Galois Theory for Multiple Modular Values

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Zeta Values

The Riemann zeta values are

$$\zeta(n) = \sum_{k\geq 1} \frac{1}{k^n} \quad \text{for } n \geq 2.$$

Euler's Theorem

$$\zeta(2n) = -\frac{B_{2n}}{2}\frac{(2\pi i)^{2n}}{(2n)!},$$

where B_{2n}'s are Bernoulli numbers.

Folklore Conjecture

The odd Riemann zeta values $\zeta(3), \zeta(5), \zeta(7), \cdots$ are algebraically independent over $\mathbb{Q}[\pi]$.

Note: Known results do not go beyond irrationality, let alone transcendence.

Multiple Zeta Values

Multiple zeta values (MZV's) are

$$\zeta(n_1, \cdots, n_r) = \sum_{1 \le k_1 < \cdots < k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}}$$

where $n_1, \cdots, n_r \ge 1$ and $n_r \ge 2$ to ensure convergence. Each MZV has a weight $n_1 + \cdots + n_r$ and a depth r.

These numbers naturally arise when one searches algebraic relations among zeta values, for example,

$$\zeta(m)\zeta(n) = \sum_{k \ge 1} \frac{1}{k^m} \sum_{l \ge 1} \frac{1}{l^n} = \Big(\sum_{k<l} + \sum_{l<k} + \sum_{k=l} \Big) \frac{1}{k^m l^n}$$
$$= \zeta(m, n) + \zeta(n, m) + \zeta(m + n).$$

All known linear relations between MZV's respect weight, but not depth. For example of this depth defect, (Gangl–Kaneko–Zagier)

$$28\zeta(3, 9) + 150\zeta(5, 7) + 168\zeta(7, 5) = \frac{5197}{691}\zeta(12).$$

Transcendence Conjecture

Conjecture (Zagier)

Let $\mathcal{Z}_n$ be the $\mathbb{Q}$-vector space of MZV's of weight n, and D_n be its dimension, then

$$\sum_{n\geq 0} D_n t^n = \frac{1}{1-t^2-t^3}$$

Conjecture (Broadhurst–Kreimer)

Let $\mathcal{Z}_{n,d}$ be the $\mathbb{Q}$-vector space of MZV's of weight n and depth d, and $D_{n,d}$ be its dimension, then

$$\sum_{n,d\geq 0} D_{n,d} s^d t^n = \frac{1+E(t)s}{1-O(t)s-S(t)s^2+S(t)s^4}$$

where $E(t)=\frac{t^2}{1-t^2}$, $O(t)=\frac{t^3}{1-t^2}$, $S(t)=\frac{t^{12}}{(1-t^4)(1-t^6)}$.

Periods: Elementary Definition and Galois Theory

A period is a complex number whose real and imaginary parts are integrals of rational differential forms, over domains defined by polynomials inequalities with rational coefficients. (Kontsevich, Zagier)

Example

$$\log(2) = \int_{1 \leq z \leq 2} \frac{dz}{z}, \qquad \zeta(2) = \int_{0<t_1<t_2<1} \frac{dt_1}{1-t_1} \frac{dt_2}{t_2},$$

MZV's are periods given by iterated integrals (next slide).

Grothendieck's framework of motives (later) suggests that there should be a Galois theory for periods: there is a motivic Galois group that acts on periods.

Iterated Integrals

Definition

Let M be a smooth manifold, PM the set of piecewise smooth paths $\gamma : [0,1] \to M$. Suppose that $\omega_1, \cdots, \omega_r$ are smooth 1-forms on M, and $\gamma \in PM$. Define the iterated integral $\int \omega_1\omega_2\cdots\omega_r$ by

$$\int_\gamma \omega_1\omega_2\cdots\omega_r = \int\cdots\int_{0\leq t_1\leq\cdots\leq t_r\leq 1} f_1(t_1)f_2(t_2)\cdots f_r(t_r)dt_1\cdots dt_r,$$

where $f_j(t)dt = \gamma^*\omega_j$.

