Problem 1 For the singlet state $\left|\psi^-\right\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$, prove that Alice and Bob's outcomes are always anti-correlated when they measure two particles respectively along the same direction.

Solution: Suppose that Alice and Bob both choose to measure spin in $\vec{v}$ direction, then the operator for them are

$$A = \vec{v} \cdot \vec{\sigma}, \quad B = \vec{v} \cdot \vec{\sigma}. \tag{1}$$

The joint operation is thus $A \otimes B$.

Suppose $|v+\rangle$ and $|v-\rangle$ are the ± 1-eigenstates of $\vec{v} \cdot \vec{\sigma}$ Then there exist complex numbers $\alpha, \beta, \gamma, \delta$ such that

$$|0\rangle = \alpha|v+\rangle + \beta|v-\rangle, \quad |1\rangle = \gamma|v+\rangle + \delta|v-\rangle. \tag{2}$$

Substituting them into the expression of singlet state, we obtain

$$\frac{|01\rangle - |10\rangle}{\sqrt{2}} = (\alpha\delta - \beta\gamma)\frac{|v+\rangle|v-\rangle - |v-\rangle|v+\rangle}{\sqrt{2}} \tag{3}$$

But $\alpha\delta - \beta\gamma$ is the determinant of the unitary matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, and thus is equal to a phase factor $e^{i\theta}$ for some real θ. (this is because that the eigenvalue of a unitary matrix is of the form $\lambda_j = e^{i\theta_j}$, then $\det U = \Pi_j e^{i\theta_j} = e^{i\sum_j \theta_j}$). This means that we can write the singlet state as

$$\frac{|01\rangle - |10\rangle}{\sqrt{2}} = \frac{|v+\rangle|v-\rangle - |v-\rangle|v+\rangle}{\sqrt{2}} \tag{4}$$

up to an unobservable global phase factor.

As a result, if a measurement of $\vec{v} \cdot \vec{\sigma}$ is performed on both qubits, i.e., Alice and Bob perform

$$A \otimes B = \vec{v} \cdot \vec{\sigma} \otimes \vec{v} \cdot \vec{\sigma} \tag{5}$$

on singlet state, they always obtain the respective outcomes a, b and $a \times b = -1$, they are anti-correlated. This completes the proof. ■

Problem 2 PPT (Positive Partial Transposition) criterion is a strong separability criterion for quantum state, which is very convenient and practical for entanglement detection.

(1) Describe the PPT (Positive Partial Transposition) criterion and the realignment criterion.

(2) For the 2-qubit state $\rho = p\left|\phi^-\right\rangle\left\langle\psi^-\right| + (1-p)\frac{\mathbb{I}}{4}$, where, $0 \le p \le 1$, $\left|\phi^-\right\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$, calculate the p's lower bound when ρ is entangled state using PPT criterion and realignment criterion respectively.

Solution:

(1) PPT criterion states that, if ρ is separable, then the partial transpose ρ^{T_A} has no negative eigenvalues. Namely, if the partial transpose of a density operator ρ has negative eigenvalue, then the density operator is entangled.

Realignment criterion states that, for any bipartite separable state ρ, the sum of all singular values of the realignment of ρ, denoted as $\tilde{\rho}$ must satisfy that $\|\tilde{\rho}\| = \sum_i \lambda_i \le 1$. Namely, if $\sum_i \lambda_j > 1$, then ρ is entangled.

(2) For the state $\rho = p\left|\phi^-\right\rangle\left\langle\psi^-\right| + (1-p)\frac{\mathbb{I}}{4}$, if we choose the ordered basis as $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, the density matrix is of the form

$$\rho = \begin{pmatrix} \frac{1+p}{4} & 0 & 0 & -\frac{p}{2} \\ 0 & \frac{1-p}{4} & 0 & 0 \\ 0 & 0 & \frac{1-p}{4} & 0 \\ -\frac{p}{2} & 0 & 0 & \frac{1+p}{4} \end{pmatrix} \tag{6}$$

Problem 2 continued on next page...

- For PPT criterion, notice that $(\rho^{T_A})_{ij;kl} = \rho_{kj;il}$, thus the partial transpose is as

$$\rho^{T_A} = \begin{pmatrix} \frac{1+p}{4} & 0 & 0 & 0 \\ 0 & \frac{1-p}{4} & -\frac{p}{2} & 0 \\ 0 & -\frac{p}{2} & \frac{1-p}{4} & 0 \\ 0 & 0 & 0 & \frac{1+p}{4} \end{pmatrix} \tag{7}$$

The eigenvalues of ρ^{T_A} are $\left\{\frac{1}{4}(1-3p), \frac{p+1}{4}, \frac{p+1}{4}, \frac{p+1}{4}\right\}$. If ρ is entangled, ρ^{T_A} has negative eigenvalue, we thus have $\frac{1}{4}(1-3p) < 0$ this implies that $1 \geq p > \frac{1}{3}$. Notice that PPT criterion sufficient and necessary for two-qubit case, the entangled range for ρ_p is thus sufficient and necessary.

