Iterated Integrals and Relative Unipotent Completions

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Iterated Integrals

Definition

Let M be a smooth manifold, PM the set of piecewise smooth paths $\gamma : [0,1] \to M$. Suppose that $\omega_1, \cdots, \omega_r \in E^1(M)$ are smooth 1-forms on M, and $\gamma \in PM$. Define the *iterated integral* $\int \omega_1\omega_2\cdots\omega_r$ by

$$\int_\gamma \omega_1\omega_2\cdots\omega_r = \int \cdots \int_{0\le t_1\le\cdots\le t_r\le 1} f_1(t_1)f_2(t_2)\cdots f_r(t_r)dt_1\cdots dt_r,$$

where $f_j(t)dt = \gamma^*\omega_j$.

Remark

Usual integrals can only detect abelian information in the fundamental group $\pi_1(M,x)$, as

$$\int_{\alpha\beta}\omega = \int_\alpha \omega + \int_\beta \omega = \int_{\beta\alpha}\omega,$$

while iterated integrals can detect non-abelian information, e.g. commutators, in $\pi_1(M,x)$

$$\int_{\alpha\beta\alpha^{-1}\beta^{-1}} \omega_1\omega_2 = \int_{\alpha\beta}\omega_1\omega_2 - \int_{\beta\alpha}\omega_1\omega_2 = \begin{vmatrix} \int_\alpha \omega_1 & \int_\beta \omega_1 \\ \int_\alpha \omega_2 & \int_\beta \omega_2 \end{vmatrix}.$$

Basic Properties of Iterated Integrals

Property 1: Change of Variables

$$\int_\gamma f^*\omega_1 \cdots f^*\omega_r = \int_{f\circ\gamma} \omega_1 \cdots \omega_r.$$

Property 2: Shuffle Product Formula (Ree 1958)

$$\int_\gamma \omega_1 \cdots \omega_r \cdot \int_\gamma \omega_{r+1} \cdots \omega_{r+s} = \sum_\sigma \int_\gamma \omega_{\sigma(1)} \cdots \omega_{\sigma(r+s)}$$

where σ runs over all the shuffles of type (r, s).

Property 3: Deconcatenation Formula $\rightarrow$ Coproduct

$$\int_{\alpha\beta} \omega_1 \cdots \omega_r = \sum_{i=0}^{r} \int_\alpha \omega_1 \cdots \omega_i \int_\beta \omega_{i+1} \cdots \omega_r.$$

Property 4: Reverse Path Formula $\rightarrow$ Antipode

$$\int_{\gamma^{-1}} \omega_1 \cdots \omega_r = (-1)^r \int_\gamma \omega_r \cdots \omega_1.$$

De Rham Theorems

Classical de Rham Theorem

Let M be a smooth manifold. There is a canonical isomorphism induced by integration

$$H^j_{\mathrm{dR}}(M) \otimes \mathbb{R} \xrightarrow{\sim} H^j(M;\mathbb{R})$$
$$[\omega] \mapsto (\gamma \mapsto \int_\gamma \omega)$$

π_1-de Rham Theorem (Chen)

Let M be a smooth manifold, $x \in M$ a base point, and $\Gamma := \pi_1(M, x)$ its fundamental group. There is an isomorphism

$$\left\{ \begin{array}{c} \text{Homotopy invariant} \\ \text{iterated integrals on } M \end{array} \right\} \xrightarrow{\sim} \varinjlim_n Hom_{\mathbb{Q}}(\mathbb{Q}[\Gamma]/J^{n+1}, \mathbb{R}) =: \mathcal{O}(\Gamma^{\mathrm{un}})_{\mathbb{R}}$$
$$\int \omega_1 \cdots \omega_r \mapsto (\gamma \mapsto \int_\gamma \omega_1 \cdots \omega_r)$$

where J is the kernel of the augmentation $\epsilon : \mathbb{Q}[\Gamma] \to \mathbb{Q}, \gamma \mapsto 1, \forall \gamma \in \Gamma$. Γ^{un} is the unipotent completion of Γ, and $\mathcal{O}(\Gamma^{\mathrm{un}})$ its coordinate ring over $\mathbb{Q}$.

