Boundary Supersymmetry of (1+1)D SPT Phase

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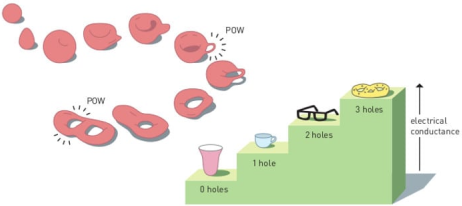
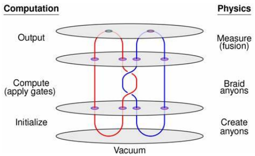
Outline

1. Topologically ordered phase
   - Big picture
   - Topological quantum field theory
   - 2d topological order
2. SPT phase
3. Boundary supersymmetry of SPT phase
   - Supersymmetric quantum mechanics
   - Boundary SUSY of (1+1)D SPT phase

Big picture for topological order

TQFT $\Leftrightarrow$ Topological order $\Leftrightarrow$ Topological quantum computation





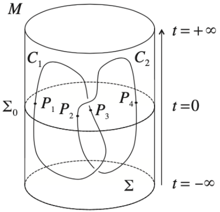
- TQFT is low energy effective theory of topological order
- Topological order can be used for quantum computation

Topological quantum field theory

- Topological quantum field theory Witten, Edward, "Topological quantum field theory", Commun. Math. Phys. 117, 353-386 (1988)

$$\text{Chern-Simons:} \quad CS[A] = \frac{\kappa}{2}\int_M dx^3 \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho . \tag{1}$$

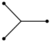
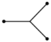
$$\text{Wilson loop:} \quad W(L) = \prod_{i=1}^{r} \exp\left(iQ_i \oint_{C_i} A \right) \tag{2}$$

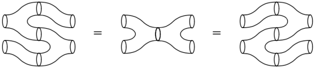


Consider the three-dimensional space M being made up of a spatial disk Σ defined though the infinite time line. A disc Σ_0 at $t=0$ introduces a set of points P_1, P_2, P_3 and P_4 where the link made out of the loops C_1 and C_2 intersects the disk.

Topological quantum field theory

- Atiyah's axiomatic approch M. Atiyah, "Topological quantum field theories", Publications Mathématiques de l'IHÉS 68 (68): 175–186 (1988)

Principle	Feynman diagram	2D cobordism	Algebraic operation (in a $\Bbbk$-algebra A)	
merging			multiplication	$A \otimes A \to A$
creation			unit	$\Bbbk \to A$
splitting			comultiplication	$A \to A \otimes A$
annihilation			counit	$A \to \Bbbk$



There is an associated Hilbert space A to each circle, 2D codordism is morphisms between these circles.

Topological quantum field theory

- Modern point of view: a TQFT is a modular functor B. Bakalov, A. A. Kirillov, "Lectures on Tensor Categories and Modular Functors", American Mathematical Society (November 20, 2000)
  Topological object and gluing operation form a modular category $\mathbf{Cob}_2$; all vector spaces form a modular category $\mathbf{Vect}$. A TQFT is a modular functor between them

$$F : \mathbf{Cob}_2 \to \mathbf{Vect}. \tag{3}$$

Principle	Feynman diagram	2D cobordism	Algebraic operation (in a $\Bbbk$-algebra A)	
merging			multiplication	$A \otimes A \to A$
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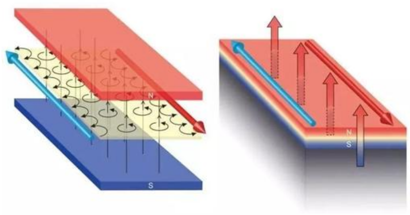
splitting			comultiplication	$A \to A \otimes A$
annihilation			counit	$A \to \Bbbk$

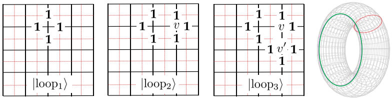
2d topological order

Topologically ordered phase

Topologically ordered phase is an equivalence class $\{\mathcal{H}, H\}$ which is a microscopic realization of TQFT.

