ & & \frac{1}{2} & \\ 0 & & & -\frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & 0 \\ & 0 & & \\ & & 0 & \\ 0 & & & -1 \end{pmatrix}$$

$\left(\text{Note } P^2 = I \otimes I \right)$

Define the permutation operator $P: V \otimes V \to V \otimes V$ through $P(v \otimes w) = w \otimes v$. Then

$$P w_1 = w_1 , \quad P w_2 = w_3 , \quad P w_3 = w_2 , \quad P w_4 = w_4$$

so we can write

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

thus the product states (70) are eigenstates of L_0 with eigenvalues $(m_1 + m_2)$. Our aim is to obtain the decomposition of $V_{l_1} \otimes V_{l_2}$ into irreducible $o(3)$-modules:

$$V_{l_1} \otimes V_{l_2} = \bigoplus_l V_l.$$

Firstly, the state with maximum L_0 eigenvalue $(l_1 + l_2)$ is $\psi_0 = |l_1, l_1\rangle \otimes |l_2, l_2\rangle$, which is an $o(3)$ *highest-weight state* since:

$$L_+\psi_0 = (L_+ |l_1, l_1\rangle) \otimes |l_2, l_2\rangle + |l_1, l_1\rangle \otimes (L_+ |l_2, l_2\rangle) = 0.$$

Thus the $o(3)$-module $V_{l_1+l_2}$ occurs in $V_{l_1} \otimes V_{l_2}$.

Secondly, two eigenstates of L_0 have eigenvalue $l_1 + l_2 - 1$, namely $|l_1, l_1 - 1\rangle \otimes |l_2, l_2\rangle$ and $|l_1, l_1\rangle \otimes |l_2, l_2 - 1\rangle$. A linear combination of these must occur in $V_{l_1+l_2}$, and another, denoted ψ_1, must belong to the orthocomplement $V^{\perp}_{l_1+l_2}$, of $V_{l_1+l_2}$. Then $V^{\perp}_{l_1+l_2}$ is an $o(3)$-module and $l_1 + l_2 - 1$ is the highest weight in this space, in other words

$$L_0\psi_1 = (l_1 + l_2 - 1)\psi_1, \qquad L_+\psi_1 = 0,$$

so the $o(3)$-module $V_{l_1+l_2-1}$ must also occur in the space $V_{l_1} \otimes V_{l_2}$.

Continuing in this manner, we eventually arrive at the decomposition:

$$V_{l_1} \otimes V_{l_2} = \bigoplus_{l=|l_1-l_2|}^{l_1+l_2} V_l. \tag{71}$$

This result is also known as the addition rule for angular momenta.

- For $l \geqslant \frac{1}{2}$, we have: $V_l \otimes V_{1/2} = V_{l+1/2} \oplus V_{l-1/2}$, for example $V_{1/2} \otimes V_{1/2} = V_1 \oplus V_0$.
- For $l \geqslant 1$, we have: $V_l \otimes V_1 = V_{l+1} \oplus V_l \oplus V_{l-1}$, for example $V_1 \otimes V_1 = V_2 \oplus V_1 \oplus V_0$.

3.16 Two-dimensional quantum oscillator

The Hamiltonian reads

$$\begin{aligned} H &= \frac{\vec{p}.\vec{p}}{2m} + \frac{m\omega^2\vec{q}.\vec{q}}{2} \\ &= \frac{p_1^2 + p_2^2}{2m} + \frac{m\omega^2(q_1^2 + q_2^2)}{2}. \end{aligned}$$

For $j = 1, 2$ set

$$\begin{aligned} b_j &= \sqrt{\frac{m\omega}{2}}q_j + i\sqrt{\frac{1}{2m\omega}}p_j, \\ b_j^\dagger &= \sqrt{\frac{m\omega}{2}}q_j - i\sqrt{\frac{1}{2m\omega}}p_j, \end{aligned}$$

Check :

$$[a^j{}_k, a^l{}_p] = [b_j^\dagger b_k, b_l^\dagger b_p] = b_j^\dagger [b_k, b_l^\dagger b_p] + [b_j^\dagger, b_l^\dagger b_p] b_k$$
$$= b_j^\dagger b_l^\dagger \cancel{[b_k, b_p]} + b_j^\dagger [b_k, b_l^\dagger] b_p + b_l^\dagger [b_j^\dagger, b_p] b_k + \cancel{[b_j^\dagger, b_l^\dagger]} b_p b_k$$
$$= \delta^l_k b_j^\dagger b_p - \delta^j_p b_l^\dagger b_k$$
$$= \delta^l_k a^j{}_p - \delta^j_p a^l{}_k .$$

which satisfy

$$[b_1, b_2] = [b_1^\dagger, b_2^\dagger] = 0,$$
$$[b_j, b_k^\dagger] = \delta_{jk} I.$$

The Hamiltonian can now be expressed as

$$\# \omega(N + I) = \quad H = \omega(b_1^\dagger b_1 + b_2^\dagger b_2 + I).$$

Defining $a^j{}_k = b_j^\dagger b_k$, these operators satisfy the $gl(2)$ commutation relations

$$[a^j{}_k, a^l{}_p] = \delta^l_k a^j{}_p - \delta^j_p a^l{}_k.$$

$$[N, a^l{}_p] = [a^j{}_j, a^l{}_p] = \delta^l_j a^j{}_p - \delta^j_p a^l{}_j = a^l{}_p - a^l{}_p = 0$$

We make a transformation to a new set of operators

$$[h, e] = e \qquad [h, f] = -f \qquad [e, f] = h$$

$$N = a^1{}_1 + a^2{}_2, \; = a^j{}_j$$
$$\tfrac{1}{\sqrt{2}} b_1^\dagger b_2 = \quad e = \frac{1}{\sqrt{2}} a^1{}_2,$$
$$f = \frac{1}{\sqrt{2}} a^2{}_1,$$
$$h = \frac{1}{2}\left(a^1{}_1 - a^2{}_2\right).$$

The set $\{e, f, h\}$ satisfy the commutation relations (26) of the $o(3)$ Lie algebra and N is central; it commutes with all other elements. (Note that N does not equal the operator L^2 discussed earlier.)

We can now determine that the following identification is in fact an $o(3)$-module isomorphism

$$|l, m\rangle = C_{lm} (b_1^\dagger)^{(m+l)} (b_2^\dagger)^{(l-m)} |0\rangle \tag{72}$$

where

$$C_{lm} = \frac{1}{\sqrt{(l+m)!(l-m)!}}.$$

$$\sum_{k=1}^{n} \delta_{kr} A_r = A_k$$

In this case $|0\rangle$ is the vacuum state satisfying

$$b_1 |0\rangle = b_2 |0\rangle = 0.$$

To see that the isomorphism holds, we begin by noting that for all j and k,

$$a^j{}_k |0\rangle = b_j^\dagger b_k |0\rangle = 0.$$

By induction it can be shown

$$[a^j{}_k, (b_m^\dagger)^n] = n \delta_{km} b_j^\dagger (b_m^\dagger)^{(n-1)},$$

AB = [A, B] + BA

and specifically

$$\begin{aligned}
\left[{a^1}_1,\, (b^\dagger_m)^n\right] &= n\delta_{1m}(b^\dagger_1)^n, \\
\left[{a^2}_2,\, (b^\dagger_m)^n\right] &= n\delta_{2m}(b^\dagger_2)^n.
\end{aligned}$$

Next, we can show that

$$\begin{aligned}
{a^1}_1 \left|l,\, m\right\rangle &= C_{lm}{a^1}_1(b^\dagger_1)^{(l+m)}(b^\dagger_2)^{(l-m)}\left|0\right\rangle \\
&= C_{lm}\left((b^\dagger_1)^{(l+m)}{a^1}_1 + (l+m)(b^\dagger_1)^{(l+m)}\right)(b^\dagger_2)^{(l-m)}\left|0\right\rangle \\
&= C_{lm}(b^\dagger_1)^{(l+m)}{a^1}_1\left|0\right\rangle + (l+m)C_{lm}(b^\dagger_1)^{(l+m)}(b^\dagger_2)^{(l-m)}\left|0\right\rangle \\
&= (l+m)\left|l,\, m\right\rangle!
\end{aligned}$$

$(b_2^\dagger)^{l-m}$ — $(b_2^\dagger)^{l-m}$

Similarly we can establish ${a^2}_2 \left|l,\, m\right\rangle = (l-m)\left|l,\, m\right\rangle$. It then follows that

$$\begin{aligned}
h\left|l,\, m\right\rangle &= \frac{1}{2}({a^1}_1 - {a^2}_2)\left|l,\, m\right\rangle \\
&= m\left|l,\, m\right\rangle, \\
N\left|l,\, m\right\rangle &= ({a^1}_1 + {a^2}_2)\left|l,\, m\right\rangle \\
&= 2l\left|l,\, m\right\rangle.
\end{aligned}$$

Next we find

$$\begin{aligned}
{a^1}_2\left|l,\, m\right\rangle &= C_{lm}{a^1}_2(b^\dagger_1)^{(l+m)}(b^\dagger_2)^{(l-m)}\left|0\right\rangle \\
&= C_{lm}(b^\dagger_1)^{(l+m)}\left((b^\dagger_2)^{(l-m)}a^1_2 + (l-m)b^\dagger_1(b^\dagger_2)^{(l-m-1)}\right)\left|0\right\rangle \\
&= (l-m)C_{lm}(b^\dagger_1)^{(l+m+1)}(b^\dagger_2)^{(l-m-1)}\left|0\right\rangle \\
&= \frac{(l-m)C_{lm}}{C_{l(m+1)}}\left|l,\, m+1\right\rangle
\end{aligned}$$

and similarly ${a^2}_1\left|l,\, m\right\rangle = \dfrac{(l+m)C_{lm}}{C_{l(m-1)}}\left|l,\, m-1\right\rangle$. Evaluating

$$\begin{aligned}
e\left|l,\, l\right\rangle &= \frac{1}{\sqrt{2}}{a^1}_2\frac{1}{\sqrt{(2l)!}}(b^\dagger_1)^{2l}\cancel{\left|l,\, l\right\rangle}\ \left|0\right\rangle \\
&= \frac{1}{\sqrt{2(2l)!}}(b^\dagger_1)^{(2l+1)}b_2\left|0\right\rangle \\
&= 0
\end{aligned}$$

shows that $\left|l,\, l\right\rangle$ is a highest-weight state. Likewise one can show that $\left|l,\, -l\right\rangle$ is a lowest weight state.

Finally, $H\left|l,\, m\right\rangle = \omega(N+I)\left|l,\, m\right\rangle = \omega(2l+1)\left|l,\, m\right\rangle$ so the energy levels of the system are $\omega,\, 2\omega,\, 3\omega\ldots.$. Moreover, we can determine the degeneracies of these levels. For each l, there are $2l+1$ states $\left|l,\, m\right\rangle$ which have energy $E = (2l+1)\omega$. Or more concisely, each energy level $n\omega$ with $n = 1, 2, 3, \ldots$ has degeneracy n.

$\underline{L} = m\underline{r} \times \underline{v}$ $\quad L^2 = \underline{L} \cdot \underline{L} = L_1^2 + L_2^2 + L_3^2$

Recall, Casimir $C = h^2 + ef + fe$, can show that $C = L^2$

3.17 Quantising angular momentum

Let

$$\mathbf{r} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$

denote the position vector for some particle with momentum

$$\mathbf{p} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}.$$

Recall that we have the commutation relations ($[x, y] = xy - yx$)

$$[q_1, p_1] = [q_2, p_2] = [q_3, p_3] = iI,$$

with all other commutators vanishing; e.g.

$$[q_1, p_2] = 0.$$

In analogy with the classical definition of angular momentum we set

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}.$$

In component form this reads

$$L_a = \sum_{b,c} \varepsilon_{abc} q_b p_c \tag{73}$$

where ε_{abc} is the Levi-Civita tensor; viz.

ε_{123} $$\varepsilon_{xyz} = 1, \quad \varepsilon_{abc} = -\varepsilon_{bac} = -\varepsilon_{acb} = -\varepsilon_{cba}.$$

Exercise 20. *Show that the components of the angular momentum operator as given by (73) satisfy the $o(3)$ commutation relations*

$$[L_a, L_b] = i \sum_{c=1}^{3} \varepsilon_{abc} L_c. \tag{74}$$

In principle, the half-odd integer values of angular momentum do not arise in the form

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}.$$

Rather, they correspond to an internal structure of particles which we refer to as *spin*. In such a case it is more common to use the symbol $\mathbf{S}$ instead of $\mathbf{L}$, and label coordinates with x, y, and z instead of 1, 2, and 3. Particles with half-odd integer spin are called *fermions*, while those of integer spin are called *bosons*. In these cases the weight of a state $|\Phi\rangle$ simply measures the spin of the state of the particle.

In order to illustrate this difference, consider the case of a spin-1/2 particle. For a fixed choice of co-ordinates there are two basis states

$$|\uparrow\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The spin operators are represented via the Pauli matrices, up to a scalar factor,

$$\pi(S^x) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \pi(S^y) = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \pi(S^z) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfying the relations (74). Now suppose that we rotate the $\mathbf{k} - \mathbf{i}$ plane by an angle θ to define a new co-ordinate system. Now we have

$$\tilde{S}^z = S^z \cos\theta + S^x \sin\theta$$

and using the Pauli matrices

$$\begin{aligned} \tilde{S}^z &= \frac{1}{2}\cos\theta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{2}\sin\theta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}. \end{aligned} \tag{75}$$

Diagonalising this matrix we obtain the eigenvectors

$$|\tilde{\uparrow}\rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} = \cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2}|\downarrow\rangle \tag{76}$$

with eigenvalue $+1/2$ and

$$|\tilde{\downarrow}\rangle = \begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix} = -\sin\frac{\theta}{2}|\uparrow\rangle + \cos\frac{\theta}{2}|\downarrow\rangle$$

with eigenvalue $-1/2$.