- For realignment criterion, notice that the realigned density matrix is defined as $\tilde{\rho}_{ij;kl} = \rho_{ik;jl}$, thus we have

$$\tilde{\rho} = \begin{pmatrix} \frac{1+p}{4} & 0 & 0 & \frac{1-p}{4} \\ 0 & -\frac{p}{2} & 0 & 0 \\ 0 & 0 & -\frac{p}{2} & 0 \\ \frac{1-p}{4} & 0 & 0 & \frac{1+p}{4} \end{pmatrix} \tag{8}$$

The singular values of $\tilde{\rho}$ are $\left\{\frac{1}{2}, \frac{p}{2}, \frac{p}{2}, \frac{p}{2}\right\}$, then $\|\tilde{\rho}\| = \sum_i \lambda_i = 1/2 + 3p/2 = \frac{3p+1}{2}$. If ρ is entangled, we must have $\|\tilde{\rho}\| > 1$, which implies that $1 \geq p > \frac{1}{3}$.

We see that the result of two criteria match well. ■

Problem 3 (1) Calculate the amount of entanglement of the state $\rho = \lambda \left|\phi^+\right\rangle \left\langle\phi^+\right| + (1-\lambda) \left|\psi^+\right\rangle \left\langle\psi^+\right|, (0 \leq \lambda \leq 1)$ with negativity measure, where $\left|\phi^-\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$, $\left|\psi^+\right\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

(2) Derive the value scope for λ when the state ρ is entangled using negativity measure.

Solution:

(1) The corresponding density matrix is of the form

$$\rho = \frac{1}{2}\begin{pmatrix} \lambda & 0 & 0 & -\lambda \\ 0 & 1-\lambda & 1-\lambda & 0 \\ 0 & 1-\lambda & 1-\lambda & 0 \\ -\lambda & 0 & 0 & \lambda \end{pmatrix} \tag{9}$$

The partial transpose is of the form

$$\rho^{T_A} = \frac{1}{2}\begin{pmatrix} \lambda & 0 & 0 & 1-\lambda \\ 0 & 1-\lambda & -\lambda & 0 \\ 0 & -\lambda & 1-\lambda & 0 \\ 1-\lambda & 0 & 0 & \lambda \end{pmatrix} \tag{10}$$

Recall that the negativity of ρ is defined

$$N(\rho) = \frac{\|\rho^{T_A}\|_1 - 1}{2} \tag{11}$$

where $\|\rho^{T_A}\|_1$ is the 1-norm, namely the sum of all singular values of ρ.

Since ρ is Herimitian square matrix, the singular value is just the absolute value of the eigenvalues. The eigenvalues of ρ^{T_A} are $\left\{\frac{1}{2}, \frac{1}{2}, \frac{2\lambda-1}{2}, \frac{1-2\lambda}{2}\right\}$, thus the singular values are $\left\{\frac{1}{2}, \frac{1}{2}, \left|\frac{2\lambda-1}{2}\right|, \left|\frac{1-2\lambda}{2}\right|\right\}$, this further implies that $\|\rho^{T_A}\|_1 = 1 + \left|\frac{2\lambda-1}{2}\right| + \left|\frac{1-2\lambda}{2}\right|$. Thus the amount of entanglement of ρ is:

$$
\begin{aligned}
N(\rho) &= \frac{\left\|\rho^{T_A}\right\| - 1}{2} = (\left|\frac{2\lambda-1}{2}\right| + \left|\frac{1-2\lambda}{2}\right|)/2 \\
&= \begin{cases} \lambda - \frac{1}{2}, & \lambda > \frac{1}{2} \\ 0, & \lambda = \frac{1}{2} \\ \frac{1}{2} - \lambda, & \lambda < \frac{1}{2} \end{cases}
\end{aligned} \tag{12}
$$

(2) The state is entangled if $N(\rho) > 0$, from the derivation of (1) we see that if $\lambda \neq 1/2$, the state is entangled, thus the entanglement range is $[0, 1/2) \cup (1/2, 1]$. ■

Problem 4 (1) Describe the definition of the Entanglement Witness (EW).

(2) For the three-qubit GHZ state,

$$
|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)
$$

prove that the entanglement witness $\mathcal{W} = \frac{1}{2}\mathbf{I} - |GHZ\rangle\langle GHZ|$ detects three-qubit entanglement around it.

(3) A mixed state $\rho = p\frac{\mathbf{I}}{8} + (1-p)|GHZ\rangle\langle GHZ|(0 \leq p \leq 1)$, calculate the p 's upper bound when ρ is entangled state using the EW given above.

Solution:

(1) The entanglement witness is the observables which can be used to distinguish the entangled state from separable states. By definition, the operator W is called an entanglement witness, if it satisfies

- For any separable state ρ_{AB}, $\mathrm{Tr}\left(W\rho_{AB}\right) \geq 0$.
- For at least one entangled state ρ^e_{AB}, $\mathrm{Tr}(W\rho^e_{AB}) < 0$. This condition is equivalent to say that W has at least one negative eigenvalue.