Iterated Integrals in a Special Case

Example

Let $M = \mathbb{P}^1 - \{0, 1, \infty\} = \mathbb{C} - \{0, 1\}$, 1-forms $\omega_0 = \frac{dz}{z}, \omega_1 = \frac{dz}{1-z}$ on M, and γ be the straight line from 0 to 1. Kontsevich observed that

$$\int_\gamma \underbrace{\omega_1 \overbrace{\omega_0 \cdots \omega_0}^{n_1 - 1}} \cdots \underbrace{\omega_1 \overbrace{\omega_0 \cdots \omega_0}^{n_r - 1}} = \zeta(n_1, \cdots, n_r). \leftarrow \text{multiple zeta values}$$

$$\begin{aligned}
\int_\gamma \omega_1 \omega_0 &= \iint\limits_{0 \le t_1 \le t_2 \le 1} \frac{dt_1}{1 - t_1} \frac{dt_2}{t_2} = \int_0^1 \frac{dt_2}{t_2} \int_0^{t_2} \frac{1}{1 - t_1} dt_1 \\
&= \int_0^1 \frac{dt_2}{t_2} \int_0^{t_2} (1 + t_1 + t_1^2 + \cdots) dt_1 = \int_0^1 \frac{dt_2}{t_2} \left(t_1 + \frac{t_1^2}{2} + \frac{t_1^3}{3} + \cdots \right) \Bigg|_0^{t_2} \\
&= \int_0^1 \frac{dt_2}{t_2} \left(t_2 + \frac{t_2^2}{2} + \frac{t_2^3}{3} + \cdots \right) = \int_0^1 \left(1 + \frac{t_2}{2} + \frac{t_2^2}{3} + \cdots \right) dt_2 \\
&= \left(t_2 + \frac{t_2^2}{2^2} + \frac{t_2^3}{3^2} + \cdots \right) \Bigg|_0^1 = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \zeta(2). \leftarrow \text{zeta values}
\end{aligned}$$

Riemann ζ function

$$\zeta(s) := \sum_{k=1}^{\infty} \frac{1}{k^s}$$

converges for $Re(s) > 1$.
Evaluating Riemann ζ function at integer $s \geq 2$, we obtain zeta values.

Even Zeta Values (Euler, 1734)

$$\zeta(2n) = \frac{(-1)^{n-1} B_{2n}}{2 \cdot (2n)!} (2\pi)^{2n} \in \mathbb{Q}^* \pi^{2n}$$

where B_{2n} are Bernoulli numbers, defined by $\sum_{n \geq 0} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}$.

Bernoulli Numbers B_n

Note that when $n \geq 3$ is odd, $B_n = 0$. The first few nonzero Bernoulli numbers are listed below.

n	B_n	n	B_n	n	B_n	n	B_n	n	B_n
0	1	2	$\frac{1}{6}$	6	$\frac{1}{42}$	10	$\frac{5}{66}$	14	$\frac{7}{6}$
1	$-\frac{1}{2}$	4	$-\frac{1}{30}$	8	$-\frac{1}{30}$	12	$-\frac{691}{2730}$	$\cdots$	$\cdots$

Odd Zeta Values

Folklore Conjecture

$$\pi, \zeta(3), \zeta(5), \zeta(7), \zeta(9), \cdots$$

are algebraically independent over $\mathbb{Q}$.

Theorem (Apéry 1979)

$$\zeta(3) \notin \mathbb{Q}.$$

Theorem (Ball–Rivoal 1999)

There exists infinitely many

$$\zeta(2n+1) \notin \mathbb{Q}.$$

Questions (Open)

$$\zeta(5) \in \mathbb{Q}? \qquad \zeta(3) \in \mathbb{Q}\pi^3?$$

Multiple Zeta Values (MZVs)

$$\zeta(n_1, \cdots, n_r) := \sum_{1 \le k_1 < \cdots < k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}}, \quad n_r \ge 2$$

$$\textit{weight} := n_1 + \cdots + n_r$$

$$\textit{depth} := r$$

Define

$$Z_n := \mathbb{Q}\text{-vector space generated by all weight } n \text{ MZVs.}$$

Set $Z_0 := \mathbb{Q}$.

$$Z := \sum_{n \ge 0} Z_n \subseteq \mathbb{R}$$

is a $\mathbb{Q}$-algebra. Conjectural algebra basis: $\pi^2, \zeta(3), \zeta(5), \zeta(7), \zeta(9), \cdots$.