A curious situation occurs when we put $\theta = 2\pi$. We find that

$$\begin{aligned} |\tilde{\uparrow}\rangle &= -|\uparrow\rangle, \\ |\tilde{\downarrow}\rangle &= -|\downarrow\rangle. \end{aligned}$$

In other words, rotating the universe by 2π causes the states of all spin-1/2 particles to change by the phase factor -1. This is an example of a *Berry phase*, and is a distinguishing feature between intrinsic spin and our usual notion of angular momentum.

For later use we set $S^{\pm} = S^x \pm iS^y$ for which the following commutation relations hold

$$[S^z, S^{\pm}] = \pm S^{\pm}, \qquad [S^+, S^-] = 2S^z. \tag{77}$$

Note that not all representations of $o(3)$ fit into the above description.

$$S^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Exercise 21. *Given the commutation relation $[b, b^\dagger] = I$, show that*

$$\begin{aligned} S^+ &= -\frac{1}{2}(b^\dagger)^2, \\ S^- &= \frac{1}{2}b^2, \\ S^z &= \frac{1}{4}(2N + I) \end{aligned}$$

$N = b^+ b$

$$S^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Fock space $F = \text{span}\{ |0\rangle, |1\rangle, |2\rangle, \ldots \}$

Recall $\Delta(a) = a \otimes I + I \otimes a$, $\Delta(I) = I \otimes I$

$\Delta^{(3)} = (id \otimes \Delta)\Delta(a) = a \otimes I \otimes I + I \otimes a \otimes I + I \otimes I \otimes a$

$= (\Delta \otimes id)\Delta(a)$

provides an $o(3)$ representation acting on Fock space $\mathcal{F}$ given by (25), by showing that the commutation relations (77) are preserved. What are the highest-weight and-lowest weight states?

Exercise 22. *For any constant N, show that*

$$\begin{aligned} S^z &= x\frac{\mathrm{d}}{\mathrm{d}x} - \frac{N}{2} \\ S^+ &= Nx - x^2\frac{\mathrm{d}}{\mathrm{d}x}, \\ S^- &= \frac{\mathrm{d}}{\mathrm{d}x}, \end{aligned}$$

provides an $o(3)$ representation by showing that the commutation relations (77) are preserved. Assuming that the representation acts on the space of polynomial functions of x, what are the highest-weight and lowest-weight states?

3.18 Coupling angular momentum states

Suppose that we have N identical particles of spin l. Mathematically we represent the total space of states by a tensor product

$$V_l \otimes V_l \otimes .... \otimes V_l \qquad - N \text{ copies}$$

which is the vector space with basis

$$|l,\, \mathbf{m}\rangle = |l,\, m_1\rangle \otimes |l,\, m_2\rangle \otimes .... \otimes |l,\, m_N\rangle\,. \tag{78}$$

The action of the spin operators S^a, $a = x,\, y,\, z$ is given by

$$\begin{aligned} \Delta^{(N)}(S^a) &= \sum_{j=1}^{N} I \otimes I \otimes \cdots \otimes \underbrace{S^a}_{jth} \otimes \cdots \otimes\ I \\ &= \sum_{j=1}^{N} S^a_j \end{aligned}$$

and it can be shown that $\Delta^{(N)}$ determines a Lie algebra homomorphism; i.e.

$$\left[\Delta^{(N)}(S^a),\, \Delta^{(N)}(S^b)\right] = i\varepsilon_{abc}\Delta^{(N)}(S^c).$$

For example, for the state (78) above

$$\Delta^{(N)}(S^z)\,|l,\, \mathbf{m}\rangle = \sum_{i=1}^{N} m_i\,|l,\, \mathbf{m}\rangle$$

so the total $\mathbf{k}$ component of spin is simply the sum of the individual $\mathbf{k}$ components of spin. For the square of the spin, the situation is more complicated. For example, for just

2 particles

$$\begin{aligned}\Delta(S^2) &= \Delta(\mathbf{S}.\mathbf{S})\\ &= \sum_{a=x,y,z} (S^a \otimes I + I \otimes S^a)(S^a \otimes I + I \otimes S^a)\\ &= S^2 \otimes I + I \otimes S^2 + 2 \sum_{a=x,y,z} S^a \otimes S^a\end{aligned}$$

or equivalently

$$S^2 \equiv S_1^2 + S_2^2 + 2\mathbf{S}_1.\mathbf{S}_2. \tag{79}$$

The allowed values for the square of the spin of two coupled particles are given from the general result

$$V_l \otimes V_k = \bigoplus_{j=|l-k|}^{l+k} V_j.$$

In other words, if we have two particles with the square of the spins given by $l(l+1)$ and $k(k+1)$ respectively, these particles may be coupled so that the allowed values of the square of the spin for the two particle system lie in the set

$$\{j(j+1): \quad j = |l-k|, |l-k|+1, ..., l+k-1, l+k\}.$$

We will not go into the proof of this general result but for later use it is instructive to consider the simplest case of the coupling of two spin-$1/2$ particles; viz.

$$V_{1/2} \otimes V_{1/2} = V_1 \oplus V_0. \tag{80}$$

The explicit matrix representatives of the spin operators are

$$A \otimes B = \left(\begin{array}{c|c} A_{11}B & A_{12}B \\ \hline A_{21}B & A_{22}B \end{array}\right)$$

$$S^x = \frac{1}{2}\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right),$$

$$S^y = \frac{1}{2}\left(\begin{array}{cc|cc} 0 & -i & -i & 0 \\ i & 0 & 0 & -i \\ \hline i & 0 & 0 & -i \\ 0 & i & i & 0 \end{array}\right),$$

$$S^z = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

$P(\underline{v}\otimes\underline{w}) = \underline{w}\otimes\underline{v}$

where the correspondence between ket vectors and column vectors is

$$|\uparrow\rangle\otimes|\uparrow\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad |\uparrow\rangle\otimes|\downarrow\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad |\downarrow\rangle\otimes|\uparrow\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad |\downarrow\rangle\otimes|\downarrow\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

A basis for the spin-1 space of states is given by

$$\begin{gathered}|\uparrow\rangle\otimes|\uparrow\rangle\\ \tfrac{1}{\sqrt{2}}\left(|\uparrow\rangle\otimes|\downarrow\rangle + |\downarrow\rangle\otimes|\uparrow\rangle\right)\\ |\downarrow\rangle\otimes|\downarrow\rangle\end{gathered} \tag{81}$$

while for the spin-0 space there is the single vector

$$\frac{1}{\sqrt{2}}\left(|\uparrow\rangle\otimes|\downarrow\rangle - |\downarrow\rangle\otimes|\uparrow\rangle\right). \tag{82}$$

Qualitatively, if we have two spin-1/2 particles they may be coupled together so the resultant system has either spin-1 or spin-0. The spin-0 system can only occupy one state which is that represented by (82), while a spin-1 system can occupy one of three eigenstates of S^z (spin 1, 0 -1) which are represented by (81).

4 Representation theory of $gl(n)$

Recall that $gl(n)$ is spanned by the n^2 abstract operators $\{a^i{}_j\}$ satisfying the $gl(n)$ commutation relations

$$[a^i{}_j, a^k{}_l] = \delta^k{}_j a^i{}_l - \delta^i{}_l a^k{}_j.$$

In particular, the subspace of *diagonal generators* $a^k{}_k$ form a maximal commutative subalgebra

$$[a^j{}_j,\, a^k{}_k] = 0 \qquad \forall j,\, k$$

and thus span an n-dimensional abelian subalgebra $\mathcal{L}_0$, which will play a role analogous to L_0 in the $o(3)$ case. For a reductive Lie algebra a maximal commutative subalgebra is called a *Cartan subalgebra*, usually represented as H.

4.1 Weights and roots for $gl(n)$

Let V be a $gl(n)$-module. A vector $v \in V$ is called a *weight vector* if it is an eigenvector of all elements of the Cartan subalgebra, viz. $\exists\{\lambda_k\} \in \mathbb{C}$ such that:

$$a^k{}_k v = \lambda_k v, \qquad k = 1, \ldots, n.$$

We call $\lambda = (\lambda_1, \ldots, \lambda_n)$ the *weight* of v, which can be expanded in terms of the basis $\{\epsilon_k : k = 1, ..., n\}$ as

$$\lambda = \sum_{k=1}^{n} \lambda_k \epsilon_k.$$

(Note $P^2 = I \otimes I$)

Define the permutation operator $P : V \otimes V \to V \otimes V$ through $P(v \otimes w) = w \otimes v$. Then

$$P w_1 = w_1 , \quad P w_2 = w_3 , \quad P w_3 = w_2 , \quad P w_4 = w_4$$

so we can write

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Define a new basis

$$u_1 = w_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$u_2 = \frac{1}{\sqrt{2}} \left(w_2 + w_3 \right) = \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

$$u_4 = \frac{1}{\sqrt{2}} \left(w_2 - w_3 \right) = \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

$$u_3 = w_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Now consider

$$f u_1 = \left(f \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \left(f \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = u_2$$

Similarly $f u_2 = u_3$, $f u_3 = 0$

Also
$$f u_4 = \frac{1}{\sqrt{2}}\left(\left(f\begin{pmatrix}1\\0\end{pmatrix}\right)\otimes\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix}\otimes\left(f\begin{pmatrix}0\\1\end{pmatrix}\right) - \left(f\begin{pmatrix}0\\1\end{pmatrix}\right)\otimes\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\otimes\left(f\begin{pmatrix}1\\0\end{pmatrix}\right)\right)$$
$$= 0$$

In this new basis we have the representations
$$\tilde{\pi}(f) = \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right) = \tilde{\pi}(e^+)$$

$$\tilde{\pi}(h) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right).$$

The 4-dim representation decomposes into the direct sum of two irreducible representations.

Casimir invariant. Recall $[h,e] = e$, $[h,f] = -f$, $[e,f] = h$.

Set
$$C = h^2 + ef + fe \in U(o(3))$$
$$= h^2 + [e,f] + fe + fe = h^2 + h + 2fe.$$
$$= h^2 + ef + [f,e] + ef = h^2 - h + 2ef.$$

The Casimir invariant has the property that
$$[C,e] = [C,f] = [C,h] = 0.$$

Next, $\pi(c) = \pi(h^2) + \pi(ef) + \pi(fe)$

$$= \frac{1}{4} I + \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{3}{4} I.$$

$$(\pi \otimes \pi)(c) = \pi(c) \otimes I + I \otimes \pi(c) + 2\big(\pi(h) \otimes \pi(h) + \pi(e) \otimes \pi(f) + \pi(f) \otimes \pi(e)\big)$$

$$= \frac{3}{2} I \otimes I + 2\left(\begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & +\frac{1}{4} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right)$$

$$= I + P$$

$$(A \otimes I)(I \otimes B) = A \otimes B$$

$$(I \otimes B)(A \otimes I) = A \otimes B$$

e.g. $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$A \otimes B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$A \otimes I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad I \otimes B \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Two-dimensional quantum oscillator

$H = H_1 + H_2$, eigenvalues $E_i = \omega(n_i + \frac{1}{2})$

so eigenvalues of H are $E = \omega(n_1 + n_2 + 1)$, $n_i = 0, 1, 2, ..$

$E = \omega$: $|0\rangle \otimes |0\rangle \equiv |0\rangle$ $\qquad E = \omega(N + I)$

$E = 2\omega$: $|0\rangle \otimes |1\rangle$, $|1\rangle \otimes |0\rangle$

$E = 3\omega$: $|0\rangle \otimes |2\rangle$, $|1\rangle \otimes |1\rangle$, $|2\rangle \otimes |0\rangle$

$E = 4\omega$: $|0\rangle \otimes |3\rangle$, $|1\rangle \otimes |2\rangle$, $|2\rangle \otimes |1\rangle$, $|3\rangle \otimes |0\rangle$

etc.

In general $|n\rangle \otimes |m\rangle = \frac{1}{\sqrt{n!}}(b^+)^n|0\rangle \otimes \frac{1}{\sqrt{m!}}(b^+)^m|0\rangle$

$$= \frac{1}{\sqrt{n!\,m!}}(b_1^+)^n (b_2^+)^m |0\rangle$$

$E = \omega$: $|0, 0\rangle$

$E = 2\omega$: $|\frac{1}{2}, \frac{1}{2}\rangle$ $\quad |\frac{1}{2}, -\frac{1}{2}\rangle$.