- Topologically protected ground state space
- Topological entanglement entropy
- Boundary state and bulk-boundary correspondence
- Anyonic mutual statistics



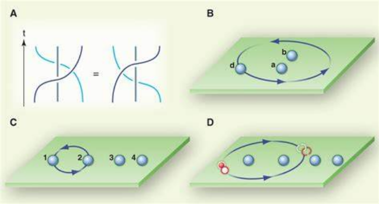


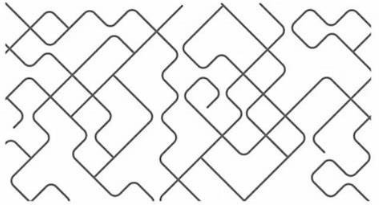
2d topological order: anyon model

Anyon model $\Leftrightarrow$ Modular tensor category $\mathcal{C}$.

- Topological charge $\{\mathbf{1}, a, b, \cdots\}$
- Fusion and splitting of charges $a \times b = \sum_c N^c_{ab} c$
- Braidings $R^c_{a,b}$
- Topological spin $\theta_a, \theta_b, \cdots$

Kitaev quantum double model;
String-net model
Trivial phase is characterized by **Hilb** $= \{\mathbf{1}\}$, the category of all finite dimensional Hilbert spaces, which only has one topological trivial charge $\mathbf{1}$.

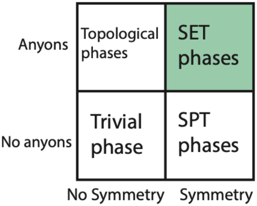


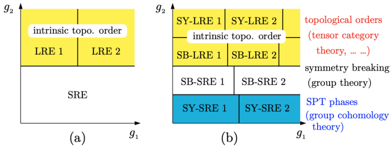


Topological order with symmetry

Symmetry enriched topological phase (SET) =anyon model $\mathcal{C}$ + symmetry G.

- Excitations: anyons+symmetric charge
- Mathematical structure: G-crossed braided fusion category $\mathcal{C}_G^\times$.
- Symmetry protected topological/trivial (SPT) phase is a special case of SET phase with only trivial anyon $\mathbf{1}$. Thus, it's characterized by $\mathbf{Hilb}_G^\times$.





SPT phase

SPT phases are gapped short-range entangled phase with symmetry G.

- d-dimensional bosonic SPT phases are characterized by group cohomology theory $H^{1+d}(G, U_T(1))$.
- Breaking symmetry G, the resulting phase is a trivial topological phase **Hilb**.
- Well-known examples: Haldane spin-1 chain; topological insulator; topological superconductor.

Phys. Rev. B 87, 155114 (2013); Rev. Mod. Phys. 82, 3045 (2010); Rev. Mod. Phys. 83, 1057 (2011).