Question and Failed Answer

Vector space basis for Z_n? Good reasons to think a basis consists of

$$\zeta(n_1, \cdots, n_r)\pi^{2m}$$

with all n_i are odd. However, there are exceptional relations such as

$$28\zeta(3,9) + 150\zeta(5,7) + 168\zeta(7,5) = \frac{5197}{691}\zeta(12).$$

Zagier's Conjecture

Conjecture (Zagier 1994)

1. Define a sequence $\{d_n\}$ satisfying

$$\begin{cases} d_n = d_{n-2} + d_{n-3} & n \geq 3 \\ d_0 = 1, d_1 = 0, d_2 = 1 \end{cases}$$

   Then $\dim_{\mathbb{Q}}(Z_n) = d_n, \quad \forall n \geq 0.$

2. $$Z = \bigoplus_{n \geq 0} Z_n.$$

Remark

1. is known for $n \leq 4$.

$n = 3$: $\zeta(3) = \zeta(1,2)$. (Euler)
$n = 4$: $\zeta(4), \zeta(1,3), \zeta(2,2), \zeta(1,1,2)$ pairwise $\mathbb{Q}$-linearly dependent. (next page)
$n = 5$: $\iff$ $\zeta(2,3), \zeta(3,2)$ $\mathbb{Q}$-linearly independent. (open)

2. is closely related to transcendence.

Example

$$\zeta(3) = \int_{[0,1]} \omega_1\omega_0\omega_0 = \int_{[0,1]} \omega_1\omega_1\omega_0 = \zeta(1,2).$$

$$\implies \text{Zagier's Conjecture is true when } n = 3.$$

$$\zeta(4) = \int_{[0,1]} \omega_1\omega_0\omega_0\omega_0 = \int_{[0,1]} \omega_1\omega_1\omega_1\omega_0 = \zeta(1,1,2)$$

$$\begin{aligned}\zeta(2)\cdot\zeta(2) &= \int_{[0,1]} \omega_1\omega_0 \cdot \int_{[0,1]} \omega_1\omega_0 \\ &= 4\int_{[0,1]} \omega_1\omega_1\omega_0\omega_0 + 2\int_{[0,1]} \omega_1\omega_0\omega_1\omega_0 \\ &= 4\zeta(1,3) + 2\zeta(2,2)\end{aligned}$$

Note that $\zeta(m)\cdot\zeta(n) = \zeta(m,n) + \zeta(n,m) + \zeta(m+n)$. Take $m = n = 2$, we get

$$\zeta(2)\cdot\zeta(2) = 2\zeta(2,2) + \zeta(4)$$

and thus

$$\zeta(4) = 4\zeta(1,3).$$

Therefore,

$$\zeta(4) = \zeta(1,1,2) = 4\zeta(1,3) = \frac{4}{3}\zeta(2,2) \implies \text{Zagier's Conjecture is true when } n = 4.$$

Conjecture (Hoffman 1997)

$$\zeta(n_1, \cdots, n_r), \qquad n_i \in \{2, 3\}$$

form a $\mathbb{Q}$-basis of Z. (Set $\zeta() := 1$.)

Hoffman's Conjecture implies Zagier's Conjecture.

Define for $n \geq 1$

$$D_n := \{(n_1, \cdots, n_r) \big| n_1 + \cdots + n_r = n, \quad n_i = 2 \text{ or } 3\}.$$

Set $|D_0| := 1$. It is easy to check that $|D_1| = 0$, $|D_2| = 1$, and

$$|D_n| = |D_{n-2}| + |D_{n-3}|$$

for $n \geq 3$. It follows that $|D_n| = d_n, \quad \forall n \geq 0$.

Motivic Results

Theorem (Goncharov, Terasoma 2002)

$$\dim_{\mathbb{Q}}(Z_n) \leq d_n.$$

This takes advantage of deep results of motives.

Theorem (Brown 2012)

$$\zeta^{mot}(n_1, \cdots, n_r), \qquad n_i \in \{2, 3\}$$

are linearly independent over $\mathbb{Q}$.

The motivic version of Hoffman's Conjecture is true.

Corollary

The motivic version of Zagier's Conjecture is true, i.e.

$$\dim_{\mathbb{Q}}(Z_n^{mot}) = d_n, \quad \forall n \geq 0$$

where

$$Z_n^{mot} := \mathbb{Q}\text{-vector space generated by all weight } n \text{ motivic MZVs}$$

Periods and Motivic Periods

Definition (Kontsevich–Zagier 2001)

Periods are numbers arising as integrals of rational differential forms, over domains defined by polynomials inequalities with rational coefficients. They look like $\int_\gamma \omega$.