$E = 3\omega$: $|1, 1\rangle$, $|1, 0\rangle$, $|1, -1\rangle$

$E = 4\omega$: $|\frac{3}{2}, \frac{3}{2}\rangle$, $|\frac{3}{2}, \frac{1}{2}\rangle$, $|\frac{3}{2}, -\frac{1}{2}\rangle$, $|\frac{3}{2}, -\frac{3}{2}\rangle$

Check:

$$[c,e] = [h^2,e] + [ef,e] + [fe,e]$$

$$= h[h,e] + [h,e]h + e[f,e] + \cancel{[e,e]}f$$
$$+ f\cancel{[e,e]} + [f,e]e$$

$$= he + eh + e(-h) + (-h)e = 0.$$

Now consider the action under $\Delta : L \to U(L) \otimes U(L)$,

$\cancel{\Delta(A)}$ which extends naturally to $\Delta : U(L) \to U(L) \otimes U(L)$

i.e. $\Delta(c) = \Delta(h^2) + \Delta(ef) + \Delta(fe)$

$$= \Delta(h)\Delta(h) + \Delta(e)\Delta(f) + \Delta(f)\Delta(e)$$
$$= (h \otimes I + I \otimes h)(h \otimes I + I \otimes h)$$
$$+ (e \otimes I + I \otimes e)(f \otimes I + I \otimes f)$$
$$+ (f \otimes I + I \otimes f)(e \otimes I + I \otimes e)$$

$$= h^2 \otimes I + 2h \otimes h + I \otimes h^2$$

$$I \otimes ef$$
$$+ ef \otimes I + f \otimes e + e \otimes f + \cancel{ef \otimes I}$$

$$+ fe \otimes I + e \otimes f + f \otimes e + I \otimes fe$$

$$= C \otimes I + I \otimes C + 2(h \otimes h + e \otimes f + f \otimes e)$$

Three-dimensional quantum oscillator

$H = H_1 + H_2 + H_3$, eigenvalues $E_i = \omega(n_i + \frac{1}{2})$

so eigenvalues of H are $E = \omega(n_1 + n_2 + n_3 + \frac{3}{2})$

$n_j = 0, 1, 2, \ldots$

$E = \frac{3}{2}\omega$, $|0\rangle \otimes |0\rangle \otimes |0\rangle \equiv |0\rangle$

$E = \frac{5}{2}\omega$, $|0\rangle \otimes |0\rangle \otimes |1\rangle$, $|0\rangle \otimes |1\rangle \otimes |0\rangle$, $|1\rangle \otimes |0\rangle \otimes |0\rangle$

$E = \frac{7}{2}\omega$, $|2\rangle \otimes |0\rangle \otimes |0\rangle$, $|0\rangle \otimes |2\rangle \otimes |0\rangle$, $|0\rangle \otimes |0\rangle \otimes |2\rangle$

$|1\rangle \otimes |1\rangle \otimes |0\rangle$, $|1\rangle \otimes |0\rangle \otimes |1\rangle$, $|0\rangle \otimes |1\rangle \otimes |1\rangle$

- This system is associated to the $gl(3)$ Lie algebra. $[a^j{}_k, a^p{}_q] = \delta^p{}_k a^j{}_q - \delta^j{}_q a^p{}_k$

Fundamental / vector / rank-1 symmetric

$\Pi : gl(3) \to \mathrm{End}(\mathbb{C}^3)$ $\quad \Pi(a^j{}_k) = e_{jk}$

$\Pi(a^1{}_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\Pi(a^2{}_1) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\Pi(a^3{}_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$[a^1{}_3, a^3{}_1] = a^1{}_1 - a^3{}_3$

$\cancel{[a^1{}_1 - a^3{}_3]}$ $[a^1{}_1 - a^3{}_3, a^1{}_3] = [a^1{}_1, a^1{}_3] - [a^3{}_3, a^1{}_3]$

$= 2a^1{}_3$

$$P(\underline{v} \otimes \underline{w}) = \underline{w} \otimes \underline{v}$$

where the correspondence between ket vectors and column vectors is

$$|\uparrow\rangle \otimes |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |\uparrow\rangle \otimes |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle \otimes |\uparrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle \otimes |\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

A basis for the spin-1 space of states is given by

$$\begin{gathered} |\uparrow\rangle \otimes |\uparrow\rangle \\ \tfrac{1}{\sqrt{2}}\left(|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle\right) \\ |\downarrow\rangle \otimes |\downarrow\rangle \end{gathered} \tag{81}$$

while for the spin-0 space there is the single vector

$$\frac{1}{\sqrt{2}}\left(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle\right). \tag{82}$$

Qualitatively, if we have two spin-1/2 particles they may be coupled together so the resultant system has either spin-1 or spin-0. The spin-0 system can only occupy one state which is that represented by (82), while a spin-1 system can occupy one of three eigenstates of S^z (spin 1, 0 -1) which are represented by (81).

4 Representation theory of $gl(n)$

Recall that $gl(n)$ is spanned by the n^2 abstract operators $\{a^i{}_j\}$ satisfying the $gl(n)$ commutation relations

$$[a^i{}_j, a^k{}_l] = \delta^k{}_j a^i{}_l - \delta^i{}_l a^k{}_j.$$

In particular, the subspace of *diagonal generators* $a^k{}_k$ form a maximal commutative subalgebra

$$[a^j{}_j,\, a^k{}_k] = 0 \qquad \forall j,\, k$$

and thus span an n-dimensional abelian subalgebra $\mathcal{L}_0$, which will play a role analogous to L_0 in the $o(3)$ case. For a reductive Lie algebra a maximal commutative subalgebra is called a *Cartan subalgebra*, usually represented as H.

4.1 Weights and roots for $gl(n)$

Let V be a $gl(n)$-module. A vector $v \in V$ is called a *weight vector* if it is an eigenvector of all elements of the Cartan subalgebra, viz. $\exists\{\lambda_k\} \in \mathbb{C}$ such that:

$$a^k{}_k v = \lambda_k v, \qquad k = 1, \ldots, n.$$

We call $\lambda = (\lambda_1, \ldots, \lambda_n)$ the *weight* of v, which can be expanded in terms of the basis $\{\epsilon_k : k = 1, ..., n\}$ as

$$\lambda = \sum_{k=1}^{n} \lambda_k \epsilon_k.$$

$\varepsilon_1 = (1, 0, 0, \ldots)$ $\lambda = (\lambda_1, \lambda_2, \ldots)$

$\varepsilon_2 = (0, 1, 0, \ldots)$ $= \lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2 + \ldots$

We impose a partial ordering on the weights given by the *lexical ordering*, viz. $\lambda \geq \mu$ if and only if the first non-zero component of the weight $\lambda - \mu$ is positive, e.g.:

$$(2, -1, 0) \geq (1, 3, -2) \geq (1, 0, 0) \geq (0, 2, 0) \geq (0, 1, -5) \geq (0, 0, -1) \geq (-2, 0, -1).$$

In order to define this ordering, we have assumed $\lambda_k \in \mathbb{R}$, ~~which will later be shown to be true.~~

Now, the $gl(n)$ generators $a^i{}_j$ are *themselves* weight vectors under the adjoint representation:

$$\mathrm{ad}(a^k{}_k) \circ a^i{}_j = [a^k{}_k, a^i{}_j] = (\delta^i{}_k - \delta^j{}_k) a^i{}_j.$$

In terms of the fundamental weights ϵ_i, $(i = 1, \ldots, n)$, the eigenvector $a^i{}_j$ has weight $\epsilon_i - \epsilon_j$. The non-zero weights of the adjoint representation are called *roots*, and the set of them is called Φ:

$$\Phi = \{\epsilon_i - \epsilon_j \mid i, j = 1, \ldots, n, \ i \neq j\}.$$

We call Φ the *root system* for $gl(n)$. It has $n(n-1)$ elements. A root $\alpha \in \Phi$ is called *positive* (respectively *negative*) if it is greater than (respectively less than) 0 under the lexical ordering. The set of the roots is thus $\Phi = \Phi^+ \cup \Phi^-$, where $\Phi^- = -\Phi^+$ (each set has half the roots).

$$\Phi^+ = \{\epsilon_i - \epsilon_j \mid i, j = 1, \ldots, n, \ i < j\}, \qquad \Phi^- = \{\epsilon_i - \epsilon_j \mid i, j = 1, \ldots, n, \ i > j\}.$$

Now consider the set:

$$\Delta = \{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid i = 1, \ldots, n-1\} \subseteq \Phi^+.$$

Every root in Φ^+ (respectively Φ^-) is a positive (respectively negative) $\mathbb{Z}$-linear combination of roots in Δ, that is for $i < j$ we have:

$$\begin{aligned} \epsilon_i - \epsilon_j &= (\epsilon_i - \epsilon_{i+1}) + (\epsilon_{i+1} - \epsilon_{i+2}) + \cdots + (\epsilon_{j-2} - \epsilon_{j-1}) + (\epsilon_{j-1} - \epsilon_j) \\ &= \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} \in \Phi^+. \end{aligned}$$

The roots in Δ are called *simple roots* - these are the positive roots that cannot be expressed as a sum of two positive roots.

4.2 The Cartan-Weyl decomposition for $gl(n)$

If v is a weight vector of a $gl(n)$-module V, of weight $\lambda = (\lambda_1, \ldots, \lambda_n)$, then the vector $a^i{}_j v$ is also a weight vector, of weight $\lambda + (\epsilon_i - \epsilon_j)$:

$$\begin{aligned} a^k{}_k (a^i{}_j v) &= [a^k{}_k, a^i{}_j] v + a^i{}_j a^k{}_k v = (\delta^i{}_k a^k{}_j - \delta^k{}_j a^i{}_k) v + a^i{}_j \lambda_k v \\ &= (\delta^i{}_k - \delta^k{}_j)(a^i{}_j v) + \lambda_k (a^i{}_j v) \\ &= [\lambda_k + \delta^k{}_i - \delta^k{}_j](a^i{}_j v) \end{aligned}$$

$AB = BA + [A, B]$

Hence, if $i < j$, it is useful to visualise this as a vector equation:

$$\begin{pmatrix} a^1_1 \\ \vdots \\ a^i{}_i \\ \vdots \\ a^j{}_j \\ \vdots \\ a^n{}_n \end{pmatrix} (a^i{}_j v) = \begin{pmatrix} \lambda_1 + 0 - 0 \\ \vdots \\ \lambda_i + 1 - 0 \\ \vdots \\ \lambda_j + 0 - 1 \\ \vdots \\ \lambda_n + 0 - 0 \end{pmatrix} (a^i{}_j v) = (\lambda + \epsilon_i - \epsilon_j)(a^i{}_j v).$$

A generator of the form $a^i{}_j$ with $i < j$ is called a *raising* generator, since it increases the weight of a weight vector, and corresponds to a positive root. Similarly, a generator of the form $a^i{}_j$ with $i > j$ is called a *lowering* generator, since it decreases the weight of a weight vector, and corresponds to a negative root.

The vector space spanned by the raising (resp. lowering) generators is denoted by $\mathcal{L}_+$ (resp. $\mathcal{L}_-$). The $\mathcal{L}_\pm$ are in fact subalgebras of $gl(n)$:

$$[\mathcal{L}_+, \mathcal{L}_+] \subseteq \mathcal{L}_+, \qquad [\mathcal{L}_-, \mathcal{L}_-] \subseteq \mathcal{L}_-.$$

(e.g. $[a^1{}_2, a^2{}_3] = a^1{}_3$, etc.) Also, $\mathcal{L}_0$ is a subalgebra of $gl(n)$. We thus obtain the following ~~Lie algebra~~ direct sum decomposition (the *Cartan-Weyl decomposition*):

$$gl(n) = \mathcal{L}_- \oplus \mathcal{L}_0 \oplus \mathcal{L}_+.$$

Observe that $\mathcal{L}_0, \mathcal{L}_\pm$ are analogous to $L_0, L_\pm$ in the representation theory of $o(3)$. If $U_0, U_\pm$ denote the universal enveloping algebras of $\mathcal{L}_0, \mathcal{L}_\pm$ respectively, then the PBW theorem implies that the universal enveloping algebra U of $gl(n)$ may be written:

$$U = U_- U_0 U_+.$$

We choose $\{a^i{}_j : 1 \leq i, j \leq n\}$ as a basis for $gl(n)$. Then a basis for U is given by all products of basis elements of $gl(n)$, ordered with lowering generators on the left, raising generators on the right, and Cartan generators in the middle.

4.3 Highest weights and irreducible modules for $gl(n)$

Given a $gl(n)$-module V, we say that a weight ν occurs in V if there exists a non-zero $v \in V$ of weight ν. The corresponding *weight space* of V is the vector space V_ν of all such vectors:

$$V_\nu = \{v \in V : a^k{}_k v = \nu_k v, \ k = 1, \ldots, n\}.$$

We denote the *multiplicity* of the weight ν in V by $\dim(V_\nu)$. We have:

Theorem 3 (Cartan). *Every finite-dimensional irreducible $gl(n)$ module V admits a basis of weight vectors. Moreover, V admits a* unique *vector of highest weight λ. The weight λ occurs with unit multiplicity in V, and all other weights in V have weight strictly less than λ.*

Thus we can *classify* and uniquely label *all* the finite-dimensional irreducible $gl(n)$-modules V in terms of their *highest weights* λ, and usually refer to the module as $V(\lambda)$. We prove the theorem in a similar fashion to Theorem 2.