SPT phase: examples

SPT order	Symmetry	Classification	Chain end/SPT probe
1B spin-1 Haldane phase	$SO(3)$	$\mathcal{H}^2(SO(3), \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2$	Spin 1/2
1B spin-1 Haldane phase	Z_2^T	$\mathcal{H}^2(Z_2^T, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2$	Kramer doublet
1B symmetry gapped phases	G	$\mathcal{H}^2(G, \mathbb{R}/\mathbb{Z})$	Projective representation of G
1F insulator w/ coplanar spin order	$U^f(1) \rtimes Z_2^T$	$\mathbb{Z}_2$	Kramers doublet
1F topological superconductor	Z_4^T	$\mathbb{Z}_2$	Charge-0 Kramers doublet
1F G^f-SPT phases	G^f	$\mathcal{H}^2(G^f, \mathbb{R}/\mathbb{Z})$	Projective representation of G^f
2B Z_n-SPT states	Z_n	$\mathcal{H}^3(Z_n, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n$	Z_n dislocation has fractional statistics/Z_n charge
2B SPT insulator	$U(1)$	$\mathcal{H}^3(U(1), \mathbb{R}/\mathbb{Z}) = \mathbb{Z}$	Even-integer Hall conductance
2B T-symmetric SPT insulator	$U(1) \rtimes Z_2^T$	$\mathcal{H}^3(U(1) \rtimes Z_2^T, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2$	π flux has Kramers doublet
2B spin quantum Hall states	$SO(3)$	$\mathcal{H}^3(SO(3), \mathbb{R}/\mathbb{Z}) = \mathbb{Z}$	Quantized spin Hall conductance
2B T-symmetric SPT spin liquid	$Z_2^T \times SO(3)$	$\mathcal{H}^3(Z_2^T \times SO(3), \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2$	
2B G-SPT states	G	$\mathcal{H}^3(G, \mathbb{R}/\mathbb{Z})$	
2F quantum spin Hall states	$U^f(1) \times U^f(1)$	$\mathbb{Z}$	Spin-charge Hall conductance
2F topological insulator	$[U^f(1) \rtimes Z_4^T]/Z_2$	$\mathbb{Z}_2$	π flux carries charge-0 Kramers doublet
2F topological superconductor	Z_4^T	$\mathbb{Z}_2$	π flux carries charge-even Kramers doublet
2F G^f-SPT states	G^f without T	Chiral central charge $c = 0$ modular extensions of sRep(G^f)	
3B T-symmetric SPT states	Z_2^T	$\mathcal{H}^4(Z_2^T, \mathbb{R}/\mathbb{Z}) \oplus \mathbb{Z}_2 = \mathbb{Z}_2^2$	
3B T-symmetric SPT insulator	$U(1) \rtimes Z_2^T$	$\mathcal{H}^4(U(1) \rtimes Z_2^T, \mathbb{R}/\mathbb{Z}) \oplus \mathbb{Z}_2 = \mathbb{Z}_2^3$	A monople is a fermion
3B T-symmetric SPT spin liquid	$Z_2^T \times SO(3)$	$\mathcal{H}^4(Z_2^T \times SO(3), \mathbb{R}/\mathbb{Z}) \oplus \mathbb{Z}_2 = \mathbb{Z}_2^4$	
3B G-SPT states	G without T	$\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z})$	
3B G-SPT states	G with T	$\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z}) \oplus \mathbb{Z}_2$	
3F topological insulator	$[U^f(1) \rtimes Z_4^T]/Z_2$	$\mathbb{Z}_2$	A monople carries half-integer charge
3F topological superconductor	Z_4^T	$\mathbb{Z}_{16}$	

Boundary supersymmetry of SPT phase

Boundary supersymmetry of SPT phase

The boundaries of all non-trivial $1+1$ dimensional intrinsically fermionic SPT phases, protected by finite group on-site symmetries (unitary or anti-unitary), are supersymmetric quantum mechanical systems.

- signatures of space-time SUSY in particle interactions has not yet been detected in particle colliders.
- proposals exist for their emergence and detection in condensed matter and cold-atomic systems, all these proposals require some kind of fine tuning of parameters.
- a new setting for emergence and detection of the simplest version of SUSY quantum mechanics without any fine-tuning.

A. Prakash, and J. Wang, arXiv:2011.12320; Z.-C. Gu Phys. Rev. Research 2, 033290 (2020)

SUSY quantum mechanics

SUSY quantum mechanics

For $\mathcal{N} = N$ SUSY quantum mechanics, we have N anticommuting Hermitian generators $Q_i, i = 1, \cdots, N$,

$$\{Q_i, Q_j\} = H\delta_{ij}; Q_i = Q_i^\dagger; \quad [H, Q_i] = 0; \quad H = \frac{2}{N}\sum_{i=1}^{N} Q_i^2.$$

Witten, E. (1982). "Constraints on supersymmetry breaking". Nuclear Physics B, 202(2), 253–316; Witten, E. (1981). Dynamical breaking of supersymmetry. Nuclear Physics B, 188(3), 513–554.

