MIXED HODGE STRUCTURES ON FUNDAMENTAL GROUPS

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Contents

This is the lecture notes for a graduate course given at Oxford in Michaelmas term 2019. References can be found on the course website. The objective is to study mixed Hodge structures on fundamental groups. Most topics and the presentation are taken from a course I have learned from Richard Hain in Duke University. Since this is hastily typed up, any typos and mistakes are made by me. I would appreciate any corrections and suggestions. Please send them to luom@maths.ox.ac.uk.

Introduction

The first interesting topological invariant of a space X one learns in algebraic topology is usually the fundamental group, also known as the first homotopy group. For each point $x \in X$, define

$$P_x X := \left\{\gamma \middle| \gamma : [0,1] \to X \text{ (piecewise) smooth}, \gamma(0) = \gamma(1) = x\right\}$$

to be the loop space. With the appropriate[1] topology, its path connected component $\pi_0(P_x X, \overline{x})$ can be identified with the fundamental group $\pi_1(X, x)$. Higher homotopy groups can be inductively defined as

$$\pi_{k+1}(X, x) \cong \pi_k(P_x X, \overline{x}).$$

We want to study these homotopy groups via differential forms, but issues need to be overcome:

- $\pi_3(S^2) \cong \mathbb{Z}$, but $E^3(S^2) = 0$
- π_1 is non-abelian, but $\int_{\alpha\beta} \omega = \int_\alpha \omega + \int_\beta \omega = \int_{\beta\alpha} \omega$. Question: how to detect its commutators?

[1] compact open.

Kuo-Tsai Chen has discovered a generalization of the usual line integral as follows. Given a path γ and differential 1-forms $\omega_1, \cdots, \omega_r$ on X. Define an *iterated line integral* by

$$\int_\gamma \omega_1 \cdots \omega_r = \int \cdots \int_{0 \le t_1 \le \cdots \le t_r \le 1} f_1(t_1) f_2(t_2) \cdots f_r(t_r) dt_1 \cdots dt_r \tag{0.1}$$

where $\gamma^* \omega_j = f_j(t)dt$. It is a time ordered integral.

Example 0.1. Suppose that we have lines $l_1, \cdots, l_r$ and a path γ on a plane. For simplicity, we assume that the path $\gamma(t)$ transversely passes each line exactly once, at time $t = a_j$. For each line l_j, we have a generalized differential form (also known as current or distribution) ω_j, such that

$$\epsilon_j := \int_\gamma \omega_j = \#(l_j \cdot \gamma) = \pm 1,$$

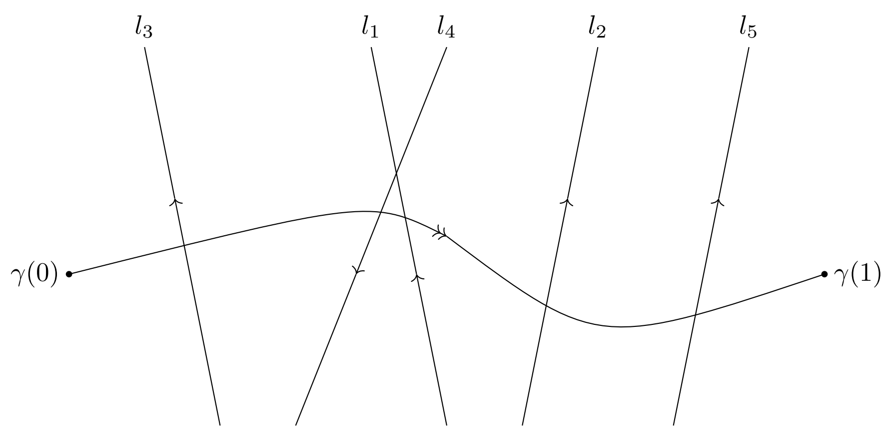
which is the intersection number of l_j with γ and the sign depends on orientation. More precisely, we have $\gamma^* \omega_j = f_j(t)dt$ with

$$f_j(t) = \epsilon_j \, \delta(t - a_j),$$

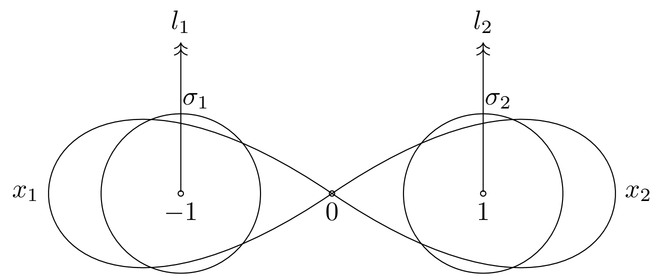
where $\delta(t - a_j)$ is the Dirac delta function centered at $t = a_j$. Now we have

$$\int_\gamma \omega_1 \cdots \omega_r = \begin{cases} \epsilon_1 \cdots \epsilon_r & \text{if } a_1 < a_2 < \cdots < a_r \\ 0 & \text{otherwise} \end{cases}$$

For the path illustrated below $\int_\gamma \omega_1 \omega_3 = 0$, $\int_\gamma \omega_3 \omega_1 \omega_2 = -1$, $\int_\gamma \omega_3 \omega_4 \omega_1 \omega_2 \omega_5 = 1$.



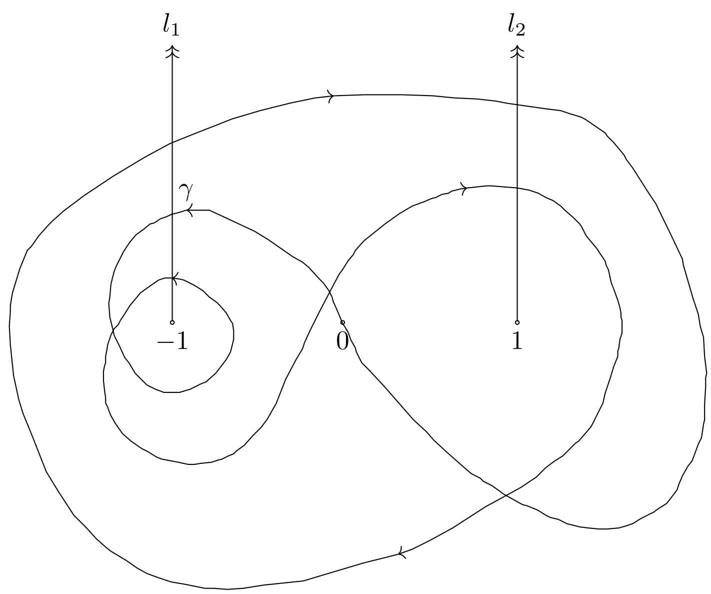
Example 0.2. Let $X = \mathbb{C} - \{-1, 1\}$. The homology group is $H_1(X; \mathbb{Z}) = \mathbb{Z} \cdot a_1 \oplus \mathbb{Z} \cdot a_2$ where a_j is the homology class $[\sigma_j]$. The cohomology group $H^1(X; \mathbb{Z})$ is generated by the two (non-compact) cycles l_1 and l_2 pictured below.



The homology class of a cycle $\gamma : S^1 \to X$ is

$$[\gamma] = \#(l_1 \cdot \gamma)a_1 + \#(l_2 \cdot \gamma)a_2.$$

From the previous example, by using iterated integrals, one knows the order and direction in which a path γ passes through l_1 and l_2, thus we know its homotopy class in $\pi_1(X, 0) \approx < x_1, x_2 >$. The homotopy class of the path γ illustrated below is $x_1^2 x_2^{-1} x_1^{-1} x_2^{-1}$.



Remark 0.3. Usual integrals (of differential forms/currents) detect non-trivial homology classes of an oriented manifold X by an *abelian* intersection theory, while iterated integrals detect non-trivial elements of $\pi_*(X)$ by a *non-abelian* intersection theory.

1. Iterated Integrals

1.1. **Path space.** For X a smooth manifold, define its path space

$$PX := \{\text{(piecewise) smooth paths on X}\}.$$

Its topology is given by the compact open topology.

Given topological spaces X, Y, Z, one can define $X^Y := \{\text{continuous maps } Y \to X\}$. The compact open topology has universal mapping property:

$$(Z \xrightarrow{\alpha} X^Y \text{ continuous}) \iff (Y \times Z \xrightarrow{\phi_\alpha} X \text{ continuous})$$

where $\phi_\alpha(y, z) = \alpha(z)(y)$.

We have an evaluation map

$$\begin{aligned} PX &\xrightarrow{e} X \times X \\ \gamma &\mapsto (\gamma(0), \gamma(1)) \end{aligned}$$

Define subspaces of PX:

$$P_{x,y}X := e^{-1}(x, y); \quad P_x X := e^{-1}(x, x).$$

1.2. **Differential forms on PX.** Question: When is $U \xrightarrow{\alpha} PX (\subset X^I)$ smooth? Shall we just ask $\phi_\alpha : I \times U \to X$ to be smooth?

Chen in his Bulletin paper defines "differentiable spaces".

Definition 1.1. A differentiable space is a set M and a collection of maps

$$(\alpha : U \to M \quad \text{where } U \text{ is open in } \mathbb{R}^d)$$

called "plots", such that

(1) every constant map $U \to M$ is a plot
(2) if $U \xrightarrow{\alpha} M$ is a plot and $V \subseteq \mathbb{R}^e$ is open and $\theta : V \to U$ is smooth, then $\alpha \circ \theta$ is a plot
(3) "local property": $\phi|_{\text{open covering}}$ is a plot $\implies$ ϕ is a plot

Example 1.2. PX is a differentiable space. Plots:

$$\alpha : U \to PX \text{ is a plot}$$

$\iff$ $\exists$ a partition $0 = t_0 < \cdots < t_n = 1$ s.t. $\phi_\alpha : I \times U \to X$ restricted to each $[t_{j-1}, t_j] \times U$ is smooth.

Definition 1.3. M, N are differentiable spaces. We say $F : M \to N$ is smooth if for each plot $\alpha : U \to M$, $F \circ \alpha$ is a plot.

Remark 1.4. (1) every plot is smooth
(2) every manifold is a differentiable space

Definition 1.5. A k-form on a differentiable space M is a family (ω_α) for every plot $U \xrightarrow{\alpha} M$, where $\omega_\alpha \in E^k(U)$ such that if $\theta : V \to U$ is smooth, then $\theta^*\omega_\alpha = \omega_{\alpha\circ\theta}$.

Denote the k-forms on M by $E^k(M)$. For $\omega \in E^k(M)$, $\eta \in E^l(M)$, define $d\omega \in E^{k+1}(M)$ by $d\omega = (d\omega_\alpha)$, and $\omega \wedge \eta \in E^{k+l}(M)$ by $\omega \wedge \eta = (\omega_\alpha \wedge \eta_\alpha)$. Therefore, $E^\bullet(M)$ is a differential graded algebra (d.g.a.).

$$\begin{tikzcd} M \arrow[r, "F"] & N \\ & U \arrow[u, "F\circ\alpha"'] \arrow[ul, bend left, "\alpha"] \end{tikzcd}$$

If $F : M \to N$ is smooth, we have pull-back $F^* : E^\bullet(N) \to E^\bullet(M)$, $\omega \mapsto F^*\omega$, where $(F^*\omega)_\alpha = \omega_{F\circ\alpha} \in E^\bullet(U)$.

1.3. **Complex of iterated integrals/Chen complex $Ch^\bullet(PX)$.** Now we go back to $M = PX$. Have

$$\begin{array}{ccc} PX & & \gamma \\ \downarrow{\scriptstyle e} & & \downarrow \\ X \times X & & (\gamma(0), \gamma(1)) \end{array}$$

where $e = (p_0, p_1)$. One can check this is smooth. (In fact, p_t is smooth $\forall t \in [0,1]$). So we have

$$p_t^* : E^\bullet(X) \to E^\bullet(PX)$$

and

$$\begin{aligned} p_0^* \otimes p_1^* : E^\bullet(X) \otimes E^\bullet(X) &\to E^\bullet(PX) \\ \omega' \otimes \omega'' &\mapsto p_0^*\omega' \wedge p_1^*\omega'' \end{aligned}$$

Definition 1.6. For $\omega_j \in E^{n_j}(X)$, define

$$\int(\omega_1|\cdots|\omega_r) = \pi_*\varphi^*(1 \times \omega_1 \times \cdots \times \omega_r \times 1)$$

where

(1) $\Delta^r = \{\text{time ordered simplex}\} = \{(t_1, \cdots, t_r) : 0 \le t_1 \le \cdots \le t_r \le 1\}$
(2) $\varphi : \Delta^r \times PX \to X \times X^r \times X$ is the sampling map, $((t_1, \cdots, t_r), \gamma) \mapsto (\gamma(0), \gamma(t_1), \cdots, \gamma(t_r), \gamma(1))$. (This map is smooth)
(3) π_* denotes integration over the fiber of the projection $\pi : \Delta^r \times PX \to PX$.

Remark 1.7. If any $\omega_j \in E^0(X)$, then $\int(\omega_1|\cdots|\omega_r) = 0$. Reason: for $p_t : [0,1] \times PX \to X$, $(t, \gamma) \mapsto \gamma(t)$, and $\omega \in E^\bullet(X)$, we can write $p_t^*\omega = dt \wedge \varphi(t) + \psi(t)$. If $\omega \in E^0(X)$ then $\varphi = 0$. But

$$\begin{aligned} \varphi^*(1 \times \omega_1 \times \cdots \times \omega_r \times 1) &= \pm dt_1 \wedge \cdots \wedge dt_r \wedge \varphi_1(t_1) \wedge \cdots \wedge \varphi_r(t_r) + (\text{lower degree terms}) \\ &\qquad \downarrow{\scriptstyle \pi_*} \\ &= \textstyle\int_{\Delta^r} \varphi_1 \wedge \cdots \wedge \varphi_r \end{aligned}$$

We will always assume $n_j = \deg(\omega_j) \ge 1$.

Definition 1.8 (Variant)**.** Let $\omega', \omega'' \in E^\bullet(X)$ with degrees n', n''. Then have

$$(*) \qquad \pi_*\varphi^*(\omega' \times \omega_1 \times \cdots \times \omega_r \times \omega'') = p_0^*\omega' \wedge \int(\omega_1|\cdots|\omega_r) \wedge p_1^*\omega'' \in E^n(PX)$$

where $n = (n_1 + \cdots + n_r - r) + n' + n''$.

Definition 1.9.

$$Ch^\bullet(PX) = \{\text{iterated integrals}\} = \{\text{linear span of } (*)\text{'s in } E^\bullet(PX)\}$$

Remark 1.10. One can restrict this to subspaces $P_{x,y}X$, P_xX of PX and obtain $Ch^\bullet(P_{x,y}X)$, $Ch^\bullet(P_xX)$.

Need

(1) Show that $Ch^\bullet(PX)$ is a d.g.a.

(2) Have formulas for d, $\wedge$.

Proposition 1.11 (Formulas)**.**

(1)

$$\begin{aligned} d\int(\omega_1|\cdots|\omega_r) &= \sum_{j=1}^{r} \pm \int(\omega_1|\cdots|d\omega_j|\cdots|\omega_r) \\ &+ \sum_{j=2}^{r} \pm \int(\omega_1|\cdots|\omega_{j-1}\wedge\omega_j|\cdots|\omega_r) \\ &\pm \int(\omega_1|\cdots|\omega_{r-1})\wedge p_1^*\omega_r \pm p_0^*\omega_1 \wedge \int(\omega_2|\cdots|\omega_r) \end{aligned}$$

(2) $\int(\omega_1|\cdots|\omega_r)\wedge\int(\omega_{r+1}|\cdots|\omega_{r+s}) = \sum_{\sigma\in Sh(r,s)} \pm \int(\omega_{\sigma(1)}|\cdots|\omega_{\sigma(r+s)})$.

Proof. For (2): (follows by standard triangulation of $\Delta^r \times \Delta^s$) Definition for a shuffle of type (r,s): a permutation σ such that $\sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(r)$ and $\sigma^{-1}(r+1) < \sigma^{-1}(r+2) < \cdots < \sigma^{-1}(r+s)$. Let

$$\Delta^r = \{(t_1,\cdots,t_r) : 0 \le t_1 \le \cdots \le t_r \le 1\},$$
$$\Delta^s = \{(t_{r+1},\cdots,t_{r+s}) : 0 \le t_{r+1} \le \cdots \le t_{r+s} \le 1\}.$$

For each point $(t_1,\cdots,t_r;t_{r+1},\cdots,t_{r+s}) \in \Delta^r\times\Delta^s$, there is a shuffle σ of type (r,s) such that

$$0 \le t_{\sigma(1)} \le t_{\sigma(2)} \le \cdots \le t_{\sigma(r+s)} \le 1.$$

This σ is unique if t_j's are distinct. So

$$\begin{aligned} \Delta^r\times\Delta^s &= \bigcup_{\sigma\in Sh(r,s)} \{(t_{\sigma^{-1}(1)},\cdots,t_{\sigma^{-1}(r+s)}) : 0 \le t_1 \le \cdots \le t_{r+s} \le 1\} \\ &= \bigcup_{\sigma\in Sh(r,s)} \Delta_\sigma^{r+s}. \end{aligned}$$

For any plot $\alpha : U \to PX$, we have

$$\begin{aligned} &\alpha^*\Big(\int(\omega_1|\cdots|\omega_r)\wedge\int(\omega_{r+1}|\cdots|\omega_{r+s})\Big) \\ &= \int_{0\le t_1\le\cdots\le t_r\le 1} \varphi_1(t_1)\wedge\cdots\wedge\varphi_r(t_r) \ \wedge \ \int_{0\le t_{r+1}\le\cdots\le t_{r+s}\le 1} \varphi_{r+1}(t_{r+1})\wedge\cdots\wedge\varphi_{r+s}(t_{r+s}) \\ &= \int_{\Delta^r\times\Delta^s} \varphi_1(t_1)\wedge\cdots\wedge\varphi_{r+s}(t_{r+s}) \\ &= \sum_{\sigma\in Sh(r,s)} \int_{\Delta_\sigma^{r+s}} \varphi_1(t_1)\wedge\cdots\wedge\varphi_{r+s}(t_{r+s}) \\ &= \sum_{\sigma\in Sh(r,s)} \int_{0\le t_1\le\cdots\le t_{r+s}\le 1} \varphi_1(t_{\sigma^{-1}(1)})\wedge\cdots\wedge\varphi_{r+s}(t_{\sigma^{-1}(r+s)}) \\ &= \sum_{\sigma\in Sh(r,s)} \pm \int_{\Delta^{r+s}} \varphi_{\sigma(1)}(t_1)\wedge\cdots\wedge\varphi_{\sigma(r+s)}(t_{r+s}) \\ &= \alpha^*\Big(\sum_{\sigma\in Sh(r,s)} \pm\int(\omega_{\sigma(1)}|\cdots|\omega_{\sigma(r+s)})\Big) \end{aligned}$$

For (1): needs basic formula (essetially uses Stoke's formula) [Add proof/leave as exercise]

$$\pi_* d \pm d\pi_* = (\partial\pi)_*$$

where we have

$$\begin{tikzcd}
(j\text{-th face}) \times PX \simeq \Delta^{r-1} \times PX \arrow[r, hook] \arrow[d, "\pi_j"] & (\partial\Delta^r) \times PX \arrow[r, hook] \arrow[d, "\partial\pi"] & \Delta^r \times PX \arrow[d, "\pi"] \\
PX \arrow[r, equal] & PX \arrow[r, equal] & PX
\end{tikzcd}$$

and define $(\partial\pi)_* := \sum_{j=0}^{r} (-1)^j (\pi_j)_*$. □

Remark 1.12. Formula (1) implies that $Ch^\bullet(PX)$ is a sub-complex of $E^\bullet(PX)$. This formula becomes simpler for $Ch^\bullet(P_{x,y}X)$ and $Ch^\bullet(P_xX)$ when we restrict to subspaces of PX.