Motivic Periods

- Definition

  $[M, \omega, \gamma]$ – equivalent class of the triple (M, ω, γ)
  M – motive, or a mixed Hodge structure (MHS)
  $\omega \in M_{\mathrm{dR}}$ – a differential form in the de Rham realization M_{dR} of M
  $\gamma \in M_B^\vee$ – a topological cycle in the dual of the Betti realization M_B of M

- Roughly

  M – a vector space
  two basis of M – de Rham vs. Betti
  transition matrix – period matrix; matrix entries – periods

Period Homomorphism

There is an evaluation homomorphism

$$\mathrm{per} : P^{mot} \to P$$

$$[M, \omega, \gamma] \mapsto \int_\gamma \omega$$

$$\zeta^{mot}(n_1, \cdots, n_r) \mapsto \zeta(n_1, \cdots, n_r)$$

Motivic Algebraic Numbers

Algebraic numbers are periods.
Let $X = \operatorname{Spec} F$, where $F = \mathbb{Q}[x]/(P(x))$ for some irreducible polynomial $P(x) \in \mathbb{Q}[x]$. Then we have $\mathcal{O}_X = F$, $H_0(X(\mathbb{C})) = \mathrm{Hom}(F, \mathbb{C})$ and

$$H^0_{\mathrm{dR}}(X) = F, \qquad H^0_B(X) = \mathrm{Hom}(F, \mathbb{C})^\vee.$$

Given an algebraic number $\alpha \in \overline{\mathbb{Q}} \subset \mathbb{C}$ such that $P(\alpha) = 0$. Define

$$\begin{aligned} \sigma_\alpha : F &\to \overline{\mathbb{Q}} \\ x &\mapsto \alpha \end{aligned}$$

so that $\sigma_\alpha \in H_0(X(\mathbb{C})) = \mathrm{Hom}(F, \mathbb{C}) = H^0_B(X)^\vee$.
Motivic algebraic number α^{mot} is defined as follows:

$$\alpha^{mot} := [H^0(X), x, \sigma_\alpha]$$

with $x \in H^0_{\mathrm{dR}}(X)$ and $\sigma_\alpha \in H^0_B(X)^\vee$. Under the period homomorphism,

$$\mathrm{per}(\alpha^{mot}) = \alpha.$$

Motivic Galois Group and Motivic Coaction

Tannakian Formalism

Fiber functor provides equivalence of categories:

$$\text{Tannakian Category} \xrightarrow{\sim} \mathrm{Rep}(G)$$
$$\text{Category of Motives} \xrightarrow{\sim} \mathrm{Rep}(G^{mot})$$
$$\text{Category of Mixed Hodge Structures} \xrightarrow{\sim} \mathrm{Rep}(G_{Hodge})$$

G-module and $\mathcal{O}(G)$-comodule V

Action $\rho : G \times V \to V$ Coaction $\Delta : V \to \mathcal{O}(G) \otimes V$

Thanks to an explicit coaction formula due to Goncharov, which is in turn dual to a formula computed by Ihara much earlier, Brown developed a Galois theory for motivic MZVs and proved the motivic version of Hoffman's conjecture. In particular, we know the algebraic structure of motivic MZVs. Let $\mathcal{H}$ be the algebra of motivic multiple zeta values. There is an isomorphism

$$\phi : \mathcal{H} \xrightarrow{\sim} \mathbb{Q}\langle f_3, f_5, \cdots \rangle \otimes \mathbb{Q}[f_2]$$
$$\zeta^{mot}(2n+1) \mapsto f_{2n+1}$$
$$\zeta^{mot}(2) \mapsto f_2$$

Finer Structure with Modular Phenomenon

Zagier's Conjecture [*Another Formulation*]

Let $d_n = \dim_{\mathbb{Q}}(Z_n)$, then

$$\sum_{n=0}^{\infty} d_n s^n = \frac{1}{1 - s^2 - s^3}.$$

Broadhurst–Kreimer Conjecture (1997)

