

# MATH1072 Notes

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- 1 Lecture 1 - Introduction to Dimensional Analysis**
- 2 Lecture 2 - Dimensional Analysis**
- 3 Lecture 3 - Introduction to Differential Equations**

Generally, an ordinary differential equation (ODE) is represented as:

$$F(t, y(t), y'(t), y''(t), \dots) = 0$$

For Instance, Newton's Law

$$m \frac{d^2 r}{dt^2} = F$$

Induction Law:

$$RI + L \frac{dI}{dt} + \frac{1}{c}$$

Population:

$$\frac{dP}{dt} = rP \left(1 - \frac{P}{k}\right)$$

Maxwell's Equations:

$$\nabla \cdot \bar{E} = \frac{1}{\rho} \varepsilon_0$$

$$\nabla \cdot \bar{B} = 0$$

$$\nabla \times \bar{E} = \frac{\delta B}{\delta t}$$

$$\nabla \times \bar{B} = \mu_0 J + \mu_0 \varepsilon_0 \frac{\delta E}{\delta t}$$

Navier-Stokes: (Modelling velocity of fluids in space)

$$\frac{\delta \bar{v}}{\delta t} = \bar{v} \nabla \bar{v} = ?$$

Schrodinger Wave Equation:

$$i\hbar \frac{\delta \psi}{\delta t} = \left[ \frac{-\hbar}{2m} \nabla^2 + V(r) \right] \psi$$

$$\psi = ?$$

We generally have:

$$F(t, y(t), y'(t), \dots) = 0, \quad y(t) = ?$$

Where the order of ODE = order of the highest derivative. Take the equation  $y' = y$ , the solutions are  $y = e^t$ ,  $y = 0$  and  $y = ce^t$ . However the latter expression encapsulates the former, so  $y(t) = ce^t$  is known as the general solution where  $c \in \mathbb{R}$ . The genreal solution is not unique.

If you take an ODE and add some initial condition, then the solution is a unique result. For Instance take the ODE  $y' = y$  and say that  $y(0) = 1$ , then we achieve a unique solution of  $c = 1$ ,  $y = e^t$ .

# 4 Lecture 4 - Ordinary Differential Equations

## 4.1 Equilibrium Solutions

An equilibrium solution (steady state solution) of an ODE (if such solution exists) is a constant solution  $y(t) = c$  which satisfies the ODE (for any  $t$ ).

**Example 1.** Take  $y' = f(t, y)$ , the equilibrium solution

$$f(t, y = c) = 0$$

This implies that the slope at  $y = c$  must be 0 for a function defined on a  $(t, y)$  plane

**Example 2.**  $y' = y$ ,  $y = 0$  is an equilibrium solution.

**Example 3.**  $y' = y(1 - y)$   $y = 0$  and  $y = 1$  are equilibrium solutions

Slope fields can visualised with Mathematica using SteamPlot (or VectorPlot). The vector flow of a field corresponding to  $f(t, y)$  is  $\{1, f(t, y)\}$  since the flow in the horizontal direction ( $t$ ) has a constant rate (the flow of time) which can be set to 1, the vertical flow is  $f(t, y)$ .

## 4.2 Stability of equilibrium solutions

If the solutions starting in condition a small neighbourhood of an equilibrium solution ( $y = c$ ) converge towards the equilibrium solution for large  $t$ , then the equilibrium solution  $y = c$  is stable.

**Example 4.**  $y' = y$ , the equilibrium solution  $y = 0$  is unstable (for  $y' = -y$ ,  $y = 0$  is stable)

**Example 5.**  $y' = y(1 - y)$ ,  $y = 0$  is unstable, but  $y = 1$  is stable.

## 5 Lecture 5 - Stability of ODE's

### 5.1 Condition for stability of equilibrium solutions

Using a Taylor Series approximation of  $f(t, y)$  near the equilibrium solution  $y = c$ , assume  $f(t, y) = f(y)$  for simplicity

$$f(y) \approx f(c) + f'(c)(y - c) + \frac{1}{2}f''(c)(y - c)^2 + \dots$$

Where  $y = c$  is the equilibrium solution  $f(c) = 0$ , thus

$$y'(t) = 0 + f'(c)(y - c) + \frac{1}{2}f''(c)(y - c) + \dots$$

Take  $u = y - c$ , then

$$u' = f'(c)u + \dots$$

Implied

$$u(t) = Ae^{f'(c)t}$$

If  $f'(c) > 0$ ,  $y = c$  is unstable. If  $f'(c) < 0$ ,  $u(t) \rightarrow 0$ ,  $y(t) \rightarrow c$  thus  $y = c$  is stable.