Natural Question: What is $H^\bullet(Ch^\bullet(PX))$?

The geometric definition of iterated integrals does not help answer this question. We will provide an answer in Section 3, after an algebraic description of iterated integrals up next.

2. Bar Constructions

2.1. **Simplicial and cosimplicial objects.** Denote the category of finite ordinals by $\mathbf{\Delta}$. The objects are the finite ordinals $[n] := \{0, 1, \cdots, n\}$ with the natural order $0 < 1 < \cdots < n$, and the morphisms are order preserving functions $f : [m] \to [n]$. Note that f is not necessarily 1-1.

Remark 2.1. One can think of $[n]$ as the set of vertices of Δ^n.

We have for each $0 \le j \le n$ a face map

$$d_j : [n-1] \hookrightarrow [n]$$

$$\begin{tikzcd}
0 \arrow[d] & 1 \arrow[d] & \cdots & j-1 \arrow[d] & j \arrow[dr] & j+1 \arrow[dr] & \cdots & n-1 \arrow[dr] & \\
0 & 1 & \cdots & j-1 & j & j+1 & \cdots & n-1 & n
\end{tikzcd}$$

that is the unique order preserving injection that omits the value j, and a degeneracy map

$$s_j : [n+1] \to [n]$$

$$\begin{tikzcd}
0 \arrow[d] & 1 \arrow[d] & \cdots & j-1 \arrow[d] & j \arrow[d] & j+1 \arrow[dl] & \cdots & n \arrow[dl] & n+1 \arrow[dl] \\
0 & 1 & \cdots & j-1 & j & j+1 & \cdots & n &
\end{tikzcd}$$

that maps both j and $j+1$ to j.

Remark 2.2. Every map f can be expressed as composites of d_j's and s_j's.

Remark 2.3. Each $f : [m] \to [n]$ induces a simplicial map $|f| : \Delta^m \to \Delta^n$ such that $f(e_j) = e_{f(j)}$ and then extends linearly via barycentric coordinates (see the following *Side*).

Side.

(1) Standard n-simplex has barycentric coordinates

$$\Delta^n = \{(s_0, \cdots, s_n) : s_j \geq 0, \sum_{j=0}^{n} s_j = 1\}.$$

Vertices are $e_j = (0, \cdots, 1, \cdots, 0)$ where 1 is at the j-th position. Faces of Δ^n are given by

$$j\text{-th face: } s_j = 0, \qquad e_j \notin j\text{-th face.}$$

(2) Time ordered n-simplex

$$\Delta^n = \{(t_1, \cdots, t_n) : 0 \leq t_1 \leq \cdots \leq t_n \leq 1\}.$$

There is a 1-1 correspondence

$$(s_0, \cdots, s_n) \leftrightarrow (t_1, \cdots, t_n)$$

given by

$$s_j = t_{j+1} - t_j, \quad j = 0, \cdots, n$$

where we set $t_0 = 0$, $t_{n+1} = 1$. Conversely,

$$t_j = s_0 + \cdots + s_{j-1}, \quad j = 1, \cdots, n.$$

Faces are

$$j\text{-th face: } s_j = 0, \quad \text{i.e. } \boxed{t_j = t_{j+1}}.$$

Definition 2.4. A simplicial object in a category $\mathcal{C}$ is a contravariant functor

$$F : \mathbf{\Delta} \to \mathcal{C}.$$

Using categorical notation, it is in $\mathcal{C}^{\mathbf{\Delta}^{\text{op}}}$.

$$\text{Diagrams:} \quad \Delta^0 \underset{d_1}{\overset{d_0}{\rightrightarrows}} \Delta^1 \underset{d_2}{\overset{d_0}{\Rrightarrow}} \Delta^2 \underset{d_3}{\overset{d_0}{\Rrightarrow}} \cdots \quad \rightsquigarrow \quad \cdots F_2 \underset{d_2}{\overset{d_0}{\Rrightarrow}} F_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} F_0$$

Example 2.5 (Simplicial set)**.**

$$K_\bullet : \mathbf{\Delta} \to \underline{\text{Sets}}$$
$$[n] \mapsto K_n = \text{“set of } n\text{-simplices”}$$

Each simplicial set $K_\bullet$ has a geometric realization

$$|K_\bullet| = \Big(\coprod_{n\geq 0} K_n \times \Delta^n\Big)/\sim$$

where $\sim$ is a natural equivalence relation generated by identifications for each morphism $f : [m] \to [n]$ of $\mathbf{\Delta}$.

e.g. standard n-simplex $\Delta^n_\bullet$,

$$\Delta^n_m = \mathrm{Hom}_{\mathbf{\Delta}}([m], [n]).$$

In particular, we define $I_\bullet := \Delta^1_\bullet$.

Claim:

$$|\Delta^n_\bullet| = \Delta^n.$$

Example 2.6. Let X be a topological space, define a simplicial set $\mathrm{Simp}_\bullet X$ with

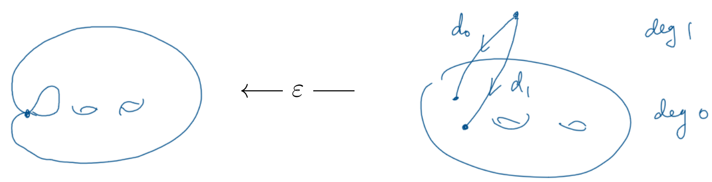
$$\mathrm{Simp}_n X = \{\sigma : \Delta^n \to X\} = \text{singular } n\text{-simplices of } X.$$

Fact:

$$|\mathrm{Simp}_\bullet X| \to X$$

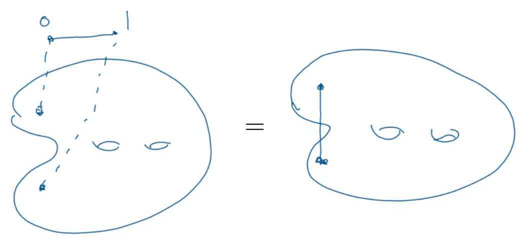
is a weakly homotopy equivalence, i.e. induces isomorphisms on all π_j's and H_j's.

Example 2.7 (Simplicial space/variety/scheme)**.** We usually use a simplicial space $X_\bullet$ to model a nodal curve X, i.e. we have a simplicial covering map



X $X_\bullet$

The geometric realization $|X_\bullet|$ of $X_\bullet$ is



which is homotopy equivalent to X.

A functor $\mathcal{C} \to \mathcal{D}$ induces $\mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}} \to \mathcal{D}^{\mathbf{\Delta}^{\mathrm{op}}}$ on the corresponding simplicial objects.

Example 2.8. We have

$$\begin{aligned} \underline{\mathrm{Sets}} &\to \underline{R - \mathrm{Mod}} \\ \Sigma = \{\sigma\} &\mapsto \bigoplus_{\sigma \in \Sigma} R = \text{free } R\text{-module on } \Sigma \\ \mathrm{Simp}_n X &\mapsto S_n(X; R) = \text{singular } n\text{-chains.} \end{aligned}$$

This induces

$$\begin{array}{ccccccc} \underline{\mathrm{Top}} & \longrightarrow & \underline{\mathrm{Sets}}^{\mathbf{\Delta}^{\mathrm{op}}} & \longrightarrow & (\underline{R - \mathrm{Mod}})^{\mathbf{\Delta}^{\mathrm{op}}} & \longrightarrow & (\text{chain complexes of } R\text{-modules}) \\ X & \longmapsto & \mathrm{Simp}_\bullet X & \longmapsto & S_\bullet(X) & \longmapsto & (S_\bullet(X), \partial) \\ & & & & (M_\bullet, d_j) & \longmapsto & (M_\bullet, d = \sum_{j=0}^n (-1)^j d_j) \end{array}$$

Definition 2.9. A cosimplicial object in $\mathcal{C}$ is a covariant functor $G : \mathbf{\Delta} \to \mathcal{C}$. Diagram: $G^0 \overset{d^0}{\underset{d^1}{\rightrightarrows}} G^1 \overset{d^0}{\underset{d^2}{\Rrightarrow}} G^2 \overset{d^0}{\underset{d^3}{\rightrightarrows\!\!\rightrightarrows}} \cdots$

Example 2.10.

$$\underline{\text{Sets}} \to \underline{R-\text{Mod}}$$
$$S \mapsto R^S = \{\text{functions } S \to R\} = (r_\sigma : \sigma \in S)$$
$$\text{Simp}_n X \mapsto S^n(X;R) = \text{singular cochains}$$

The differential on the cosimplicial R-module $S^n(X)$ is given by

$$\delta = \sum_j (-1)^j d^j$$

where d^j are coface maps.

2.2. **Cosimplicial model of PX.** For X a topological space, define

$$P^\bullet X := X^{I_\bullet} = \text{Hom}(I_\bullet, X),$$

where $I_\bullet$ is the simplicial model of the unit interval (see Example 2.5). Note that there is a natural map $PX \to ||P^\bullet X||$ which is not a homotopy equivalence *except* when X is simply connected.

We first look closely into $I_\bullet$. By definition,

$$I_\bullet := \text{Hom}_{\mathbf{\Delta}}([\bullet],[1])$$

and

$$I_n = \text{Hom}_{\mathbf{\Delta}}([n],[1]).$$

In fact, I_n consists of $(n+2)$ order preserving maps of the form

$$\begin{array}{ccc} \underbrace{0 \quad \cdots \quad j-1} & \qquad & \underbrace{j \quad \cdots \quad n} \\ \downarrow & & \downarrow \\ 0 & & 1 \end{array}$$

For each $j = 0, \cdots, n+1$, the map above corresponds to the j-th bipartition

$$0, \cdots, j-1 \mid j, \cdots, n$$

of $[n]$ (although for $j = 0, n+1$, it is not actually a bipartition). So we will also refer to elements/maps in I_n as bipartitions. These bipartitions pull back along order preserving maps. For example, for the face map

$$d_j : [n-1] \hookrightarrow [n]$$

$$\begin{tikzcd} 0 \arrow[d] & 1 \arrow[d] & \cdots & j-1 \arrow[d] & j \arrow[dr] & j+1 \arrow[dr] & \cdots & n-1 \arrow[dr] & \\ 0 & 1 & \cdots & j-1 & j & j+1 & \cdots & n-1 & n \end{tikzcd}$$

Pullback along d_j induces a map

$$I_n = \text{Hom}_{\mathbf{\Delta}}([n],[1]) \to I_{n-1} = \text{Hom}_{\mathbf{\Delta}}([n-1],[1])$$

which, expressed in terms of bipartitions, is the degeneracy map

$$s_j : [n+1] \to [n]$$

$$
\begin{tikzcd}
0 \arrow[d] & 1 \arrow[d] & \cdots & j-1 \arrow[d] & j \arrow[d] & j+1 \arrow[dl] & \cdots & n \arrow[dl] & n+1 \arrow[dl] \\
0 & 1 & \cdots & j-1 & j & j+1 & \cdots & n &
\end{tikzcd}
$$

because both the j-th and the $(j+1)$-th bipartitions of $[n]$

$$0, \cdots, j-1 \mid j, j+1, \cdots, n \qquad \text{and} \qquad 0, \cdots, j-1, j \mid j+1, \cdots, n$$

pull back to the j-th bipartition of $[n-1]$

$$0, \cdots, j-1 \mid j, \cdots, n-1.$$

From the above discussion, we have

$$P^n X = X^{I_n} = \mathrm{Hom}(I_n, X) \cong X^{n+2}$$

and the map $d_j : [n-1] \to [n]$ induces $I_n \to I_{n-1}$, which in turn induces a coface map

$$\begin{aligned} d^j : X^{I_{n-1}} &\to X^{I_n} \\ (x_0, \cdots, x_n) &\mapsto (x_0, \cdots, x_j, x_j, \cdots, x_n) \end{aligned}$$

This is the coface map that we use to define the cosimplicial model $P^\bullet X$.

Example 2.11 (Another cosimplicial space)**.** $\Delta^\bullet$ is given by

$$
\begin{tikzcd}
\Delta^0 \arrow[r, shift left, "d^0"] \arrow[r, shift right, "d^1"'] & \Delta^1 \arrow[r, shift left=2, "d^0"] \arrow[r] \arrow[r, shift right=2, "d^2"'] & \Delta^2 \arrow[r, shift left=3, "d^0"] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3, "d^3"'] & \cdots
\end{tikzcd}
$$

Note that the coface maps is given by

$$\begin{aligned} d^j : \Delta^{n-1} &\to \Delta^n \\ (t_1, \cdots, t_{n-1}) &\mapsto (t_1, \cdots, t_j, t_j, \cdots, t_n) \end{aligned}$$

which are "dual" to the face maps d_j in $\mathbf{\Delta}$.

Take

$$\Delta^\bullet \times PX \to P^\bullet X$$

with

$$\begin{aligned} \Delta^n \times PX &\to P^n X = X^{n+2} \\ ((t_1, \cdots, t_n), \gamma) &\mapsto (\gamma(0), \gamma(t_1), \cdots, \gamma(t_n), \gamma(1)) \end{aligned}$$

in degree n. This is compatible with coface maps, so we have a smooth map of cosimplicial spaces.

Applying the de Rham complex, we have

$$P^\bullet X \xrightarrow{E^\bullet} \text{simplicial d.g.a}$$

$$\begin{tikzcd}[row sep=small]
 & \vdots & \vdots \\
\deg n & X^{n+2} \arrow[u, shift left] \arrow[u, shift right, "\cdots"'] & E^\bullet(X)^{\otimes(n+2)} \arrow[d, shift left] \arrow[d, shift right, "\cdots"'] \arrow[from=u, shift left] \arrow[from=u, shift right] \\
 & \vdots \arrow[from=u, shift left] \arrow[from=u, shift right, "\cdots"'] & \vdots \arrow[d, shift left=3, "d_3"] \arrow[d, shift left=1] \arrow[d, shift right=1] \arrow[d, shift right=3, "d_0"'] \\
\deg 2 & X^4 \arrow[u, shift left=3, "d^3"'] \arrow[u, shift left=1] \arrow[u, shift right=1] \arrow[u, shift right=3, "d^0"] & E^\bullet(X)^{\otimes 4} \arrow[d, shift left=2, "d_2"] \arrow[d] \arrow[d, shift right=2, "d_0"'] \\
\deg 1 & X^3 \arrow[u, shift left=2, "d^2"'] \arrow[u] \arrow[u, shift right=2, "d^0"] & E^\bullet(X)^{\otimes 3} \arrow[d, shift left, "d_1"] \arrow[d, shift right, "d_0"'] \\
\deg 0 & X^2 \arrow[u, shift left, "d^1"'] \arrow[u, shift right, "d^0"] & E^\bullet(X)^{\otimes 2}
\end{tikzcd}$$

In particular, the coface map $d^j : X^{I_{n-1}} \to X^{I_n}$ induces the face map

$$\begin{aligned}
d_j : E^\bullet(X)^{\otimes(n+2)} &\to E^\bullet(X)^{\otimes(n+1)} \\
\omega_0 \otimes \cdots \otimes \omega_{n+1} &\mapsto \omega_0 \otimes \cdots \otimes \omega_j \wedge \omega_{j+1} \otimes \cdots \otimes \omega_{n+1}
\end{aligned}$$

because the diagonal map $\mathrm{diag} : X \to X \times X$ induces

$$\begin{tikzcd}
E^\bullet(X) \otimes E^\bullet(X) \arrow[rr, "- \wedge -"] \arrow[dr, "\sim"'] & & E^\bullet(X) \\
 & E^\bullet(X \times X) \arrow[ur, "\mathrm{diag}^*"'] &
\end{tikzcd}$$

From this, we have

$$\text{simplicial d.g.a} \longrightarrow \text{double complex}$$

$$(E^\bullet(X)^{\otimes(\bullet+2)}, d) \longmapsto (E^\bullet(X)^{\otimes(\bullet+2)}, d, \delta)$$

where d is the de Rham differential as usual, and

$$\delta = \sum_j (-1)^j d_j$$

is given by taking alternating sum of the face maps (cf. Example 2.8). One can check that d and δ commute.

Example 2.12. In this double complex, we have

$$d(\omega_0 \otimes \cdots \otimes \omega_{n+1}) = \sum_{j=0}^{n+1} \pm\, \omega_0 \otimes \cdots \otimes \omega_{j-1} \otimes d\omega_j \otimes \omega_{j+1} \otimes \cdots \otimes \omega_{n+1}$$

and

$$\delta(\omega_0 \otimes \cdots \otimes \omega_{n+1}) = \sum_{j=0}^{n} (-1)^j \omega_0 \otimes \cdots \otimes \omega_j \wedge \omega_{j+1} \otimes \cdots \omega_{n+1}.$$

These are basically differentials in the bar construction $B(E^\bullet(X), E^\bullet(X), E^\bullet(X))$, which we will discuss next (cf. the differential $d\int(\omega_1|\cdots|\omega_n)$ in Proposition 1.11 (1)).

2.3. **Bar constructions.** Both the bar construction and the reduced bar construction can be combined easily with Hodge theory. We discuss the bar construction first.