(2) To prove that $\mathcal{W}$ is an entanglement witness, we need to show that $\mathcal{W}$ satisfy the conditions for an entanglement witness.

- We need to show that for $\mathcal{W} = \frac{1}{2}\mathbf{I} - |GHZ\rangle\langle GHZ|$, $\mathrm{Tr}\left(\rho_{\text{sep}}\mathcal{W}\right) \geq 0$ for all separable states ρ_{sep}. This is equivalent to that, for all separable states, $\mathrm{Tr}\left(\rho_{\text{sep}}|GHZ\rangle\langle GHZ|\right) \leq \frac{1}{2}$. Let's try to calculate the maximum value of $\mathrm{Tr}\left(\rho_{\text{sep}}|GHZ\rangle\langle GHZ|\right) = \langle GHZ|\rho_{\text{sep}}|GHZ\rangle$. By definition, three particle separable states is probabilistic mixture of three particle product states $|\psi_i\rangle_A|\phi_i\rangle_B|\chi_i\rangle_C$, i.e.,

$$
\rho_{\text{sep}} = \sum_i p_i(|\psi_i\rangle\langle\psi_i|)_A \otimes (|\phi_i\rangle\langle\phi_i|)_B \otimes (|\chi_i\rangle\langle\chi_i|)_C. \tag{13}
$$

  The means that the maximum value can be taken over all product state

$$
\max_{\rho_{\text{sep}}} \left\{\langle GHZ|\rho_{\text{sep}}|GHZ\rangle\right\} = \max_{|\Psi_{\text{prod}}\rangle = |\psi\rangle|\phi\rangle|\chi\rangle} \left\{|\langle GHZ||\Psi_{\text{prod}}\rangle|^2\right\} \tag{14}
$$

  This implies that

$$
\max_{\rho_{sep}} \mathrm{Tr}\left(\rho_{\text{sep}}|GHZ\rangle\langle GHZ|\right) = 1/2. \tag{15}
$$

  Thus $\min_{\rho_{\text{sep}}} \mathrm{Tr}\left(\rho_{\text{sep}}\mathcal{W}\right) = 0$, $\mathrm{Tr}\left(\rho_{\text{sep}}\mathcal{W}\right) \geq 0$ for all separable states ρ_{sep}, we complete the proof.

- We can choose the entangled state as $\rho_{GHZ} = |GHZ\rangle\langle GHZ|$, it's easily checked that $\mathrm{Tr}(\mathcal{W}\rho_{GHZ}) = -1/2 < 0$, thus it detects entangled states around GHZ state. The second condition is satisfied.

Problem 4 continued on next page. . .

(3) ρ is an entangled state, then

$$\mathrm{Tr}(\rho\mathcal{W}) = \frac{p}{8}\mathrm{Tr}(\mathcal{W}) + (1-p)\langle GHZ|\mathcal{W}|GHZ\rangle = \frac{3p}{8} - \frac{1-p}{2} < 0 \tag{16}$$

this implies that $p < \frac{4}{7}$. ■

Problem 5 (1) What conditions should a good s meet?
(2) Describe the definition of distillable entanglement and entanglement cost and their relationship.
(3) Write down the monogamy of entanglement and describe its physical meanings.
Solution:
(1) A good entanglement measure $E(\cdot)$ should satisfy that,

- For any separable state ρ, there is no entanglement, thus we must have $E(\rho) = 0$;
- Monotonicity under local operation and classical communication (LOCC) operation: no increase under LOCC operations, namely $E\left(\Lambda_{LOCC}(\rho)\right) \leq E(\rho)$;
- Continuity: mathematically, E is continuous, i.e. $E(\rho) - E(\sigma) \to 0$, when $\|\rho - \sigma\| \to 0$;
- Convexity: mathematically, E is conves function, i.e. $E(\lambda\rho + (1-\lambda)\sigma) \leq \lambda E(\rho) + (1-\lambda)E(\sigma)$;
- Normalization, i.e. $E\left(P_+^d\right) = \log d$.

(2) Their definitions are as follows,

- Distillable entanglement $E_D(\rho)$ of ρ is the supremum of achievable conversion rates for converting the state ρ to the maximally entangled states asymptotically. Mathematically, it is defined as

$$E_D(\rho) := \sup\left\{r : \lim_{n\to\infty}\left[\inf_{\Psi} D(\Psi(\rho^{\otimes n}), \Phi\left(2^{rn}\right)\right] = 0\right\} \tag{17}$$

where $D(\cdot,\cdot)$ is trace distance, Ψ is a general trace preserving LOCC operation, and $\Phi(K)$ is the density operator corresponding to the maximallly entangled state in K dimensions,

$$\Phi(K) = \left|\phi_K^+\right\rangle\left\langle\phi_K^+\right|. \tag{18}$$

- Entanglement cost $E_C(\rho)$ of ρ is the infimum of achievable conversion rates for converting maximally entangled states into state ρ asymptotically. Mathematically, it's defined as

