MATH1072 Notes

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1 Lecture 1 - Introduction to Dimensional Analysis

2 Lecture 2 - Dimensional Analysis

3 Lecture 3 - Introduction to Differential Equations

Generally, an ordinary differential equation (ODE) is represented as:

$$F(t, y(t), y'(t), y''(t), ...) = 0$$

For Instance, Newton's Law

$$m\frac{d^2r}{dt^2} = F$$

Induction Law:

$$RI + L\frac{dI}{dt} + \frac{1}{c}$$

Population:

$$\frac{dP}{dt} = rP(1 - \frac{P}{k})$$

Maxwell's Equations:

$$\nabla \cdot \bar{E} = \frac{1}{\rho}\varepsilon_0$$

$$\nabla \cdot \bar{B} = 0$$

$$\nabla \times \bar{E} = \frac{\delta B}{\delta t}$$

$$\nabla \times \bar{B} = \mu_0 J + \mu_0 \varepsilon_0 \frac{\delta E}{\delta t}$$

Navier-Stokes: (Modelling velocity of fluids in space)

$$\frac{\delta \overline{v}}{\delta t} = \overline{v} \nabla \overline{v} = ?$$

Schrodinger Wave Equation:

$$i\hbar \frac{\delta \psi}{\delta t} = \left[\frac{-\hbar}{2m} \nabla^2 + V(r) \right] \psi$$

$$\psi = ?$$

We generally have:

$$F(t, y(t), y'(t), ...) = 0, \; y(t) = ?$$

Where the order of ODE = order of the higest derivative. Take the equation $y' = y$, the solutions are $y = e^t$, $y = 0$ and $y = ce^t$. However the latter expression encapsulates the former, so $y(t) = ce^t$ is known as the general solution where $c \in \mathbb{R}$. The genreal solution is not unique.

If you take an ODE and add some initial condition, then the solution is a unique result. For Instance take the ODE $y' = y$ and say that $y(0) = 1$, then we achieve a unique solution of $c = 1$, $y = e^t$.

4 Lecture 4 - Ordinary Differential Equations

4.1 Equilibrium Solutions

An equilibrium solution (steady state solution) of an ODE (if such solution exists) is a constant solution $y(t) = c$ which satisfies the ODE (for any t).

Example 1. *Take $y' = f(t, y)$, the equilibrium solution*

$$f(t, y = c) = 0$$

This implies that the slope at $y = c$ must be 0 for a function defined on a (t, y) plane

Example 2. $y' = y$, $y = 0$ *is an equilibrium solution.*

Example 3. $y' = y(1 - y)m$ $y = 0$ *and* $y = 1$ *are equilibrium solutions*

Slope fields can visualised with Mathematica using SteamPlot (or VectorPlot). The vector flow of a field corresponding to $f(t, y)$ is $\{1, f(t, y)\}$ since the flow in the horizontal direction (t) has a constant rate (the flow of time) which can be set to 1, the vertical flow is $f(t, y)$.

4.2 Stability of equilibrium solutions

If the solutions starting in condition a small neighbourhood of an equilibrium soltion ($y = c$) converge towards the equilibrium solution for large t, then the equilibrium solution $y = c$ is stable.

Example 4. *$y' = y$, the equilibrium solution $y = 0$ is unstable (for $y' = -y$, $y = 0$ is stable)*

Example 5. *$y' = y(1-y)$, $y = 0$ is unstable, but $y = 1$ is stable.*

5 Lecture 5 - Stability of ODE's

5.1 Condition for stability of equilibrium soltuions

Using a Taylor Series approximation of $f(t,y)$ near the equilibrium solution $y = c$, assume $f(t,y) = f(y)$ for simplicity

$$f(y) \approx f(c) + f'(c)(y-c) + \frac{1}{2}f''(c)(y-c)^2 + ...$$

Where $y = c$ is the equilibrium solution $f(c) = 0$, thus

$$y'(t) = 0 + f'(c)(y-c) + \frac{1}{2}f''(c)(y-c) + ...$$

Take $u = y - c$, then

$$u' = f'(c)u + ...$$

Implying

$$u(t) = Ae^{f'(c)t}$$

If $f'(c) > 0$, $y = c$ is unstable. If $f'(c) < 0$, $u(t) \to 0$, $y(t) \to c$ thus $y = c$ is stable.

5.2 Euler's Method

An iterative operation which models $y_k \approx y(k \cdot \Delta t)$. Given $y' = f(t, y)$, $y(0) = c$

$$y'(t) \equiv \lim_{\Delta \to 0} \frac{y(t+\Delta) - y(t)}{\Delta} \tag{1}$$

$$\approx \frac{y(t+\Delta) - y(t)}{\Delta} \tag{2}$$

As it is an iterative method, $y_{k+1} = y_k + f(t_k, y_k)$

$$\frac{y(t_{k+1}) - y(t_k)}{\Delta} \approx f(t_k, y_k)$$

Example 6. *Given $y' = 2t$, $y(t) =?$, $y(0) = 0$. By Euler's Method*

$$N = 1,\ \Delta = 1 \tag{3}$$

$$N = 2,\ \Delta = \frac{1}{2} \tag{4}$$

$$N = 4,\ \Delta = \frac{1}{4} \tag{5}$$

$$N = 10\ \Delta = \frac{1}{10} \tag{6}$$

6 Lecture 6 -

6.1 Error Generated in Euler's Method

Assume $y(t_k) = y_k$, the error is represented with the taylor series approximation