Given

(1) $A^\bullet$ a d.g.a. (with differential d of degree $+1$). e.g. $C^\bullet(X)$ = singular cochains, $E^\bullet(X)$ = de Rham complex.
(2) $M^\bullet, N^\bullet$ are cochain complexes
(3) $M^\bullet$ = right $A^\bullet$-module, $N^\bullet$ = left $A^\bullet$-module, such that the structure maps

$$M^\bullet \otimes A^\bullet \to M^\bullet \quad \text{and} \quad A^\bullet \otimes N^\bullet \to N^\bullet$$

are chain maps

The *bar construction* $B(M^\bullet, A^\bullet, N^\bullet)$ (will be simply denoted by $B(M, A, N)$) is a double complex, with

$$B^{-s,t}(M, A, N) = [M \otimes A^{\otimes s} \otimes N]^t$$

consisting of elements $m[a_1|\cdots|a_s]n$ such that $\deg(m) + \sum_j \deg(a_j) + \deg(n) = t$. Note that the total degree of $m[a_1|\cdots|a_s]n \in B^{-s,t}(M, A, N)$ in $B(M, A, N)$ is $(-s) + t = t - s$.

Define an endomorphism J of each graded vector space by $J : v \mapsto (-1)^{\deg v} v$. Define the bar differential by $d_B = d + \delta$ (see Example 2.12) so that

$$d_B[a_1|\cdots|a_s] = \sum_j \pm[a_1|\cdots|da_j|\cdots|a_s] + \sum_j \pm[a_1|\cdots|a_j \wedge a_{j+1}|\cdots|a_s].$$

The total differential for the double complex $B(M, A, N)$ is

$$\begin{aligned} D &= d_\otimes + d_C =: D^{(0,1)} + D^{(1,0)} \\ &= d_M \otimes 1_T \otimes 1_N + J_M \otimes d_B \otimes 1_N + J_M \otimes J_T \otimes d_N + d_C \end{aligned}$$

where the combinatorial differential d_C is defined by

$$d_C(m[a_1|\cdots|a_s]n) = (-1)^s Jm[a_1|\cdots|a_{s-1}]a_s \cdot n + m \cdot (-1)^{s \deg(a_1)} a_1[a_2|\cdots|a_s]n.$$

There is a standard filtration, called bar filtration, on $B(M, A, N)$ with

$$B_s(M, A, N) = \bigoplus_{r \le s} B^{-r,t}(M, A, N).$$

Remark 2.13. Note that

$$B(M, A, N) = \bigcup_{s \ge 0} B_s(M, A, N) = \varinjlim_s B_s(M, A, N),$$

so we have

$$H^\bullet(B(M, A, N)) = \varinjlim_s H^\bullet(B_s(M, A, N)).$$

This leads us to *Eilenberg–Moore spectral sequence* (EMss). It has pages

- $E_0^{-s,t} = [M \otimes A^{\otimes s} \otimes N]^t$ with differential $d_0 = D^{(0,1)}$
- $E_1^{-s,t} = [H^\bullet(M) \otimes H^\bullet(A)^{\otimes s} \otimes H^\bullet(N)]^t$ with differential $d_1 = D^{(1,0)}$.

Remark 2.14. E_1 is $B(H^\bullet(M), H^\bullet(A), H^\bullet(N))$ with total differential $D = D^{(1,0)}$.

With the natural map

$$B(\mathbb{R}, E^\bullet(X), \mathbb{R}) \to Ch^\bullet(P_{x,y}X)$$
$$[\omega_1|\cdots|\omega_s] \mapsto \int(\omega_1|\cdots|\omega_s)$$

in mind, terms such as $[f_1|\cdots|f_n]$ where f_j's are functions should be redundant, cf. Remark 1.7. Note that these terms have negative total degrees. The following theorem confirms our expectation.

Definition 2.15. Let $A^\bullet$ be a d.g.a. over k. It is *connected* if

$$A^j = \begin{cases} 0, & j < 0 \\ k, & j = 0. \end{cases}$$

It is *homologically connected* if $H^\bullet(A^\bullet)$ is connected, i.e.

$$H^j(A^\bullet) = \begin{cases} 0, & j < 0 \\ k, & j = 0. \end{cases}$$

For example, $A^\bullet = E^\bullet(X)$ for X a connected manifold.

Theorem 2.16. *If*

(1) *$A^\bullet$ is homologically connected*
(2) *$H^j(M^\bullet)$ and $H^j(N^\bullet)$ vanish for $j < 0$*

then $H^j(B(M, A, N)) = 0$ when $j < 0$.

To get rid of these negative degree terms, we introduce the *reduced bar construction* $\overline{B}(M, A, N)$, it will have a nicer EMss. Suppose $A^\bullet$ is connected and homologically connected, then $A^0 = k$. Write

$$A^\bullet = k \oplus \underbrace{(IA^\bullet, d)}_{\deg \geq 1}.$$

Set

$$\overline{B}^{-s}(M, A, N) = M \otimes (IA)^{\otimes s} \otimes N \subseteq B^{-s}(M, A, N) = B_s(M, A, N)$$

then $\overline{B}(M, A, N) \subseteq B(M, A, N)$ is a subcomplex. The corresponding EMss has

$$E_1^{-s} = H^\bullet(M) \otimes \underbrace{IH^\bullet(A)^{\otimes s}}_{\deg \geq s} \otimes H^\bullet(N)$$

Theorem 2.17. *If $A^\bullet$ is homologically connected, then*

$$\overline{B}(H^\bullet(M), H^\bullet(A), H^\bullet(N)) \to B(H^\bullet(M), H^\bullet(A), H^\bullet(N))$$

is a quasi-isomorphism.

Suppose that X is a manifold. Evaluating at $x, y \in X$ induces augementations $e_x, e_y : E^\bullet(X) \to \mathbb{R}$. Take $A^\bullet \subseteq E^\bullet(M)$ to be a sub dga that both augmentations restrict to non-trivial homomorphisms $A^\bullet \to \mathbb{R}$. Take $M^\bullet = N^\bullet = \mathbb{R}$ as $A^\bullet$-modules with these homomorphisms. We can form the reduced bar construction $\overline{B}(\mathbb{R}, A\bullet, \mathbb{R})$.

Define $Ch^\bullet(P_{x,y}(X); A^\bullet)$ to be the subcomplex of $Ch^\bullet(P_{x,y}X)$ spanned by iterated integrals $\int \omega_1 \cdots \omega_r$ with $\omega_j \in A^\bullet$.

Theorem 2.18. *Suppose that X is connected. If $H^0(A^\bullet) \cong \mathbb{R}$ and the natural map $H^\bullet(A^\bullet) \to H^\bullet(X)$ is injective, then the natural map*

$$\begin{aligned} \overline{B}(\mathbb{R}, A^\bullet, \mathbb{R}) &\to Ch^\bullet(P_{x,y}X; A^\bullet) \\ [\omega_1|\cdots|\omega_r] &\mapsto \int(\omega_1|\cdots|\omega_r) \end{aligned}$$

is a well defined isomorphism of dgas.

Proof. Check the formulas for the differentials. □

Corollary 2.19. *If X is connected, $A\bullet \subseteq E^\bullet(X)$ is a sub dga for which the inclusion $A^\bullet \hookrightarrow E^\bullet(X)$ induces isomorphism on cohomology, then the inclusion*

$$Ch^\bullet(P_{x,y}X; A^\bullet) \hookrightarrow Ch^\bullet(P_{x,y}X)$$

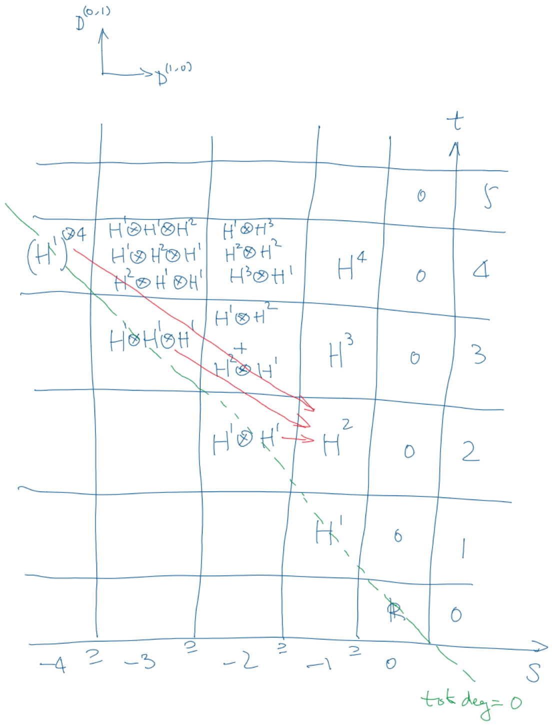
induces an isomorphism on cohomology.

Proof. By the assumption, the E_1 page on both sides should be the same, the statement follows. □

Remark 2.20. This is useful as we can take $A^\bullet$ to be

(1) minimal model (smallest sub dga that computes cohomology)—this simplifies computation.
(2) the logarithmic de Rham complex—this leads to Hodge theory.

Suppose X is connected, it is instructive to see the E_1 page of the EMss corresponding to the reduced bar construction $\overline{B}(\mathbb{R}, E^\bullet(X), \mathbb{R})$. Note that $IH^\bullet(E^\bullet(X))$ is the reduced cohomology $\widetilde{H}^\bullet(X)$ of X. In the picture below, we will simply denote $\widetilde{H}^\bullet(X)$ by $H^\bullet$.



Denote by (E_r, d_r) the r-th page (with differential) of the EMss.

Remark 2.21. A few remarks are in order.

(1) The total differential $D = D^{(0,1)} + D^{(1,0)}$, so that $d_0 = D^{(0,1)}$, $d_1 = D^{(1,0)}$.

(2) The green line has total degree 0. Here lies all iterated line integrals. Elements in $H^0(\overline{B}(\mathbb{R}, E^\bullet(X), \mathbb{R}))$ are represented by elements of the form

$$\begin{tikzcd}
0 & & \\
\omega_1 \otimes \omega_2 \arrow[u] \arrow[r] \arrow[dr, leftrightarrow, "\text{tot. deg 0 line}"'] & {} & \\
& \omega_{12} \arrow[u] \arrow[r] & 0
\end{tikzcd}$$

that are closed, i.e. applying D gives 0. For the element in the above diagram, this means $d\omega_1 = d\omega_2 = 0$ and $\omega_1 \wedge \omega_2 + d\omega_{12} = 0$, this gives rise to a homotopy invariant iterated line integral $\int(\omega_1\omega_2 + \omega_{12})$.

(3) Massey products are indicated by the red arrows, where the map $(H^1)^{\otimes(r+1)} \to H^2$ is given by the differential d_r of the r-th page. When $r = 1$, it is simply the cup product; when $r = 2$, it is the first non-trivial Massey product.

3. Chen's π_1-de Rham Theorem

In this section, we first state de Rham theorem for the loop space of a simply connected space. This allows us to compute rational homotopy groups using iterated integrals. In the non simply connected case, we state Chen's π_1-de Rham theorem.

Theorem 3.1 (Chen)**.** *If X is path connected and simply connected, then integration induces an isomorphism*

$$H^\bullet(Ch^\bullet(P_xX)) \to H^\bullet(P_xX;\mathbb{R}).$$

In fact, it is an isomorphism of Hopf algebras.

Example 3.2. Let $X = S^n$, $n \geq 2$. Take minimal model

$$A^j = \begin{cases} \mathbb{R}, & j = 0 \\ \mathbb{R}\cdot\omega, & j = n \end{cases}$$

with $\int_{S^n} \omega = 1$. As $A^\bullet \hookrightarrow E^\bullet(S^n)$ is a quasi-isomorphism, by Corollary 2.19, we have isomorphism

$$H^\bullet(Ch^\bullet(P_xS^n);A^\bullet) \xrightarrow{\cong} H^\bullet(Ch^\bullet(P_xS^n;\mathbb{R})).$$

Since

$$Ch^\bullet(P_xS^n;A^\bullet) = \bigoplus_{k\geq 0} \int (\underbrace{\omega|\cdots|\omega}_{k})$$

has total differential $D \equiv 0$ (as $d\omega = 0$, $\omega\wedge\omega = 0$), we have

$$H^\bullet(P_xS^n;\mathbb{R}) = \begin{cases} \mathbb{R}, & j = (n-1)k \\ 0, & \text{else} \end{cases}$$

Suppose that M is connected.

Definition 3.3. An element of $H^\bullet(M)$ is *decomposable* if it is in the image of the cup product mapping

$$H^{>0}(M)\otimes H^{>0}(M) \to H^\bullet(M).$$

The set of *indecomposable* elements is defined by

$$QH^\bullet(M) := H^{>0}(M)/H^{>0}(M)^{\otimes 2}.$$

When X is simply connected, P_xX is a connected H-space. Chen's de Rham theorem and the Cartan–Serre theorem imply that

Theorem 3.4. *If X is simply connected, then integration induces an isomorphism*

$$QH^j(P_xX;\mathbb{R}) \xrightarrow{\approx} \mathrm{Hom}(\pi_j(P_xX,\overline{x}),\mathbb{R}) \cong \mathrm{Hom}(\pi_{j+1}(X,x),\mathbb{R}).$$

Example 3.5. $X = S^n$, $n \geq 2$. Denote $\theta_k := \int(\underbrace{\omega|\cdots|\omega}_{k})$. By the previous example,

$$H^\bullet(P_xS^n;\mathbb{R}) = \bigoplus_{k\geq 0}\mathbb{R}\cdot\theta_k.$$

If n is odd, we have $\theta_1^2 = \int(\omega)\wedge\int(\omega) = 2\int(\omega|\omega) = 2\theta_2$, $\theta_1^k = k!\theta_k$, so the indecomposables consist of $\mathbb{R}$-linear span of θ_1.

If n is even, we have $\theta_1^2 = 0$, $\theta_1 \wedge \theta_{2m} = \theta_{2m+1}$, $\theta_2 \wedge \theta_{2m} = (m+1)\theta_{2m+2}$, so the indecomposables consist of $\mathbb{R}$-linear span of θ_1, θ_2.

By the above theorem, we have

$$\pi_j(S^n) \otimes \mathbb{R} = \begin{cases} \mathbb{R}, & j = n, \quad n \text{ odd} \\ \mathbb{R}, & j = n \text{ or } 2n-1, \quad n \text{ even} \\ 0, & \text{else} \end{cases}$$

In the non simply connected case, suppose that X is connected, $x \in X$, we have

$$\begin{tikzcd}
H^0(Ch^\bullet(P_xX)) \arrow[rr] & & H^0(P_xX;\mathbb{R}) \arrow[d, equal, "(*)"] \\
 & & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\pi_1(X,x),\mathbb{R}) \\
 & & \mathrm{Hom}^{\mathrm{cts}}_{\mathbb{R}}(\mathbb{R}\pi_1(X,x),\mathbb{R}) \arrow[u, hook] \\
\varinjlim_r H^0(Ch^\bullet_r(P_xX)) \arrow[uuu, "\approx"] \arrow[r] & \varinjlim_r \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}\pi_1(X,x)/I^{r+1},\mathbb{R}) \arrow[ur, "\approx (\text{by definition})"'] &
\end{tikzcd}$$

Remark 3.6.
1. Here $Ch^\bullet_r(P_xX) \cong B_r(\mathbb{R}, E^\bullet(X), \mathbb{R})$ denotes iterated line integrals of length $\leq r$.
2. For $(*)$: $H^0(M)$ are locally constant functions on M; when M is the path/loop space of X, locally constant functions on M are equivalent to homotopy functionals on paths/loops in X, i.e. they depend only on the homotopy class relative to end points. Therefore, these homotopy functionals descends to functions on $\pi_1(X,x)$.
3. The augmentation ideal I is the kernel of the augmentation of the group algebra
$$\mathbb{R}\pi_1(X,x) \to \mathbb{R}$$
sending $[\gamma] \mapsto 1$. Powers of I define a topology on the group algebra, making sense the notation $\mathrm{Hom}^{\mathrm{cts}}$.

Theorem 3.7 (Chen's π_1-de Rham Theorem)**.** *Integration induces an isomorphism*

$$H^0(Ch^\bullet(P_xX)) \xrightarrow{\approx} \mathrm{Hom}^{\mathrm{cts}}_{\mathbb{R}}(\mathbb{R}\pi_1(X,x),\mathbb{R})$$

Proof. See Hain [Bowdoin, §4]. □

Remark 3.8. The right hand side in the above theorem is, by definition, the coordinate ring $\mathcal{O}(\pi_1^{\mathrm{un}}(X,x)_{/\mathbb{R}})$ of the unipotent completion of $\pi_1(X,x)$ over $\mathbb{R}$. The completion can be defined over $\mathbb{Q}$ by replacing $\mathbb{R}$ with $\mathbb{Q}$.

4. Basics of Mixed Hodge Theory

The standard reference is Deligne [Hodge II]. There are other useful sources: Voisin (Textbook), Peters–Steenbrink (Survey/Monograph), Cattani–El Zein et al (Summer school collections) ...

4.1. **Pure Hodge structures.** Suppose that A is a subring of $\mathbb{R}$, for example, $\mathbb{Z}$, $\mathbb{Q}$, $K \subseteq \mathbb{R}$ a number field, or $\mathcal{O}_K$ its ring of integers.

Definition 4.1. An A-Hodge structure of weight $m \in \mathbb{Z}$ consists of a finitely generated A-module V_A and a bigrading on its complexification

$$V_{\mathbb{C}} = V_A \otimes \mathbb{C} = \bigoplus_{p+q=m} V^{p,q}$$

satisfying $\overline{V^{p,q}} = V^{q,p}$, where $\overline{(\cdot)}$ denotes the complex conjugation on $\mathbb{C}$. We say vectors $v \in V^{p,q}$ are of type (p,q).

Example 4.2 (Prototype)**.** Let X be a compact Kähler manifold (e.g. smooth projective over $\mathbb{C}$), set $A = \mathbb{Z}$ (or $\mathbb{Q}$, $\mathbb{R}$), then

$$V_{\mathbb{Z}} = H^m(X;\mathbb{Z})$$

is a $\mathbb{Z}$-Hodge structure of weight m, with

$$V_{\mathbb{C}} = H^m(X;\mathbb{C}) = \bigoplus_{p+q=m} H^{p,q}(X)$$

where

$$H^{p,q}(X) = \frac{\text{closed forms of type } (p,q)}{\text{exact forms of type } (p,q)} \subseteq H^m(X).$$

Remark 4.3. If V_A is a Hodge structure of odd weight, then

$$\dim_{\mathbb{C}} V_{\mathbb{C}} = \operatorname{rank}_A V_A \equiv 0 \mod 2.$$

Corollary 4.4. $\dim H^{2k+1}(X;\mathbb{C}) \equiv 0 \mod 2$ *for X smooth projective.*

Remark 4.5. No known topological proof that $H^1(X;\mathbb{Z})$ has even rank for X smooth projective.