$$E_C(\rho) = \inf\left\{r : \lim_{n\to\infty}\left[\inf_{\Psi} D\left(\rho^{\otimes n}, \Psi\left(\Phi\left(2^{rn}\right)\right)\right)\right] = 0\right\} \tag{19}$$

where $D(\cdot,\cdot)$ is trace distance, Ψ is a general trace preserving LOCC operation, and $\Phi(K)$ is the density operator corresponding to the maximallly entangled state in K dimensions,

$$\Phi(K) = \left|\phi_K^+\right\rangle\left\langle\phi_K^+\right|. \tag{20}$$

For pure state $\rho = |\psi\rangle\langle\psi|$, we know that entanglement cost and distillable entanglement are the same $E_C(|\psi\rangle) = E_D(|\psi\rangle)$.

(3) Monogamy relation of entanglement states that: for any tripartite state of systems A, B_1, B_2 and entanglement measure $E(\cdot)$, we have

$$E\left(A : B_1\right) + E\left(A : B_2\right) \leq E\left(A : B_1B_2\right) \tag{21}$$

Problem 5 continued on next page. . .

This means that if A and B are maximally entangled, then A and C must not be entangled.

If the above tripartite monogamy relation holds in general, i.e. not only for qubits, then it can be immediately generalized by induction to the multipartite case:

$$E\left(A:B_1\right)+E\left(A:B_2\right)+\cdots+E\left(A:B_N\right)\leq E\left(A:B_1B_2\cdots B_N\right). \tag{22}$$

Squashed-entanglement satisfy the entanglement monogamy relation.

$$E_{\mathrm{sq}}(A:B)+E_{\mathrm{sq}}(A:C)\leqslant E_{\mathrm{sq}}(A:BC). \tag{23}$$

With the same spirit, there are also some other kind of expressions of the the monogamy relation of entanglement. For instance, consider concurrence (we know that $\mathcal{C}(\rho)=1$ means the ρ is maximally entangled and $\mathcal{C}(\rho)=0$ means the ρ is not entangled), consider Coffman-Kundu-Wootters monogamy inequality is as

$$\mathcal{C}_{AB}^2+\mathcal{C}_{AC}^2\leq\mathcal{C}_{A:(BC)}^2 \tag{24}$$

This means that when A and B are maximally entangled, then $\mathcal{C}_{AB}^2=0$, the relation implies that $\mathcal{C}_{AC}^2=0$, i.e., A and C must not be entangled. For negativity, similar result holds.

Problem 6 The four Bell states have the following mathematical expressions on the basis {0,1} (the eigenstates of σ_z),

$$\left|\Phi^{\pm}\right\rangle=\frac{1}{\sqrt{2}}(|00\rangle\pm|11\rangle)$$
$$\left|\Psi^{\pm}\right\rangle=\frac{1}{\sqrt{2}}(|01\rangle\pm|10\rangle)$$

(1) Prove that the four Bell states can be transformed to each other using single qubit rotations $\{I,\sigma_x,\sigma_y,\sigma_z\}$.

(2) Give the representation of the four Bell states on the basis $\{+,-\}$ (the eigenstates of σ_x).

Solution:

(1) Recall that

$$\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix},\sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{25}$$

Then it's easily checked that

$$\Phi^+\begin{cases}\xrightarrow{\sigma_x\otimes I}\left|\Psi^+\right\rangle\\\xrightarrow{\sigma_y\otimes I}-i\left|\Psi^-\right\rangle\\\xrightarrow{\sigma_z\otimes I}\left|\Phi^-\right\rangle\end{cases}\qquad\Phi^-\begin{cases}\xrightarrow{\sigma_x\otimes I}-\left|\Psi^-\right\rangle\\\xrightarrow{\sigma_y\otimes I}i\left|\Psi^+\right\rangle\\\xrightarrow{\sigma_z\otimes I}\left|\Phi^+\right\rangle\end{cases} \tag{26}$$

$$\Psi^+\begin{cases}\xrightarrow{\sigma_x\otimes I}\left|\Phi^+\right\rangle\\\xrightarrow{\sigma_y\otimes I}-i\left|\Phi^-\right\rangle\\\xrightarrow{\sigma_z\otimes I}\left|\Psi^-\right\rangle\end{cases}\qquad\Psi^-\begin{cases}\xrightarrow{\sigma_x\otimes I}-\left|\Phi^-\right\rangle\\\xrightarrow{\sigma_y\otimes I}i\left|\Phi^+\right\rangle\\\xrightarrow{\sigma_z\otimes I}\left|\Psi^+\right\rangle\end{cases} \tag{27}$$

(2) Notice that we can directly replace

$$|0\rangle=\frac{1}{\sqrt{2}}(|+\rangle+|-\rangle),\quad|1\rangle=\frac{1}{\sqrt{2}}(|+\rangle-|-\rangle). \tag{28}$$

Then we obtain

$$\begin{cases}\left|\Phi^+\right\rangle\to\left|\tilde{\Phi}^+\right\rangle=\frac{1}{\sqrt{2}}(|++\rangle+|--\rangle)\\\left|\Phi^-\right\rangle\to\left|\tilde{\Psi}^+\right\rangle=\frac{1}{\sqrt{2}}(|+-\rangle+|-+\rangle)\\\left|\Psi^+\right\rangle\to\left|\tilde{\Phi}^-\right\rangle=\frac{1}{\sqrt{2}}(|++\rangle-|--\rangle)\\\left|\Psi^-\right\rangle\to-\left|\tilde{\Psi}^-\right\rangle=-\frac{1}{\sqrt{2}}(|+-\rangle-|-+\rangle)\end{cases} \tag{29}$$

This completes the problem. ■

Problem 7 (1) Describe the physical meanings of von Neumann entropy.

