Problem 1 Please write down a group of universial quantum logical gate set.

Solution:

We have the following universal quantum logical gate set, for example

- 45-degree rotation of Z gate T, Hadamard gate H, CNOT.
- Hadamard gate H, phase gate S, controlled phase gate $\Lambda(S)$
- Hadamard gate H, phase gate S, controlled controlled not gate $\Lambda^2(\sigma_x)$

Problem 2 Please write down the DiVincenzo criterion that quantum computer implementation must satisfy.

Solution:

- Scalability: A scalable physical system with well characterized parts, usually qubits.
- Initialization: The ability to initialize the system in a simple fiducial state.
- Control: The ability to control the state of the computer using sequences of elementary universal gates.
- Stability: Decoherence times much longer than gate times, together with the ability to suppress decoherence through error correction and fault-tolerant computation.
- Measurement:The ability to read out the state of the computer in a convenient product basis.

Problem 3 Please write down the matrix form of the C-NOT gate, and the state resulting from the application of this gate to four Bell states.

Solution:

This matrix for C-NOT gate is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Denote the four Bell states as

$$\left|\beta_{xy}\right\rangle = \frac{|0, y\rangle + (-1)^x |1, \bar{y}\rangle}{\sqrt{2}}$$

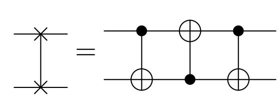
then

$$\begin{aligned} \mathrm{C-NOT}\left(\left|\beta_{00}\right\rangle\right) &= (|0\rangle + |1\rangle)|0\rangle/\sqrt{2} \\ \mathrm{C-NOT}\left(\left|\beta_{01}\right\rangle\right) &= (|0\rangle + |1\rangle)|1\rangle/\sqrt{2} \\ \mathrm{C-NOT}\left(\left|\beta_{10}\right\rangle\right) &= (|0\rangle - |1\rangle)|0\rangle/\sqrt{2} \\ \mathrm{C-NOT}\left(\left|\beta_{11}\right\rangle\right) &= (|0\rangle - |1\rangle)|1\rangle/\sqrt{2} \end{aligned}$$

Problem 4 Please construct the quantum SWAP gate to swap two qubits using the C-NOT gate.

Solution:

The construction is as follows



Problem 4 continued on next page. . .

To see that this circuit accomplishes the swap operation, note that the sequence of gates has the following sequence of effects on a computational basis state $|a,b\rangle$,

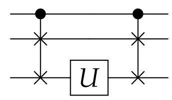
$$\begin{aligned}|a,b\rangle &\longrightarrow |a,a\oplus b\rangle\\ &\longrightarrow |a\oplus(a\oplus b),a\oplus b\rangle = |b,a\oplus b\rangle\\ &\longrightarrow |b,(a\oplus b)\oplus b\rangle = |b,a\rangle\end{aligned}$$

where all additions are done modulo 2 .

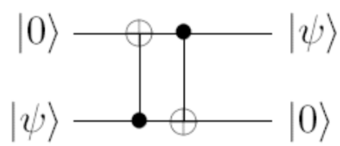
Problem 5 Suppose you are given a box which performs a unitary gate U on a one-qubit input state. In addition, you are given $|u\rangle$, and eigenstate of U with eigenvalue one ($U|u\rangle = |u\rangle$). Provide a quantum circuit which peforms a controlled-U gate(control qubit $|q\rangle$ and target qubit $|\psi\rangle$), using this box, $|u\rangle$ and quantum Fredkin (i.e. controlled-swap) gates.

Solution:

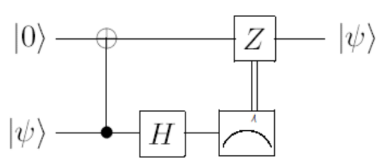
It's as follows



Problem 6 An unknown qubit in the state $|\psi\rangle$ can be swapped with a second qubit which is prepared in the state $|0\rangle$ using only two controlled-not gates, with the circuit



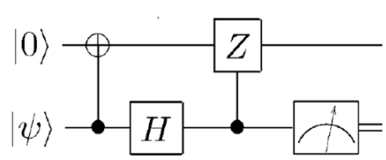
Show that the circuit below, which use only a single cnot gate, with measurement and a classically controlled single qubit operation, also accomplish the same task(use circuit equivalences):



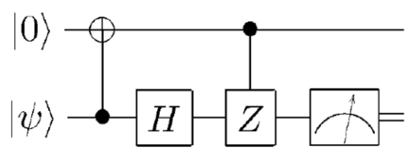
Solution:

The proof is as follows.

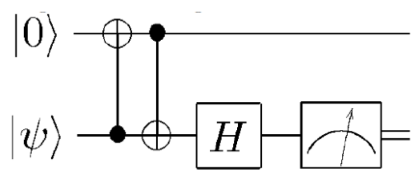
Classical control operation after measurement is equivalent to quantum control operation before measurement, hence the circuit is equivalent to



Controlled-Z operation is symmetric between control bit and target bit, hence the circuit is equivalent to



Commuting controlled-Z through Hadamard we get controlled-not

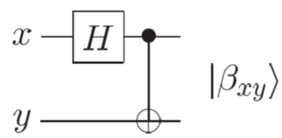


This completes the proof.

Problem 7 Please design a quantum circuit which converts the state $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ into four Bell states.

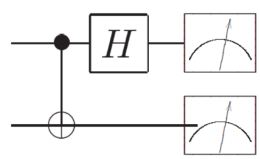
Solution:

The circuit is as follows



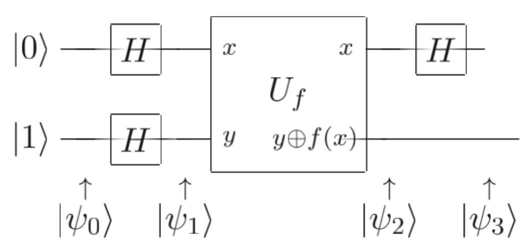
Problem 8 Please design a quantum circuit to perform full Bell state measurement, i.e. to distinguish four Bell states by projective measurement at $|0\rangle, |1\rangle$ basis.

Solution: The circuit is as follows



Problem 9 Please draw the quantum circuit of Deutsch algorithm, and analysis how it works.

Solution: The circuit is as follows



The algorithm works as follows

- The input state is $|\psi_0\rangle = |01\rangle$.
- After Hadamard gates, we obtain $|\psi_1\rangle = |+\rangle|-\rangle$.
- After U_f, we obtain

$$|\psi_2\rangle = \begin{cases} \pm\left[\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right], & f(0) = f(1) \\ \pm\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right], & f(0) \neq f(1). \end{cases} \tag{1}$$

- Then after Hadamard gates, we have

$$|\psi_3\rangle = \begin{cases} \pm|0\rangle\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right] & \text{if } f(0)=f(1) \\ \pm|1\rangle\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right] & \text{if } f(0)\neq f(1) \end{cases} \tag{2}$$

By measuring the first bit, we obtain the answer.

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