$$\Big| y_{k+1} - y(t_k + \Delta) \Big| = y(t_{k+1}) = y(t_k) + y'(t_k)\Delta + \frac{1}{2}y''(t_k)\Delta^2 + ...$$

$$y_{k+1} = y_k + f(t_k, y_k)\Delta$$

Thus

$$\text{error} = |y_{k+1} - y(t_k + \Delta)| \propto \Delta^2$$

However generalized for N steps,

$$\begin{aligned} N \cdot |y_{k+1} - y(t_k + \Delta)| &\propto \Delta^2 \cdot N \\ &\propto \Delta^2 \cdot \frac{1}{\Delta} \propto \Delta \end{aligned}$$

$$y_{k+1} = y_k + \frac{1}{2}\Big[f(t_k, y_k) + f(t_{k+1}, y_{k+1})\Big]\Delta$$

7 Lecture 8

$$y' = f(t, y), y(0) = y_0$$

$$t \in [0, t_{max}]$$

1. Euler's Method

$$y_{k+1} = y_k + f(t_k, y_k)\Delta$$

   Where total error $\propto \Delta$

2. Heun Method (Revised Euler's Method)

$$\begin{cases} y_{k+1} = y_k + f(t_k, y_k)\Delta \\ y_{k+1} = y_k + \frac{1}{2}[f(t_k, y_k) + f(t_k + \Delta, y_{k+1})] \end{cases}$$

   Where total error $\propto \Delta^2$. Note that Heun Method will possibly show in Assignment 3.

7.1 Runge Kutta Method

$$
\begin{aligned}
p_1 &= f(t_k, y_k)\Delta \\
p_2 &= f(t_k + \frac{\Delta}{2}, y_k + \frac{p_1}{2})\Delta \\
p_3 &= f(t_k + \frac{\Delta}{2}, y_k + \frac{p_2}{2})\Delta \\
p_4 &= f(t_k + \frac{\Delta}{2}, y_k + p_3)\Delta
\end{aligned}
$$

for

$$y_{k+1} = y_k + \frac{1}{6}p_1 + \frac{1}{3}p_2 + \frac{1}{3}p_3 + \frac{1}{6}p_4 + ...$$

7.2 Adaptive Step Size

Fixed step size is mostly inefficient in most cases, so we use an adaptive step size for numerical methods to achieve better approximations.

7.3 Coupled Systems

$$
\begin{aligned}
y' &= f(t, y_1, y_2) \\
x' &= g(t, y_1, y_2)
\end{aligned}
$$

$$x(t) =?\ y(t) =?$$

In the context of Euler's Method

$$
\begin{cases}
x_{k+1} = x_k + f(t_k, x_k, y_k)\Delta \\
y_{k+1} = y_k + g(t_k, x_k, y_k)\Delta
\end{cases}
$$

7.4 Analytical ODE Solutions

7.4.1 Linear First Order ODE's

By definition,

$$y(t) = f(t)y(t) + g(t)$$

is generally the standard form of a linear first order ODE. Note that

$$y'(t) = f(t)y(t)$$

is a special case of a linear first order ODE that is seperable.

Example 7.

$$ty' + y = t\cos t$$

Observe that, by the product rule for differentiation, $ty' + y = (ty)'$.

$$\begin{aligned}
(ty)' &= t\cos t \\
\int (ty)'dy &= \int t\cos t dt \\
ty &= t\sin t - \int \sin t dt \\
ty &= t\sin t + \cos t + c \\
\therefore y &= \frac{t\sin t + \cos t}{t} + \frac{c}{t}
\end{aligned}$$

8 Lecture 9 - Applications of First Order ODEs

Common applications of first order ODEs are

8.1 Radioactive Decay

The particles of a radioactive material decay spontaneously in a stochastic process. The total mass of the radioactive atoms decrease with time. We can represent this as

$$\frac{dM}{dt} = -kM$$

For $M(t)$ to represent the mass of the radioactive material over time t.

Clearly the solution to this is

$$M(t) = M_0 e^{-kt}$$

Note that there is a limitation to this ODE Model. We assume that the mass changes continuously in time. Whereas in reality it changes in discrete steps following individual decay events. However for the macroscopic mass, the number of particles is much larger, so we can neglect the discrete jumps and assume a continuous deterministic model. Which works well for most cases.

The lifetime of particles varies, however we can categorize them with their average lifetime. We do this by asking how long it takes for a particle to reduce to half of the initial value. To obtain a more accurate value for average lifetime, consider grouping the lifetime of particles into discrete "bins". If the number of particles with lifetime in $[t_j, t_j + \Delta]$ is N_j, then the average lifetimes is represented as

$$\sum N_j = N_0$$

$$\tau \approx \frac{\sum t_j N_j}{\sum N_j}$$

If the number of particles decreases exponentially

$$N(t) = N_0 e^{-kt}$$

Then the number of particles lost in an interval $[t_j, t_j + \Delta]$ is

$$N_j = N[t_j] - N[t_j + \Delta]$$

Or

$$N_j = \frac{N(t_j) - N(t_j + \Delta)}{\Delta} \cdot \Delta \approx -N'(t)\Delta$$

Taking $\Delta \to 0$. The sum for calculating the average lifetime turns into an integral.

$$\tau = \frac{1}{N_0} \int_0^\infty t(-N'(t))\, dt = \int_0^\infty t e^{-kt}\, dt = \frac{1}{k}$$

8.2 Protein Synthesis and Degradation

Seriously who cares.