(2) Prove that $S(\rho) \leq \log D$, where D is the number of the non-zero eigenvalues of ρ

(3) Prove the subadditivity of the von Neumann entropy

$$|S(A) - S(B)| \leq S(A, B) \leq S(A) + S(B)$$

(4) Prove the concavity of the von Neumann entropy

$$S\left(\sum_i p_i \rho_i\right) \geq \sum_i p_i S\left(\rho_i\right)$$

(5) Prove that the two body pure state $\left|\psi_{AB}\right\rangle$ is a entangled state if and only if $S(B \mid A) < 0$, in which $S(B \mid A) = S(B, A) - S(A), S(\cdot)$ is the von Neumann entropy.

Solution:

(1) Like its classical counterpart Shannon entropy, the von Neumann entropy $S(\rho)$ quantizes the quantum information of each character of the quantum source characterized by a quantum state ρ.

(2) The proof is completely the same as the proof for Shannon entropy by diagonalizing the density matrix.

$$S(\rho) = -\operatorname{tr}(\rho \log \rho) = -\sum_i p_i \log p_i = \sum_{i=1}^{D} p_i \log \frac{1}{p_i} \leq \log\left(\sum_{i=1}^{D} p_i \frac{1}{p_i}\right). \tag{30}$$

Where in the last step, we need to use the concavity of logarithmic function $\log(x)$, we regard $x_i = 1/p_i$, then

$$\log\left(\sum_i p x_i\right) \geq \sum_i p_i \log x_i. \tag{31}$$

Thus we complete the proof.

(3) We first prove that $S(A, B) \leq S(A) + S(B)$. Recall that relative entropy between arbitrary two quantum states are nonnegative. Consider the relative entropy of ρ_{AB} and $\rho_A \otimes \rho_B$

$$\begin{aligned} S\left(\rho_{AB} \| \rho_A \otimes \rho_B\right) &= \operatorname{tr}\left(\rho_{AB} \log \rho_{AB}\right) - \operatorname{tr}\Big(\rho_{AB} \log\left(\rho_A \otimes \rho_B\right)\Big) \\ &= -S\left(\rho_{AB}\right) - \operatorname{tr}\left(\rho_{AB} \log \rho_A\right) - \operatorname{tr}\left(\rho_{AB} \log \rho_B\right) \\ &= -S\left(\rho_{AB}\right) + S\left(\rho_A\right) + S\left(\rho_B\right) \\ &\geq 0 \end{aligned} \tag{32}$$

This directly implies that $S(A, B) \leq S(A) + S(B)$

For $|S(A) - S(B)| \leq S(A, B)$, consider a purification of $\rho_{AB} = tr_C |\phi\rangle_{ABC}\langle\phi|$, apply subadditivity to ρ_{BC}, we can get that

$$S(B, C) \leq S(B) + S(C) \tag{33}$$

since $S(B, C) = S(A), S(C) = S(A, B)$, so we get that

$$S(A, B) \geq S(A) - S(B) \tag{34}$$

Similarly, $S(A, B) \geq S(B) - S(A)$. To summarize, we have

$$|S(A) - S(B)| \leq S(A, B). \tag{35}$$

This completes the proof.

(4) By Applying subadditivity of von Neumann entropy to the state

$$\rho_{AB} = \sum_i p_i \rho_i \otimes |i\rangle\left\langle i\right|_B \tag{36}$$

where $|i\rangle$ is a set of orthonormal states for B, we obtain that

$$S\left(\rho_{AB}\right) \leq S\left(\rho_A\right) + S\left(\rho_B\right) = S\left(\sum_i p_i \rho_i\right) + H\left(p_i\right) \tag{37}$$

By invoking joint entropy theorem which states that $S(\sum_i p_i \rho_i \otimes |i\rangle\langle i|) = H(p_i) + \sum_i p_i S(\rho_i)$, we obtain that

$$S\left(\rho_{AB}\right) = S\left(\sum_i \rho_i \otimes p_i |i\rangle\langle i|_B\right) = \sum_i p_i S(\rho_i) + H\left(p_i\right). \tag{38}$$

Using this and equation (37) we obtain

$$S\left(\sum_i p_i \rho_i\right) \geq \sum_i p_i S\left(\rho_i\right). \tag{39}$$