Proof. Let V be a finite-dimensional irreducible $gl(n)$ module. From linear algebra, the $a^i{}_i$ must have at least one non-zero eigenvector $v \in V$, say of weight ν. Since V is irreducible, $V = Uv$, and as U is spanned by all generator products:

$$a^{i_1}{}_{j_1} a^{i_2}{}_{j_2} \cdots a^{i_k}{}_{j_k}$$

hence V is spanned by all vectors $a^{i_1}{}_{j_1} a^{i_2}{}_{j_2} \cdots a^{i_k}{}_{j_k} v$. This latter vector is a weight vector, of weight:

$$\nu + (\epsilon_{i_1} - \epsilon_{j_1}) + (\epsilon_{i_2} - \epsilon_{j_2}) + \cdots + (\epsilon_{i_k} - \epsilon_{j_k}).$$

(Since $a^k{}_l v$ has weight $\nu + (\epsilon_k - \epsilon_l)$, then $a^p{}_q a^k{}_l v$ has weight $\nu + (\epsilon_k - \epsilon_l) + (\epsilon_p - \epsilon_q)$.) Therefore, V has a basis of weight vectors. Since V is finite-dimensional, there must exist a weight vector $v_+ \in V$ with weight λ which is maximal under the lexical ordering. Then:

$$a^k{}_k v_+ = \lambda_k v_+, \qquad \mathcal{L}_+ v_+ = (0)$$

Thus $\mathcal{U}_+ v_+ = \mathbb{C} v_+$, so $\mathcal{U}_0 v_+ = \mathbb{C} v_+$, so in fact $V = \mathcal{U} v_+ = \mathcal{U}_- \mathcal{U}_0 \mathcal{U}_+ v_+ = \mathcal{U}_- v_+$. Now observe:

$$\begin{aligned} \mathcal{U}_- &= \mathbb{C} \oplus \mathcal{L}_- \oplus \mathcal{L}_-^2 \oplus \mathcal{L}_-^3 \oplus \cdots = \mathbb{C} \oplus \{\mathcal{L}_- \oplus \mathcal{L}_-^2 \oplus \mathcal{L}_-^3 \oplus \cdots\} \\ &= \mathbb{C} \oplus \{\mathbb{C} \oplus \mathcal{L}_- \oplus \mathcal{L}_-^2 \oplus \cdots\} \mathcal{L}_- = \mathbb{C} \oplus \mathcal{U}_- \mathcal{L}_- \end{aligned}$$

Then $V = \mathcal{U}_- v_+ = \mathbb{C} v_+ \oplus \mathcal{U}_- \mathcal{L}_- v_+$, where $\mathcal{U}_- \mathcal{L}_- v_+$ is spanned by weight vectors of weight strictly less than λ. Thus the maximal weight λ is unique. Then λ occurs with unit multiplicity in V, and all other weights $\mu \in V$ satisfy $\mu < \lambda$. $\square$

Theorem 4. *Let V be a finite-dimensional $gl(n)$-module.*

1. *If V is cyclically generated by a highest-weight vector v_+, i.e. $V = \mathcal{U} v_+$ (V is* standard cyclic*), then V is irreducible.*
2. *V is irreducible if and only if it admits a unique (up to a scalar multiple) maximal weight vector.*

In fact, this theorem applies to any semi-simple Lie algebra.

4.4 Limitations on possible highest weights

We have seen that the finite-dimensional $o(3)$-modules have highest weights $l \in \frac{1}{2}\mathbb{Z}^+$. Here, we obtain analogous conditions on the highest weights λ of the finite-dimensional irreducible $gl(n)$-modules. To this end, corresponding to each of the $\frac{n}{2}(n-1)$ pairs of distinct integers $1 \leq i < j \leq n$, we have an $o(3)$-subalgebra of $gl(n)$, with generators:

$$L_+ = a^i{}_j, \qquad L_0 = \frac{1}{2}(a^i{}_i - a^j{}_j), \qquad L_- = a^j{}_i.$$

Finite-dimensional, irreducible representations of $gl(n)$

Recall $gl(n)$ has commutation relations

$$[a^j{}_k, a^l{}_m] = \delta^l{}_k a^j{}_m - \delta^j{}_m a^l{}_k$$

Note, that $I_1 = \sum_{k=1}^{n} a^k{}_k$ is central i.e.

$$[I_1, a^j{}_k] = 0 \quad \text{for all } 1 \le j, k \le n.$$

Examples: For $n=3$, we can identify the subalgebras $\{a^1{}_2, a^2{}_1, a^1{}_1 - a^2{}_2\}$

$$\{a^2{}_3, a^3{}_2, a^2{}_2 - a^3{}_3\}$$

$$\{a^1{}_3, a^3{}_1, a^1{}_1 - a^3{}_3\}$$

Consider the fundamental representation $\pi: gl(3) = \mathrm{End}(\mathbb{C}^3)$ given by $\pi(a^j{}_k) = e_{jk}$ e.g.

$$\pi(a^1{}_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \pi(a^3{}_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

upper triangular

lower triangular

$$\pi(a^2{}_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

diagonal

Note that for all $1 \le j, k \le n$

$$[a^j{}_j, a^k{}_k] = 0$$

The set $\{a^j{}_j : j = 1, \ldots, n\}$ is the Cartan subalgebra

The strategy is to label ~~states~~ in simultaneous eigenstates according to the eigenvalues of the elements in the Cartan subalgebra.

$$\pi(a^2_2)\begin{pmatrix}1\\0\\0\end{pmatrix} = 0, \quad \pi(a^2_2)\begin{pmatrix}0\\1\\0\end{pmatrix} = 1\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\pi(a^2_2)\begin{pmatrix}0\\0\\1\end{pmatrix} = 0.$$ Similar results for $\pi(a^1_1)$ and $\pi(a^3_3)$

The vector $\begin{pmatrix}1\\0\\0\end{pmatrix}$ is said to have weight $(1,0,0)$

The vector $\begin{pmatrix}0\\1\\0\end{pmatrix}$ is said to have weight $(0,1,0)$

The vector $\begin{pmatrix}0\\0\\1\end{pmatrix}$ is said to have weight $(0,0,1)$

Next, consider the adjoint representation of $gl(3)$.

Consider $\mathrm{ad}(a^j_k)\circ a^l_m = \delta^l_k a^j_m - \delta^j_m a^l_k$

In particular $\mathrm{ad}(a^j_j)\circ a^l_m = \delta^l_j a^j_m - \delta^j_m a^l_j$

$$= \delta^l_j a^l_m - \delta^j_m a^l_m$$

$$= (\delta^l_j - \delta^j_m)\, a^l_m$$

Consider the vector a^1_2: $\quad [a^1_1, a^1_2] = a^1_2$

$$[a^2_2, a^1_2] = -a^1_2$$

$$[a^3_3, a^1_2] = 0.$$

The "vector" a^1_2 has weight $(1,-1,0)$

" a^1_3 " $(1,0,-1)$

a^2_3 $(0,1,-1)$

a^1_1 $(0,0,0)$

" a^2_2 " $(0,0,0)$

a^3_3 $(0,0,0)$

a^2_1 $(-1,1,0)$

a^3_2 $(0,-1,1)$

a^3_1 $(-1,0,1)$

To order the weights we use "lexical", "lexicographic" "lexicographical" ordering. In this case this gives in order from highest to lowest:

$(1,0,-1)$, $(1,-1,0)$, $(0,1,-1)$, $(0,0,0)\times 3$

$(0,-1,1)$, $(-1,1,0)$, $(-1,0,+1)$

Note that $(1,-1,0)+(0,1,-1)=(1,0,-1)$,

and $(0,-1,1)+(-1,1,0)=(-1,0,1)$

The weight $(1,-1,0)$ and $(0,1,-1)$ are simple positive weights

Notation; In general, we identify

$$(2,1,1,\dot{0}) \equiv \underbrace{(2,1,1,0,0,\ldots,0)}_{n \text{ entries}} = 2\varepsilon_1+\varepsilon_2+\varepsilon_3$$

It is also customary to use the notation

$$\varepsilon_1 = (1,\dot{0})$$

$$\varepsilon_2 = (0,1,\dot{0}) \qquad \varepsilon_3 = (0,0,1,\dot{0}) \text{ etc.}$$

The adjoint module of $gl(3)$ has a lowest weight vector $a^3{}_1$ with weight $(-1,0,1)$. The following are zero-weight vectors

$$[a^1{}_3, a^3{}_1] = a^1{}_1 - a^3{}_3$$

$$[a^1{}_2, [a^2{}_3, a^3{}_1]] = [a^1{}_2, a^2{}_1] = a^1{}_1 - a^2{}_2$$

$$[a^2{}_3, [a^1{}_2, a^3{}_1]] = -[a^2{}_3, a^3{}_2] = a^3{}_3 - a^2{}_2$$

$$AB = BA + [A,B]$$

For $gl(n)$, let v_0 denote a highest weight vector with weight $\lambda = (\lambda_1, \lambda_2, \dots \lambda_n)$, $n \geq 4$

Consider $v_1 = a^4_3 v_0$ has weight $\lambda - \varepsilon_3 + \varepsilon_4$

$v_2 = a^2_1 a^4_3 v_0$ has weight $\lambda - \varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4$

$v_3 = a^3_2 a^2_1 a^4_3 v_0$ has weight $\lambda - \varepsilon_1 + \varepsilon_4$

$v_4 = a^1_4 a^3_2 a^2_1 a^4_3 v_0$ has weight λ

$$\begin{aligned}
v_4 &= a^1_4 a^3_2 a^2_1 a^4_3 v_0 \\
&= a^3_2 a^1_4 a^2_1 a^4_3 v_0 \\
&= a^3_2 a^2_1 a^1_4 a^4_3 v_0 + a^3_2 [a^1_4, a^2_1] a^4_3 v_0 \\
&= a^3_2 a^2_1 a^1_4 a^4_3 v_0 - a^3_2 a^2_4 a^4_3 v_0 \\
&= a^3_2 a^2_1 a^4_3 a^1_4 v_0 + a^3_2 a^2_1 [a^1_4, a^4_3] v_0 \\
&\neq \qquad - a^3_2 a^4_3 a^2_4 v_0 \\
&\qquad - a^3_2 [a^2_4, a^4_3] v_0
\end{aligned}$$

$\vdots$

$$\cancel{(a^2_2 - a^3_3)}\, v_0 = \cancel{(\lambda_2 - \lambda_3)}\, v_0$$

Thus we can *classify* and uniquely label *all* the finite-dimensional irreducible $gl(n)$-modules V in terms of their *highest weights* λ, and usually refer to the module as $V(\lambda)$. We prove the theorem in a similar fashion to Theorem 2.

Proof. Let V be a finite-dimensional irreducible $gl(n)$ module. From linear algebra, the $a^i{}_i$ must have at least one non-zero eigenvector $v \in V$, say of weight ν. Since V is irreducible, $V = Uv$, and as U is spanned by all generator products:

$$a^{i_1}{}_{j_1} a^{i_2}{}_{j_2} \cdots a^{i_k}{}_{j_k}$$

hence V is spanned by all vectors $a^{i_1}{}_{j_1} a^{i_2}{}_{j_2} \cdots a^{i_k}{}_{j_k} v$. This latter vector is a weight vector, of weight:

$$\nu + (\epsilon_{i_1} - \epsilon_{j_1}) + (\epsilon_{i_2} - \epsilon_{j_2}) + \cdots + (\epsilon_{i_k} - \epsilon_{j_k}).$$

(Since $a^k{}_l v$ has weight $\nu + (\epsilon_k - \epsilon_l)$, then $a^p{}_q a^k{}_l v$ has weight $\nu + (\epsilon_k - \epsilon_l) + (\epsilon_p - \epsilon_q)$.) Therefore, V has a basis of weight vectors. Since V is finite-dimensional, there must exist a weight vector $v_+ \in V$ with weight λ which is maximal under the lexical ordering. Then:

$$a^k{}_k v_+ = \lambda_k v_+, \qquad \mathcal{L}_+ v_+ = (0)$$

Thus $\mathcal{U}_+ v_+ = \mathbb{C} v_+$, so $\mathcal{U}_0 v_+ = \mathbb{C} v_+$, so in fact $V = \mathcal{U} v_+ = \mathcal{U}_- \mathcal{U}_0 \mathcal{U}_+ v_+ = \mathcal{U}_- v_+$. Now observe:

$$\begin{aligned} \mathcal{U}_- &= \mathbb{C} \oplus \mathcal{L}_- \oplus \mathcal{L}_-^2 \oplus \mathcal{L}_-^3 \oplus \cdots = \mathbb{C} \oplus \{\mathcal{L}_- \oplus \mathcal{L}_-^2 \oplus \mathcal{L}_-^3 \oplus \cdots\} \\ &= \mathbb{C} \oplus \{\mathbb{C} \oplus \mathcal{L}_- \oplus \mathcal{L}_-^2 \oplus \cdots\} \mathcal{L}_- = \mathbb{C} \oplus \mathcal{U}_- \mathcal{L}_- \end{aligned}$$

Then $V = \mathcal{U}_- v_+ = \mathbb{C} v_+ \oplus \mathcal{U}_- \mathcal{L}_- v_+$, where $\mathcal{U}_- \mathcal{L}_- v_+$ is spanned by weight vectors of weight strictly less than λ. Thus the maximal weight λ is unique. Then λ occurs with unit multiplicity in V, and all other weights $\mu \in V$ satisfy $\mu < \lambda$. □

Theorem 4. *Let V be a finite-dimensional $gl(n)$-module.*

1. *If V is cyclically generated by a highest-weight vector v_+, i.e. $V = \mathcal{U} v_+$ (V is* standard cyclic*), then V is irreducible.*

2. *V is irreducible if and only if it admits a unique (up to a scalar multiple) maximal weight vector.*

In fact, this theorem applies to any semi-simple Lie algebra.