Example 4.6 (Non-example)**.** There is a diffeomorphism

$$S^3 \times \mathbb{R} \approx \mathbb{C}^2 \backslash \{(0,0)\}$$
$$(\xi, \lambda) \mapsto e^{\lambda}\xi$$

The $\mathbb{Z}$-action on $S^3 \times \mathbb{R}$

$$n : (\xi,\lambda) \mapsto (\xi, \lambda + n)$$

corresponds to the $\mathbb{Z}$-action on $\mathbb{C}^2\backslash\{(0,0)\}$

$$n : (z,w) \mapsto e^n(z,w) = (e^n z, e^n w),$$

which is free and properly discontinuous. We thus get a quotient

$$X := \mathbb{Z}\backslash(\mathbb{C}^2\backslash\{(0,0)\}) \approx S^3 \times S^1.$$

This is a compact complex manifold (in fact a Hopf manifold). Its Betti numbers b_j are

$\deg j$	b_j
0	1
1	1
2	0
3	1
4	1

By the previous corollary, X is not Kähler, and thus not projective.

Suppose that we have a family

$$\begin{tikzcd} X_t \arrow[d, dash] & \subset & \mathcal{X} \arrow[d, dash] \\ t & \in & T \end{tikzcd}$$

where T is a contractible smooth complex manifold (e.g. T = disk), and X_t is smooth projective for all $t \in T$. Then we have

$$\begin{tikzcd} \mathcal{X} \arrow[d, dash] & \approx & X_0 \times T \simeq X_0 \arrow[d, dash] \\ T \arrow[rr, equal] & & T \end{tikzcd}$$

and natural isomorphisms

$$H^m(X_t) \cong H^m(X_0).$$

Fix $X := X_0$ and its $H^m(X;\mathbb{C})$ as a reference vector space, then

$$H^{p,q}(X_t) \subseteq H^m(X;\mathbb{C})$$

is a subspace, varying with t, of dimension

$$h_t^{p,q} := \dim H^{p,q}(X_t) = \text{const} =: h^{p,q}.$$

This gives rise to a map to the Grassmannian

$$\varphi^{p,q} : T \to \mathrm{Gr}_{h^{p,q}}(H^m(X;\mathbb{C})).$$

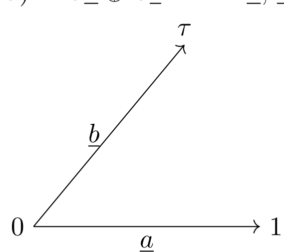
Question: Is this map holomorphic?

Note that we have $\varphi^{p,q} = \overline{\varphi^{q,p}}$. All $\varphi^{p,q}$ being holomorphic would imply that all $\varphi^{p,q}$ are constant. But this is NOT THE CASE!

Take $T = \mathfrak{h}$ the upper half plane and a family

$$\begin{tikzcd} E_\tau := \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau) \arrow[d, dash] & \subset & \mathcal{E} \arrow[d, dash] \\ \tau & \in & \mathfrak{h} \end{tikzcd}$$

We have $H^{1,0}(E_\tau) \subseteq H^1(X;\mathbb{C}) = \mathbb{C}\check{\underline{a}} \oplus \mathbb{C}\check{\underline{b}}$ where $\check{\underline{a}}$, $\check{\underline{b}}$ are duals of $\underline{a}$, $\underline{b}$.



The canonical 1-form $\omega_\tau \in H^1(E_\tau)$ can be written as

$$\omega_\tau = dz = \check{\underline{a}} + \tau\check{\underline{b}},$$

which varies holomorphically in the family.

In general, define the *Hodge filtration*

$$F^p V_{\mathbb{C}} = \bigoplus_{s \geq p} V^{s,m-s}.$$

These vary holomorphically in families!

Definition 4.7 (Hodge structure defined by $F^\bullet$)**.** An A-Hodge structure of weight m consists of a finitely generated A-module V_A, whose complexification $V_{\mathbb{C}} = V_A \otimes \mathbb{C}$ has a decreasing filtration (called the Hodge filtration)

$$\cdots \supseteq F^p \supseteq F^{p+1} \supseteq \cdots$$

satisfying

$$F^p \bigoplus \overline{F^{m-p+1}} \approx V_{\mathbb{C}}$$

for all p.

Remark 4.8. This definition is equivalent to the previous one. One just needs to note that

$$V^{p,q} = F^p \cap \overline{F^q}.$$

Definition 4.9 (1-dimensional Hodge structures/Tate twists)**.** Define $A(p)$ the Hodge structure of weight $-2p$, whose underlying A-module is V_A. Its complexification $V_{\mathbb{C}} = V^{-p,-p}$. It is commonly written as

$$A(p) = (2\pi i)^p \cdot A \subseteq \mathbb{C},$$

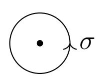
for example, $\mathbb{Z}(p) = (2\pi i)^p\ \mathbb{Z} \subseteq \mathbb{C}$.

Remark 4.10. In the context of periods, for $A(p)$, we pick a Betti basis e^B for V_A and a de Rham basis e^{dR} for $V_{\mathbb{C}}$, then we have

$$\begin{aligned} V_A \otimes \mathbb{C} &\xrightarrow{\approx} V_{\mathbb{C}} \\ e^B &\mapsto (2\pi i)^p e^{\mathrm{dR}}. \end{aligned}$$

Example 4.11 (Basic example: $H^1(\mathbb{G}_m) \cong \mathbb{Z}(-1)$)**.** We view this in the context of periods.

Betti: We have $\sigma \in H_1^B(\mathbb{G}_m)$, and its dual $e^B = \check{\sigma} \in H_B^1(\mathbb{G}_m)$.



DR: We have $e^{\mathrm{dR}} = \frac{dz}{z} \in H^1_{\mathrm{dR}}(\mathbb{G}_{m/\mathbb{Q}})$. And we have

$$\begin{array}{ccc} H_B^1(\mathbb{G}_m) \otimes \mathbb{C} & \xleftarrow[\approx]{\text{comp}} & H^1_{\mathrm{dR}}(\mathbb{G}_m) \otimes \mathbb{C} \\[2em] (\gamma \mapsto \int_\gamma \omega) & \longleftarrow\!\!\shortmid & \omega \qquad \forall \omega \in H^1_{\mathrm{dR}}, \gamma \in H_1^B \\[2em] (\sigma \mapsto \int_\sigma \frac{dz}{z} = 2\pi i) = 2\pi i \check{\sigma} & \longleftarrow\!\!\shortmid & \frac{dz}{z} \\[2em] e^B = \check{\sigma} = (\sigma \mapsto 1) & \longmapsto & (2\pi i)^{-1}\frac{dz}{z} = (2\pi i)^{-1} e^{\mathrm{dR}} \end{array}$$

Definition 4.12. A *polarization* on a $\mathbb{Q}$-Hodge structure of weight m is a non-degenerate $(-1)^m$-symmetric bilinear form

$$S : V_{\mathbb{Q}} \otimes V_{\mathbb{Q}} \to \mathbb{Q}$$

satisfying the Riemann–Hodge bilinear relations:

(1) $S(V^{p,q}, \overline{V^{r,s}}) = 0$ unless $p = r$, $q = s$

(2) $i^{p-q}S(v,\overline{v}) > 0$ for all $0 \neq v \in V^{p,q}$

Example 4.13. Let C be a compact Riemann surface, then $V = H^1(C)$ is a Hodge structure of weight 1. Define

$$S(u,v) = \int_C u \wedge v$$

for $u, v \in V$. It is a polarization on V:

$$i \int_C \omega \wedge \overline{\omega} > 0$$

for any $0 \neq \omega \in H^0(\Omega^1_C)$.

Theorem 4.14 (Hodge)**.** *If X is smooth projective, then $H^m(X)$ is a polarizable Hodge structure.*

The category of $\mathbb{Q}$-polarized Hodge structures ($\mathbb{Q}$-PHS) is semisimple.

Proposition 4.15. *If (V,S) is a $\mathbb{Q}$-PHS of weight m and $A \subseteq V$ a sub Hodge structure, then*

(1) *$S|_A$ is non-degenerate, so that $V_{\mathbb{Q}} = A_{\mathbb{Q}} \oplus A^{\perp}_{\mathbb{Q}}$*
(2) *$V = A \oplus A^{\perp}$ as $\mathbb{Q}$-PHS*

There is a natural correspondence between PHS of weight one and the first cohomology of abelian varieties.

Proposition 4.16. *Suppose X, Y are abelian varieties and $H^1(X;\mathbb{Q}) \cong H^1(Y;\mathbb{Q})$ as $\mathbb{Q}$-Hodge structures, then X is isogenous to Y.*

Theorem 4.17. *If X, Y are abelian varieties, and*

$$H^1(Y;\mathbb{Q}) \subseteq H^1(X;\mathbb{Q})$$

then X is isogenous to $Y \times Z$ where Z is an abelian variety associated to $H^1(Y;\mathbb{Q})^{\perp}$ in $H^1(X;\mathbb{Q})$.

Remark 4.18. Every abelian variety is isogenous to a product of simple abelian varieties.

4.2. **Mixed Hodge structures.** Based on knowledge on l-adic cohomology and guided by the theory of motives, Deligne in [Hodge I] indicates that analogously a natural mixed Hodge structure can be put on Betti cohomology. Even limit mixed Hodge structure is discussed at the end (loc. cit.).

Definition 4.19. An ($\mathbb{Z}$-)mixed Hodge structure (MHS) consists of

(1) a finitely generated $\mathbb{Z}$-module $V_{\mathbb{Z}}$
(2) an increasing filtration (*weight filtration*) $W_\bullet$ on $V_{\mathbb{Q}} = V_{\mathbb{Z}} \otimes \mathbb{Q}$

$$\cdots \subseteq W_n V_{\mathbb{Q}} \subseteq W_{n+1} V_{\mathbb{Q}} \subseteq \cdots$$

(3) a decreasing filtration (*Hodge filtration*) $F^\bullet$ on $V_{\mathbb{C}} = V_{\mathbb{Z}} \otimes \mathbb{C}$

$$\cdots \supseteq F^p V_{\mathbb{C}} \supseteq F^{p+1} V_{\mathbb{C}} \supseteq \cdots$$

such that $\operatorname{Gr}^W_n V = W_n V / W_{n-1} V$ with induced $F^\bullet$

$$F^p \operatorname{Gr}^W_n V_{\mathbb{C}} := (F^p \cap W_n + W_{n-1})/W_{n-1} = (F^p \cap W_n)/(F^p \cap W_{n-1})$$

is a ($\mathbb{Q}$-)Hodge structure of weight n.

Remark 4.20. One can define A-mixed Hodge structures similarly. Unless specified, we will mostly work with $\mathbb{Z}$-mixed Hodge structures in the rest of this section, e.g. we will simply denote $H^m(X;\mathbb{Z})$ by $H^m(X)$.

Theorem 4.21 (Deligne)**.** *The category A-MHS of A-mixed Hodge structures is an abelian tensor category. If A is a field (e.g. $\mathbb{Q}$, $\mathbb{R}$), then the category A-MHS is tannakian. The functors $\mathrm{Gr}^W_\bullet$, $\mathrm{Gr}^\bullet_F$, $\mathrm{Gr}^\bullet_F \mathrm{Gr}^W_\bullet$ are exact.*

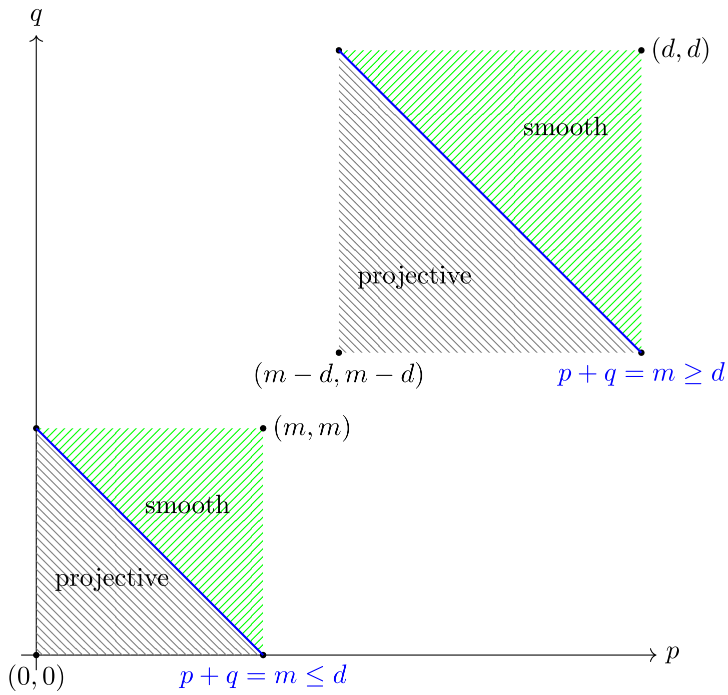
Remark 4.22. The most essential piece of the theorem is to show that the category A-MHS is abelian. This is done by constructing a natural splitting called Deligne splitting. See Deligne [Hodge II, Lemme (1.2.8)] or Griffiths–Schmid [Survey, Lemma (1.12)].

Theorem 4.23 (Deligne)**.** *If X is a complex algebraic variety, then $H^\bullet(X)$ has a natural $\mathbb{Z}$-mixed Hodge structure.*

Remark 4.24. Naturality is on algebraic morphisms, NOT on continuous maps.

Remark 4.25. If X is smooth, then weights on $H^m(X)$ are at least m; if X is projective, then weights on $H^m(X)$ are at most m.

The following diagram indicates possible types (p,q) for $H^m(X)$, with $d = \dim_{\mathbb{C}} X$.

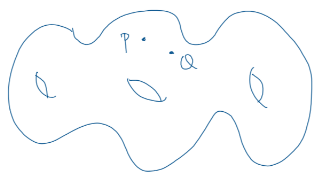


From this we get a table of weights on $H^m(X)$, with $d = \dim_{\mathbb{C}} X$

	general	smooth	projective
$m \le d$	$[0, 2m]$	$[m, 2m]$	$[0, m]$
$m \ge d$	$[2m-2d, 2d]$	$[m, 2d]$	$[2m-2d, m]$

It is instructive to first learn a couple of ad hoc elementary examples picked from Durfee.

Example 4.26 (smooth curve)**.** Let $X = \overline{X} - D$ be a smooth curve, where $\overline{X}$ is a smooth projective curve of genus g, and $D = \{P, Q\}$ is a divisor with $P \neq Q$, $P, Q \in \overline{X}$. We now describe the natural mixed Hodge structure on $H^1(X)$.



We have Gysin sequence

$$0 \to H^1(\overline{X}) \to H^1(X) \xrightarrow{\mathrm{Res}} H^0(D)(-1) \xrightarrow{\deg} H^2(\overline{X})$$

where $H^0(D) = H^0(P) \oplus H^0(Q)$, Res denotes the residue maps at P and Q, deg denotes the degree map on divisors. There is a copy of $\mathbb{Z}$ generated by $(1, -1)$ inside $H^0(D)$ whose degree is zero. One pulls back this from the sequence and obtains a short exact sequence

$$0 \to H^1(\overline{X}) \xrightarrow{\alpha} H^1(X) \xrightarrow{\mathrm{Res}} \mathbb{Z}(-1) \to 0 \tag{4.1}$$

There exists $\omega_{P,Q} \in H^1(X)$ that maps to the generator $(1, -1)$, i.e.

$$\mathrm{Res}_P(\omega_{P,Q}) = 1, \quad \mathrm{Res}_Q(\omega_{P,Q}) = -1.$$

To describe the mixed Hodge structure on $H^1 = H^1(X)$ is equivalent to describing the two filtrations. For the weight filtration, applying $\mathrm{Gr}^W_\bullet$ to (4.1), one easily gets

$$W_0 H^1_{\mathbb{Q}} = 0, \quad W_1 H^1_{\mathbb{Q}} = \mathrm{Im}\,\alpha, \quad W_2 H^1_{\mathbb{Q}} = H^1_{\mathbb{Q}}.$$

For the Hodge filtration, applying $\mathrm{Gr}^\bullet_F$ to (4.1) we have $F^0 H^1_{\mathbb{C}} = H^1_{\mathbb{C}}$, and $F^1 = F^1 H^1_{\mathbb{C}} \subseteq H^1(X; \mathbb{C})$ needs to satisfy

$$F^1 \cap H^1(\overline{X}; \mathbb{C}) = H^{1,0}(\overline{X}) \quad \text{and} \quad \mathrm{Res}\, F^1 = \mathbb{C}.$$

Define

$$F^1 = F^1 H^1(X; \mathbb{C}) := H^0(\Omega^1_{\overline{X}}) \oplus \mathbb{C} \cdot \omega_{P,Q}.$$

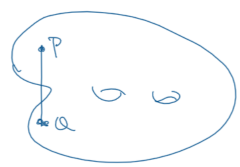
This is well defined, and satisfies the above condition. In general, we would define

$$F^1 := H^0(\Omega^1_{\overline{X}}(P + Q)) = H^0(\Omega^1_{\overline{X}}(\log D)).$$

Example 4.27 (nodal curve)**.** Let C be a nodal curve, obtained by identifying two distinct points P, Q on a smooth projective curve X. As in Example 2.7, we replace C by a simplicial variety $X_\bullet$ whose geometric realization $|X_\bullet|$ is homotopic to C. The degree 0 component X_0 of $X_\bullet$ is X; the degree 1 component X_1 is a point.