(5) Since $\left|\psi_{AB}\right\rangle$ is a pure state, we known that $S(A,B) = 0$. If $\left|\psi_{AB}\right\rangle$ is an entangled state, taking its Schmidt decomposition, we have

$$\left|\psi_{AB}\right\rangle = \sum_i \sqrt{p_i}\left|i_A\right\rangle\left|i_B\right\rangle, \tag{40}$$

the number of nozero p_i is equal or great than 2. From the Schmidt decomposition we have

$$\rho_A = \sum_i p_i \left|i_A\right\rangle\left\langle i_A\right|, \quad S(A) \quad = -\sum_i p_i \log p_i > 0 \tag{41}$$

Thus we have

$$S(B \mid A) = S(A,B) - S(A) = -S(A) < 0. \tag{42}$$

The completes the problem. ■

Problem 8 Prove that $\left|\psi^-\right\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ is invariant under transformation $U(\theta,\vec{n}) \otimes U(\theta,\vec{n})$ where $U(\theta,\vec{n}) = e^{-\frac{i}{2}\theta\vec{n}\cdot\vec{\sigma}}$

Solution: Recall that

$$U(\theta,\vec{n}) = e^{-\frac{i}{2}\theta\cdot\vec{n}\cdot\vec{\sigma}} = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}\vec{n}\cdot\vec{\sigma} \tag{43}$$

from which we have

$$U(\theta,\vec{n}) \otimes U(\theta,\vec{n}) = \cos^2\frac{\theta}{2}I \otimes I - i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\left(n\cdot\vec{\sigma}_B + n\cdot\vec{\sigma}_A\right) - \sin^2\frac{\theta}{2}(\vec{n}\cdot\vec{\sigma})_A \otimes (\vec{n}\cdot\vec{\sigma})_B. \tag{44}$$

We have

$$(\vec{n}\cdot\vec{\sigma})_A \otimes (\vec{n}\cdot\vec{\sigma})_B \left|\psi^-\right\rangle = -\left|\psi^-\right\rangle. \tag{45}$$

And notice that

$$\cos^2\frac{\theta}{2}I \otimes I\left|\psi^-\right\rangle = \cos^2\frac{\theta}{2}\left|\psi^-\right\rangle, \tag{46}$$

$$(\vec{n}\cdot\vec{\sigma}_A + \vec{n}\cdot\vec{\sigma}_B)\left|\psi^-\right\rangle = 0 \tag{47}$$

We have

$$U(\theta,\vec{n}) \otimes U(\theta,\vec{n})\left|\psi^-\right\rangle = \left|\psi^-\right\rangle. \tag{48}$$

We thus completes the proof. ■

Problem 9 The entropy of quantum state, expressed as a density matrix ρ, is $S(\rho) = -\operatorname{tr}(\rho \log_2 \rho)$; in terms of its eigenvalues λ_k, this is $S(\rho) = -\Sigma_k \lambda_k \log_2 \lambda_k$. A state ρ is a pure state if and only if $\operatorname{tr}\left(\rho^2\right) = 1$. Prove that this is equivalent to $S(\rho) = 0$. You may use the fact ρ is a valid density matrix if and only if $\operatorname{tr}(\rho) = 1$ and ρ is a positive operator (i.e. its eigenvalues are ≥ 0).

Solution: Let's first prove that: ρ pure $\Leftrightarrow \operatorname{Tr}\left(\rho^2\right) = 1 \Rightarrow S(\rho) = 0$. If $\operatorname{Tr}\left(\rho^2\right) = 1$

$$\Sigma_k \lambda_k^2 = \Sigma_k \lambda_k = 1 \tag{49}$$

But since $0 \leq \lambda_k \leq 1, \forall k$, $\lambda_k^2 \leq \lambda_k$, the only way the saturate the equality is that there exist one k such that $\lambda_k = 1$. This implies that ρ only has one nonzero eigenvalue, i.e., $\rho = |\psi\rangle\langle\psi|$ for some $|\psi\rangle$, it's a pure state. Thus $S(\rho) = 0$.

For the other direction, $S(\rho) = 0 \Rightarrow \operatorname{Tr}\left(\rho^2\right) = 1 \Leftrightarrow \rho$ pure. Consider

$$S(\rho) = -\Sigma_k \lambda_k \log_2 \lambda_k = 0 \tag{50}$$

since $0 \leq \lambda_k \leq 1, \forall k$, we know that $\lambda_k \log_2 \lambda_k \leq 0$, for all k, Therefore, the only way for the above condition to be satisfied is for $\lambda_k = 0, 1$, for all k, this implies that $\operatorname{tr}\left(\rho^2\right) = 1$, i.e., ρ is a pure state.

To summarize, the following three statements are equivalent: (i) ρ pure; (ii) $\operatorname{Tr}\left(\rho^2\right) = 1$; (iii) $S(\rho) = 0$. ■

Problem 10 For the 2-qubit state

$$\rho = p\left|\Psi^-\right\rangle\left\langle\Psi^-\right| + (1-p)\frac{I}{4},$$

where $0 \leq p \leq 1$, $\left|\Psi^-\right\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$ calculate the EOF (Entanglement of Formation) of ρ.