4.4 Limitations on possible highest weights

We have seen that the finite-dimensional $o(3)$-modules have highest weights $l \in \frac{1}{2}\mathbb{Z}^+$. Here, we obtain analogous conditions on the highest weights λ of the finite-dimensional irreducible $gl(n)$-modules. To this end, corresponding to each of the $\frac{n}{2}(n-1)$ pairs of distinct integers $1 \leq i < j \leq n$, we have an $o(3)$-subalgebra of $gl(n)$, with generators:

$$L_+ = a^i{}_j, \qquad L_0 = \frac{1}{2}(a^i{}_i - a^j{}_j), \qquad L_- = a^j{}_i.$$

Now let v_+ be the maximal weight vector of the finite dimensional irreducible $gl(n)$-module $V(\lambda)$. Then:

$$L_+v_+ = a^i{}_jv_+ = 0, \qquad L_0v_+ = \frac{1}{2}(\lambda_i - \lambda_j)v_+.$$

(Compare with $L_+\psi_l^l = 0$; and $L_0\psi_l^l = l\psi_l^l$.) Therefore v_+ must be the highest-weight vector for an irreducible $o(3)$-module contained in $V(\lambda)$, so that: $\frac{1}{2}(\lambda_i - \lambda_j) \in \frac{1}{2}\mathbb{Z}^+$, hence $\lambda_i - \lambda_j \in \mathbb{Z}^+$ for $i < j$. Thus the highest weights λ must have components satisfying:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n, \qquad \lambda_i - \lambda_j \in \mathbb{Z}. \tag{83}$$

A weight λ satisfying (83) is called a *dominant weight*, written $\lambda \in D^+$.

For example, let V be the n-dimensional $gl(n)$ module with basis $\{v^k\}$ satisfying $a^i{}_jv^k = \delta^k{}_jv^i$. Now $a^i{}_iv^k = \delta^k{}_iv^i = \delta^i{}_kv_k$ hence v_k is an eigenvector, indeed it is a weight vector, with weight ϵ_k. Clearly v_1 is the unique highest-weight vector, with weight ϵ_1. In fact ϵ_1 is the unique dominant weight in $V(\epsilon_1)$, and hence $V(\epsilon_1)$ must be irreducible.

It can be shown that corresponding to *each* dominant weight λ, there exists a finite-dimensional irreducible $gl(n)$-module, labelled $V(\lambda)$. The problem of the *explicit* construction (cf. the $o(3)$ case) of $gl(n)$ representations is in general non-trivial. The construction is aided by using a (*symmetry adapted*) basis (e.g. the *Gel'fand-Tsetlin basis*).

Let ρ be the half-sum of the positive roots of $gl(n)$, viz.:

$$\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha.$$

With this, we have *Weyl's dimension formula*

$$\dim[V(\lambda)] = \prod_{\alpha\in\Phi^+}\frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)}.$$

which will not be proved, but is useful to know.

Exercise 23. *Explicitly determine ρ for $gl(n)$ and then use Weyl's formula to calculate* $\dim[V(\lambda)]$ *for the following choices:*

(i) $\lambda = \epsilon_1$

(ii) $\lambda = 2\epsilon_1$

(iii) $\lambda = \epsilon_1 + \epsilon_2$

(iv) $\lambda = \epsilon_1 - \epsilon_n$

Exercise 24. *Define elements A^m_{ij} of the universal enveloping algebra $U(gl(n))$ recursively by*

$$A^m_{ij} = \sum_{k=1}^{n} a^i{}_kA^{m-1}_{kj}$$
$$A^1_{ij} = a^i{}_j.$$

Generally, if $W_{\pm}$ has basis $\{w_\alpha\}$ and $a \in L$ then on $W \otimes W$

$a(w_\alpha \otimes w_\beta \pm w_\beta \otimes w_\alpha) = (aw_\alpha) \otimes w_\beta \pm (aw_\beta) \otimes w_\alpha$

$w_\alpha \otimes (aw_\beta) \pm w_\beta \otimes aw_\alpha$

– the action of $a \in L$ preserves the "permutation symmetry"

Show that these elements satisfy

$$[a^i{}_j,\, A^m_{kl}] = \delta^k_j A^m_{il} - \delta^i_l A^m_{kj}.$$

Show that

$$I_m = \sum_{k=1}^{n} A^m_{kk}$$

are Casimir invariants; i.e.

$$[I_m,\, a^i{}_j] = 0 \qquad 1 \le i, j \le n.$$

Determine the eigenvalue of the Casimir invariant I_2 on any irreducible module with highest weight

$$\lambda = (\lambda_1,\, \lambda_2,\, \ldots,\, \lambda_n).$$

- Considering $gl(n)$, let e_i, $i = 1, \ldots, n$ be the usual elementary basis of $V = \mathbb{C}^n$. Then V constitutes an irreducible $gl(n)$-module with the definition:

$$a^i{}_j e_k = \delta_{jk} e_i.$$

Observe that e_i is a weight vector of weight ϵ_i, and $\epsilon_1 = (1, \dot{0}_{n-1})$, corresponding to the weight of e_1, is the unique highest-weight vector of V: we call V the fundamental *vector representation* of $gl(n)$.

Now let $V \otimes V$ be the tensor product space with basis $e_i \otimes e_j$ (the *rank two tensor representation*), which becomes a $gl(n)$-module with the action:

$$a^i{}_j(e_k \otimes e_q) = (a^i{}_j e_k) \otimes e_q + e_k \otimes (a^i{}_j e_q) = \delta_{jk}(e_i \otimes e_q) + \delta_{jq}(e_k \otimes e_i).$$

We write the Clebsch-Gordan decomposition: $V = V^+ \oplus V^-$, where:

$$\begin{aligned} V^+ &= \operatorname{span}\{e_i \otimes e_j + e_j \otimes e_i \mid i \le j,\ i, j = 1, \ldots, n\}, \\ V^- &= \operatorname{span}\{e_i \otimes e_j - e_j \otimes e_i \mid i < j,\ i, j = 1, \ldots, n\}. \end{aligned}$$

Here, V^+ is irreducible, with highest weight $(2, \dot{0}_{n-1})$ and highest-weight vector $e_1 \otimes e_1$; and V^- is irreducible, with highest weight $(1, 1, \dot{0}_{n-2})$, and highest-weight vector $\frac{1}{\sqrt{2}}(e_1 \otimes e_2 - e_2 \otimes e_1)$.

- More generally, consider the rank k tensor product module:

$a(v_\alpha \otimes v_\beta \otimes \ldots \otimes v_\gamma)$
$= (av_\alpha) \otimes v_\beta \otimes \ldots \otimes v_\gamma$
$+ v_\alpha \otimes (av_\beta) \otimes \ldots \otimes v_\gamma$
$+ \ldots$
$+ v_\alpha \otimes v_\beta \otimes \ldots (av_\gamma)$

$$V^k = \underbrace{V \otimes \cdots \otimes V}_{k \text{ fold product}}.$$

Let V^+ be the subspace spanned by the symmetrised vectors $(i_1, i_2, \ldots, i_k = 1, \ldots, n)$:

$$\frac{1}{\sqrt{k!}} \sum_{\pi \in S_k} e_{i_{\pi(1)}} \otimes e_{i_{\pi(2)}} \otimes \cdots \otimes e_{i_{\pi(k)}}, \qquad i_1 \le i_2 \le \cdots \le i_k.$$

– $v^1 \otimes v^1 \otimes \ldots \otimes v^1$ highest weight

this is completely symmetric

e.g. $v^1 \otimes v^2 \otimes v^3 + v^1 \otimes v^3 \otimes v^2 + v^2 \otimes v^1 \otimes v^3 + v^2 \otimes v^3 \otimes v^1 + v^3 \otimes v^2 \otimes v^1 + v^3 \otimes v^1 \otimes v^2$

Similarly, let V^- be the subspace spanned by the antisymmetrised vectors ($i_1, i_2, \ldots, i_k = 1, \ldots, n$):

$$\frac{1}{\sqrt{k!}} \sum_{\pi \in S_k} \operatorname{sn}(\pi) e_{i_{\pi(1)}} \otimes e_{i_{\pi(2)}} \otimes \cdots \otimes e_{i_{\pi(k)}}, \qquad i_1 < i_2 < \cdots < i_k.$$

Then $V^\pm$ are irreducible $gl(n)$-modules with highest weights $(k, \dot{0}_{n-1})$ and $(\dot{1}_k, \dot{0}_{n-k})$ respectively. The representations afforded by $V^\pm$ are respectively called the *symmetric* and *antisymmetric rank k tensor representations* of $gl(n)$. In general the complete reduction of V^k into irreducible tensors of various symmetries can be achieved through the use of *Young diagrams.*

Exercise 25. *The Lie algebra $gl(n)$ with basis $\{a^i{}_j\}_{i,j=1}^n$ has a $gl(n-1)$ subalgebra with basis $\{a^i{}_j\}_{i,j=1}^{n-1}$. Viewing $gl(n)$ as a $gl(n-1)$-module under the commutator, determine the decomposition into irreducible submodules and give the highest weight for each submodule.*

4.5 The boson and fermion calculi

In many areas of quantum systems, especially many-body systems of identical particles, it is convenient to formulate problems in terms of canonical boson and/or fermion operators. For bosonic systems a set of n canonical boson operators $\{b_j,\, b_j^\dagger : 1 \le j \le n\}$ satisfy the commutation relations

$$[b_j,\, b_k] = [b_j^\dagger,\, b_k^\dagger] = 0, \qquad [b_j,\, b_k^\dagger] = \delta_{jk} I \tag{84}$$

It is usual to refer to the $\{b_j^\dagger\}$ as *creation operators* and the $\{b_j\}$ as *annihilation operators.*

Exercise 26. *Show that the canonical boson operators realise the $gl(n)$ algebra by setting*

$$a^j{}_k = b_j^\dagger b_k$$

The operator:

$$N = \sum_{j=1}^n a^j{}_j = \sum_{i=1}^n b_j^\dagger b_j,$$

is referred to as the boson *number operator*; it is the first order Casimir invariant of $gl(n)$.

For canonical fermion operators $\{c_j,\, c_j^\dagger : 1 \le j \le n\}$ the situation is similar except now the following *anti-commutation* relations are satisfied

$$\{c_j,\, c_k\} = \{c_j^\dagger,\, c_k^\dagger\} = 0, \qquad \{c_j,\, c_k^\dagger\} = \delta_{jk} I \tag{85}$$

where the anti-commutator $\{\, ,\, \}$ is realised as

$$\{A,\, B\} = AB + BA.$$

For $gl(n)$, let v_0 denote a highest-weight vector with weight $\lambda = (\lambda_1, \ldots, \lambda_n)$, $n \geq 4$

Consider $v_1 = a^4_2 v_0$ has weight $\lambda - \varepsilon_2 + \varepsilon_4$

$v_2 = a^3_1 v_1$ " " $\lambda - \varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4$

$v_3 = a^2_3 v_2$ " " $\lambda - \varepsilon_1 + \varepsilon_4$

$v_4 = a^1_4 v_3$ " " λ

$$
\begin{aligned}
v_4 &= a^1_4 a^2_3 a^3_1 a^4_2 v_0 \\
&= a^2_3 a^1_4 a^3_1 a^4_2 v_0 \\
&= a^2_3 a^3_1 a^1_4 a^4_2 v_0 - a^2_3 a^3_4 a^4_2 v_0 \\
&= \cancel{a^2_3 a^3_1 a^4_2 a^1_4 v_0} + \cancel{a^2_3 a^3_1 a^1_2 v_0} \\
&\qquad - \cancel{a^2_3 a^4_2 a^3_4 v_0} - a^2_3 a^3_2 v_0 \\
&= -\cancel{a^3_2 a^2_3 v_0} - (a^2_2 - a^3_3) v_0 \\
&= (\lambda_3 - \lambda_2) v_0
\end{aligned}
$$

$$AB = BA + [A,B]$$

For $gl(n)$, let v_0 denote a highest weight vector with weight $\lambda = (\lambda_1, \lambda_2, \dots \lambda_n)$, $n \geq 4$

Consider $v_1 = a^4_3 v_0$ has weight $\lambda - \varepsilon_3 + \varepsilon_4$

$v_2 = a^2_1 a^4_3 v_0$ has weight $\lambda - \varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4$

$v_3 = a^3_2 a^2_1 a^4_3 v_0$ has weight $\lambda - \varepsilon_1 + \varepsilon_4$

$v_4 = a^1_4 a^3_2 a^2_1 a^4_3 v_0$ has weight λ

$$v_4 = a^1_4 a^3_2 a^2_1 a^4_3 v_0$$
$$= a^3_2 a^1_4 a^2_1 a^4_3 v_0$$

$a^1_4 a^2_1 a^4_3 v_0$ has weight $\lambda + \varepsilon_2 - \varepsilon_3 > \lambda$ so $a^1_4 a^2_1 a^4_3 v_0 = 0$.

$$= a^3_2 a^2_1 a^1_4 a^4_3 v_0 + a^3_2 [a^1_4, a^2_1] a^4_3 v_0$$
$$= a^3_2 a^2_1 a^1_4 a^4_3 v_0 - a^3_2 a^2_4 a^4_3 v_0$$
$$= a^3_2 a^2_1 a^4_3 \cancel{a^1_4} v_0 + a^3_2 a^2_1 [a^1_4, a^4_3] v_0$$
$$= a^3_2 a^2_1 \cancel{a^1_3} v_0 - a^3_2 a^4_3 \cancel{a^2_4} v_0$$
$$- a^3_2 \cancel{a^2_3} v_0 - a^3_2 [a^2_4, a^4_3] v_0$$
$$= 0$$

$$\cancel{(a^2_2 - a^3_3) v_0 = (\lambda_2 - \lambda_3) v_0}$$

$\mathfrak{gl}(3)$ $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ For $j<k$ $a^j_k v^1 = 0$

Example $\mathfrak{gl}(n)$ modules.