$C =$  $X =$ 

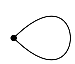
$X_\bullet =$ 

$|X_\bullet| = (X_0 \coprod X_1 \times [0,1])/ \sim =$ 

We have a "cofibration sequence"

$$X \hookrightarrow |X_\bullet| (\simeq C) \to \Gamma$$

where

$\Gamma =$ 

is the dual graph of C/nerve of its (simplicial) covering. This induces a short exact sequence

$$0 \to H^1(\Gamma) \xrightarrow{\alpha} H^1(C) \to H^1(X) \to 0$$

where $H^1(\Gamma) \cong \mathbb{Z}(0)$. To describe the mixed Hodge structure on $H^1 = H^1(C)$, we describe the two filtrations. For the weight filtration, we have

$$W_{-1} H^1_{\mathbb{Q}} = 0, \quad W_0 H^1_{\mathbb{Q}} = \operatorname{Im} \alpha, \quad W_1 H^1_{\mathbb{Q}} = H^1_{\mathbb{Q}}.$$

For the Hodge filtration, we have

$$F^0 H^1_{\mathbb{C}} = H^1_{\mathbb{C}}, \quad F^1 H^1_{\mathbb{C}} = \text{classes } \omega \in H^0(\Omega^1_X) \text{ in } H^1_{\mathbb{C}}.$$

Before moving on to the construction of mixed Hodge structures, we end with an example of MHS on a fundamental group.

Example 4.28 (MHS on $\pi_1^{\mathrm{un}}(\mathbb{C}\backslash S, x)$)**.** Suppose that $S = \{a_1, \cdots, a_n\}$ is a finite set of points in the complex plane $\mathbb{C}$, and $x \in \mathbb{C}\backslash S$. Let

$$A = \mathbb{C}\langle\langle X_1, \cdots, X_n\rangle\rangle$$

be power series in noncommuting indeterminates X_j's. Set $\omega_j = \frac{dz}{z-a_j}$ and

$$\Omega = \sum_{j=1}^{n} \omega_j X_j \in H^0(\Omega^1(\mathbb{C}\backslash S)) \otimes A.$$

We have

$$T := 1 + \int(\Omega) + \int(\Omega|\Omega) + \cdots$$

which is an A-valued iterated integral on $\mathbb{C}\backslash S$ (as all $\int(\omega_{j_1}|\cdots|\omega_{j_r})$ are closed). This induces an isomorphism

$$\Theta : \mathbb{C}\pi_1(\mathbb{C}\backslash S, x)^\wedge \xrightarrow{\approx} \mathbb{C}\langle\langle X_1, \cdots, X_n\rangle\rangle.$$

of completed Hopf algebras with augmentation ideals I and $I_A = (X_1, \cdots, X_n)$. We may view this as a comparison isomorphism from Betti to de Rham for the fundamental group (cf. Theorem 3.7).

Define $W_\bullet$ and $F^\bullet$ on A by giving X_j type $(-1,-1)$, so that $X_J = X_{j_1} \cdots X_{j_r}$ is of type $(-r,-r) = (-|J|, -|J|)$. Define

$$W_m A = \text{span of } X_J\text{: } -2|J| \leq m, \text{ i.e. } |J| \geq -\frac{m}{2}$$

and

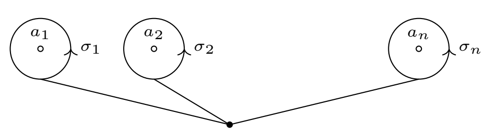
$$F^p A = \text{span of } X_J\text{: } -|J| \geq p, \text{ i.e. } |J| \leq -p$$

These filtrations carry over to the left side of Θ. On the weighted graded pieces, we have

$$\begin{aligned}
&\mathrm{Gr}^W_{-2m}\, \mathbb{Q}\pi_1^\wedge \cong I^m/I^{m+1} \xrightarrow{\Theta_m} \mathrm{Gr}^W_{-2m}\, A \cong I_A^m/I_A^{m+1} \cong V^{\otimes m} \quad \text{where } V = \oplus_{j=1}^n \mathbb{C} \cdot X_j \\
&\quad (\gamma_1 - 1) \cdots (\gamma_m - 1) \mapsto (T(\gamma_1) - 1) \cdots (T(\gamma_m) - 1) \equiv \int_{\gamma_1} \Omega \cdots \int_{\gamma_m} \Omega \mod I_A^{m+1} \\
&\quad (\sigma_{j_1} - 1) \cdots (\sigma_{j_m} - 1) \mapsto (2\pi i)^m X_{j_1} \cdots X_{j_m}
\end{aligned}$$

as

$$\Theta_1(\sigma_j) = T(\sigma_j) - 1 \equiv \int_{\sigma_j} \omega_j X_j = 2\pi i X_j \mod I_A^2.$$



Therefore, we have

$$\mathrm{Gr}^W_{-2m} \cong \bigoplus_{|J|=m} \mathbb{Z}(m).$$

5. Mixed Hodge Structures on Fundamental Groups

In this section, we will put mix Hodge structures on $H^\bullet(X)$ and $H^\bullet(Ch^\bullet(P_{x,y}X))$. The basic setup is

$$\boxed{\text{MHC}} + \boxed{\text{DR Thm}} \to \boxed{\text{MHS}}$$

More specifically, Deligne [Hodge II]

- defines a Mixed Hodge Complex (MHC)
- $H^\bullet$(MHC) is a MHS
- constructs MHC that computes $H^\bullet(X)$

thus putting a MHS on $H^\bullet(X)$.

Hain [Big Red]

- defines multiplicative MHC &
- modules over multiplicative MHC
- $B(M, A, N)$ is MHC if M, A, N are

thus putting a MHS on $H^\bullet(Ch^\bullet(P_{x,y}X))$.

5.1. **Strictness.**

Definition 5.1. A filtered morphism $\varphi : (V_1, F_\bullet) \to (V_2, F_\bullet)$ is strict with respect to filtrations $F_\bullet$ if

$$\operatorname{Im}\varphi \cap F_m V_2 = \operatorname{Im}(F_m V_1)$$

for all m, i.e. if $v \in F_m V_2$ and $v \in \operatorname{Im}\varphi$, then $\exists u \in F_m V_1$, such that $\varphi(u) = v$.

Proposition 5.2. *If $(A^\bullet, F^\bullet)$ is a filtered complex, then*

$$d \text{ is strict with respect to } F^\bullet \iff \{{}_F E_r\} \text{ degenerates at } E_1$$

Proof. We have

$$E_0^s = \operatorname{Gr}_F^s A^\bullet, \qquad E_1^{s,t} = H^{s+t}(\operatorname{Gr}_F^s A^\bullet)$$

and

$$\begin{tikzcd} E_1^{s,t} \arrow[r] \arrow[d, equal] & E_1^{s+1,t} \arrow[d, equal] \\ H^{s+t}(\operatorname{Gr}_F^s) & H^{s+t+1}(\operatorname{Gr}_F^s) \end{tikzcd}$$

Given $\alpha \in H^{s+t}(\operatorname{Gr}_F^s)$, it is represented by $a \in F^s$ such that $da \in F^{s+1}$. Then $d_1\alpha = [da] \in H^{s+t+1}(\operatorname{Gr}_F^{s+1})$ and

$$d_1\alpha = 0 \iff \exists b \in F^{s+1}, \text{ such that } db \equiv da \mod F^{s+2}.$$

One concludes that

$$d_1 = 0 \iff \forall s \ \&\ a \in F^s, \text{ s.t. } da \in F^{s+1}, \exists b \in F^{s+1}, \text{ s.t. } db \equiv da \mod F^{s+2}.$$

(i.e. we have $d(a-b) \in F^{s+2}$, with $a - b \in F^s$. This leads us to the next page E_2.) Suppose $d_1 = 0$ so

$$E_2^{s,t} = E_1^{s,t} = H^{s+t}(\operatorname{Gr}_F^s)$$

then $\alpha \in E_2^{s,t}$ is represented by $a \in F^s$ such that $da \in F^{s+2}$ (for example, $(a - b)$ we just found above). So

$$d_2\alpha = 0 \iff \exists c \in F^{s+2}, \text{ s.t. } dc \equiv da \mod F^{s+3}.$$

Then (given $d_1 = 0$),

$$d_2 = 0 \iff \forall s \;\&\; a \in F^s,\ \text{s.t. } da \in F^{s+2}, \exists c \in F^{s+2},\ \text{s.t. } dc \equiv da \mod F^{s+3}.$$

etc.

It is then easy to see that strictness is equivalent to

$$\forall a \in F^s,\ \text{s.t. } da \in F^{s+t}, \exists \underbrace{b+c+d+\cdots}_{t-\text{terms}} \text{ s.t. } da \equiv d(b+c+d+\cdots) \mod F^{s+t+1}$$

which, in turn, is equivalent to

$$d_1 = d_2 = \cdots = 0,$$

i.e. $\{{}_F E_r\}$ degenerates at E_1. □

5.2. **Hodge to DR spectral sequence.** Let X be a complex manifold. Its de Rham complex

$$E^m_{\mathbb{C}}(X) = \bigoplus_{p+q=m} E^{p,q}(X)$$

with differential $d = \partial + \overline{\partial}$. Define

$$F^p E^m(X) = \bigoplus_{s \geq p} E^{s,m-s}(X).$$

This filtration gives rise to *Hodge to DR spectral sequence* with

$$E_1^{p,q} = H^{p+q}(E^{p,0} \xrightarrow{\overline{\partial}} E^{p,1} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} E^{p,n} \to 0) = H^{p,q}_{\overline{\partial}}(X)$$

where $n = \dim_{\mathbb{C}} X$. There is a fine/acyclic resolution of the sheaf Ω^p_X of differentials:

$$0 \to \Omega^p_X \to \boxed{\mathcal{E}^{p,0}_X \xrightarrow{\overline{\partial}} \mathcal{E}^{p,1}_X \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{E}^{p,n}_X \to 0}$$

By sheaf theory, we have

$$H^q(X, \Omega^p_X) \cong H^{p,q}_{\overline{\partial}}(X).$$

So we have

$$E_1^{p,q} = H^q(X, \Omega^p_X) \implies H^{p+q}(X; \mathbb{C}).$$

Note that for X compact, using Hermitian metric on X we have

$$H^{p,q}_{\overline{\partial}}(X) \cong \mathcal{H}^{p,q}_{\overline{\partial}}(X)$$

where $\mathcal{H}^{p,q}_{\overline{\partial}}(X)$ denotes harmonic forms for the Laplacian $\Delta_{\overline{\partial}}$; and using Riemannian metric on X

$$H^{p+q}(X; \mathbb{C}) \cong \mathcal{H}^{p+q}_d(X)$$

where $\mathcal{H}^{p+q}_d(X)$ denotes harmonic forms for the Laplacian Δ_d.

If X is compact Kähler, we have Kähler identities that give

$$\Delta_d = 2\Delta_{\overline{\partial}}.$$

This implies that

$$\mathcal{H}^m_d(X) = \bigoplus_{p+q=m} \mathcal{H}^{p,q}_{\overline{\partial}}(X).$$

By dimension counting, Hodge to DR spectral sequence degenerates at E_1 for X compact Kähler. By Proposition 5.2, the differential d on $E^\bullet(X)$ is strict with respect to $F^\bullet$.

We provide an easy application for this strictness.

Example 5.3 (Hodge filtration $F^\bullet$ on $H^0(\overline{B}_2(X))$)**.** Let X be compact Kähler. Fix a base point $x \in X$. Then

$$\overline{B}_2(X) := \overline{B}_2(\mathbb{C}, E^\bullet(X), \mathbb{C}) = \{\sum a_{jk}(\omega_j|\omega_k) + (\xi)\}$$

and $H^0(\overline{B}_2(X))$ consists of $\sum a_{jk}(\omega_j|\omega_k) + (\xi)$ with

$$\sum a_{jk}\omega_j \wedge \omega_k + d\xi = 0$$

i.e. $\sum a_{jk}[\omega_j] \otimes [\omega_k] \in K$ where

$$0 \to K \to H^1(X) \otimes H^1(X) \xrightarrow{\cup} H^2(X)$$

cf. Remark 2.21 (2). We have a short exact sequence

$$0 \to H^1(X) \to H^0(\overline{B}_2(X)) \to K \to 0$$

$$\varphi \mapsto \int(\varphi)$$

$$\sum a_{jk}(\omega_j|\omega_k) + (\xi) \mapsto \sum a_{jk}[\omega_j] \otimes [\omega_k]$$

For holomorphic 1-forms $\omega_1, \omega_2 \in F^1$, we have $\omega_1 \otimes \omega_2 \in F^2$. As $\omega_1 \wedge \omega_2 = 0$ we have

$$\int(\omega_1|\omega_2) \in F^2 H^0(\overline{B}_2(X)).$$

For holomorphic 1-forms $\omega_j, \omega_k \in F^1$, suppose $[\omega_j] \cup [\overline{\omega}_k] = 0$, then as $\omega_j \wedge \overline{\omega}_k \in F^1$, by strictness, we can find $\xi \in F^1$ such that

$$\omega_j \wedge \overline{\omega}_k + d\xi = 0.$$

This gives rise to

$$\int(\omega_j|\overline{\omega}_k) + (\xi) \in F^1 H^0(\overline{B}_2(X)).$$

5.3. **Mixed Hodge Complex (MHC).** Let $k \subseteq \mathbb{R}$ be a subfield.

Definition 5.4. A k-MHC is a pair of complexes $K_k^\bullet$ and $K_{\mathbb{C}}^\bullet$ with filtrations

$$\mathbf{K} = ((K_k, W_\bullet), (K_{\mathbb{C}}, W_\bullet, F^\bullet))$$

and

(1) a fixed $W_\bullet$-filtered quasi-isomorphism
$$(K_k^\bullet \otimes \mathbb{C}, W_\bullet) \leftarrow (K_1^\bullet, W_\bullet) \to (K_2^\bullet, W_\bullet) \to \cdots \leftarrow (K_{\mathbb{C}}^\bullet, W_\bullet)$$
(2) $({}_W E_0(K_{\mathbb{C}}), d_0)$ is strictly compatible with $F^\bullet$
(3) ${}_W E_1^{l,m}(\mathbf{K})$ has HS of weight m with respect to induced $F^\bullet$

Remark 5.5. (1) implies that we have isomorphisms on ${}_W E_1$, and

$$(H^\bullet(K_k), W_\bullet) \otimes \mathbb{C} \cong (H^\bullet(K_{\mathbb{C}}), W_\bullet).$$

This makes sense of k-structure (weight filtration) of the HS in (3), while the Hodge filtration $F^\bullet$ comes from (2).

Theorem 5.6 (Deligne)**.** *If* $\mathbf{K}$ *is a* k*-MHC, then* $H^\bullet(\mathbf{K})$ *is a* k*-MHS with*

(1) *induced* $F^\bullet$
(2) $W_m H^j(K_k) = \mathrm{Im}\{H^j(W_{m-j}(K_k)) \to H^j(K_k)\}$
(3) ${}_W E_2 = {}_W E_\infty$*, i.e. weight spectral sequence degenerates at* E_2*.*

Example 5.7. For X smooth projective over $\mathbb{C}$, to construct MHS on $H^\bullet(X)$, we define the complex part of a MHC

$$K_{\mathbb{C}}^\bullet = E_{\mathbb{C}}^\bullet(X)$$

with

$$W_{-1} = 0, \quad W_0 = E_{\mathbb{C}}^\bullet(X).$$

Then we have

$$W_j H^j(K_{\mathbb{C}}) = \operatorname{Im} H^j(W_0 K_{\mathbb{C}}) = H^j(K_{\mathbb{C}})$$

and $W_{j-1}H^j(K_{\mathbb{C}}) = 0$. This is consistent with our expectations (Example 4.2).

5.4. **Logarithmic de Rham complex.** For X a smooth variety over $\mathbb{C}$, but not necessarily projective, we construct logarithmic de Rham complex.

By resolution of singularities (Hironaka), we can write

$$X = \overline{X} - D$$

where D is a divisor with normal crossings, i.e. at each $P \in \overline{X}$, we have local holomorphic coordinates $(z_1, \cdots, z_n)$ on a neighborhood U of P, and $k \leq n$, such that $U \cap D$ is given by

$$z_1 \cdots z_k = 0.$$

We say D has simple normal crossings if each of its components is smooth. One can always blow up a divisor with normal crossings to get a divisor with simple normal crossings.

Now we construct C^∞-logarithmic de Rham complex $E^\bullet(\overline{X} \log D)$ for D with normal crossings. It has two steps:

(1) Construct sheaf of logarithmic differentials $\Omega_{\overline{X}}^\bullet(\log D)$. Define

$$\Omega_{\overline{X}}^\bullet(\log D) := \wedge^\bullet \Omega_{\overline{X}}^1(\log D).$$

For sheaf $\Omega_{\overline{X}}^1(\log D)$, locally on U as above,

$$\Omega_{\overline{X}}^1(\log D)(U)$$

is a free $\mathcal{O}(U)$-module generated by $\Omega^1(U)$ and $\frac{dz_j}{z_j}$ for $j = 1, \cdots, k$. Define weight filtration $W_\bullet$ so that

$$W_r \Omega_{\overline{X}}^m(\log D)(U)$$

have elements of the form

$$\sum_{\substack{|J|=m \\ s \leq r}} f_J(z) \frac{dz_{j_1}}{z_{j_1}} \wedge \cdots \wedge \frac{dz_{j_s}}{z_{j_s}} \wedge dz_{j_{s+1}} \wedge \cdots \wedge dz_{j_m} \tag{5.1}$$

where $f_J(z)$ is holomorphic. Suppose that $D = \bigcup_{j=1}^N D_j$ is a union of its components, and for a subset $J \subseteq \{1, \cdots, N\}$, $D_J = \bigcap_{j \in J} D_j$. We have residue maps

$$\operatorname{Res} : \operatorname{Gr}_r^W \Omega_{\overline{X}}^m(\log D)(U) \xrightarrow{\approx} \bigoplus_{\text{codim } r \text{ components } D_J} \Omega_{\overline{X}}^{m-r}(D_J) \tag{5.2}$$

with element (5.1) mapping to

$$\sum_{|J|=r} f_J(z) dz_{j_{r+1}} \wedge \cdots \wedge d_{z_{j_m}}.$$

(2) Construct C^∞-sheaf of logarithmic differential forms

$$\mathcal{E}^{p,q}_{\overline{X}}(\log D) := \mathcal{E}^{0,q}_{\overline{X}} \otimes \Omega^p_{\overline{X}}(\log D).$$

This is a double complex, we denote the associated total complex by $\mathcal{E}^\bullet_{\overline{X}}(\log D)$. A (p,q)-form, locally on U, can be written as

$$\sum \varphi_{J,K} \frac{dz_{j_1}}{z_{j_1}} \wedge \cdots \wedge \frac{dz_{j_r}}{z_{j_r}} \wedge dz_{j_{r+1}} \wedge \cdots \wedge dz_{j_p} \wedge d\overline{z}_{k_1} \wedge \cdots d\overline{z}_{k_q}.$$

Definition 5.8. Define the logarithmic de Rham complex

$$E^\bullet(\overline{X}\log D) := \Gamma(\overline{X}, \mathcal{E}^\bullet_{\overline{X}}(\log D)).$$

For weight filtration, W_r is generated locally by terms with at most r logarithmic differential forms $\frac{dz_j}{z_j}$. For Hodge filtration, count the number of dz's.