Solution:

This is the 2-qubit Werner state, to calculate the entanglement of formation $E_F(\rho)$, we can first calculate the concurrence $\mathcal{C}(\rho)$ and then use the famous relation between concurrence and entanglement of formation

$$E_F(\rho) = H\left(\frac{1 + \sqrt{1 - \mathcal{C}^2(\rho)}}{2}\right), \tag{51}$$

here $H(x) = -x \log x - (1-x) \log x$ is the Shannon entropy function.

The density matrix of ρ in the ordered basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ is

$$\rho = \begin{pmatrix} \frac{1-p}{4} & 0 & 0 & 0 \\ 0 & \frac{1+p}{4} & \frac{-p}{2} & 0 \\ 0 & \frac{-p}{2} & \frac{1+p}{4} & 0 \\ 0 & 0 & 0 & \frac{1-p}{4} \end{pmatrix} \tag{52}$$

And since

$$\sigma_y \otimes \sigma_y = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{53}$$

then from definition that $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$, we see that $\tilde{\rho} = \rho$. The square roots of eigenvalues for $\rho\tilde{\rho} = \rho^2$ are nothing but the eigenvalue of ρ. Four eigenvalues are (in decreasing order)

$$\lambda_4 = \frac{1+3p}{4}, \lambda_3 = \lambda_2 = \lambda_1 = \frac{1-p}{4}. \tag{54}$$

Then by definition, the concurrence is

$$\mathcal{C}(\rho) = \max\{0, \lambda_4 - \lambda_3 - \lambda_2 - \lambda_1\}. \tag{55}$$

we see that

$$\begin{cases} \mathcal{C}(\rho) = 0, & \text{for } \quad 0 \le p \le 1/3 \\ \mathcal{C}(\rho) = \frac{3p-1}{2}, & \text{for } 1/3 < p \le 1 \end{cases} \tag{56}$$

From this, we see that

$$\begin{cases} H(\rho) = 0, & \text{for } \quad 0 \le p \le 1/3 \\ H(\rho) = H(\frac{1+\sqrt{1-(\frac{3p-1}{2})^2}}{2}) & \text{for } 1/3 < p \le 1 \end{cases} \tag{57}$$

Thus we complete the calculation. ■

Problem 11 Consider the state $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right), \rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$. Calculate the Von Neumann entropy of ρ_A.

Solution: By direct calculation

$$\rho_A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \tag{58}$$

this implies that

$$S(\rho_A) = -\left(\frac{1}{2}\log\left(\frac{1}{2}\right) + \frac{1}{2}\log\left(\frac{1}{2}\right)\right) = 1. \tag{59}$$

The maximally mixed state has the maximal entropy. ■

Problem 12 Give a noisy entanglement state with purity F for the singlet state $|\Psi^-\rangle$,

$$W_F = F|\Psi^-\rangle\langle\Psi^-| + \frac{1-F}{3}|\Psi^+\rangle\langle\Psi^+| + \frac{1-F}{3}|\Phi^+\rangle\langle\Phi^+| + \frac{1-F}{3}|\Phi^-\rangle\langle\Phi^-|$$

Supposing $F = \frac{3}{5}$, please design a two-way LOCC purification protocol that can obtain the singlet state $|\Psi^-\rangle$ with as high fidelity as possible from the above mixed state in five steps.

Solution:

The solution here I give is referred to the paper "*Purification of Noisy Entanglement and Faithful Teleportation via Noisy Channels*" by C. Bennett *et al.*

Notice that Bell diagonal state (also called symmetric Werner state by some author)

$$W_F = F|\Psi^-\rangle\langle\Psi^-| + \frac{1-F}{3}|\Psi^+\rangle\langle\Psi^+| + \frac{1-F}{3}|\Phi^+\rangle\langle\Phi^+| + \frac{1-F}{3}|\Phi^-\rangle\langle\Phi^-|. \tag{60}$$

has the fidelity with $|\Psi^-\rangle$ as $F(|\Psi^-\rangle, W_F) = F$. Alice and Bob share two pairs of W_F states respectively, i.e. W_F^{12} and W_F^{34}, with 1 and 3 in Alice's side, 2 and 4 in Bob's side. Then the two-way purification protocol works as follows:

- Alice and Bob make unilateral transformation σ_y, namely $\sigma_y \otimes I$ for Alice and $I \otimes \sigma_y$ for Bob, on their two pairs of W_F states. Notice that $\sigma_y|0\rangle = i|1\rangle$ and $\sigma_y|1\rangle = -i|0\rangle$. Thus under $\sigma_y \otimes I$ we have

  $$|\Psi^-\rangle \to -i|\Phi^+\rangle, \quad |\Psi^+\rangle \to -i|\Phi^-\rangle, \quad |\Phi^-\rangle \to i|\Psi^+\rangle, \quad |\Phi^+\rangle \to -i|\Psi^-\rangle. \tag{61}$$

  Similarly for $I \otimes \sigma_y$ (with only minus sign difference). This means that after the unilateral transformation, the state becomes

  $$W_F \xrightarrow{\sigma_y \otimes I} W_F' = F|\Phi^+\rangle\langle\Phi^+| + \frac{1-F}{3}|\Phi^-\rangle\langle\Phi^-| + \frac{1-F}{3}|\Psi^-\rangle\langle\Psi^-| + \frac{1-F}{3}|\Psi^+\rangle\langle\Psi^+|. \tag{62}$$

Problem 12 continued on next page...

