MATH1072 Notes

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1 Lecture 1 - Introduction to Dimensional Analysis

2 Lecture 2 - Dimensional Analysis

3 Lecture 3 - Introduction to Differential Equations

Generally, an ordinary differential equation (ODE) is represented as:

$$F(t, y(t), y'(t), y''(t), ...) = 0$$

For Instance, Newton's Law

$$m\frac{d^2r}{dt^2} = F$$

Induction Law:

$$RI + L\frac{dI}{dt} + \frac{1}{c}$$

Population:

$$\frac{dP}{dt} = rP(1 - \frac{P}{k})$$

Maxwell's Equations:

$$\begin{aligned}
\nabla \cdot \bar{E} &= \frac{1}{\rho}\varepsilon_0 \\
\nabla \cdot \bar{B} &= 0 \\
\nabla \times \bar{E} &= \frac{\delta B}{\delta t} \\
\nabla \times \bar{B} &= \mu_0 J + \mu_0 \varepsilon_0 \frac{\delta E}{\delta t}
\end{aligned}$$

Navier-Stokes: (Modelling velocity of fluids in space)

$$\frac{\delta \overline{v}}{\delta t} = \overline{v}\nabla\overline{v} = ?$$

Schrodinger Wave Equation:

$$i\hbar\frac{\delta\psi}{\delta t} = \Big[\frac{-\hbar}{2m}\nabla^2 + V(r)\Big]\psi$$

$$\psi = ?$$

We generally have:

$$F(t, y(t), y'(t), ...) = 0,\ y(t) = ?$$

Where the order of ODE = order of the higest derivative. Take the equation $y' = y$, the solutions are $y = e^t$, $y = 0$ and $y = ce^t$. However the latter expression encapsulates the former, so $y(t) = ce^t$ is known as the general solution where $c \in \mathbb{R}$. The genreal solution is not unique.

If you take an ODE and add some initial condition, then the solution is a unique result. For Instance take the ODE $y' = y$ and say that $y(0) = 1$, then we achieve a unique solution of $c = 1$, $y = e^t$.

4 Lecture 4 - Ordinary Differential Equations

4.1 Equilibrium Solutions

An equilibrium solution (steady state solution) of an ODE (if such solution exists) is a constant solution $y(t) = c$ which satisfies the ODE (for any t).

Example 1. *Take $y' = f(t, y)$, the equilibrium solution*

$$f(t, y = c) = 0$$

This implies that the slope at $y = c$ must be 0 for a function defined on a (t, y) plane

Example 2. *$y' = y$, $y = 0$ is an equilibrium solution.*

Example 3. *$y' = y(1 - y)m$ $y = 0$ and $y = 1$ are equilibrium solutions*

Slope fields can visualised with Mathematica using SteamPlot (or VectorPlot). The vector flow of a field corresponding to $f(t, y)$ is $\{1, f(t, y)\}$ since the flow in the horizontal direction (t) has a constant rate (the flow of time) which can be set to 1, the vertical flow is $f(t, y)$.

4.2 Stability of equilibrium solutions

If the solutions starting in condition a small neighbourhood of an equilibrium soltion ($y = c$) converge towards the equilibrium solution for large t, then the equilibrium solution $y = c$ is stable.

Example 4. *$y' = y$, the equilibrium solution $y = 0$ is unstable (for $y' = -y$, $y = 0$ is stable)*

Example 5. *$y' = y(1 - y)$, $y = 0$ is unstable, but $y = 1$ is stable.*

5 Lecture 5 - Stability of ODE's

5.1 Condition for stability of equilibrium soltuions

Using a Taylor Series approximation of $f(t, y)$ near the equilibrium solution $y = c$, assume $f(t, y) = f(y)$ for simplicity

$$f(y) \approx f(c) + f'(c)(y - c) + \frac{1}{2} f''(c)(y - c)^2 + ...$$

Where $y = c$ is the equilibrium solution $f(c) = 0$, thus

$$y'(t) = 0 + f'(c)(y - c) + \frac{1}{2} f''(c)(y - c) + ...$$

Take $u = y - c$, then

$$u' = f'(c)u + ...$$

Implying

$$u(t) = Ae^{f'(c)t}$$

If $f'(c) > 0$, $y = c$ is unstable. If $f'(c) < 0$, $u(t) \to 0$, $y(t) \to c$ thus $y = c$ is stable.

