Problem 1 1. $U = |0\rangle\langle 0| \otimes I_2 + |1\rangle\langle 1| \otimes \sigma_x$ is a C-NOT gate, calculate the following in terms of $I_2, \sigma_x, \sigma_y, \sigma_z$
(1) $U(\sigma_x \otimes I_2)U^\dagger$;
(2) $U(\sigma_z \otimes I_2)U^\dagger$;
(3) $U(\sigma_x \otimes \sigma_x)U^\dagger$;
(4) $U(\sigma_z \otimes \sigma_z)U^\dagger$

Solution: By direct calculation, we have
(1) $\sigma_x \otimes \sigma_x$;
(2) $\sigma_z \otimes I_2$;
(3) $\sigma_x \otimes I_2$;
(4) $I_2 \otimes \sigma_z$.

Problem 2 Find a parity check matrix H for the [6,2] repetition code defined by the generator matrix G. Then verify that $HG = 0$.

$$G = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

Solution: This can be constructed as follows

$$H = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Problem 3 Please give a parity check matrix H for the [7,4] Hamming code, and write down its distance.

Solution: the parity check matrix is

$$H = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

The distance of the code is 3.

Problem 4 Please write down the difference between quantum error correction and classical error correction.

Solution: There are three main differences:

No cloning: One might try to implement the repetition code quantum mechanically by duplicating the quantum state three or more times. This is forbidden by the no-cloning theorem. Even if cloning were possible, it would not be possible to measure and compare the three quantum states output from the channel.

Errors are continuous: A continuum of different errors may occur on a single qubit. Determining which error occurred in order to correct it would appear to require infinite precision, and therefore infinite resources.

Measurement destroys quantum information: In classical error-correction we observe the output from the channel, and decide what decoding procedure to adopt. Observation in quantum mechanics generally destroys the quantum state under observation, and makes recovery impossible.

Problem 5 (1) Please write down the quantum error-correction condition.

(2) Consider the three qubit bit flip code, with corresponding projector $P = |000\rangle\langle 000| + |111\rangle\langle 111|$. The noise process that this code protects against has operation elements

$$\left\{ \sqrt{(1-p)^3} I, \sqrt{p(1-p)^2} X_1, \sqrt{p(1-p)^2} X_2, \sqrt{p(1-p)^2} X_3 \right\}$$

where p is the probability that one bit flips. Verify the quantum error-correction conditions for this code and noise process.

Solution: (1) For a quantum code with projector Π_C, the error $\mathcal{E}$ with Kraus operators $\{E_i\}$ can be corrected if and only if

$$\Pi_C E_i E_j^\dagger \Pi_C = \alpha_{ij} \Pi_C \tag{1}$$

where α_{ij} is a Hermitian matrix.

(2) The operation elements of the noise process is $\{E_0, E_1, E_2, E_3\}$, where $E_0 = \sqrt{(1-p)^3} I$ and $E_i = \sqrt{p(1-p)^2} X_i$ for $i = 1, 2, 3$. If $i \neq j, PX_i^\dagger X_j P = 0$, if $i = j, PX_i^\dagger X_j P = P^2 = P$. So $PE_i^\dagger E_j P = \alpha_{ij} P$, in which $\alpha_{00} = (1-p)^3, \alpha_{ii} = p(1-p)^2$ for $i = 1, 2, 3$ and $\alpha_{ij} = 0 (i \neq j)$.$\alpha$ is an Hermitian matrix, so the three qubit bit flip code and noise process satisfies the quantum error-correction conditions.

Problem 6 Consider the three qubit phase flip code, with corresponding projector $P = |+++\rangle\langle +++| + |---\rangle\langle ---|$. Verify that this code satisfies the quantum error- correction conditions for the set of error operators $\{I, Z_1, Z_2, Z_3\}$.

Solution: Since $PE_i^\dagger E_j P = \alpha_{ij} P$, in which $\alpha_{ii} = 1$ and $\alpha_{ij} = 0 (i \neq j)$.$\alpha$ is an Hermitian matrix.

Problem 7 (1) For 4-qubit GHZ state $|\psi\rangle = (|0011\rangle + |1100\rangle)/\sqrt{2}$, please write down its linearly independent stabilizers.

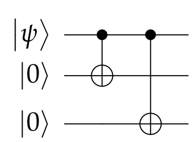
(2) For 4-qubit cluster state $|\psi\rangle = (|+\rangle|0\rangle|+\rangle|0\rangle + |+\rangle|0\rangle|-\rangle|1\rangle + |-\rangle|1\rangle|-\rangle|0\rangle + |-\rangle|1\rangle|+\rangle|1\rangle)/2$, please write down its linearly independent stabilizers.

