Appendices

Appendix II

Gamma Function

1. **(a)** $\Gamma(5) = \Gamma(4+1) = 4! = 24$

(b) $\Gamma(7) = \Gamma(6+1) = 6! = 720$

(c) Using Example 1 in the text,

$$-2\sqrt{\pi} = \Gamma\left(-\frac{1}{2}\right) = \Gamma\left(-\frac{3}{2}+1\right) = -\frac{3}{2}\Gamma\left(-\frac{3}{2}\right).$$

Thus, $\Gamma(-3/2) = 4\sqrt{\pi}/3$.

(d) Using (c)

$$\frac{4\sqrt{\pi}}{3} = \Gamma\left(-\frac{3}{2}\right) = \Gamma\left(-\frac{5}{2}+1\right) = -\frac{5}{2}\Gamma\left(-\frac{5}{2}\right).$$

Thus $\Gamma(-5/2) = -8\sqrt{\pi}/15$

2. If $t = x^5$, then $dt = 5x^4\,dx$ and $x^5\,dx = \frac{1}{5}t^{1/5}\,dt$. Now

$$\int_0^\infty x^5 e^{-x^5}\,dx = \int_0^\infty \frac{1}{5}t^{1/5}e^{-t}\,dt = \frac{1}{5}\int_0^\infty t^{1/5}e^{-t}\,dt = \frac{1}{5}\Gamma\left(\frac{6}{5}\right) = \frac{1}{5}(0.92) = 0.184.$$

3. If $t = x^3$, then $dt = 3x^2\,dx$ and $x^4\,dx = \frac{1}{3}t^{2/3}\,dt$. Now

$$\int_0^\infty x^4 e^{-x^3}\,dx = \int_0^\infty \frac{1}{3}t^{2/3}e^{-t}\,dt = \frac{1}{3}\int_0^\infty t^{2/3}e^{-t}\,dt = \frac{1}{3}\Gamma\left(\frac{5}{3}\right) = \frac{1}{3}(0.89) \approx 0.297.$$

4. If $t = -\ln x = \ln\dfrac{1}{x}$ then $dt = -\dfrac{1}{x}\,dx$. Also $e^t = \dfrac{1}{x}$, so $x = e^{-t}$ and $dx = -x\,dt = -e^{-t}\,dt$. Thus

$$\int_0^1 x^3\left(\ln\frac{1}{x}\right)^3 dx = \int_\infty^0 (e^{-t})^3 t^3(-e^{-t})\,dt = \int_0^\infty t^3 e^{-4t}\,dt = \int_0^\infty \left(\frac{1}{4}u\right)^3 e^{-u}\left(\frac{1}{4}du\right) \qquad \boxed{u = 4t}$$

$$= \frac{1}{256}\int_0^\infty u^3 e^{-u}du = \frac{1}{256}\Gamma(4) = \frac{1}{256}(3!) = \frac{3}{128}.$$

5. Since $e^{-t} \geq e^{-1}$ for $0 \leq t \leq 1$,

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt > \int_0^1 t^{x-1}e^{-t}dt \geq e^{-1}\int_0^1 t^{x-1}dt = \frac{1}{e}\left(\frac{1}{x}t^x\right)\bigg|_0^1 = \frac{1}{ex}$$

for $x > 0$. As $x \to 0^+$, we see that $\Gamma(x) \to \infty$.

6. For $x > 0$

$$\begin{aligned}\Gamma(x+1) &= \int_0^\infty t^x e^{-t} dt \qquad \boxed{u = t^x,\ du = xt^{x-1}\,dt;\quad dv = e^{-t}\,dt,\ v = -e^{-t}} \\ &= -t^x e^{-t}\Big|_0^\infty - \int_0^\infty xt^{x-1}(-e^{-t})\,dt = x\int_0^\infty t^{x-1}e^{-t}dt = x\Gamma(x).\end{aligned}$$