1. $\pi(a^j_k) = e_{jk} \in \mathrm{End}(\mathbb{C}^n)$. In module notation, let $\{v^l : l=1,\dots,n\}$ denote the standard basis for $\mathbb{C}^n \cong V$. Then $a^j_k v^l = \delta^l_k v^j$. The highest weight is $\lambda = \varepsilon_1$

2. $\pi^*(a^j_k) = -e_{kj} \in \mathrm{End}(\mathbb{C}^n)$. In module notation, let $\{v_l : l=1,\dots,n\}$ denote the basis for $V^* \cong \mathbb{C}^n$. Then $a^j_k v_l = -\delta^j_l v_k$. The highest weight is $\lambda = -\varepsilon_n$.

3. Adjoint action: $a^j_k \circ (a^p_q) = [a^j_k, a^p_q]$

The highest weight is

$$\lambda = \varepsilon_1 - \varepsilon_n.$$

$$\begin{pmatrix} 0 & & & * \\ & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Tensor products: Consider $V \otimes V$ with basis $\{v^j_{\cancel{j}} \otimes v^k_{\cancel{k}} : 1 \le j,k \le n\}$ – (dimension n^2).

Recall that on a tensor product ($a \in L$, $v \in V$, $w \in W$) the action is defined as

$$a(v \otimes w) = av \otimes w + v \otimes aw$$

$$a^j_k(v^p_{\cancel{p}} \otimes v^q_{\cancel{q}}) = a^j_k v^p \otimes v^q + v^p \otimes a^j_k v^q$$
$$= \delta^p_k v^j \otimes v^q + \delta^q_k v^p \otimes v^j$$

Note : $a^j{}_j (v^p \otimes v^q) = \delta^p_j v^j \otimes v^q + \delta^q_j v^p \otimes v^j$.

We see that $v^p \otimes v^q$ has weight $\varepsilon_p + \varepsilon_q$

For example in $gl(3)$, the possible weights in $V \otimes V$ are $(2,0,0)$, $(0,2,0)$, $(0,0,2)$
$(1,1,0)$, $(0,1,1)$, $(1,0,1)$

The only candidates for highest weights are $(2,0,0)$ and $(1,1,0)$.

Generally speaking, if v^+ satisfies $a^j{}_k v^+ = 0$ for all $j<k$, and w^+ satisfies $a^j{}_k w^+ = 0$ for all $j<k$, then $a^j{}_k (v^+ \otimes w^+) = a^j{}_k v^+ \otimes w^+ + v^+ \otimes a^j{}_k w^+$
$= 0$

In $V \otimes V$, $v^1 \otimes v^1$ is a highest-weight vector of weight $2\varepsilon_1$, $v^1 \otimes v^2 - v^2 \otimes v^1$ is a highest-weight vector of weight $\varepsilon_1 + \varepsilon_2$.

symmetric

We write $V(\varepsilon_1) \otimes V(\varepsilon_1) = V(2\varepsilon_1) \oplus V(\varepsilon_1 + \varepsilon_2)$

— holds for $gl(n)$

antisymmetric.

e.g. For $gl(3)$, we have

$$\underset{\text{dim } 3}{V(\varepsilon_1)} \otimes V(\varepsilon_2) = \underset{\text{dim } 6}{V(2\varepsilon_1)} \oplus \underset{\text{dim } 3}{V(\varepsilon_1+\varepsilon_2)}$$

For $V(2\varepsilon_1)$, a basis is

$v^1 \otimes v^1$, $v^2 \otimes v^2$, $v^3 \otimes v^3$,

$v^1 \otimes v^2 + v^2 \otimes v^1$, $v^1 \otimes v^3 + v^3 \otimes v^1$,

$v^2 \otimes v^3 + v^3 \otimes v^2$.

For $V(\varepsilon_1+\varepsilon_2)$ a basis is

$v^1 \otimes v^2 - v^2 \otimes v^1$, $v^1 \otimes v^3 - v^3 \otimes v^1$,

$v^2 \otimes v^3 - v^3 \otimes v^2$.

Note that for the antisymmetric module with highest weight

$$(1,1,0) = (1,1,1) + (0,0,-1)$$

Up to tensoring with a 1-dimensional module $V(\varepsilon_1+\varepsilon_2)$ and $V(-\varepsilon_3)$ are isomorphic.

Aside: For $gl(n)$:

A basis for $V(\varepsilon_1+\varepsilon_2)$ is

$\{v^j \otimes v^k - v^k \otimes v^j : j \neq k\}$

$\frac{1}{2}n(n-1)$ is the dimension.

A basis for $V(2\varepsilon_1)$ is

$\{v^j \otimes v^k + v^k \otimes v^j : j \neq k\}$

$\cup \{v^j \otimes v^j : j = 1, \dots, n\}$

$\frac{1}{2}n(n+1)$ is the dimension here

Consider a one-dimensional $gl(n)$-module V_0. Let $\{v\}$ denote ~~the~~ a basis, define the action

$$a^j_k v = \mu \delta^j_k v.$$ Exercise: check that this action preserves the commutation relations.

$\mu \in \mathbb{R}$.

The vector v has weight $(\mu, \mu, \dots, \mu)$.

Then take the tensor product

$$V(\lambda) \otimes V_0 = V(\lambda')$$ where if

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$$

then $\lambda' = (\lambda_1 + \mu, \lambda_2 + \mu, \dots, \lambda_n + \mu)$

Let $w \in V(\lambda)$ then $a^j_k (w \otimes v)$

$$= a^j_k w \otimes v + w \otimes \mu \delta^j_k v$$

If we have two ~~repr~~ modules $V(\lambda)$, $V(\lambda')$ such that $\lambda - \lambda' = \mu(\varepsilon_1, \dots, \varepsilon_n)$ for some $\mu \in \mathbb{R}$.

Recall, $gl(n)$ has a central element $I_1 = \sum_{j=1}^{n} a^j_j$.

~~Aside from the action of I_1, the action~~

The action of $sl(n) \subset gl(n)$ is the same on $V(\lambda)$ and $V(\lambda')$.

$$I_1 = \sum_{j=1}^{n} a^j{}_j$$

Finally for tensor products, what happens with

$V(\varepsilon_1) \otimes V(-\varepsilon_n)$

Weights in $V(\varepsilon_1)$ are ε_j $j=1,\dots,n$

Weights in $V(-\varepsilon_n)$ are $-\varepsilon_k$, $k=1,\dots,n$

The possible weights in $V(\varepsilon_1) \otimes V(-\varepsilon_n)$ are of the form

$\varepsilon_j - \varepsilon_k$. There is definitely a vector, viz. $v^1 \otimes v_n$ of weight $\varepsilon_1 - \varepsilon_n$ which is a highest weight vector. Recalling the actions,

in $V(\varepsilon_1)$ $\quad a^j{}_k v^\lambda = \delta^\lambda_k v^j \quad (a^j{}_k v^1 = 0 \quad j<k)$

in $V(-\varepsilon_n)$ $\quad a^j{}_k v_\lambda = -\delta^j_\lambda v_k$, we see that (using Einstein notation)

$$\begin{aligned} a^j{}_k (v^\lambda \otimes v_\lambda) &= a^j{}_k v^\lambda \otimes v_\lambda + v^\lambda \otimes a^j{}_k v_\lambda \\ &= \delta^\lambda_k v^j \otimes v_\lambda - \delta^j_\lambda v^\lambda \otimes v_k \\ &= v^j \otimes v_k - v^j \otimes v_k = 0. \end{aligned}$$

Notation: $V(\varepsilon_1) \otimes V(-\varepsilon_n) = V(\varepsilon_1 - \varepsilon_n) \oplus V(0)$

Note $V(\varepsilon_1 - \varepsilon_n)$ is self-dual.

Considering $V(\varepsilon_1 - \varepsilon_n) \otimes V(\varepsilon_1 - \varepsilon_n)$, we can check (Einstein summation adopted)

$$a^j_k (a^p_q \otimes a^q_p)$$
$$= [a^j_k, a^p_q] \otimes a^q_p + a^p_q \otimes [a^j_k, a^q_p]$$
$$= (\delta^p_k a^j_q - \delta^j_q a^p_k) \otimes a^q_p + a^p_q \otimes (\delta^q_k a^j_p - \delta^j_p a^q_k)$$
$$= a^j_q \otimes a^q_k - a^p_k \otimes a^j_p + a^p_k \otimes a^j_p - a^j_q \otimes a^q_k$$
$$= 0,$$

Suppose $\bar{L} \subseteq L$ where $\bar{L}$ is a subalgebra of L. The problem of decomposing ~~and~~ an L-module V into irreducible $\bar{L}$-modules is known as a **branching rule**: $V(\lambda) \downarrow \bigoplus_j V(\bar{\lambda}_j)$ where λ denotes a ~~g~~ highest weight for an L-module, and ~~λ_j~~ $\bar{\lambda}_j$ are highest weights for $\bar{L}$-modules. E.g. consider $gl(n-k) \subseteq gl(n)$ generated by $\{a^j_k : 1 \le j, k \le n-k\}$. For the fundamental module of $gl(n)$

$$V(\varepsilon_1) \downarrow V(\varepsilon_1) \oplus k\, V(0)$$

~~$V(\varepsilon$~~) For $gl(n-k)$, $V(\varepsilon_1) = \operatorname{span}\{v^l : l = 1, \dots n-k\}$

Note $\dim(V(2\varepsilon_1)) = \frac{1}{2}n(n+1)$

Next, consider the $gl(n)$-module $V(2\varepsilon_1)$. What is the decomposition into $gl(n-1)$-modules, where $gl(n-1) = \text{span}\{a^j_k : 1 \le j, k \le n-1\}$. Observe, the weights in $V(2\varepsilon_1)$ have the form $2\varepsilon_j$, $j=1,\dots,n$ and $\varepsilon_j + \varepsilon_k$. The vector of weight $2\varepsilon_1$ is a highest weight for $gl(n-1)$. Note that the $gl(n)$ weight $\varepsilon_j + \varepsilon_n$, has weight ε_j with respect to $gl(n-1)$. In particular, $\varepsilon_1 + \varepsilon_n$ corresponds to the highest weight ε_1 for $gl(n-1)$.

$$\underbrace{V(2\varepsilon_1)}_{\dim \frac{1}{2}n(n+1)} \downarrow \underbrace{V(2\varepsilon_1)}_{\dim \frac{1}{2}(n-1)n} \oplus \underbrace{V(\varepsilon_1)}_{\dim n-1} \oplus \underbrace{V(0)}_{\dim 1},$$

where $V(0) = \text{span}\{w\}$ where w has $gl(n)$-weight $(0,0,\dots,2) = 2\varepsilon_n$.

In $gl(m)$ for $j<k$ set

$$e = a^j{}_k, \quad f = a^k{}_j, \quad h = a^j{}_j - a^k{}_k$$

satisfying

$$[e,f] = h, \quad [h,e] = 2e, \quad [h,f] = -2f.$$

Alternatively, for $j<k<l$ set

$$\bar{e} = \sqrt{2}\left(a^j{}_k - a^k{}_l\right), \quad \bar{f} = \sqrt{2}\left(a^k{}_j - a^l{}_k\right), \quad \bar{h} = 2\left(a^j{}_j - a^l{}_l\right)$$

satisfying

$$[\bar{e},\bar{f}] = \bar{h}, \quad [\bar{h},\bar{e}] = 2\bar{e}, \quad [\bar{h},\bar{f}] = -2\bar{f}$$

Consider $V(\varepsilon_1)$ for $gl(3)$, $V(\varepsilon_1) = \text{span}\{v^1, v^2, v^3\}$.

We have for $e = a^1{}_2$, $f = a^2{}_1$, $h = a^1{}_1 - a^2{}_2$

$$hv^1 = v^1, \quad hv^2 = -v^2, \quad hv^3 = 0.$$

For $\bar{e} = \sqrt{2}\left(a^1{}_2 - a^2{}_3\right)$, $\bar{f} = \sqrt{2}\left(a^2{}_1 - a^3{}_2\right)$, $\bar{h} = 2\left(a^1{}_1 - a^3{}_3\right)$

$$\bar{h}v^1 = 2v^1, \quad \bar{h}v^2 = 0, \quad \bar{h}v^3 = -2v^3$$

and, more generally:

$$a^i{}_i c^\dagger_{j_1} c^\dagger_{j_2} \cdots c^\dagger_{j_k} |0\rangle = (\delta^{j_1}{}_i + \delta^{j_2}{}_i + \cdots + \delta^{j_k}{}_i) c^\dagger_{j_1} c^\dagger_{j_2} \cdots c^\dagger_{j_k} |0\rangle \,.$$

It follows that the state (87) is an eigenstate of the number operator N with eigenvalue k, and is moreover a $gl(n)$ weight state, of weight:

$$\epsilon_{j_1} + \epsilon_{j_2} + \cdots + \epsilon_{j_k}.$$

We conclude that the states (87) are linearly independent, since their $gl(n)$ weights are distinct, and thus constitute a basis for $\mathcal{F}$. We will denote by $\mathcal{F}_k$ (for $k = 0, \ldots, l$) the space of eigenstates of N with eigenvalue k, referred to as the *space of k fermion states.* We then have a vector space decomposition:

$$\mathcal{F} = \bigoplus_{k=0}^{n} \mathcal{F}_k,$$

where $\mathcal{F}_0$ is the one-dimensional space spanned by $|0\rangle$. Clearly the dimension of $\mathcal{F}_k$ is given by the number of k-tuples $(j_1, j_2, \ldots, j_k)$ such that $j_1 < j_2 < \cdots < j_k$ and $j_1, j_2, \ldots, j_k = 1, \ldots, n$, viz.: $\dim(\mathcal{F}_k) = \left(\begin{array}{c} n \\ k \end{array} \right)$. In particular, $\mathcal{F}_n$ is one-dimensional, and spanned by the state:

$$\psi_0 = c^\dagger_1 c^\dagger_2 \cdots c^\dagger_n |0\rangle \,, \tag{88}$$

and $\dim(\mathcal{F}) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) = (1+1)^n = 2^n$.