Define $Z_{n,r} :=$ $\mathbb{Q}$-vector space generated by all weight n, depth r MZVs. Then

$$\sum_{n,r \geq 0} \dim_{\mathbb{Q}}(Z_{n,r}) s^n t^r = \frac{1 + \mathbb{E}(s)t}{1 - \mathbb{O}(s)t + \mathbb{S}(s)t^2 - \mathbb{S}(s)t^4}$$

where

$$\mathbb{E}(s) = \frac{s^2}{1 - s^2}, \mathbb{O}(s) = \frac{s^3}{1 - s^2}, \mathbb{S}(s) = \frac{s^{12}}{(1 - s^4)(1 - s^6)}.$$

Remark

The series $\mathbb{E}(s)$ and $\mathbb{O}(s)$ are the generating series for the dimensions of the spaces of even and odd zeta values respectively, and $\mathbb{S}(s)$ is the generating series for the dimensions of the space of cusp forms for the full modular group $\mathrm{SL}_2(\mathbb{Z})$.

Example [*Depth Defect*] (Gangl–Kaneko–Zagier 2006)

$$28\zeta(3,9) + 150\zeta(5,7) + 168\zeta(7,5) = \frac{5197}{691}\zeta(12).$$

A Natural Idea to Include Modular Forms

Cases	$\mathbb{P}^1 - \{0,1,\infty\} = \mathcal{M}_{0,4}$ i.e. moduli space of genus 0 curves with 4 marked points	$\mathcal{M}_{1,1}$ i.e. moduli space of genus 1 curves with 1 marked points
π_1	$\pi_1(\mathbb{P}^1 - \{0,1,\infty\}) \cong \langle x_0, x_1 \rangle = F_2$	$\pi_1(\mathcal{M}_{1,1}) \cong \mathrm{SL}_2(\mathbb{Z})$
Completions of π_1	Unipotent Completion $\pi_1^{\mathrm{un}}(\mathbb{P}^1 - \{0,1,\infty\})$	Relative Completion $\mathcal{G}^{\mathrm{rel}} := \pi_1^{\mathrm{rel}}(\mathcal{M}_{1,1})$

What is ... Relative Completion?

Cases	Example 1	Example 2
Inputs: Γ discrete group	$F_2 \cong \pi_1(\mathbb{P}^1 - \{0,1,\infty\})$	$\mathrm{SL}_2(\mathbb{Z}) \cong \pi_1(\mathcal{M}_{1,1})$
R reductive group $/\mathbb{Q}$	$\mathbb{1}$	SL_2
$\rho : \Gamma \to R(\mathbb{Q})$ Zariski dense	trivial	$\mathrm{SL}_2(\mathbb{Z}) \hookrightarrow \mathrm{SL}_2(\mathbb{Q})$
Output: $\mathcal{G}$	$F_2^{\mathrm{un}} \cong \pi_1^{\mathrm{un}}(\mathbb{P}^1 - \{0,1,\infty\})$	$\mathcal{G}^{\mathrm{rel}}$

A Natural Idea to Include Modular Forms

In general, the output $\mathcal{G}$ is the *relative completion* of Γ with respect to $\rho : \Gamma \to R(\mathbb{Q})$. It is a pro-algebraic group defined over $\mathbb{Q}$, which is an extension

$$1 \to \mathcal{U} \to \mathcal{G} \to R \to 1$$

of R by a pro-unipotent group $\mathcal{U}$. When R is the trivial group $\mathbb{1}$, then $\mathcal{G} = \mathcal{U}$ is the *unipotent completion* of Γ, which we usually denote by Γ^{un}.
Universal property of relative completion is similar to that of unipotent completion:

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[dr, "\phi"'] & \Gamma^{\mathrm{un}}(\mathbb{Q}) \arrow[d, dashed, "\exists !"] \\ & G_\phi(\mathbb{Q}) \end{tikzcd}$$

(a) unipotent

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[dr, "\varphi"'] & \mathcal{G}^{\mathrm{rel}}(\mathbb{Q}) \arrow[d, dashed, "\exists !"] \\ & G_\varphi(\mathbb{Q}) \end{tikzcd}$$

(b) relative

Here

$$G_\phi \cong \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{while} \quad G_\varphi \cong \begin{pmatrix} R\text{-reps} & * & * & * \\ 0 & R\text{-reps} & * & * \\ 0 & 0 & R\text{-reps} & * \\ 0 & 0 & 0 & R\text{-reps} \end{pmatrix}.$$