Solution: (1) The stabilizers can be chosen as following: $g_1 = ZZII, g_2 = -IZZI, g_3 = IIZZ, g_4 = XXXX$.

(2) The stabilizers can be chosen as following: $g_1 = XZII, g_2 = ZXZI, g_3 = IZXZ, g_4 = IIZX$.

Problem 8 Please draw the quantum circuit of the 3 -qubit bit flip code, and certify that it can encode the qubit $a|0\rangle + b|1\rangle$ to $a|000\rangle + b|111\rangle$.

Solution: The quantum circuit is as follows



The encoding process is

$$\begin{aligned} &|\psi\rangle|0\rangle|0\rangle = (a|0\rangle + b|1\rangle)|0\rangle|0\rangle \\ &\longrightarrow (a|00\rangle + b|11\rangle)|0\rangle \\ &\longrightarrow (a|000\rangle + b|111\rangle) \end{aligned} \tag{2}$$

Problem 9 For 9 -qubit Shor code, its logical bit code is $|0\rangle_L = (|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)/2\sqrt{2}$, $|1\rangle_L = (|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)/2\sqrt{2}$.

(1) Please give all the generators of the stabilizers;

(2) Please draw the encoding quantum circuit;

Problem 9 continued on next page. . .

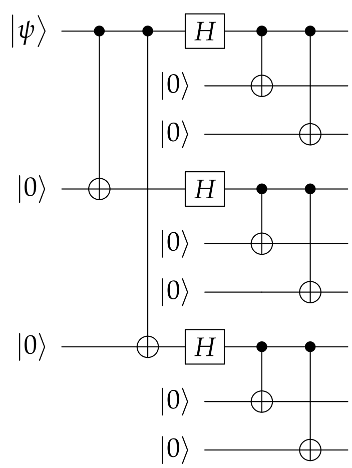
(3) For a bit/phase flip error of a certain bit, how to detect and correct it? Please take the bit flip error and phase flip error for example, write down the program of error detection and correction.

Solution:

(1) The stabilizers are as follows

$$\begin{aligned} g_1 &= ZZIIIIIII \\ g_2 &= IZZIIIIII \\ g_3 &= IIIZZIIII \\ g_4 &= IIIIZZIII \\ g_5 &= IIIIIIZZI \\ g_6 &= IIIIIIIZZ \\ g_7 &= XXXXXXIII \\ g_8 &= IIIXXXXXX \end{aligned} \tag{3}$$

(2) the encoding quantum circuit are as follows



(3) For the bit flip error, suppose that the error is in the first qubit. Then we choose to measure the stabilizer $g_1, \cdots, g_6$, from the outcomes we can infer the position where the bit flip error occurs, thus we can do σ_x to correct it.

For phase flip error, suppose that the error is in the first qubit, by measuring the stabilizers g_7, g_8, we can infer that the phase flip error is in the first block, thus we can do σ_z on arbitrary qubit in the first block to correct it.

Problem 10 Please write down the imaging algorithm of single-photon camera.

Solution:

Step 0: calibration

Calibrate $\mathbf{B}$ and also identify set of "hot pixels" $\mathcal{H}$.

Step 1: estimate scene reflectivity

Combine photon-count likelihood with spatial correlation to solve regularized ML estimation (convex).

$\underset{\mathbf{A}}{\text{minimize}} \left[\sum_{(i,j)\notin\mathcal{H}} \mathcal{L}\left(\mathbf{A}_{i,j}; \mathbf{C}_{i,j}, \mathbf{B}_{i,j}\right)\right] + \tau_A\, \text{pen}(\mathbf{A})$ subject to $\mathbf{A}_{i,j} \geq 0$.

Step 2: censor noise photons

Use OMP $(\mathbf{T}_k, N_k)$ to locate photon clusters and reject photon arrival times not near to them by pulsewidth

Step 3: estimate scene depth

Combine arrival-time likelihood with spatial correlation to solve regularized ML estimation (convex).

$\underset{\mathrm{D}}{\text{minimize}} \left[\sum_{(i,j)\notin\mathcal{H} T_{i,j}\in\mathbf{U}_{i,j}} \mathcal{L}\left(\mathbf{D}_{i,j}; T_{i,j}\right)\right] + \tau_D\, \text{pen}(\mathbf{D})$ subject to $\mathrm{D}_{i,j} \geq 0$.