### 5.2 Euler's Method

An iterative operation which models  $y_k \approx y(k \cdot \Delta t)$ . Given  $y' = f(t, y)$ ,  $y(0) = c$

$$y'(t) \equiv \lim_{\Delta \rightarrow 0} \frac{y(t + \Delta) - y(t)}{\Delta} \quad (1)$$

$$\approx \frac{y(t + \Delta) - y(t)}{\Delta} \quad (2)$$

As it is an iterative method,  $y_{k+1} = y_k + f(t_k, y_k)$

$$\frac{y(t_{k+1}) - y(t_k)}{\Delta} \approx f(t_k, y_k)$$

**Example 6.** Given  $y' = 2t$ ,  $y(t) = ?$ ,  $y(0) = 0$ . By Euler's Method

$$N = 1, \Delta = 1 \quad (3)$$

$$N = 2, \Delta = \frac{1}{2} \quad (4)$$

$$N = 4, \Delta = \frac{1}{4} \quad (5)$$

$$N = 10, \Delta = \frac{1}{10} \quad (6)$$

## 6 Lecture 6 -

### 6.1 Error Generated in Euler's Method

Assume  $y(t_k) = y_k$ , the error is represented with the Taylor series approximation

$$\begin{aligned} \left| y_{k+1} - y(t_k + \Delta) \right| &= y(t_{k+1}) - y(t_k) - y'(t_k)\Delta - \frac{1}{2}y''(t_k)\Delta^2 - \dots \\ y_{k+1} &= y_k + f(t_k, y_k)\Delta \end{aligned}$$

Thus

$$\text{error} = |y_{k+1} - y(t_k + \Delta)| \propto \Delta^2$$

However generalized for  $N$  steps,

$$\begin{aligned} N \cdot |y_{k+1} - y(t_k + \Delta)| &\propto \Delta^2 \cdot N \\ &\propto \Delta^2 \cdot \frac{1}{\Delta} \propto \Delta \end{aligned}$$

$$y_{k+1} = y_k + \frac{1}{2} \left[ f(t_k, y_k) + f(t_{k+1}, y_{k+1}) \right] \Delta$$

## 7 Lecture 8

$$\begin{aligned} y' &= f(t, y), y(0) = y_0 \\ t &\in [0, t_{max}] \end{aligned}$$

### 1. Euler's Method

$$y_{k+1} = y_k + f(t_k, y_k) \Delta$$

Where total error  $\propto \Delta$

### 2. Heun Method (Revised Euler's Method)

$$\begin{cases} y_{k+1} = y_k + f(t_k, y_k) \Delta \\ y_{k+1} = y_k + \frac{1}{2} [f(t_k, y_k) + f(t_k + \Delta, y_{k+1})] \end{cases}$$

Where total error  $\propto \Delta^2$ . Note that Heun Method will possibly show in Assignment 3.

## 7.1 Runge Kutta Method

$$\begin{aligned} p_1 &= f(t_k, y_k) \Delta \\ p_2 &= f\left(t_k + \frac{\Delta}{2}, y_k + \frac{p_1}{2}\right) \Delta \\ p_3 &= f\left(t_k + \frac{\Delta}{2}, y_k + \frac{p_2}{2}\right) \Delta \\ p_4 &= f\left(t_k + \frac{\Delta}{2}, y_k + p_3\right) \Delta \end{aligned}$$

for

$$y_{k+1} = y_k + \frac{1}{6} p_1 + \frac{1}{3} p_2 + \frac{1}{3} p_3 + \frac{1}{6} p_4 + \dots$$

## 7.2 Adaptive Step Size

Fixed step size is mostly inefficient in most cases, so we use an adaptive step size for numerical methods to achieve better approximations.

## 7.3 Coupled Systems

$$y' = f(t, y_1, y_2)$$

$$x' = g(t, y_1, y_2)$$

$$x(t) = ? \quad y(t) = ?$$

In the context of Euler's Method

$$\begin{cases} x_{k+1} = x_k + f(t_k, x_k, y_k)\Delta \\ y_{k+1} = y_k + g(t_k, x_k, y_k)\Delta \end{cases}$$

## 7.4 Analytical ODE Solutions

### 7.4.1 Linear First Order ODE's

By definition,

$$y'(t) = f(t)y(t) + g(t)$$

is generally the standard form of a linear first order ODE. Note that

$$y'(t) = f(t)y(t)$$

is a special case of a linear first order ODE that is separable.

**Example 7.**

$$ty' + y = t \cos t$$

Observe that, by the product rule for differentiation,  $ty' + y = (ty)'$ .

$$(ty)' = t \cos t$$

$$\int (ty)' dy = \int t \cos t dt$$

$$ty = t \sin t - \int \sin t dt$$

$$ty = t \sin t + \cos t + c$$

$$\therefore y = \frac{t \sin t + \cos t}{t} + \frac{c}{t}$$