SUSY quantum mechanics: example

Example: SUSY quantum harmonic oscillator

Consider Hamonic oscillator $H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 q^2$, with $a, a^\dagger$ creating and annihlation operators, by introducing fermonic operators

$$\begin{aligned} \psi &= \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \psi^\dagger &= \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{4}$$

we see that $\{\psi^\dagger, \psi\} = 1, \quad \{\psi^\dagger, \psi^\dagger\} = \{\psi, \psi\} = 0$. Then we definie $Q = a\psi^\dagger$ and $Q^\dagger = a^\dagger\psi$, the final SUSY Hamiltonian is

$$H = QQ^\dagger + Q^\dagger Q = \left(-\frac{d^2}{dx^2} + \frac{x^2}{4}\right) I - \frac{1}{2}\left[\psi, \psi^\dagger\right]. \tag{5}$$

Witten, E. (1982). "Constraints on supersymmetry breaking". Nuclear Physics B, 202(2), 253–316; Witten, E. (1981). Dynamical breaking of supersymmetry. Nuclear Physics B, 188(3), 513–554.

Boundary SUSY of (1+1)D SPT phase: model study

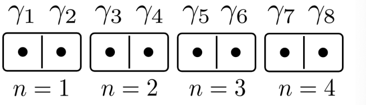
$(1+1)D$ superconductor

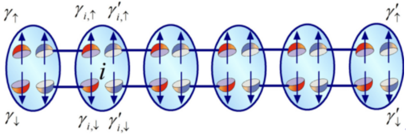
$(1+1)D$ interaction topological superconductor with time-reversal symmetry $\mathcal{T}$ satisfying $\mathcal{T}^2 = +\mathbb{1}$ form a group $\mathbb{Z}_8$. Different SPTs are classified by $\nu = 0, \cdots, 7 \in \mathbb{Z}_8$.

L. Fidkowski and A. Kitaev, Phys. Rev. B 83, 075103 (2011); L. Fidkowski and A. Kitaev, Phys. Rev. B 81, 134509 (2010).

The $\nu = 2$ case

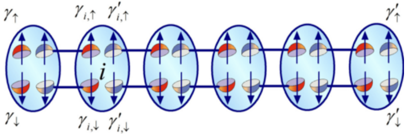
This case can be realized by two copies of Kiteav chain:





Boundary SUSY of (1+1)D SPT phase: model study

The $\nu = 2$ case: bulk



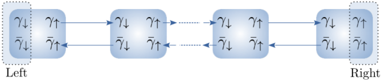
$$\begin{aligned} H &= -\mathrm{i} \textstyle\sum_{j=1}^{L} \left(\gamma_{\uparrow,j}\gamma_{\downarrow,j+1} - \bar{\gamma}_{\uparrow,j}\bar{\gamma}_{\downarrow,j+1}\right) \\ \mathcal{T} &= \mathcal{K} \textstyle\prod_{j=1}^{L} \left(\gamma_{\downarrow,j}\bar{\gamma}_{\uparrow,j}\right), P_f = \textstyle\prod_{j=1}^{L} \left(\mathrm{i}\bar{\gamma}_{\downarrow,j}\gamma_{\downarrow,j}\right)\left(\mathrm{i}\bar{\gamma}_{\uparrow,j}\gamma_{\uparrow,j}\right) \end{aligned} \tag{6}$$

$$\begin{aligned} &\text{time-reversal}: \quad \mathcal{T}: \gamma_\sigma \mapsto \tau^z_{\sigma\sigma'}\gamma_{\sigma'}, \bar{\gamma}_\sigma \mapsto \tau^z_{\sigma\sigma'}\bar{\gamma}_{\sigma'}, \quad \mathrm{i} \mapsto -\mathrm{i} \\ &\text{fermion parity}: \quad P_f: \gamma_\sigma \mapsto -\gamma_\sigma, \quad \bar{\gamma}_\sigma \mapsto -\bar{\gamma}_\sigma, \quad \mathrm{i} \mapsto \mathrm{i} \end{aligned} \tag{7}$$

A. Prakash, and J. Wang, arXiv:2011.12320; Z.-C. Gu Phys. Rev. Research 2, 033290 (2020)

Boundary SUSY of (1+1)D SPT phase: model study

The $\nu = 2$ case: boundary



Relabeling $\gamma_{\downarrow,1} \equiv \gamma, \bar\gamma_{\downarrow,1} \equiv \bar\gamma$, we see

$$\hat{\mathcal{T}} = \mathcal{K}\gamma, \quad \hat{P}_f = i\bar\gamma\gamma \tag{8}$$

$$\hat{\mathcal{T}}^2 = \hat{P}_f^2 = \mathbb{1}, \quad \hat{\mathcal{T}}\hat{P}_f = -\hat{P}_f\hat{\mathcal{T}} \tag{9}$$