- Alice and Bob perform the bilateral controlled-NOT operations CNOT on their two pairs of W_F' states with 1 and 2 as 'source' particles and 3 and 4 as 'target' particles. For clarity, we explain here what we mean by bilateral controlled-NOT operations: CNOT operation is done for 3 conditioned to 1 and CNOT operation is done for 4 conditioned to 2. More precisely, we have

$$\mathsf{CNOT}_{13} \otimes \mathsf{CNOT}_{24}(W_F' \otimes W_F'). \tag{63}$$

Recall that

$$\mathsf{CNOT}|00\rangle = |00\rangle, \quad \mathsf{CNOT}|01\rangle = |01\rangle, \quad \mathsf{CNOT}|10\rangle = |11\rangle, \quad \mathsf{CNOT}|11\rangle = |10\rangle. \tag{64}$$

For Bell states $|\psi\rangle_{12} \otimes |\varphi\rangle_{34}$ with $|\psi\rangle_{12}$ source state and $|\varphi\rangle_{34}$ the target state. Then for the operation

$$\mathsf{CNOT}_{13} \otimes \mathsf{CNOT}_{24}(|\text{source}_{\text{in}}\rangle_{12} \otimes |\text{target}_{\text{in}}\rangle_{34}) = |\text{source}_{\text{out}}\rangle_{12} \otimes |\text{target}_{\text{out}}\rangle_{34} \tag{65}$$

we have the following result:

$\text{source}_{\text{in}}$	$\text{target}_{\text{in}}$	$\text{source}_{\text{out}}$	$\text{target}_{\text{out}}$
$\Phi^\pm$	Φ^+	$\Phi^\pm$	Φ^+
$\Psi^\pm$	Φ^+	$\Psi^\pm$	Ψ^+
$\Psi^\pm$	Ψ^+	$\Psi^\pm$	Φ^+
$\Phi^\pm$	Ψ^+	$\Phi^\pm$	Ψ^+
$\Phi^\pm$	Φ^-	$\Phi^\mp$	Φ^-
$\Psi^\pm$	Φ^-	$\Psi^\mp$	Ψ^-
$\Psi^\pm$	Ψ^-	$\Psi^\mp$	Φ^-
$\Phi^\pm$	Ψ^-	$\Phi^\mp$	Ψ^-

(66)

Based on this, they choose to measure the output state $|\text{target}_{\text{out}}\rangle_{34}$ of two target particles along the z-axis. If their z-spins are parallel, keep the correspond out source state; otherwise, discard the out source state.

Notice the measurements along the Z axis can only distinguish Φ from Ψ but can't distinguish $-$ from $+$), we can only kept all Φ target state or Ψ target state, here we choose keep the Φ target state, namely, we keep the 1,3,5,7 rows' source states. Notice that this implies that resulted source state is a mixed state of Bell states with their respective probabilities.

If we set $G = \frac{1-F}{3}$, we obtain that

$$\rho = \frac{1}{F^2 + 5G^2 + 2FG}\left((F^2 + G^2)|\Phi^+\rangle\langle\Phi^+| + 2FG|\Phi^-\rangle\langle\Phi^-| + 2G^2|\Psi^-\rangle\langle\Psi^-| + 2G^2|\Psi^+\rangle\langle\Psi^+|\right) \tag{67}$$

Notice that, now the fidelity between the resulting state and $|\Phi^+\rangle$ is

$$F' = \frac{F^2 + G^2}{F^2 + 5G^2 + 2FG} = \frac{F^2 + \frac{1}{9}(1-F)^2}{F^2 + \frac{2}{3}F(1-F) + \frac{5}{9}(1-F)^2}. \tag{68}$$

This is a recurrence relation when we repeat this step.

Notice that because $F'(F)$ is continuous and exceeds F over the entire range $\frac{1}{2} < F < 1$, iteration of the above step can make the fidelity between Φ^+ and the resulted state arbitrarily high.

- The last step is the implement unilateral σ_y operation to rotate Φ^+ into Ψ^-, i.e.,

$$\rho = F'|\Phi^+\rangle\langle\Phi^+| + \cdots \mapsto F'|\Psi^-\rangle\langle\Psi^-| + \cdots \tag{69}$$

Thus we can distill Bell diagonal states of arbitrarily high purity $F_{\text{out}} < 1$ from a supply of input mixed states W_F of any purity $F_{\text{in}} > \frac{1}{2}$, (here $F = 3/5 > 1/2$ satisfy this condition). ■