Remark 5.9. If one uses the direct image $j_*\Omega^\bullet_X$ for the inclusion $j : X \to \overline{X}$, then its weight filtration can be defined by counting the order of poles. It is $W_\bullet$-filtered quasi-isomorphic to the logarithmic de Rham complex, see Deligne [Hodge II, (3.1.10)–(3.1.11)]. This explains in Example 5.7, the weight filtration is concentrated in weight 0 as there are no poles.

Theorem 5.10 (Deligne)**.** *$E^\bullet(\overline{X}\log D)$ is a MHC.*

Remark 5.11. Actually, we have only constructed the complex part $K_\mathbb{C}$ of $\mathbf{K}$. The similar construction for the k-part K_k is omitted.

Set

$$D^{[m]} := \coprod_{|J|=m} D_J.$$

For example, $D^{[1]} = \coprod_j D_j$, $D^{[2]} = \coprod_{j<k} D_j \cap D_k$, etc. Suppose D has simple normal crossings, then all $D^{[j]}$ are smooth. By the residue maps (5.2), we have

$${}_W E_1^{-l,m} = \underbrace{H^{m-2l}(D^{[l]})(-l)}_{\text{HS of weight } m}$$

for the weight spectral sequence for $E^\bullet(\overline{X}\log D)$. The differential d_1 are given by Gysin maps[2]. And this spectral sequence degenerates at E_2.

---

[2]This is the degree map that showed up in the Gysin sequence in Example 4.26. In general, if Y is a divisor of X with $\dim X = d$, the inclusion $Y \hookrightarrow X$ induces $H_{2d-j}(Y) \to H_{2d-j}(X)$; Poincaré duality then gives $H^{j-2}(Y)(d-1) \to H^j(X)(d)$, and thus the Gysin map $H^{j-2}(Y)(-1) \to H^j(X)$.

$$
\begin{array}{ccccccc}
H^0(D^{[3]})(-3) & \to & H^2(D^{[2]})(-2) & \to & H^4(D^{[1]})(-1) & \to & H^6(\overline{X}) \\
 & & H^1(D^{[2]})(-2) & \to & H^3(D^{[1]})(-1) & \to & H^5(\overline{X}) \\
 & & H^0(D^{[2]})(-2) & \to & H^2(D^{[1]})(-1) & \to & H^4(\overline{X}) \\
 & & 0 & & H^1(D^{[1]})(-1) & \to & H^3(\overline{X}) \\
 & & 0 & & H^0(D^{[1]})(-1) & \to & H^2(\overline{X}) \\
 & & 0 & & 0 & & H^1(\overline{X}) \\
 & & 0 & & 0 & & H^0(\overline{X})
\end{array}
\qquad
\begin{array}{l}
d_0 \\
\uparrow \\
\cdot \to d_1
\end{array}
$$

Corollary 5.12. *For X smooth, $H^\bullet(X) = H^\bullet(E^\bullet(\overline{X}\log D))$ has a MHS.*

Remark 5.13. One can check that the MHS on $H^1(X)$ is the same as the one we saw in Example 4.26.

Example 5.14. Let $X = \overline{X} - D$ be a smooth variety over $\mathbb{C}$ as before. Fix a base point $x \in X$, we have augmentation

$$e_x : E^\bullet(\overline{X}\log D) \to \mathbb{C}$$

by evaluating at x. This leads to

$$\overline{B}(E^\bullet(\overline{X}\log D)) := \overline{B}(\mathbb{C}, E^\bullet(\overline{X}\log D), \mathbb{C}),$$

the complex part of a MHC. The Hodge filtration is obvious, counting numbers of dz's. The weight filtration is the convolution of $W_\bullet$ on $E^\bullet(\overline{X}\log D)$ with the length filtration on the (reduced) bar construction. For example,

$$(\frac{dz}{z}|\frac{dw}{w}|d\xi) \in F^3$$

and it is in W_{2+3}, where $2 = 1 + 1 + 0$ comes from $W_\bullet$ on $E^\bullet(\overline{X}\log D)$ counting the number of logarithmic forms; 3 comes from the length.

Theorem 5.15 (Hain)**.** *This is a MHC.*

Remark 5.16. Again the k-part of this MHC is omitted.

Corollary 5.17. *For X smooth, $H^\bullet(Ch^\bullet(P_xX)) = H^\bullet(\overline{B}(E^\bullet(\overline{X}\log D)))$ has a MHS.*

One can look back at Example 4.28 to check the understanding of the weight and Hodge filtrations.

6. Deligne's Canonical Extension

6.1. Variations of Hodge structures.

Definition 6.1. A variation of Hodge structures (VHS), $\mathbb{V}$ over a smooth (complex) variety T, of weight m consists of

(1) a $\mathbb{Q}$-local system $\mathbb{V}_{\mathbb{Q}}$ over T, with
$$\mathbb{V}_{\mathbb{C}} := \mathbb{V}_{\mathbb{Q}} \otimes \mathbb{C}.$$
The corresponding holomorphic vector bundle
$$\mathcal{V} := \mathbb{V} \otimes \mathcal{O}_T$$
over T has a flat connection
$$\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_T.$$
(2) Need holomorphic sub-bundles (*Hodge sub-bundles*) $\mathcal{F}^p$ of $\mathcal{V}$
$$\cdots \supseteq \mathcal{F}^p \supseteq \mathcal{F}^{p+1} \supseteq \cdots$$
such that $\cup_p \mathcal{F}^p = \mathcal{V}$, $\cap_p \mathcal{F}^p = 0$.
(3) For each $t \in T$, the fiber
$$V_t := (\mathbb{V}_{\mathbb{Q},t}, (\mathbb{V}_{\mathbb{C},t}, \mathcal{F}^\bullet_t))$$
is a Hodge structure of weight m.
(4) Griffiths transversality:
$$\nabla \mathcal{F}^p \subseteq \mathcal{F}^{p-1} \otimes \Omega^1_T.$$
(Think of this as $\nabla : \mathcal{F}^p(\mathcal{V}) \to \mathcal{F}^p(\mathcal{V} \otimes \Omega^1_T)$)

Example 6.2. Let $f : X \to T$ be a family of smooth projective varieties, i.e. X_t are smooth projective for any $t \in T$. Take
$$\mathbb{V}_{\mathbb{Q}} = R^m f_* \mathbb{Q}.$$
This is a VHS of weight m with fiber V_t and
$$\mathbb{V}_{\mathbb{Q},t} = H^m(X_t; \mathbb{Q}).$$

Definition 6.3. A polarized variation of Hodge structures (PVHS) of weight m over T is a VHS
$$\mathbb{V} = (\mathbb{V}_{\mathbb{Q}}, \mathcal{V}, \mathcal{F}^p)$$
with a $(-1)^m$-symmetric flat bilinear form
$$Q : \mathbb{V} \otimes \mathbb{V} \to \mathbb{Q}$$
such that $Q_t : V_t \otimes V_t \to \mathbb{Q}$ polarizes V_t.

Remark 6.4. All VHS coming from algebraic geometry are polarizable.

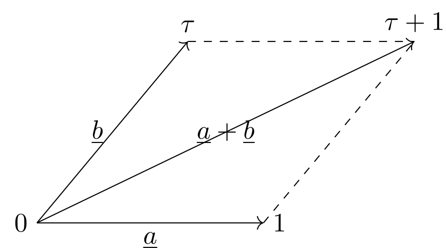
Remark 6.5. The category of PVHS over T is semi-simple (globally). However, it is locally (quasi-)unipotent, cf. Thm. 7.1 (1).

Families usually come with singular fibers. It is very interesting to find out the information near such fibers. We will focus on the 1-dimensional families locally, use the following example as a working example, eventually compute limit mixed Hodge structures in the next section.

Example 6.6 (Canonical Example)**.** We will consider a VHS over $T = \mathbb{D}^*$ a punctured (q-)disk. This comes from considering families of elliptic curves over the upper half plane $\mathfrak{h}$, which is the universal covering space of $\mathbb{D}^*$.

$$\begin{array}{ccc} \mathbb{H}_{\mathfrak{h}} & \to & \mathbb{H}_{\mathbb{D}^*} \\ \Big| & & \Big| \\ \mathfrak{h} & \to & \mathbb{D}^* \\ & & \\ \tau & \mapsto & q = e^{2\pi i \tau} \end{array}$$

(1) $\mathbb{Q}$-local system $\mathbb{H}_{\mathfrak{h}}$: For any $\tau \in \mathfrak{h}$, define $\Lambda_\tau := \mathbb{Z} \oplus \mathbb{Z}\tau$, $E_\tau := \mathbb{C}/\Lambda_\tau$. We have
$$H_1(E_\tau; \mathbb{Z}) \cong \Lambda_\tau \cong \mathbb{Z}\underline{a} \oplus \mathbb{Z}\underline{b} = H_{\mathbb{Z}}.$$
After being tensored with $\mathbb{Q}$, these give rise to a $\mathbb{Q}$-local system $\mathbb{H}_{\mathfrak{h}}$ over $\mathfrak{h}$, whose corresponding vector bundle $\mathcal{H}_{\mathfrak{h}}$ is trivialized by the flat sections $\underline{a}$ and $\underline{b}$.



(2) Monodromy action on the fiber: We consider the monodromy determined by a counter-clockwise loop around 0 in $\mathbb{D}^*$, which corresponds to $\tau \mapsto \tau + 1$ in $\mathfrak{h}$. Therefore, the monodromy is given by
$$\begin{pmatrix} \underline{b} \\ \underline{a} \end{pmatrix} \mapsto \begin{pmatrix} \underline{a} + \underline{b} \\ \underline{a} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \underline{b} \\ \underline{a} \end{pmatrix}.$$
Note that $\underline{a}$ is invariant under the monodromy.

(3) Poincaré duality identifies H_1 with H^1: Using the intersection form $(\underline{a} \cdot \underline{b}) = 1 = -(\underline{b} \cdot \underline{a})$, we define the Poincaré duality
$$\begin{aligned} P : H_1 &\to H^1 \\ u &\mapsto (u \cdot -) \\ \underline{a} &\mapsto (\underline{a} \cdot -) = \check{\underline{b}} \\ \underline{b} &\mapsto (\underline{b} \cdot -) = -\check{\underline{a}} \end{aligned}$$

(4) $\mathbb{H}_{\mathbb{D}^*}$, $\mathcal{H}_{\mathbb{D}^*}$ and its sections: Define a section of $\mathcal{H}_{\mathfrak{h}}$ by
$$\underline{w} := 2\pi i \; dz = 2\pi i(\check{\underline{a}} + \tau \check{\underline{b}}).$$
By Poincaré duality, we can write it as
$$\underline{w} = 2\pi i(-\underline{b} + \tau \underline{a}).$$
One can check that it is invariant under the monodromy action as
$$(-(\underline{a} + \underline{b}) + (\tau + 1)\underline{a}) = -\underline{b} + \tau \underline{a}.$$

Therefore the vector bundle $\mathcal{H}_{\mathbb{D}^*}$, corresponding to $\mathbb{H}_{\mathbb{D}^*}$, is trivialized by $\underline{a}$ and $\underline{w}$:

$$\mathcal{H}_{\mathbb{D}^*} = \mathcal{O}_{\mathbb{D}^*}\underline{a} \oplus \mathcal{O}_{\mathbb{D}^*}\underline{w}.$$

The Hodge subbundle

$$F^1\mathcal{H}_{\mathbb{D}^*} = \mathcal{O}_{\mathbb{D}^*}\underline{w}$$

is generated by $\underline{w}$.

(5) Connection ∇ on $\mathcal{H}_{\mathbb{D}^*}$: On $\mathcal{H}_{\mathfrak{h}}$, sections $\underline{a}$ and $\underline{b}$ are flat, i.e. $\nabla\underline{a} = \nabla\underline{b} = 0$. And we compute

$$\nabla\underline{w} = \nabla 2\pi i(\tau\underline{a} - \underline{b}) = 2\pi i\ d\tau \cdot \underline{a} = \underline{a}\frac{dq}{q}.$$

Therefore, we have connection

$$\nabla = d + \underline{a}\frac{\partial}{\partial\underline{w}}\frac{dq}{q}$$

on $\mathcal{H}_{\mathbb{D}^*}$.

Remark 6.7. As we will see in Example 6.12, we can extend $\mathcal{H}_{\mathbb{D}^*}$ to $\mathbb{D}$ by

$$\mathcal{H}_{\mathbb{D}} := \mathcal{O}_{\mathbb{D}}\underline{a} \oplus \mathcal{O}_{\mathbb{D}}\underline{w}$$

with the same connection ∇. This is Deligne's canonical extension. Note that the connection has regular singularity at $q = 0$ and nilpotent residue

$$\underline{a}\frac{\partial}{\partial\underline{w}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

cf. Remark 6.11.

6.2. **Deligne's canonical extension.** The standard reference is Deligne [ODE]. Careful expositions have also been given by Conrad (notes on Riemann–Hilbert, available on course website) and Hain (last note on his course website). We will mostly follow Hain, adopting his notations.

Suppose that we have a local system of $\mathbb{C}$-vector spaces

$$\mathbb{V} \to \mathbb{D}^*$$

over a punctured disk, and we assume $1 \in \mathbb{D}^*$ (by rescaling). Denote the corresponding flat vector bundle by $\mathcal{V}$. Denote the fiber over $t \in \mathbb{D}^*$ by V_t, and the monodromy

$$h_t : V_t \to V_t$$

is determined by a counter-clockwise loop around 0 in $\mathbb{D}^*$. The local system is determined by any one of these. In particular, it is determined by

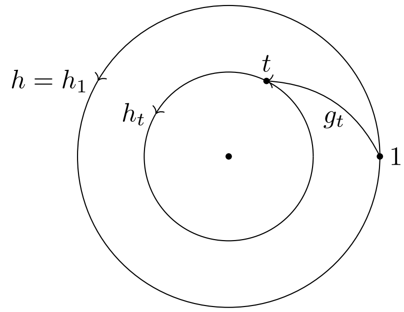
$$h := h_1 \in \operatorname{Aut} V_1.$$

By flatness, we have parallel transport

$$g_t : V_1 \to V_t$$

between fibers, with $g_1 = \mathrm{id}$ and

$$h_t = g_t h g_t^{-1}.$$



Choose a logarithm of h, set

$$N = \frac{1}{2\pi i} \log h.$$

Deligne chooses N such that its eigenvalues λ satisfy

$$0 \leq \mathrm{Re}(\lambda) < 1.$$

In case h is unipotent, there is a canonical choice such that $\mathrm{Re}(\lambda) \equiv 0$, with the finite sum

$$N = \frac{1}{2\pi i} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(h - \mathrm{id})^n}{n} \in \mathrm{End}\, V_1.$$

Set $N_t = g_t N g_t^{-1} \in \mathrm{End}\, V_t$, then $e^{2\pi i N_t} = h_t$. For $v \in V_1$, let $v(t)$ be the flat section of $\mathbb{V}$ over a neighborhood U of t, then $v(t) = g_t v$. Set

$$\varphi(t) := g_t t^{-N} v = t^{-N_t} v(t) \in V_t.$$

A priori, $\varphi(t)$ is multi-valued on $\mathbb{D}^*$, but it is single valued: at $t = 1$, when we analytically continue along the unit circle, we have

$$t^{-N} = e^{-N \log t} \mapsto e^{-N(\log t + 2\pi i)} = e^{-N \log t} \cdot e^{-2\pi i N} = t^{-N} \cdot h^{-1}$$

and $v \mapsto hv$ so that

$$t^{-N} v \mapsto t^{-N} \cdot h^{-1} h v = t^{-N} v.$$

We trivialize $\mathcal{V}$ over $\mathbb{D}^*$ by

$$\begin{aligned} V_1 \times \mathbb{D}^* &\xrightarrow{\approx} \mathcal{V} \\ (v, t) &\mapsto \varphi(t) = g_t t^{-N} v \end{aligned}$$

We call this Deligne's trivialization.

Proposition 6.8. *The pullback of the connection on $\mathcal{V}$ to $V_1 \times \mathbb{D}^*$ is*

$$\nabla = d - N \frac{dt}{t}$$

Proof. Since g_t is a flat section of $\mathrm{Aut}(\mathcal{V})$, we have

$$\nabla(g_t t^{-N} v) = g_t d(t^{-N}) v = g_t(-N t^{-N-1} dt) v = -N(g_t t^{-N} v) \frac{dt}{t}.$$

$\square$

Definition 6.9. Fix a choice of N, Deligne's canonical extension to $\mathbb{D}$ of $\mathcal{V}$ over $\mathbb{D}^*$ is the extension

$$\begin{tikzcd}
\mathcal{V} \arrow[r, "\approx", no head] \arrow[d, no head] & V_1 \times \mathbb{D}^* \arrow[r, hook] \arrow[d, no head] & V_1 \times \mathbb{D} \arrow[d, no head] \\
\mathbb{D}^* \arrow[r, equal] & \mathbb{D}^* \arrow[r, hook] & \mathbb{D}
\end{tikzcd}$$

using Deligne's trivialization above.

Remark 6.10. Since the extension is a trivial bundle, the central fiber V_0 is well defined up to a unique isomorphism (Proposition 6.13), we can write the extension as

$$\begin{tikzcd}
\mathcal{V} \arrow[r, hook] \arrow[d, no head] & V_0 \times \mathbb{D} \arrow[d, no head] \\
\mathbb{D}^* \arrow[r, hook] & \mathbb{D}
\end{tikzcd}$$

and N should be regarded as an element of $\mathrm{End}(V_0)$.

Remark 6.11. In the case when the monodromy is unipotent, and N is nilpotent, Deligne's canonical extension is characterized by the properties:

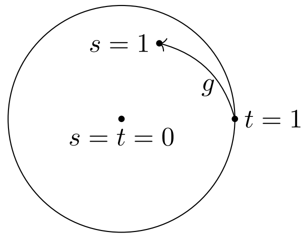
(1) $(\mathcal{V}, \nabla)$ has a regular singularity at 0:
$$\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_{\mathbb{D}}(\log 0).$$

(2) $\mathrm{Res}_0 \nabla$ is nilpotent.