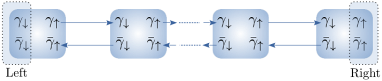
$$\begin{aligned} \hat{\mathcal{T}} &: \gamma \mapsto +\gamma, \bar\gamma \mapsto +\bar\gamma, \quad \mathrm{i} \mapsto -\mathrm{i} \\ \hat{P}_f &: \gamma \mapsto -\gamma, \bar\gamma \mapsto -\bar\gamma, \quad \mathrm{i} \mapsto \quad \mathrm{i} \end{aligned} \tag{10}$$

The only Hamiltonian which consistent with symmeties is $H = c\mathbb{1}$.

A. Prakash, and J. Wang, arXiv:2011.12320; Z.-C. Gu Phys. Rev. Research 2, 033290 (2020)

Boundary SUSY of (1+1)D SPT phase: model study

The $\nu = 2$ case: boundary SUSY



Define two Hermitian super-charges $\hat{Q}_+ \equiv \sqrt{c}\gamma$ and $\hat{Q}_- \equiv \sqrt{c}\bar{\gamma}$ which satisfy the $\mathcal{N} = 2$ SUSY QM algebra,

$$\left\{\hat{Q}_\alpha, \hat{Q}_\beta\right\} = 2\hat{H}\delta_{\alpha\beta}, \left[\hat{H}, \hat{Q}_\alpha\right] = \left\{\hat{P}_f, \hat{Q}_\alpha\right\} = 0$$

The complex super-charge can be defined as $Q = \hat{Q}_+ + i\hat{Q}_-$ and $\bar{Q} = \hat{Q}_+ - i\hat{Q}_-$

$$\hat{\mathcal{T}} : \begin{pmatrix} Q \\ \bar{Q} \end{pmatrix} \mapsto \begin{pmatrix} \bar{Q} \\ Q \end{pmatrix} \tag{11}$$

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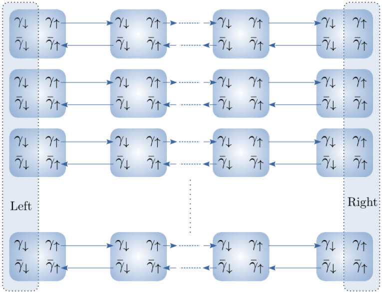
A. Prakash, and J. Wang, arXiv:2011.12320; Z.-C. Gu Phys. Rev. Research 2, 033290 (2020)

Boundary SUSY of (1+1)D SPT phase: model study

The $\nu = 2$ case: boundary SUSY



Since $\nu = 2 \in \mathbb{Z}_8$, it's invaraint modulo 8, stacking $4N$ copies to original one does not change the SPT phase. In this case, SUSY become a little bit complicated, but the boundary is still $\mathcal{N} = 2$ SUSY quantum mechanics.

A. Prakash, and J. Wang, arXiv:2011.12320; Z.-C. Gu Phys. Rev. Research 2, 033290 (2020)

Boundary SUSY of (1+1)D SPT phase: general result

Boundary SUSY of (1+1)D SPT phase

Any boundary Hamiltonian of a system belonging to a non-trivial (1+1) D SPT phase protected by finite on-site unitary or anti-unitary symmetries can be expressed as a supersymmetric quantum mechanical system if and only if the SPT phase is intrinsically fermionic.

Proof outline:
(I) Prove that an SPT phase is not intrinsically fermionic if and only if $\hat{P}_f$ commutes with all elements of symmetry group $\hat{G}_f$
(II) Prove that if $\hat{P}_f$ does not commute with all elements of symmetry group $\hat{G}_f$, then boundary Hamiltonian $\hat{H}$ is supersymmetric.

A. Prakash, and J. Wang, arXiv:2011.12320; Z.-C. Gu Phys. Rev. Research 2, 033290 (2020)

Thank You for Your Attentions