Examples

Let $M = \mathbb{P}^1(\mathbb{C}) - \{0,1,\infty\}$, 1-forms $\omega_0 = \frac{dz}{z}, \omega_1 = \frac{dz}{1-z}$ on M, and γ be the straight line from 0 to 1. Then

$$\int_\gamma \omega_1 \overbrace{\omega_0\cdots\omega_0}^{n_1-1}\cdots\omega_1\overbrace{\omega_0\cdots\omega_0}^{n_r-1} = \zeta(n_1,\cdots,n_r).$$

Periods: Algebraic and π_1-de Rham Theorems

Algebraic de Rham Theorem (Grothendieck)

Let $X_{/\mathbb{Q}}$ be a smooth variety. There is a comparison isomorphism

$$\mathrm{comp}_{B,dR} : H^j_{dR}(X_{/\mathbb{Q}}) \otimes \mathbb{C} \xrightarrow{\sim} H^j_B(X(\mathbb{C}); \mathbb{Q}) \otimes \mathbb{C}$$

The comparison isomorphism is induced by integration. Periods are manifestations of different $\mathbb{Q}$-structures on both sides.

π_1-de Rham Theorem (Chen)

Let M be a complex manifold, $p \in M$ a base point, and $\pi_1 := \pi_1(M, p)$ its fundamental group. There is an isomorphism

$$\left\{ \begin{array}{c} \text{Closed iterated} \\ \text{integrals on } M \end{array} \right\} \xrightarrow{\sim} \varinjlim Hom(\mathbb{C}[\pi_1]/J^{n+1}, \mathbb{C}) =: \mathcal{O}(\pi_{1/\mathbb{Q}}^{\mathrm{un}}) \otimes \mathbb{C}$$

One can develop an algebraic π_1-de Rham theory; the cases of elliptic curves and the modular curve have been worked out (L.).

Motivic Periods

Let $\mathcal{T}$ be a tannakian category over $\mathbb{Q}$, with objects $V = (V_B, V_{dR}, \mathrm{comp})$, two fiber functors

$$\omega_B, \omega_{dR} : \mathcal{T} \to \mathrm{Vec}_{\mathbb{Q}}, \quad V \mapsto V_B, V_{dR}$$

and a functorial isomorphism $V_{dR} \otimes \mathbb{C} \xrightarrow{\sim} V_B \otimes \mathbb{C}$. Then $\mathcal{T}$ is equivalent to the category of representations of an affine, i.e. pro-algebraic, group $G_{\mathcal{T}}^{dR}$ defined over $\mathbb{Q}$.

1. There is a ring of *motivic periods* $P_{\mathcal{T}}^{\mathfrak{m}}$ generated by symbols

$$[V, \omega, \sigma] \qquad V \in \mathcal{T}, \omega \in V_{dR}, \sigma \in V_B^{\vee}$$

2. The Galois group $G_{\mathcal{T}}^{dR}$ acts on motivic periods $P_{\mathcal{T}}^{\mathfrak{m}}$.
3. There is a period homomorphism to complex numbers:

$$\mathrm{per} : P_{\mathcal{T}}^{\mathfrak{m}} \to \mathbb{C}$$

Remark: One can enrich the category by putting extra structures on objects, e.g. mixed Hodge structures $\rightsquigarrow$ category $\mathcal{H}$.