Example 6.12 (Canonical Example, cont'd)**.** In our Example 6.6 (2), in terms of flat sections $\underline{a}$ and $\underline{b}$, the monodromy
$$h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
is unipotent. And we choose canonically
$$N = \frac{1}{2\pi i} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(h - \mathrm{id})^n}{n} = \frac{1}{2\pi i}(h - \mathrm{id})$$
where $\mathrm{id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Note that as $(h - \mathrm{id})^2 = 0$, $N^2 = 0$. In the local coordinate $t = q$,
$$q^{-N} = e^{-N \log q} = \sum_{n=0}^{\infty} \frac{(-N \log q)^n}{n!} = \mathrm{id} + (-N \log q) = \mathrm{id} - N \cdot 2\pi i \tau$$
since $q = e^{2\pi i \tau}$ and $\tau = \frac{\log q}{2\pi i}$. We have
$$q^{-N}\underline{a} = \underline{a}, \quad \text{and} \quad q^{-N}\underline{b} = \underline{b} - \tau \underline{a} = -\frac{\underline{w}}{2\pi i}.$$
Up to a constant multiple, the sections $\underline{a}$ and $\underline{w}$ that we considered before give rise to Deligne's trivialization, and we obtain Deligne's canonical extension by simply extending them across 0, cf. Remark 6.7.

Proposition 6.13. *Deligne's canonical extension does not depend on the choice of holomorphic coordinate t in $\mathbb{D}$.*

Proof. Suppose that s is another holomorphic coordinate in $\mathbb{D}$ centered at 0. Write
$$s(t) = t f(t)$$
with $f(t)$ holomorphic and $f(0) \neq 0$. Without loss of generality, assume $s = 1$ is in $\mathbb{D}^*$ (otherwise rescale or swap $s \leftrightarrow t$).



Denote by g the parallel transport

$$g : V_{t=1} \to V_{s=1}.$$

Set

$$\begin{aligned} h' &= ghg^{-1} \in \operatorname{Aut} V_{s=1}, \\ N' &= gNg^{-1} \in \operatorname{End} V_{s=1}, \\ g'_s &= g_{t(s)}g^{-1} : V_{s=1} \to V_s. \end{aligned}$$

We have the trivialization

$$\begin{tikzcd} V_{s=1} \times \mathbb{D}^* \arrow[rr, "\approx"] \arrow[dr, no head] & & \mathcal{V} \arrow[dl, no head] \\ & \mathbb{D}^* & \end{tikzcd}$$

given by

$$(w, s) \longmapsto g'_s s^{-N'} w.$$

Define

$$\Phi : V_{t=1} \times \underbrace{\mathbb{D}_t}_{t\text{-disk}} \to V_{s=1} \times \underbrace{\mathbb{D}_s}_{s\text{-disk}}$$

by

$$(v, t) \mapsto (gf(t)^N v, s(t)).$$

This is holomorphic at $t = 0$ and $f(0) \neq 0$, so Φ is an isomorphism of holomorphic vector bundles.

Restricting to punctured disks, we have the commutative diagram

$$\begin{tikzcd} V_{t=1} \times \mathbb{D}^*_t \arrow[rr, "\Phi|_{\mathbb{D}^*}"] \arrow[dr, "\approx"'] & & V_{s=1} \times \mathbb{D}^*_s \arrow[dl, "\approx"] \\ & \mathcal{V} & \end{tikzcd}$$

with

$$\begin{tikzcd} (v, t) \arrow[rr, mapsto] \arrow[dr, mapsto] & & (w, s) = (gf(t)^N v, s(t)) \arrow[dl, mapsto] \\ & g_t t^{-N} v = g'_s s^{-N'} w & \end{tikzcd}$$

since

$$\begin{aligned} g'_s s(t)^{-N'}(gf(t)^N v) &= g_t g^{-1} s(t)^{-N'} gf(t)^N v \\ &= g_t s(t)^{-N} f(t)^N v \\ &= g_t t^{-N} v. \end{aligned}$$

□

For any choice of coordinate t, centered at $0 \in \mathbb{D}$,

$$\lim_{P \to 0} t(P)^{-N_P} v(P)$$

exists in V_0, where $P \in \mathbb{D}^*$, $P \mapsto v(P)$ is a flat section of $\mathbb{V}$, and N_P the monodromy at P. This induces parallel transport

$$V_P \xrightarrow{\approx} V_0$$

depending on t, i.e. the map

$$\begin{aligned} H^0(\widetilde{\mathbb{D}^*}, \pi^*\mathbb{V}) &\to V_0 \\ v &\mapsto \lim_{P \to 0} t(P)^{-N_P} v(P) \end{aligned}$$

is an isomorphism for each choice of t, where $\pi : \widetilde{\mathbb{D}^*} \to \mathbb{D}^*$ is the universal covering map. In fact, this gives a $\mathbb{Q}$-structure on V_0:

$$\begin{aligned} H^0(\widetilde{\mathbb{D}^*}, \pi^*\mathbb{V}_{\mathbb{Q}}) \otimes \mathbb{C} &\to V_0 \\ v \otimes 1 &\mapsto \lim_{P \to 0} t(P)^{-N_P} v(P) \end{aligned}$$

Proposition 6.14. *This $\mathbb{Q}$-structure depends only on the tangent vector $\partial/\partial t \in T_0\mathbb{D}$. Denote it by $V_{\partial/\partial t}$. If $\partial/\partial s = \lambda \partial/\partial t$, then the $\mathbb{Q}$-structure $V_{\partial/\partial s}$ on V_0 is*

$$V_{\partial/\partial s} = \lambda^N V_{\partial/\partial t}$$

where $N = \mathrm{Res}_0 \nabla$.

Proof. Suppose that s is another holomorphic coordinate in $\mathbb{D}$ centered at 0. Write

$$s(t) = tf(t)$$

with $f(t)$ holomorphic and $f(0) \neq 0$. Taking differential at 0, we have

$$ds = f(0)\ dt.$$

Taking dual, we get

$$\partial/\partial s = f(0)^{-1} \partial/\partial t \quad \text{and} \quad \lambda = f(0)^{-1}.$$

Let $v \in H^0(\widetilde{\mathbb{D}^*}, \pi^*\mathbb{V}_{\mathbb{Q}})$, then

$$\begin{aligned} \lim_{P \to 0} s(P)^{-N_P} v(P) &= \lim_{P \to 0} f(t(P))^{-N_P} t(P)^{-N_P} v(P) \\ &= f(0)^{-N} \lim_{P \to 0} t(P)^{-N_P} v(P) \qquad \text{need } \lim_{P \to 0} N_P = N \\ &= \lambda^N \lim_{P \to 0} t(P)^{-N_P} v(P) \end{aligned}$$

i.e.

$$V_{\partial/\partial s} = \lambda^N V_{\partial/\partial t}.$$

□

7. Limit Mixed Hodge Structures

The standard reference is Schmid [VHS]. Recall the definition for PVHS from the beginning of last section. We summarize results in [VHS] as follows.

Theorem 7.1 (Schmid). *If $\mathbb{V} \to \mathbb{D}^*$ is a PVHS, then*

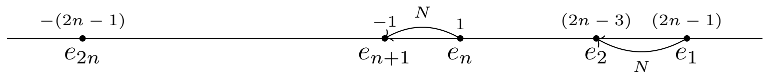
1. *the monodromy h is quasi-unipotent (Remark 6.5), i.e. $\exists n, m$, s.t. $(h^n - \mathrm{id})^m = 0$. This is proved by Landman in geometric situations, and Borel abstractly. Without loss of generality, we will assume that h is unipotent. One can base change to a finite cover of $\mathbb{D}^*$ and pull back the local system:*
$$\begin{array}{ccc} p^*\mathbb{V} & \longrightarrow & \mathbb{V} \\ \Big| & & \Big| \\ \mathbb{D}^* & \xrightarrow{p} & \mathbb{D}^* \end{array}$$
where p is the power map $t \mapsto t^n$.
2. *(Nilpotent Orbit Theorem) If monodromy is unipotent, the Hodge sub-bundles extend to sub-bundles of the canonical extension of $\mathcal{V}$ to $\mathbb{D}$.*
3. *(SL_2 Orbit Theorem) For each non-zero tangent vector $\vec{v} \in T_0\mathbb{D}$, e.g. $\vec{v} = \partial/\partial t$,*
$$(V_{\partial/\partial t}, V_0, F^\bullet, M_\bullet)$$
is a MHS. Here the $\mathbb{Q}$-structure $V_{\partial/\partial t}$ on V_0 is given by Proposition 6.14, the Hodge filtration $F^\bullet$ is given by (2), and the weight filtration is the **monodromy weight filtration** *$M_\bullet$ that we will define next.*

7.1. **Monodromy weight filtration.** Suppose V is a finite dimensional vector space over a field of characteristic zero, and $N : V \to V$ is a nilpotent endomorphism.

Proposition 7.2. *There is a unique filtration $W_\bullet = W_\bullet(N)$ on V such that*

1. *$NW_k \subseteq W_{k-2}$*
2. *The induced map*
$$N^k : \mathrm{Gr}^W_k V \xrightarrow{\approx} \mathrm{Gr}^W_{-k} V$$
on the associated graded is an isomorphism.

Proof. Since N is nilpotent, it has Jordan canonical form with Jordan blocks
$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$
It suffices to prove existence and uniqueness for a single Jordan block, such as above. Suppose it is a $m \times m$-matrix. We can find basis $\{e_j\}$, $j = 1, \cdots, m$, so that
$$Ne_j = e_{j+1}, \quad j < m$$
$$Ne_m = 0$$
The following diagrams indicate the weight filtration: For $m = 2n$,

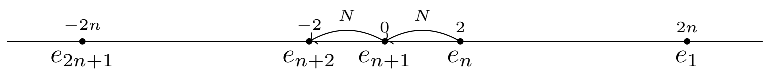


The uniqueness is given by

$$W_{2n-2l} = W_{2n-1-2l} = N^l(V), \quad 0 \leq l \leq 2n-1,$$

$$W_{2n-1} = W_{2n} = W_{2n+1} = \cdots = V \quad \text{and} \quad W_{-2n} = W_{-2n-1} = \cdots = 0.$$

For $m = 2n+1$,



The uniqueness is given by

$$W_{2n+1-2l} = W_{2n-2l} = N^l(V), \quad 0 \leq l \leq 2n,$$

$$W_{2n} = W_{2n+1} = W_{2n+2} = \cdots = V \quad \text{and} \quad W_{-2n-1} = W_{-2n-2} = \cdots = 0.$$

$\square$

Remark 7.3. The weight filtration $W_\bullet = W_\bullet(N)$ is centered at 0.

Recall that if the monodromy h is unipotent, we have a nilpotent endomorphism N on the central fiber, see Remark 6.10.

Definition 7.4. For a PVHS (over $\mathbb{D}^*$) of weight m, i.e. each fiber has HS of weight m, define the monodromy weight filtration

$$M_\bullet := W_{\bullet - m}$$

on the central fiber where $W_\bullet = W_\bullet(N)$ is the filtration defined in the last proposition for the nilpotent endomorphism N.

Remark 7.5. The monodromy weight filtration $M_\bullet = W_{\bullet-m}$ is centered at m.

Remark 7.6. In general, as can be seen from the examples in the next subsection, the monodromy weight filtration of the limit MHS does not depend on the choice of tangent vector.

Example 7.7. We have a nilpotent endomorphism

$$N = \underline{a}\frac{\partial}{\partial \underline{w}}$$

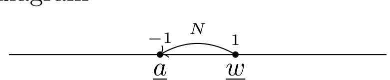
on the central fiber

$$H_0 = \mathbb{C}\underline{a} \oplus \mathbb{C}\underline{w}$$

of the canonical extension of $\mathcal{H}_{\mathbb{D}^*}$. So

$$N\underline{w} = \underline{a}, \quad N\underline{a} = 0,$$

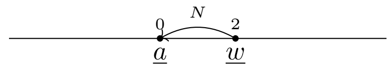
this gives the following diagram



for the weight filtration $W_\bullet = W_\bullet(N)$ given by

$$W_{-2} = 0, \quad W_{-1} = W_0 = \mathbb{C}\underline{a}, \quad W_1 = H_0.$$

Since $\mathbb{H}_{\mathbb{D}^*}$ is a PVHS of weight 1, we have the following diagram



for the monodromy weight filtration $M_\bullet = W_{\bullet-1}$ given by

$$M_{-1} = 0, \quad M_0 = M_1 = \mathbb{C}\underline{a}, \quad M_2 = H_0.$$

So we have

$$\mathrm{Gr}_2^M = \mathbb{C}\underline{w}, \quad \mathrm{Gr}_0^M = \mathbb{C}\underline{a}.$$

7.2. **Examples of limit mixed Hodge structures.** Now we are ready to compute limit mixed Hodge structures for the PVHS $\mathbb{H}_{\mathbb{D}^*}$.

Example 7.8 (Limit MHS at $\partial/\partial q$)**.** By the description before Proposition 6.14, the $\mathbb{Q}$-structure (even $\mathbb{Z}$-structure)

$$H_{\partial/\partial q} = \mathbb{Z}\mathbf{a} \oplus \mathbb{Z}\mathbf{b}$$

where

$$\mathbf{a} = q^{-N}\underline{a} = \underline{a}$$
$$\mathbf{b} = q^{-N}\underline{b} = -\frac{\underline{w}}{2\pi i}$$

is computed in Example 6.12. We view $H_{\partial/\partial q}$ as a Betti $\mathbb{Q}$-structure $H^B_{\partial/\partial q}$, and here N is the Betti version

$$N^B = \frac{1}{2\pi i}\underline{a}\frac{\partial}{\partial \underline{b}}.$$

It is different from the de Rham version

$$N^{\mathrm{dR}} = \underline{a}\frac{\partial}{\partial \underline{w}}$$

appeared in Example 7.7. In fact, by Proposition 6.8, the residue of the connection is $-N^B$, so we have

$$N^{\mathrm{dR}} = -N^B.$$

With the central fiber

$$H_0 = \mathbb{C}\underline{a} \oplus \mathbb{C}\underline{w}, \quad F^1H_0 = \mathbb{C}\underline{w}$$

and the weight filtration $M_\bullet$ we just computed in Example 7.7,

$$\mathrm{Gr}_2^M = \mathbb{C}\underline{w}, \quad \mathrm{Gr}_0^M = \mathbb{C}\underline{a},$$

it is easy to see that (by Remark 4.10)

$$H_{\partial/\partial q} = \mathbb{Z}(0) \oplus \mathbb{Z}(-1)$$

where the weight 0 part $\mathbb{Z}(0)$ is generated by $\mathbf{a} = \underline{a}$, and weight 2 part $\mathbb{Z}(-1)$ by $\mathbf{b} \mapsto -(2\pi i)^{-1}\underline{w}$.

Remark 7.9. To be very precise, N^B and N^{dR} give rise to monodromy weight filtrations $M^B_\bullet$ on $H^B_{\partial/\partial q}$ and $M^{\mathrm{dR}}_\bullet$ on H_0, respectively. These monodromy weight filtrations are compatible under the comparison isomorphism.

Example 7.10 (Limit MHS at $\lambda\partial/\partial q$)**.** We use local coordinate $t = q/\lambda$, since then we have

$$\partial/\partial t = \lambda\partial/\partial q.$$

As the previous example, we compute the $\mathbb{Q}$-structure using $N = N^B = \frac{1}{2\pi i}\underline{a}\frac{\partial}{\partial \underline{b}}$. We have

$$t^{-N} = (q/\lambda)^{-N} = e^{-N\log(q/\lambda)} = \mathrm{id} - N\log(q/\lambda) = \mathrm{id} + \frac{\log\lambda}{2\pi i}\underline{a}\frac{\partial}{\partial \underline{b}} - \frac{\log q}{2\pi i}\underline{a}\frac{\partial}{\partial \underline{b}}$$

and the $\mathbb{Q}$-structure $H_{\lambda\partial/\partial q} = \mathbb{Z}\mathbf{a} \oplus \mathbb{Z}\mathbf{b}$ with

$$\begin{aligned} \mathbf{a} &= t^{-N}\underline{a} = \underline{a} \\ \mathbf{b} &= t^{-N}\underline{b} = \underline{b} + \log\lambda\frac{\underline{a}}{2\pi i} - \tau\underline{a} = -\frac{\underline{w}}{2\pi i} + \log\lambda\frac{\underline{a}}{2\pi i} \end{aligned}$$

Then

$$\underbrace{\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix}}_{\text{Betti basis}} = \underbrace{\begin{pmatrix}\underline{a} & \underline{w}\end{pmatrix}}_{\text{DR basis}} \begin{pmatrix} 1 & \frac{\log\lambda}{2\pi i} \\ 0 & -\frac{1}{2\pi i}\end{pmatrix}.$$

We have the central fiber

$$H_0 = \mathbb{C}\underline{a} \oplus \mathbb{C}\underline{w}, \quad F^1H_0 = \mathbb{C}\underline{w}$$

and the weight filtration $M_\bullet$ with

$$\mathrm{Gr}_2^M = \mathbb{C}\underline{w}, \quad \mathrm{Gr}_0^M = \mathbb{C}\underline{a}.$$

The weight 0 part $\mathbb{Z}(0)$ is again generated by $\mathbf{a} = \underline{a}$, and the quotient of $H_{\lambda\partial/\partial q}$ by $\mathbb{Z}(0)$ is isomorphic to $\mathbb{Z}(-1)$ since modulo a multiple of $\underline{a}$ we have $\mathbf{b} \mapsto -(2\pi i)^{-1}\underline{w}$. Unlike $H_{\partial/\partial q}$, this MHS does not split in general. It is an extension

$$0 \to \mathbb{Z}(0) \to H_{\lambda\partial/\partial q} \to \mathbb{Z}(-1) \to 0$$

of $\mathbb{Z}(-1)$ by $\mathbb{Z}(0)$, and corresponds to

$$\lambda \in \mathbb{C}^* \cong \mathrm{Ext}^1(\mathbb{Z}(-1), \mathbb{Z}(0)).$$

Remark 7.11. To understand this limit MHS geometrically, read Hain's KZB notes, see the picture in Section 16 (ignoring the puncture, with the small loop γ, at the identity of the elliptic curve), and Appendix B for the construction of the fibers over tangent vectors.