Where are ... Modular Forms?
By Levi, the extension

$$1 \to \mathcal{U}^{\mathrm{rel}} \to \mathcal{G}^{\mathrm{rel}} \to \mathrm{SL}_2 \to 1$$

gives

$$\mathcal{G}^{\mathrm{rel}} \cong \mathrm{SL}_2 \ltimes \mathcal{U}^{\mathrm{rel}}.$$

As $\mathcal{U}^{\mathrm{rel}}$ is pro-unipotent, it is equivalent to studying its Lie algebra $\mathfrak{u}^{\mathrm{rel}}$, which is pro-nilpotent.
The Lie algebra $\mathfrak{u}^{\mathrm{rel}}$ is isomorphic to $\mathbb{L}(H_1(\mathfrak{u}^{\mathrm{rel}}))^\wedge$, i.e. a (completed) Lie algebra that is freely and topologically generated by $H_1(\mathfrak{u}^{\mathrm{rel}})$.
There is an isomorphism

$$\begin{aligned} H_1(\mathfrak{u}^{\mathrm{rel}}) &\cong \prod_{n \geq 2} \left(H^1(\mathrm{SL}_2(\mathbb{Z}), S^{2n}H)^* \otimes S^{2n}H \right) \\ &\cong \prod_{n \geq 2} \left(S^{2n}H(2n+1) \oplus \prod_{\substack{f \text{ eigen cusp form} \\ \text{of weight } 2n+2}} V_f \otimes S^{2n}H(2n+1) \right) \end{aligned}$$

where H is the standard representation of SL_2, $S^m H$ its m-th symmetric powers, and V_f the Hodge structure associated to the cusp form f.

Modular Forms of the Second Kind

$H^1(\mathrm{SL}_2(\mathbb{Z}), S^{2n}H)$ has a natural mixed Hodge structure:

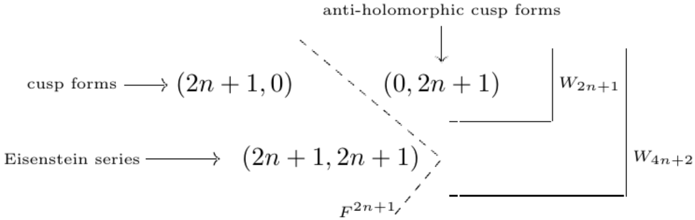


Figure: Hodge types of $H^1(\mathrm{SL}_2(\mathbb{Z}), S^{2n}H)$

Definition

Modular forms of the second kind are algebraic representatives for $H^1(\mathrm{SL}_2(\mathbb{Z}), S^{2n}H)$. (cf. differential forms of the second kind)

Theorem (L. 2018, L. 2023)

(Regularized) Iterated integrals of modular forms (of the second kind) can be constructed. They generalize those studied in part by Brown (2014) and by Manin (2005, 2006).

Theorem (Saad 2020)

Every MZV of weight n and depth r can be expressed as a $\mathbb{Q}$-linear combination of iterated integrals of the form

$$(2\pi i)^n \int_0^{i\infty} E_{2n_1+2}(\tau_1)\tau_1^{b_1}\, d\tau_1 \cdots E_{2n_s+2}(\tau_s)\tau_s^{b_s}\, d\tau_s$$

where $s \leq r$, $0 \leq b_i \leq 2n_i$, the total modular weight $m := (2n_1+2) + \cdots + (2n_s+2)$ is bounded by $n+s$, and

$$E_{2k}(\tau) = -\frac{B_{2k}}{4k} + \sum_{n\geq 1} \sigma_{2k-1}(n) q^n, \quad q := e^{2\pi i \tau}$$

is (level one) Eisenstein series of weight $2k$.

Iterated Integrals of Modular Forms

Examples (MZVs)

$$\zeta(3) = -(2\pi i)^3 \int_0^{i\infty} E_4(\tau)d\tau, \qquad \zeta(5) = -\frac{1}{12}(2\pi i)^5 \int_0^{i\infty} E_6(\tau)d\tau,$$

$$\zeta(3,5) = -\frac{5}{12}(2\pi i)^8 \int_0^{i\infty} E_6(\tau_1)d\tau_1 E_4(\tau_2)d\tau_2 + \frac{503}{2^{13}3^5 5^2 7}(2\pi i)^8.$$

$$\int E_{2n} \leftrightarrow \zeta(2n-1), \quad \int E_{2n}E_{2m} \leftrightarrow \zeta(2n-1, 2m-1)$$