Mixed Tate Motives over $\mathbb{Z}$ and $\pi_1^{\mathfrak{m}}(\mathbb{P}^1 - \{0, 1, \infty\})$

1. The category $\mathsf{MTM}(\mathbb{Z})$ of mixed Tate motives over $\mathbb{Z}$ has been constructed (Deligne–Goncharov, Levine, Voevodsky, Borel, ...).
2. Its de Rham Tannaka group $G^{dR}_{\mathsf{MTM}(\mathbb{Z})}$ is an extension
$$1 \to U^{dR}_{\mathsf{MTM}(\mathbb{Z})} \to G^{dR}_{\mathsf{MTM}(\mathbb{Z})} \to \mathbb{G}_m \to 1$$
of the multiplicative group by a pro-unipotent group whose graded Lie algebra is freely generated by $\sigma_3^{dR}, \sigma_5^{dR}, \cdots$ in degrees $-3, -5, \cdots$.
3. This affine group $G^{dR}_{\mathsf{MTM}(\mathbb{Z})}$ acts faithfully on the de Rham realization $\pi_1^{dR}(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{1}_0, -\vec{1}_1)$ of the motivic fundamental groupoid $\pi_1^{\mathfrak{m}}(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{1}_0, -\vec{1}_1)$ (Brown).
4. The (co)action can be explicitly computed on motivic MZV's. This leads to a proof of Zagier's Conjecture for motivic MZV's (Brown).

Mixed Modular Motives and Relative Completion of $\mathrm{SL}_2(\mathbb{Z})$

1. We have to work in the category $\mathcal{H}$ of motivic periods enriched with mixed Hodge structures.
2. Replace $\mathbb{P}^1 - \{0, 1, \infty\}$ by $\mathrm{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{h}$ where $\mathfrak{h}$ denotes the upper half plane, and take relative (unipotent) completion with respect to $\mathrm{SL}_2(\mathbb{Z}) \hookrightarrow \mathrm{SL}_2(\mathbb{Q})$. This is a pro-algebraic group $\mathcal{G}^{rel}$ over $\mathbb{Q}$ which is an extension

$$1 \to \mathcal{U}^{rel} \to \mathcal{G}^{rel} \to \mathrm{SL}_2 \to 1$$

of SL_2 by a pro-unipotent group $\mathcal{U}^{rel}$ whose Lie algebra is freely generated by

$$\prod_{n \geq 0} H^1(\mathrm{SL}_2(\mathbb{Z}), S^n\mathrm{H})^\vee \otimes S^n\mathrm{H}$$

where H is the standard reprensentation of SL_2 and $S^n\mathrm{H}$ its n-th symmetric powers. It has a mixed Hodge structure (Hain).

Mixed Modular Motives and Relative Completion of $\mathrm{SL}_2(\mathbb{Z})$

3. By Eichler–Shimura, Zucker, as a $\mathbb{R}$-mixed Hodge structure

$$H^1(\mathrm{SL}_2(\mathbb{Z}), S^n\mathrm{H}) \otimes \mathbb{R} = \mathbb{R}(-n-1) \oplus \bigoplus_f V_f$$

where f runs through a basis for Hecke eigen cusp forms of weight $(n+2)$, and V_f the rank 2 motive associated with f.

4. The periods of the relative completion $\mathcal{G}^{rel}$, called 'multiple modular values' by Brown, are (regularized) iterated integrals of modular forms.

5. In the simplest case (length 1 iterated integrals) for cusp forms, these iterated integrals reduce to classical Eichler integrals and provide periods of cusp forms, i.e. their critical L-values.

Mixed Modular Motives and Relative Completion of $\mathrm{SL}_2(\mathbb{Z})$

6. For non-critical L-values of cusp forms, the simplest case corresponds to a non-trivial extension $\mathrm{Ext}^1(\mathbb{Q}, V_\Delta(12))$, where Δ is the Ramanujan cusp form of weight 12 and V_Δ its associated motive. This explains the identity found by Gangl–Kaneko–Zagier. In general, these extensions constructed from cusp forms provide relations between iterated integrals of Eisenstein series, which include MZV's. This provides a geometric explanation of the depth defect between MZV's.
7. Brown, and previously Manin, only studied totally holomorphic multiple modular values that are (regularized) iterated integrals of holomorphic modular forms. An explicit $\mathbb{Q}$-de Rham theory for the relative completion of $\mathrm{SL}_2(\mathbb{Z})$ constructs (regularized) iterated integrals of modular forms of the second kind, which provide all multiple modular values (L.).

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