7.3. **Limits of VMHS.** Unlike PVHS, for a variation of mixed Hodge structures (VMHS), the existence of limit is not automatic. One needs local admissiblility. Suppose locally we have a local system of MHS

$$\begin{array}{c} (\mathbb{V}, W_\bullet) \\ \Big| \\ \mathbb{D}^* \end{array}$$

and the corresponding vector bundle

$$\begin{array}{c} (\mathcal{V}, F^\bullet) \\ \Big| \\ \mathbb{D}^* \end{array}$$

extends across 0, and the central fiber

$$(V_0, W_\bullet)$$

comes with a nilpotent operator N compatible with $W_\bullet$. Note that

$$H^0(\widetilde{\mathbb{D}^*}, \pi^*\mathbb{V}) \xrightarrow{\approx} V_0$$

and $W_\bullet$ is from VMHS. A particular ingredient for local admissibility is the relative weight filtration of N on $(V_0, W_\bullet)$.

Definition 7.12. $M_\bullet$ is a *relative weight filtration* of N on $(V, W_\bullet)$ if the induced filtration on $\mathrm{Gr}^W_m V$ is $W_\bullet(N)$ reindexed to center at m and $N(M_r) \subseteq M_{r-2}$.

Remark 7.13. For generic nilpotent N on $(V, W_\bullet)$, there is no such $M_\bullet$. For example, see Hain–Matsumoto [UMEM, Appendix A, Example A.5.]

Definition 7.14. An *admissible VMHS* over a smooth algebraic curve $T = \overline{T} - \Sigma$ is a graded polarizable VMHS which is locally admissible at each $s \in \Sigma$.

Example 7.15 (Admissible VMHS)**.**

(1) (Steenbrink–Zucker) For locally topologically trivial familiy of smooth algebraic varieties $f : X \to T$, take

$$\mathbb{V} = R^m f_* \mathbb{Q}.$$

(2) (Gullen–Navarro–Puerta) For locally topologically trivial familiy of varieties (not necessarily smooth; could be simplicial, singular, ...) $f : X \to T$, take

$$\mathbb{V} = R^m f_* \mathbb{Q}.$$

(3) (Hain) For locally topologically trivial familiy of smooth varieties

$$\begin{tikzcd} X \arrow[d, "f"'] \\ T \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

with σ a smooth section. Take local system $\mathbb{V}$ whose fiber at $t \in T$ is

$$V_t := \mathcal{O}(\pi_1^{\mathrm{un}}(X_t, \sigma(t))).$$

8. Regularised Periods

8.1. **Asymptotics of periods.** Suppose that we have dual local systems

$$\begin{tikzcd} \mathbb{V} \arrow[d, dash] & \check{\mathbb{V}} \arrow[d, dash] \\ \mathbb{D}^* & \mathbb{D}^* \end{tikzcd}$$

e.g. $\{H_j(X_t)\}$, $\{H^j(X_t)\} = R^j f_* \mathbb{Q}$ for a family $f : X \to T$. Assume the monodromy is unipotent, and we have Deligne's canonical extensions

$$\begin{tikzcd} \mathcal{V} \arrow[d, dash] & \check{\mathcal{V}} \arrow[d, dash] \\ \mathbb{D} & \mathbb{D} \end{tikzcd}$$

endowed with connection ∇, $\check{\nabla}$, respectively. We have

$$\mathrm{Res}_0 \nabla = -N, \quad \mathrm{Res}_0 \check{\nabla} = \check{N}.$$

Proposition 8.1. *Suppose that $\gamma(t)$ is a flat section of $\mathbb{V}$, and $\omega(t)$ is a holomorphic section of $\check{\mathcal{V}}$, then*

$$\int_{\gamma(t)} \omega(t) := \langle \gamma(t), \omega(t) \rangle$$

is a polynomial

$$\sum_{j=0}^{d} a_j(t)(\log t)^j$$

in $\log t$, where $a_j(t) \in \mathcal{O}(\mathbb{D})$. Furthermore,

$$\lim_{t \to 0} \langle t^{-N_t} \gamma(t), \omega(t) \rangle = a_0(0).$$

Remark 8.2.

(1) The first statement follows from another characterization of Deligne's canonical extension (unipotent case): flat sections of $\mathcal{V}$, $\check{\mathcal{V}}$ have logarithmic growth at 0. See ending remark in Conrad's notes, or higher dimensional generalization in Deligne [ODE, Ch. II, Thm 5.2].
(2) The second statement is how we will compute regularised periods.

Proof. Suppose

$$\{\gamma_1(t), \cdots, \gamma_m(t)\}$$

is a basis of $H^0(\widetilde{\mathbb{D}^*}, \pi^*\mathbb{V})$, and

$$\{\omega_1(t), \cdots, \omega_m(t)\}$$

is a framing of $\check{\mathcal{V}}$ over $\mathbb{D}$. Let

$$\{\varphi_j(t) = t^{-N_t}\gamma_j(t)\}, \quad j = 1, \cdots, m$$

be Deligne's framing[3] of $\mathcal{V}$ over $\mathbb{D}$, and

$$\{\check{\varphi}_1, \cdots, \check{\varphi}_m\}$$

the dual framing of $\check{\mathcal{V}}$, i.e.

$$\langle \varphi_j, \check{\varphi}_k \rangle = \delta_{jk}.$$

Identifying the central fiber V_0 with

$$\bigoplus_{j=1}^{m} \mathbb{C}\varphi_j,$$

then

$$N = -\operatorname{Res}_0 \nabla$$

acts on V_0, cf. Proposition 6.8. We write

$$(N\varphi_1, \cdots, N\varphi_m) = (\varphi_1, \cdots, \varphi_m)A$$

where $A \in M_m(\mathbb{C})$ is nilpotent, then

$$N^k \cdot (\varphi_1, \cdots, \varphi_m) = (\varphi_1, \cdots, \varphi_m)A^k$$

and

$$t^N \cdot (\varphi_1, \cdots, \varphi_m) = (\varphi_1, \cdots, \varphi_m)t^A.$$

We have

$$(\omega_1(t), \cdots, \omega_m(t)) = (\check{\varphi}_1, \cdots, \check{\varphi}_m)\Phi(t)$$

[3]This is the framing we used in Deligne's trivialization.

where $\Phi \in \mathrm{GL}_m(\mathcal{O}(\mathbb{D}))$. So

$$\begin{aligned}\langle \begin{pmatrix}\gamma_1(t)\\ \vdots \\ \gamma_m(t)\end{pmatrix}, (\omega_1(t), \cdots, \omega_m(t))\rangle \approx & \langle \begin{pmatrix}t^N\varphi_1\\ \vdots \\ t^N\varphi_m\end{pmatrix}, (\check{\varphi}_1, \cdots, \check{\varphi}_m)\Phi(t)\rangle \\ &= t^{A^T}\langle \begin{pmatrix}\varphi_1\\ \vdots \\ \varphi_m\end{pmatrix}, (\check{\varphi}_1, \cdots, \check{\varphi}_m)\rangle \Phi(t) \\ &= t^{A^T}\Phi(t) \\ &= e^{A^T \log t}\Phi(t) \in M_m(\mathcal{O}(\mathbb{D})[\log t])\end{aligned}$$

since A is nilpotent, $e^{A^T \log t}$ is a polynomial in $\log t$. If

$$\gamma(t) = c_1(t)\gamma_1(t) + \cdots + c_m(t)\gamma_m(t),$$
$$\omega(t) = f_1(t)\omega_1(t) + \cdots + f_m(t)\omega_m(t),$$

then

$$\langle \gamma(t), \omega(t)\rangle \approx (c_1(t), \cdots, c_m(t))e^{A^T \log t}\Phi(t)\begin{pmatrix}f_1(t)\\ \vdots \\ f_m(t)\end{pmatrix}.$$

By the Remark below, we can write

$$\langle \gamma(t), \omega(t)\rangle = \sum_{j=0}^{d} a_j(t)(\log t)^j.$$

Then taking the limit as $t \to 0$, we have

$$\begin{aligned}\lim_{t\to 0}\langle t^{-N_t}\gamma(t), \omega(t)\rangle = & \lim_{t\to 0}\langle t^{-N}\gamma(t), \omega(t)\rangle \\ &= \lim_{t\to 0}(c_1(t), \cdots, c_m(t))\Phi(t)\begin{pmatrix}f_1(t)\\ \vdots \\ f_m(t)\end{pmatrix} \\ &= (c_1(0), \cdots, c_m(0))\Phi(0)\begin{pmatrix}f_1(0)\\ \vdots \\ f_m(0)\end{pmatrix} \\ &= a_0(0).\end{aligned}$$

□

Remark 8.3. Here we are using the fine results of nilpotent orbit theorem, indicated by $\approx, =$ above. Namely, we have used (the nilpotent orbits)

$$t^N\varphi_j, \quad j = 1, \cdots, m$$

to approximate flat sections $\gamma_j(t)$ of $\mathbb{V}$. Their distance $|t^N\varphi_j - \gamma_j(t)|$ is asymptotically bounded by

$$C|t|^\alpha|\log t|^\beta$$

as $t \to 0$, cf. Schmid [VHS, (4.9)]. This also implies that the limit

$$\lim_{t\to 0} t^{-N_t}\gamma(t) = \lim_{t\to 0} t^{-N}\gamma(t)$$

is well defined, which was used implicitly in the last section when computing limit MHS.

8.2. **Regularising iterated integrals.** Suppose X is a smooth projective curve, D is an effective divisor on X. Let $\vec{v} \in T_P X$ be a non-zero tangent vector at $P \in Supp(D)$. For $Q \in X - D$, and $\omega_1, \cdots, \omega_r \in H^0(X, \Omega^1_X(\log D))$, we will regularise

$$\int_{\vec{v}}^{Q} (\omega_1|\cdots|\omega_r).$$

Set

$$A = \mathbb{C}\langle X_1, \cdots, X_n\rangle / I^{r+1}.$$

Embed

$$A \hookrightarrow \mathrm{End}(A)$$

by left multiplication. The A-valued 1-form

$$\Omega = \omega_1 X_1 + \cdots + \omega_n X_n \in H^0(\Omega^1_X(\log D)) \otimes A$$

satisfies $d\Omega = \Omega \wedge \Omega = 0$, thus defines a flat connection

$$\nabla = d + \Omega$$

on the trivial bundle

$$\begin{array}{c} A \times X \\ \Big| \\ X \end{array}$$

with regular singularities along D. Note that $N = -\operatorname{Res}_P \Omega$ with

$$\operatorname{Res}_P \Omega := \sum_{j=1}^{n} \operatorname{Res}_P \omega_j X_j$$

is nilpotent in A.

Define

$$T(z) := \langle 1 + \int(\Omega) + \int(\Omega|\Omega) + \cdots + \int \underbrace{(\Omega|\cdots|\Omega)}_{r}, \gamma_{z,Q}\rangle$$

where $\gamma_{z,Q}$ is a path from z to Q. Then, by formula of d for iterated integrals (Proposition 1.11 (1)), we have

$$dT = -\Omega T,$$

i.e. T is a flat section for $\nabla = d + \Omega$.

Choose a holomorphic coordinate t at P such that

$$\vec{v} = \partial/\partial t.$$

Set

$$\Omega_P := \sum_{j=1}^{n} \overline{\omega}_j X_j (= \operatorname{Res}_P \Omega \frac{dt}{t})$$

where $\overline{\omega}_j = \operatorname{Res}_P \omega_j \frac{dt}{t}$. We will view Ω_P as an A-valued 1-form on the tangent space $T_P X$, cf. Remark 8.4. Since

$$\int_1^t \underbrace{(\frac{dt}{t}|\cdots|\frac{dt}{t})}_{m} = \frac{1}{m!}(\log t)^m,$$

for a path $\gamma_{1,t}$ from 1 to t, we have

$$\langle 1+\int(\Omega_P)+\int(\Omega_P|\Omega_P)+\cdots,\gamma_{1,t}\rangle=\sum_{m=0}^{\infty}\int_1^t\underbrace{(\Omega_P|\cdots|\Omega_P)}_{m}$$

$$=\sum_{m=0}^{\infty}\frac{(\mathrm{Res}_P\,\Omega\log t)^m}{m!}=e^{\mathrm{Res}_P\,\Omega\log t}=t^{\mathrm{Res}_P\,\Omega}=t^{-N}.$$

The regularised iterated integrals are coefficients of

$$\lim_{t\to 0}t^{-N}T(t).$$

For example,

$$\int_{\vec{v}}^{Q}(\omega_1|\cdots|\omega_r)=\text{coefficient of }X_1\cdots X_r.$$

This implies the formula

$$\int_{\vec{v}}^{Q}(\omega_1|\cdots|\omega_r)=\lim_{t\to 0}\sum_{j=0}^{r}\underbrace{\int_1^t(\overline{\omega}_1|\cdots|\overline{\omega}_j)}_{\substack{\text{coeff. of }X_1\cdots X_j\\ \text{in }t^{-N}}}\underbrace{\int_t^Q(\omega_{j+1}|\cdots|\omega_r)}_{\substack{\text{coeff. of }X_{j+1}\cdots X_r\\ \text{in }T(t)}}.$$

Remark 8.4. This formula looks much like the composition of paths formula for usual iterated integrals:

$$\int_{\gamma_1\gamma_2}(\omega_1|\cdots|\omega_r)=\sum_{j=0}^{r}\int_{\gamma_1}(\omega_1|\cdots|\omega_j)\int_{\gamma_2}(\omega_{j+1}|\cdots|\omega_r).$$

Therefore, we can think of the path from $\vec{v}$ to Q as a composition of paths: a path from $\vec{v}$ to P in the tangent space T_PX, followed by a path from P to Q in X, cf. Brown [MMV, Definition 4.4 & Figure 1], which was originated from Deligne [*Le groupe fondamental de la droite projective moins trois points*, 15.9].

Example 8.5 (Regularised periods)**.**

(1) Let $X=\mathbb{P}^1$, $D=\{0,\infty\}$, $Q\in\mathbb{C}^*$, $\vec{v}=\partial/\partial z$ at $0\in\mathbb{P}^1$. Then

$$\int_{\vec{v}}^{Q}\frac{dz}{z}=\lim_{t\to 0}\left(\int_1^t\frac{dz}{z}+\int_t^Q\frac{dz}{z}\right)=\lim_{t\to 0}\int_1^Q\frac{dz}{z}=\log Q.$$

(2) Let X,D,Q be the same as (1), $\vec{v}=\lambda\partial/\partial z$. Take $t=z/\lambda$, then $\dfrac{dt}{t}=\dfrac{dz}{z}$, and

$$\int_{\vec{v}}^{Q}\frac{dz}{z}=\lim_{t\to 0}\left(\int_{t=1\leftarrow z=\lambda}^{t}\frac{dz}{z}+\int_t^Q\frac{dz}{z}\right)=\log Q-\log\lambda.$$

(3) (Drinfel'd Associator $\Phi(X_0,X_1)$)

This is the regularisation of

$$\langle T,\mathrm{dch}\rangle$$

where dch is the path

$$0 \longrightarrow 1$$

in $\mathbb{P}^1 - \{0, 1, \infty\}$ from

$$\vec{\mathsf{v}}_0 := \partial/\partial x \in T_0\mathbb{P}^1$$

to

$$\vec{\mathsf{v}}_1 := -\partial/\partial x \in T_1\mathbb{P}^1$$

and

$$T = 1 + \int(\Omega) + \int(\Omega|\Omega) + \cdots$$

where

$$\Omega = \frac{dz}{z}X_0 - \frac{dz}{1-z}X_1$$

is the 1-form that defines the KZ connection

$$\nabla_{KZ} = d + \Omega$$

on

$$\begin{array}{c} \mathbb{C}\langle\langle X_0, X_1\rangle\rangle \times \mathbb{P}^1 \\ \big| \\ \mathbb{P}^1 \end{array}$$

with regular singularities at $0, 1, \infty$.

Since

$$\operatorname{Res}_0 \frac{dz}{z} = 1, \quad \operatorname{Res}_0(-\frac{dz}{z-1}) = 0,$$
$$\operatorname{Res}_1 \frac{dz}{z} = 0, \quad \operatorname{Res}_1(-\frac{dz}{z-1}) = 1,$$

we have 1-forms

$$\Omega_0 := \operatorname{Res}_0 \Omega \frac{dx}{x} = \frac{dx}{x}X_0, \quad \Omega_1 := \operatorname{Res}_1 \Omega \frac{dx}{x} = \frac{dx}{x}X_1$$

on the tangent spaces $T_0\mathbb{P}^1$ and $T_1\mathbb{P}^1$ respectively. Since

$$\int_1^t(\Omega_0) = (\int_1^t \frac{dx}{x})X_0 = (\log t)X_0, \quad \int_s^1(\Omega_1) = (\int_s^1 \frac{dx}{x})X_1 = (-\log s)X_1,$$

we have

$$\begin{aligned} \Phi(X_0, X_1) &= \lim_{\substack{t\to 0\\ s\to 0}} e^{(\log t)X_0}\langle T, [t, 1-s]\rangle e^{(-\log s)X_1} \\ &= \lim_{\substack{t\to 0\\ s\to 0}} t^{X_0}\langle T, [t, 1-s]\rangle s^{-X_1}. \end{aligned}$$

Here the limit is taken with $s, t > 0$, and $[t, 1-s]$ is the path that traverses the interval. If we write

$$\Phi(X_0, X_1) = \sum_{\substack{w \text{ word}\\ \text{in } X_0, X_1}} c(w)w,$$

then we have

- $c(X_0) = c(X_1) = 0 \implies c(X_0^p) = c(X_1^p) = 0, \quad p > 0.$
- $c(X_1X_0^{k_1-1}\cdots X_1X_0^{k_r-1}) = (-1)^r\zeta(k_1, \cdots, k_r)$
- $c(X_0^pX_1w) = (-1)^p c(X_1(X_0^p \,ш\, w))$
- $c(wX_0X_1^p) = (-1)^p c((w \,ш\, X_1^p)X_0)$

where

$$\zeta(k_1, \cdots, k_r) := \sum_{0<n_1<\cdots<n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}$$

for integers $k_j \geq 1$, $k_r \geq 2$, and $ш$ is the shuffle product for words. The first two items can be computed directly. The last two can be checked by induction on p and using the fact that

$$c(v ш w) = c(v)c(w).$$

This fact follows from a regularised version of Proposition 1.11 (2).

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