Example (not a MZV)

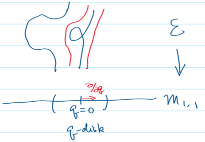
$$600\pi \int_0^{i\infty} E_4(\tau_1)\tau_1 d\tau_1 E_{10}(\tau_2)\tau_2^4 d\tau_2 + 480\pi \int_0^{i\infty} E_4(\tau_1)\tau_1^2 d\tau_1 E_{10}(\tau_2)\tau_2^3 d\tau_2$$

$$= \int_0^{i\infty} \Delta(\tau)\tau^{11} d\tau = \Lambda(\Delta, 12)$$

where $\Delta(\tau)$ is the Ramanujan cusp form of weight 12, and $\Lambda(\Delta, -)$ its completed L-function.

Geometric Explanation of Depth Defect [*Very Sketchy*]

Monodromy and Degeneration + Completions (Hain 2020)



$$\pi_1(\mathcal{M}_{1,1}) \longrightarrow \mathrm{Aut}\left(\mathcal{E}^{\times}_{\partial/\partial q}\right)$$

$$\rightsquigarrow \quad \mathfrak{u}_{1,1} \longrightarrow \mathrm{Der}\left(\mathbb{L}(a,b)\right)$$

Motivic Galois Theory + Hodge Theory (Hain–Matsumoto 2020)

$$0 \to \mathfrak{r} \to \mathbb{L}(\mathbb{E}) \to \mathfrak{u}^{\mathrm{geom}} \to 0$$

Length Two (Pollack 2009)

$$0 \to \bigoplus_n S_{2n} \to \mathbb{E} \wedge \mathbb{E} \to \mathrm{gr}_2 \mathfrak{u}^{\mathrm{geom}} \to 0$$

Depth Graded Motivic Lie Algebra (Brown 2014, Brown 2021)

$$0 \to \bigoplus_n S_{2n} \to \mathbb{D}_1 \wedge \mathbb{D}_1 \to \mathbb{D}_2 \to 0$$

Example with Concrete Computations

One of the period polynomial for Δ is

$$X^8Y^2 - 3X^6Y^4 + 3X^4Y^6 - X^2Y^8.$$

Pollack relations/Ihara–Takao relations

$$[\epsilon_4, \epsilon_{10}] - 3[\epsilon_6, \epsilon_8] = 0 \quad \text{or} \quad [\overline{\sigma}_3, \overline{\sigma}_9] - 3[\overline{\sigma}_5, \overline{\sigma}_7] = 0.$$

There is a one dimensional subspace in weight 12 generated by

$$3f_3 \wedge f_9 + f_5 \wedge f_7,$$

which is dual to the above relation. Using the correspondence

$$\mathbf{e}_{2n} \leftrightarrow \frac{2}{(2n-2)!}\epsilon_{2n}$$

so that dually we get

$$\int E_{2n}E_{2m} \leftrightarrow \frac{(2n-2)!(2m-2)!}{2^2} f_{2n-1}f_{2m-1},$$

and we are led to the following example.

Example with Concrete Computations

Example (MZV via Double Iterated Integrals of Modular Forms)

$$9\int_0^{i\infty} E_4(\tau_1)d\tau_1 E_{10}(\tau_2)d\tau_2 + 14\int_0^{i\infty} E_6(\tau_1)d\tau_1 E_8(\tau_2)d\tau_2$$
$$= -\frac{3^3\cdot 5\cdot 7}{2^6}\left(\frac{1}{9}\zeta(3,9) + 3\zeta(3)\zeta(9) + \frac{5}{3}\zeta(5)\zeta(7) - \frac{31\cdot 139}{2\cdot 691}\zeta(12)\right).$$

The left hand side corresponds to

$$9[\mathbf{e}_4^\vee, \mathbf{e}_{10}^\vee] + 14[\mathbf{e}_6^\vee, \mathbf{e}_8^\vee]$$

which in turn gives back

$$84\cdot 6!\cdot(3f_3\wedge f_9 + f_5\wedge f_7).$$

This particular linear combination of double iterated integrals of modular forms becomes a MZV, as it precisely cancels out the non-critical value $\Lambda(\Delta, 12)$ appeared previously.

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Thank you for your attention!