5.2 Euler's Method

An iterative operation which models $y_k \approx y(k \cdot \Delta t)$. Given $y' = f(t, y)$, $y(0) = c$

$$y'(t) \equiv \lim_{\Delta \to 0} \frac{y(t + \Delta) - y(t)}{\Delta} \tag{1}$$

$$\approx \frac{y(t + \Delta) - y(t)}{\Delta} \tag{2}$$

As it is an iterative method, $y_{k+1} = y_k + f(t_k, y_k)$

$$\frac{y(t_{k+1}) - y(t_k)}{\Delta} \approx f(t_k, y_k)$$

Example 6. *Given $y' = 2t$, $y(t) = ?$, $y(0) = 0$. By Euler's Method*

$$N = 1,\ \Delta = 1 \tag{3}$$
$$N = 2,\ \Delta = \frac{1}{2} \tag{4}$$
$$N = 4,\ \Delta = \frac{1}{4} \tag{5}$$
$$N = 10\ \Delta = \frac{1}{10} \tag{6}$$

6 Lecture 6 -

6.1 Error Generated in Euler's Method

Assume $y(t_k) = y_k$, the error is represented with the taylor series approximation

$$\Big| y_{k+1} - y(t_k + \Delta) \Big| = y(t_{k+1}) = y(t_k) + y'(t_k)\Delta + \frac{1}{2}y''(t_k)\Delta^2 + ...$$
$$y_{k+1} = y_k + f(t_k, y_k)\Delta$$

Thus

$$\text{error} = |y_{k+1} - y(t_k + \Delta)| \propto \Delta^2$$

However generalized for N steps,

$$\begin{aligned} N \cdot |y_{k+1} - y(t_k + \Delta)| &\propto \Delta^2 \cdot N \\ &\propto \Delta^2 \cdot \frac{1}{\Delta} \propto \Delta \end{aligned}$$

$$y_{k+1} = y_k + \frac{1}{2}\Big[f(t_k, y_k) + f(t_{k+1}, y_{k+1})\Big]\Delta$$

7 Lecture 8

$$y' = f(t, y), y(0) = y_0$$
$$t \in [0, t_{max}]$$

1. Euler's Method

   $$y_{k+1} = y_k + f(t_k, y_k)\Delta$$

   Where total error $\propto \Delta$

2. Heun Method (Revised Euler's Method)

   $$\begin{cases} y_{k+1} = y_k + f(t_k, y_k)\Delta \\ y_{k+1} = y_k + \frac{1}{2}[f(t_k, y_k) + f(t_k + \Delta, y_{k+1})] \end{cases}$$

   Where total error $\propto \Delta^2$. Note that Heun Method will possibly show in Assignment 3.

7.1 Runge Kutta Method

$$p_1 = f(t_k, y_k)\Delta$$
$$p_2 = f(t_k + \frac{\Delta}{2}, y_k + \frac{p_1}{2})\Delta$$
$$p_3 = f(t_k + \frac{\Delta}{2}, y_k + \frac{p_2}{2})\Delta$$
$$p_4 = f(t_k + \frac{\Delta}{2}, y_k + p_3)\Delta$$

for

$$y_{k+1} = y_k + \frac{1}{6}p_1 + \frac{1}{3}p_2 + \frac{1}{3}p_3 + \frac{1}{6}p_4 + ...$$

7.2 Adaptive Step Size

Fixed step size is mostly inefficient in most cases, so we use an adaptive step size for numerical methods to achieve better approximations.

7.3 Coupled Systems

$$y' = f(t, y_1, y_2)$$
$$x' = g(t, y_1, y_2)$$

$$x(t) = ?\ y(t) = ?$$

In the context of Euler's Method

$$\begin{cases} x_{k+1} = x_k + f(t_k, x_k, y_k)\Delta \\ y_{k+1} = y_k + g(t_k, x_k, y_k)\Delta \end{cases}$$

7.4 Analytical ODE Solutions

7.4.1 Linear First Order ODE's

By definition,

$$y(t) = f(t)y(t) + g(t)$$

is generally the standard form of a linear first order ODE. Note that

$$y'(t) = f(t)y(t)$$

is a special case of a linear first order ODE that is seperable.

Example 7.

$$ty' + y = t\cos t$$

Observe that, by the product rule for differentiation, $ty' + y = (ty)'$.

$$(ty)' = t\cos t$$
$$\int (ty)'dy = \int t\cos t dt$$
$$ty = t\sin t - \int \sin t dt$$
$$ty = t\sin t + \cos t + c$$
$$\therefore y = \frac{t\sin t + \cos t}{t} + \frac{c}{t}$$