Exercise 28. *For $n = 4$, there is a representation of the $o(4)$ algebra (satisfying commutation relations (34) given by fermion operators as a result of Ado's theorem, viz.*

$$a^i{}_j = c^\dagger_i c_j + c_i c^\dagger_j$$

In the decomposition $\mathcal{F} = \bigoplus_{k=0}^{4} \mathcal{F}_k$, identify those $\mathcal{F}_k$ that are irreducible $o(4)$-modules.

5 Lie algebras in quantum mechanics

In quantum mechanics, the state of a physical system is specified by a vector (often called *wavefunction*)$\psi \in \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space. Physical observables are associated with self-adjoint operators acting on $\mathcal{H}$. In particular, the *Hamiltonian H* of the system, the quantum analogue of the energy in classical mechanics, constitutes a self-adjoint operator on $\mathcal{H}$. So too do the coordinates q_i and momenta p_i of particles of the system. These are no longer commutative operators, but instead satisfy the *canonical commutation relations*:

$$[q_k, p_j] = i\hbar\delta^k_j, \qquad [q_k, q_j] = [p_k, p_j] = 0.$$

Here $2\pi\hbar$ is Planck's constant – we commonly choose units that set $\hbar$ to unity. The state ψ of a system is determined by *Schrödinger's time-dependent wave equation*:

$$i\hbar\frac{\mathrm{d}\psi}{\mathrm{d}t} = H\psi. \tag{89}$$

A physical observable A evolves in time according to:

$$i\hbar\frac{\mathrm{d}A}{\mathrm{d}t} = [A, H]. \tag{90}$$

Thus, any self-adjoint operator A on $\mathcal{H}$ which commutes with H will be a constant of the motion. According to the *correspondence principle*, a quantum Hamiltonian may be constructed from a classical one by making the substitution: q_i remains the same, except that now it is an *operator*, and p_j is replaced with the operator $-i\hbar\partial/\partial q_j$. This way, (89) may be viewed as a partial differential equation. Setting

$$\psi(\vec{q}, t) = \psi(\vec{q})e^{-iEt/\hbar}$$

then ψ solves (89) provided that $\psi(\vec{q})$ satisfies *Schrödinger's time-independent wave equation*:

$$H\psi = E\psi. \tag{91}$$

It is a trademark of quantum mechanics that (in bound-state problems), the E are always given by a discrete set (*energy levels*):

$$E_n, \qquad n = 0, 1, 2, \ldots$$

which are eigenvalues of H. The corresponding eigenspaces:

$$\mathcal{E}_n = \{\psi \in \mathcal{H} \mid H\psi = E_n\psi\}$$

are referred to as the nth *energy level* of the system (the states in which the system has energy E_n).

We define the *degeneracy* of the nth energy state as $\dim(\mathcal{E}_n)$. Lie algebras are useful in understanding degeneracies. They are also used to provide quantum numbers for labelling physical states, and to simplify calculations for numerical approximations of complex systems (e.g. molecules).

5.1 Symmetries and Lie algebras

Associated with a Hamiltonian H of a quantum system is a set $\mathcal{D}$ of symmetry operators satisfying

$$[A, H] = 0, \qquad A \in \mathcal{D}.$$

In view of (90), such operators are constants of the motion. These operators span a complex vector space which we denote L_H.

Exercise 29. *Show that if $A, B \in L_H$ then $[A, B] \in L_H$.*

From the Jacobi identity

$$[[A,B],H] = [A,[B,H]] - [B,[A,H]]$$

$$= 0 \quad \text{since} \quad [B,H] = [A,H] = 0$$

In view of the above, the elements of L_H close to form a Lie algebra. Now we define the *action* of one operator on another in terms of the commutator

$$A \circ B = [A,\, B]$$

which in particular satsifies the derivation property

$$A \circ (BC) = (A \circ B)C + B(A \circ C).$$

Powers of the action are defined recursively

$$A^m \circ B = A \circ (A^{m-1} \circ B).$$

In terms of this action we can define a transformation on B through

$$U_A(\lambda) \circ B = \exp(i\lambda A) \circ B, \qquad A^0 \circ B = B$$

where λ is a real parameter.

Exercise 30. *Show that*

$$U_A(\lambda) \circ B = \exp(i\lambda A) B \exp(-i\lambda A).$$

Hint: *Differentiate both sides of the expression.*

5.2 Single particle in a central potential

Let

$$r^2 = q_1^2 + q_2^2 + q_3^2, \qquad p^2 = p_1^2 + p_2^2 + p_3^2.$$

A Hamiltonian in a central potential has the form

$$H = \frac{p^2}{2m} + V(r). \tag{92}$$

Choosing units such that $\hbar = 1$ we have canonical commutation relations:

$$[q_j, p_k] = i\delta_k^j, \qquad [q_j, q_k] = [p_j, p_k] = 0. \tag{93}$$

$p = mv$

Classically, the system admits spherical (or rotational) symmetry, as $V = V(r)$. We seek the quantum analogue of this result. The quantum angular momentum vector operator is $\vec{L} = \vec{r} \times \vec{p}$, alternatively written:

e.g.

$$L_i = \varepsilon_{ijk} q_j p_k, \qquad \text{(implicit sum on } j \text{ and } k\text{)},$$

that is $L_1 = q_2 p_3 - q_3 p_2$, and cyclic permutations. Using (93), the L_i satisfy the $o(3)$ commutation relations:

realisation of o(3)

$$[L_1, L_2] = iL_3 \qquad \text{(and cyclic permutations)}.$$

Exercise 31. *We say that $\vec{A}$ is an $o(3)$ tensor operator if*

$$[L_j,\, A_k] = i\varepsilon_{jkl}A_l$$

where ε_{jkl} is the alternating tensor. Throughout we adopt the Einstein summation convention over repeated indices and note that

$$\varepsilon_{jkl}\varepsilon_{jmn} = \delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm}.$$

1. *Show that*
$$\varepsilon_{jkl}\varepsilon_{jkn} = 2\delta_{ln}.$$

2. *Given that $\vec{A}$ and $\vec{B}$ are tensor operators, show that*

   (i) $\vec{r}$, $\vec{p}$, $\vec{A}\times\vec{B}$ *are all tensor operators*

   (ii) $[\vec{L},\, \vec{A}\cdot\vec{B}] = 0$

   (iii) $[\vec{L}\cdot\vec{L},\, \vec{A}] = 2\vec{A} + 2i\vec{A}\times\vec{L}$

   (iv) $\vec{L}\cdot\vec{r} = \vec{r}\cdot\vec{L} = \vec{L}\cdot\vec{p} = \vec{p}\cdot\vec{L} = 0$

   (v) $\vec{L}\times\vec{A} + \vec{A}\times\vec{L} = 2i\vec{A}$

   (vi) $(\vec{A}\times\vec{L})\cdot\vec{B} + (\vec{A}\times\vec{B})\cdot\vec{L} = 2i(\vec{A}\cdot\vec{B})$

   (vii) $\vec{A}\cdot(\vec{L}\times\vec{B}) + (\vec{A}\times\vec{B})\cdot\vec{L} = 2i(\vec{A}\cdot\vec{B})$

   (viii) $(\vec{A}\times\vec{L})\times\vec{B} = \vec{L}(\vec{A}\cdot\vec{B}) - \vec{A}(\vec{L}\cdot\vec{B}) + i(\vec{A}\times\vec{B})$.

3. *Assume that*
$$[r,\, q_j] = 0,$$
$$[r,\, p_j] = \frac{iq_j}{r}.$$
For H of the form (92), deduce that
$$[L_j,\, H] = 0, \qquad j = 1, 2, 3.$$

The angular momentum vector $\vec{L}$ is a constant of the motion as it commutes with the Hamiltonian. It follows that the energy levels (eigenspaces) $\mathcal{E}_n$ of the system

$$\mathcal{E}_n = \{\psi \in \mathcal{H} \mid H\psi = E_n\psi\},$$

give rise to $o(3)$-modules. Indeed since $\psi \in \mathcal{E}_n$, we have $L_i\psi \in \mathcal{E}_n$ as:

$$H(L_i\psi) = L_iH\psi = L_iE_n\psi = E_n(L_i\psi).$$

In the absence of any further symmetries, we would expect $\mathcal{E}_n$ to give rise to an irreducible $o(3)$ module. Therefore, on the basis of $o(3)$ representation theory, we would expect the energy levels to be $(2l+1)$-fold degenerate, for $l \in \frac{1}{2}\mathbb{Z}^+$. Setting $n = 2l$, we label energy levels by $n = 0, 1, 2, 3, \ldots$, and obtain $\dim(\mathcal{E}_n) = n + 1$.)

This is generally true for spherical symmetry, but for two very important quantum mechanical systems, there is actually even greater degeneracy - the simple harmonic oscillator in three dimensions and the hydrogen atom. In following subsections, we will explore these.

5.3 Three-dimensional isotropic simple harmonic oscillator

Again, we have a central field problem, this time:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2$$

where ω is the frequency of the oscillator. The energy levels are labelled by a principal quantum number $n \in \mathbb{Z}^+$, and are $\frac{1}{2}(n+1)(n+2)$ fold degenerate, again greater than that predicted by $o(3)$ symmetry. As for the hydrogen atom, this occurs due to extra symmetries. We introduce operators:

$$b_j = \sqrt{\frac{m\omega}{2}}q_j + i\sqrt{\frac{1}{2m\omega}}p_j, \qquad b_j^\dagger = \sqrt{\frac{m\omega}{2}}q_j - i\sqrt{\frac{1}{2m\omega}}p_j.$$

These satisfy the canonical boson commutation relations:

$$[b_i, b_j^\dagger] = \delta^i_j, \qquad [b_i, b_j] = [b_i^\dagger, b_j^\dagger] = 0.$$

The operators $b^i_j = b_i^\dagger b_j$ satisfy the $gl(3)$ commutation relations:

Realisation $\pi(a^j{}_k) = b_j^+ b_k$

$$[b^i_j, b^k_l] = \delta^k_j b^i_l - \delta^i_l b^k_j.$$

$[\pi(a^i{}_k), \pi(a^p{}_q)] = \delta^p_k \pi(a^j{}_q) - \delta^j_q \pi(a^p{}_k)$

As before, the first order invariant $N = \sum_{i=1}^{n} b^i_i$ commutes with the $gl(3)$ generators. Now in terms of the b^i_j we have:

$N = \sum_{j=1}^{3} \pi(a^j{}_j)$

$$H = \omega\left(N + \frac{3}{2}I\right).$$

In particular, $[H, b^i_j] = 0$, for $i, j = 1, 2, 3$. Therefore, we expect energy levels to give rise to irreducible $gl(3)$-modules. To construct energy eigenvectors of H, we follow the approach of Section 4.6 and introduce the (normalised) vacuum state $|0\rangle$, defined such that:

$$b_i|0\rangle = 0, \qquad i = 1, 2, 3, \qquad \langle 0|0\rangle = 1.$$

The energy eigenstates are then constructed within Fock space (see Section 4.6), $\mathcal{F}_m$, which is spanned by the states:

$$\left\{ (b_1^\dagger)^{m_1}(b_2^\dagger)^{m_2}(b_3^\dagger)^{m_3}|0\rangle \,|m_i \in \mathbb{Z}^+ \right\}. \tag{94}$$

Note that:

Raising operators are $\pi(a^j{}_k) = b_j^+ b_k$ $j < k$.

$$b^i_j|0\rangle = b_i^\dagger b_j|0\rangle = 0, \qquad [b^i_j, b_k^\dagger] = \delta^k_j b_i^\dagger.$$

Exercise 32. *Show by induction that*

$$[b^i_j, (b_k^\dagger)^n] = n\delta^k_j b_i^\dagger (b_k^\dagger)^{n-1}.$$

By successive application of raising generators the state $(b_1^+)^{m_1}(b_2^+)^{m_2}(b_3^+)^{m_3}|0\rangle \mapsto (b_1^+)^{m_1+m_2+m_3}|0\rangle$

We can express bosonic Fock space as
$F = V(0) \oplus V(\varepsilon_1) \oplus V(2\varepsilon_1) \oplus \ldots = \bigoplus_{k\geq 0}^{\infty} V(k\varepsilon_1)$

From the above, it follows that states in (94) are eigenstates of the number operator $N = \sum_{i=1}^{3} b_i^\dagger b_i$, with eigenvalues $m = m_1 + m_2 + m_3$. The states in (94) are called *weight m boson states*; they form a (weight) basis for $\mathcal{F}_m$ with weight (m_1, m_2, m_3), and give rise to $gl(3)$ modules.[1]

Only the state $(b_1^\dagger)^m |0\rangle$ is maximal, as it's the only state on which *all* the raising operators will vanish. Therefore, $\mathcal{F}_m$ is an irreducible $gl(3)$ module with highest weight $(m, 0, 0)$. Observe that $\dim(\mathcal{F}_m) = \frac{1}{2}(m+1)(m+2)$ (cf. Exercise 23), and on $\mathcal{F}_m$, H has eigenvalue:

$$E_m = \omega\left(m + \frac{3}{2}\right).$$

In summary, the energy spectrum and degeneracies of the simple harmonic oscillator can be derived by exploiting the $gl(3)$ symmetry.

5.4 Non-relativistic hydrogen atom

This was one of the first systems to be studied quantum mechanically. Specifically we are interested in the Hamiltonian associated with the central field problem of a nucleus of charge Z (in units of e, the electron charge) and a single electron of mass m. The quantum analogue of the classical Hamiltonian is:

$$H = \frac{p^2}{2m} - \frac{Ze^2}{r}. \tag{95}$$

$p^2 = p_1^2 + p_2^2 + p_3^2$
$r = \sqrt{q_1^2 + q_2^2 + q_3^2}$

From spectroscopy, the energy levels are found to not be $(n+1)$-fold degenerate as predicted by $o(3)$ symmetry).[2] The different symmetry arises from the fact that this Hamiltonian admits extra constants of the motion in addition to L. These extra symmetries are often called *hidden symmetries.*

Firstly, we consider the *classical* hydrogen atom. For bound-state problems, the electron is 'captured' by the field, and the orbit is an ellipse. From Hamilton's equations:

$$\frac{\mathrm{d}q_j}{\mathrm{d}t} = \frac{\partial H}{\partial p_j} = \frac{p_j}{m}, \qquad \frac{\mathrm{d}p_j}{\mathrm{d}t} = -\frac{\partial H}{\partial q_j} = -\frac{Ze^2}{r^3}q_j,$$

thus $\vec{L} = \vec{r} \times \vec{p}$ is a constant in time, as:

$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = \left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \times \vec{p}\right) + \left(\vec{r} \times \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}\right) = \frac{1}{m}(\vec{p} \times \vec{p}) - \frac{Ze^2}{r^3}(\vec{r} \times \vec{r}) = \vec{0}.$$

$\frac{d}{dt}(\vec{A} \times \vec{B}) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$

Clearly, $\vec{L} \perp \vec{r}$ and $\vec{p}$. The inverse square law is the *only* central force giving rise to closed orbits – in this case ellipses in time, in the plane perpendicular to $\vec{L}$. There is another

[1]The 3 is the number of degrees of freedom of the system in the physical space. In general, for an oscillator with n degrees of freedom, we would work with $gl(n)$ modules.

[2]Here, n is called the *Bohr principal quantum number.*

constant of the motion, the *Rungé vector* $\vec{A}$, peculiar to the inverse square law of force, pointing along the major axis of the ellipse. As $\vec{A}$ is in the plane of the ellipse, it is a linear combination of the vectors $\vec{r}$ and $\vec{L} \times \vec{p}$. Set

$$\vec{A} = \frac{1}{Ze^2m}(\vec{L} \times \vec{p}) + \frac{\vec{r}}{r}.$$

At the operator level $\vec{L} \times \vec{p} \neq -\vec{p} \times \vec{L}$

Exercise 33. *Show by direct differentiation that*

$$\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} = \vec{0}.$$

In the light of the classical case, we consider the quantum situation. The quantum analogue of the *Rungé vector* is the *Pauli-Rungé vector*:

$$\vec{A} = \frac{1}{2Ze^2m}(\vec{L} \times \vec{p} - \vec{p} \times \vec{L}) + \frac{\vec{r}}{r}.$$

Exercise 34. *Show that*

$$[A_j,\, H] = 0$$

with H *given by (95). Note from Exercise 31 that* $\vec{A}$ *is an* $o(3)$ *tensor operator, so that*

$$[\vec{A} \cdot \vec{A},\, L_j] = 0.$$

Consider $X = \left(\begin{array}{c|c} X_1 & 0 \\ \hline 0 & X_2 \end{array}\right)$

Show that

$$\vec{A} \cdot \vec{A} = I + \frac{2}{Z^2e^4m}(I + \vec{L} \cdot \vec{L})H$$

Introducing the operators

$$a_i = \sqrt{-\frac{Z^2e^4m}{2H}}A_i,$$

Scale X "blockwise"

$X \mapsto \left(\begin{array}{c|c} \alpha_1 X_1 & 0 \\ \hline 0 & \alpha_2 X_2 \end{array}\right)$

we have

$$I = -\frac{2}{Z^2e^4m}(I + \vec{a} \cdot \vec{a} + \vec{L} \cdot \vec{L})H \qquad (96)$$

Exercise 35. *Show that*

$$[a_1, a_2] = iL_3, \qquad [L_1, L_2] = iL_3, \qquad [L_1, a_2] = ia_3 \qquad \textit{(and cyclic permutations)}$$

The above are seen to be the defining relations of $o(4)$, so we expect the energy levels of the hydrogen atom to give rise to irreducible $o(4)$-modules, not irreducible $o(3)$-modules. Now we consider representations of $o(4)$. The operators:

$$\vec{M}^{\pm} = \frac{1}{2}(\vec{L} \pm \vec{a})$$

give rise to two new $o(3)$ algebras which commute:

$$[M^+_\alpha, M^-_\beta] = 0$$

Recall, the fundamental module action
$a^j{}_k v^i = \delta^i_k v^j$

In $gl(m)$ for $j<k$ set

$e = a^j{}_k$ $[e,f] = h$

$f = a^k{}_j$ satisfying $[h,e] = 2e$

$h = a^j{}_j - a^k{}_k$ $[h,f] = -2f$.

Alternatively, for $j<k<l$ set

$\bar{e} = \sqrt{2}\left(a^j{}_k - a^k{}_l\right)$ $[\bar{e},\bar{f}] = \bar{h}$

$\bar{f} = \sqrt{2}\left(a^k{}_j - a^l{}_k\right)$ satisfying $[\bar{h},\bar{e}] = 2\bar{e}$

$\bar{h} = 2\left(a^j{}_j - a^l{}_l\right)$ $[\bar{h},\bar{f}] = -2\bar{f}$

Consider $V(\varepsilon_1)$ for $gl(3)$, $V(\varepsilon_1) = \text{span}\{v^1, v^2, v^3\}$.
We have for $e = a^1{}_2$, $f = a^2{}_1$, $h = a^1{}_1 - a^2{}_2$

$hv^1 = v^1$, $hv^2 = -v^2$, $hv^3 = 0$.

For $\bar{e} = \sqrt{2}\left(a^1{}_2 - a^2{}_3\right)$, $\bar{f} = \sqrt{2}\left(a^2{}_1 - a^3{}_2\right)$

$\bar{h} = 2\left(a^1{}_1 - a^3{}_3\right)$

$\bar{h}v^1 = 2v^1$, $\bar{h}v^2 = 0$, $\bar{h}v^3 = -2v^3$

For the $sl(2)$ branching rule, $V(\varepsilon_1) \downarrow V_1 \oplus V_0$,
for the $\overline{sl(2)}$ branching rule, $V(\varepsilon_1) \downarrow V_2$
where V_d denote an A_1-module of dimension $d+1$.

Consider

$$A\{B,C\} - [A,C]B$$
$$= A(BC + CB) - (AC - CA)B$$
$$= ABC + CAB = \{AB, C\}$$

$$A[B,C] + \{A,C\}B$$
$$= A(BC - CB) + (AC + CA)B$$
$$= ABC + CAB = \{AB, C\}$$

$F_3 = \mathrm{span}\{$
$c_1^+ c_2^+ c_3^+ |0\rangle,$
$c_1^+ c_2^+ c_4^+ |0\rangle,$
$c_2^+ c_3^+ c_4^+ |0\rangle,$
$c_1^+ c_3^+ c_4^+ |0\rangle\}$
with weights
$(1,1,1,0), (1,1,0,1)$
$(0,1,1,1), (1,0,1,1)$

Fock space for $gl(4)$:

$F_0 = \mathrm{span}\{|0\rangle\}$, with weight $(0,0,0,0)$.

$F_1 = \mathrm{span}\{c_1^+|0\rangle, c_2^+|0\rangle, c_3^+|0\rangle, c_4^+|0\rangle\}$. The weights occurring in F_1 are $(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)$

so $F_1 \cong V(\varepsilon_1)$

$F_2 = \mathrm{span}\{c_1^+ c_2^+|0\rangle, c_1^+ c_3^+|0\rangle, c_1^+ c_4^+|0\rangle,$
$c_2^+ c_3^+|0\rangle, c_2^+ c_4^+|0\rangle, c_3^+ c_4^+|0\rangle\}$.

The weights are $(1,1,0,0), (1,0,1,0), (1,0,0,1)$
$(0,1,1,0), (0,1,0,1), (0,0,1,1)$

F_3 - see above, $F_4 = \mathrm{span}\{c_1^+ c_2^+ c_3^+ c_4^+|0\rangle\}$

Examples of Exercise 31

(ii). $[L_j, A_k B_k]$

$$= A_k [L_j, B_k] + [L_j, A_k] B_k$$

$$= i \left(\varepsilon_{jkl} A_k B_l + \varepsilon_{jkl} A_l B_k \right)$$

$$= i \left(\varepsilon_{jkl} A_k B_l + \varepsilon_{jlk} A_k B_l \right)$$

$$= i \left(\varepsilon_{jkl} + \varepsilon_{jlk} \right) A_k B_l$$

$$= 0$$

(iv) $\vec{L} \cdot \vec{r} = L_j r_j$

$$= \varepsilon_{jpq} q_p p_q q_j$$

$$= \varepsilon_{jpq} \left(p_q q_p + i \delta_{pq} \right) q_j$$

$$= \varepsilon_{jpq} p_q q_p q_j + i \varepsilon_{jpp} q_j$$

$$= 0$$

Note: $\varepsilon_{jpq} p_q q_p q_j$

$$= \varepsilon_{pjq} p_q q_j q_p$$

$$= \varepsilon_{pjq} p_q q_p q_j$$

$$= - \varepsilon_{jpq} p_q q_p q_j = 0$$

Exercise 33 : $\vec{A} = \frac{1}{Ze^2 m}\left(\vec{L}\times\vec{p}\right) + \frac{1}{r}\vec{r}$

$r^2 = \vec{r}_t \cdot \vec{r}_t \Rightarrow \frac{dr}{dt} = \frac{1}{mr}\vec{p}\cdot\vec{r}$

For later use

$\vec{L}\times\vec{r} = \left(\vec{r}\times\vec{p}\right)\times\vec{r}$

$\underline{A}\times\left(\underline{B}\times\underline{C}\right) = \left(\underline{A}\cdot\underline{C}\right)\underline{B} - \left(\underline{A}\cdot\underline{B}\right)\underline{C}$

$= r^2\vec{p} - \left(\vec{p}\cdot\vec{r}\right)\vec{r}$

$$\frac{d\vec{A}}{dt} = \frac{1}{Ze^2 m}\left(\vec{L}\times\frac{d\vec{p}}{dt}\right) + \frac{1}{r^2}\left(r\frac{d\vec{r}}{dt} - \frac{dr}{dt}\vec{r}_t\right)$$

$$= \frac{-1}{Ze^2 m}\cdot\frac{Ze^2}{r^3}\left(\vec{L}_t\times\vec{r}\right) + \frac{1}{mr}\vec{p} - \frac{1}{mr^3}\left(\vec{p}\cdot\vec{r}\right)\vec{r}$$

$$= \frac{-1}{mr^3}\left(r^2\vec{p} - \left(\vec{p}\cdot\vec{r}\right)\vec{r}\right) + \frac{1}{mr}\vec{p} - \frac{1}{mr^3}\left(\vec{p}\cdot\vec{r}\right)\vec{r}$$

$$= \vec{0}$$

(Einstein summation)

Check: $[a_j, a_k] = i\varepsilon_{jk\lambda} L_\lambda$,

$[L_j, L_k] = i\varepsilon_{jk\lambda} L_\lambda$, $[L_j, a_k] = i\varepsilon_{jk\lambda} a_\lambda$

Next

$$
\begin{aligned}
4[M_j^+, M_k^\pm] &= [L_j + a_j, L_k \pm a_k] \\
&= [L_j, L_k] + [a_j, L_k] \\
&\quad \pm [L_j, a_k] \pm [a_j, a_k] \\
&= i\varepsilon_{jk\lambda} L_\lambda - \varepsilon_{kj\lambda} a_\lambda \\
&\quad \pm i\varepsilon_{jk\lambda} a_\lambda \pm i\varepsilon_{jk\lambda} L_\lambda
\end{aligned}
$$

For "+"

$$
\begin{aligned}
4[M_j^+, M_k^+] &= 2i\varepsilon_{jk\lambda} L_\lambda + 2i\varepsilon_{jk\lambda} a_\lambda \\
&= 4i\varepsilon_{jk\lambda} M_\lambda^+
\end{aligned}
$$

For "−" $4[M_j^+, M_k^-] = 0$

$$[h, e] = 2e$$

Consider a weight vector $|m_1, m_2\rangle$ such that

$$h_1 |m_1, m_2\rangle = m_1 |m_1, m_2\rangle$$

$$h_2 |m_1, m_2\rangle = m_2 |m_1, m_2\rangle$$

Consider $e_1 |m_1, m_2\rangle$

$$h_1 e_1 |m_1, m_2\rangle = (m_1 + 2) e_1 |m_1, m_2\rangle$$

$$h_2 e_1 |m_1, m_2\rangle = e_1 h_2 |m_1, m_2\rangle$$

$$= m_2 e_1 |m_1, m_2\rangle$$

Consider an $o_1(3)$ module with basis $\{|m_1\rangle\}$

This becomes an $o(4) = o(3) \oplus o(3)$ module with definition $h_2 |m_1\rangle = e_2 |m_1\rangle = f_2 |m_1\rangle = 0$.