Simulation of Bessel Functions

PRINCIPLES OF COMMUNICATION

ECS301

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1 Project Objective

In this project, I aim to numerically calculate the Bessel Values $J_n(x)$ for $n = 0, 1, 2, ...$ by applying Simpson's Rule using Equation 3. I will be using Python program to achieve the same and plot the Bessel Values against the values of x.

2 Introduction to Bessel Function

Bessel functions, generalized by Friedrich Bessel, are canonical solutions $y(x)$ of Bessel Differential Equation Equation 1

$$x^2 \frac{d^2}{dx^2} y + x \frac{d}{dx} y + (x^2 - n^2) y = 0 \tag{1}$$

where n is any arbitrary complex number that represents *order* of the Bessel Function
For each n the solution of the differential equation is defined as a power series

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{1}{2}x)^{n+2m}}{m!(n+m)!} \tag{2}$$

The function $J_n(x)$ is called a *Bessel Function of the first kind of order n*. Equation 1 has two coefficient functions, i.e. $1/x$ and $(1 - n^2/x^2)$. The series expansion Equation 2 converges for all $x > 0$. The function $J_n(x)$ can also be expressed as integral forms

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin\theta - n\theta) d\theta \tag{3}$$

or,

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(jx \sin\theta - jn\theta) d\theta \tag{4}$$

Bessel functions play a crucial role in various fields of science and engineering, including communication systems. Introduced by Friedrich Bessel in the early 19th century, these mathematical functions are particularly valuable in analyzing systems with cylindrical symmetry, such as circular waveguides, antennas, and signal processing in communication systems.

In the realm of communication systems, Bessel functions find application in the analysis of electromagnetic fields, especially in scenarios where waves propagate with cylindrical symmetry. They provide a mathematical framework for understanding the behavior of signals in waveguides, antennas, and other transmission mediums commonly employed in communication technologies.

Bessel functions are especially relevant in the context of signal processing and modulation techniques. In communication systems, signals are often modulated onto carrier waves for transmission. The mathematical properties of Bessel functions help characterize the modulation process, enabling engineers to optimize transmission efficiency, mitigate interference, and ensure reliable communication.

Moreover, Bessel functions appear in the analysis of the impulse response of linear time-invariant (LTI) systems, which are fundamental in understanding the behavior of communication channels. By studying the response of these systems to impulse inputs, engineers can predict how signals will propagate through communication channels, aiding in the design and optimization of communication systems.

2.1 Simpson's Rule

Simpson's Rule is a method used for numerical integration or numerical approximation of definite integrals. It's a technique to approximate the area under a curve by approximating it as a series of parabolic arcs.

The basic idea behind Simpson's Rule is to fit a quadratic (parabolic) curve between three adjacent points on the curve and then use the area under this parabola to approximate the area under the curve. By doing this for consecutive pairs of intervals and summing up these areas, you can approximate the total area under the curve.

The formula for Simpson's Rule can be expressed as:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

Where:
- a and b are the lower and upper limits of integration.
- n is the number of intervals (must be even for Simpson's Rule).
- h is the width of each interval, given by $h = \frac{b-a}{n}$.
- $x_0, x_1, x_2, \ldots, x_n$ are the evenly spaced points where $x_0 = a$ and $x_n = b$, with $x_i = a + ih$.
- $f(x_0), f(x_1), f(x_2), \ldots, f(x_n)$ are the function values at the corresponding points.

Simpson's Rule generally gives more accurate results than the simpler Trapezoidal Rule for the same number of function evaluations, especially when the function being integrated is smooth and well-behaved. However, it requires the number of intervals to be even, and it can be more computationally expensive since it involves more function evaluations per interval.

2.2 Properties of Bessel Function

There are some properties of Bessel function $J_n(x)$ which can be listed down for better understanding of application of Bessel Functions:

1.. $J_n(x) = (-1)^n J_{-n}(x)$

2. $J_n(x) = (-1)^n J_n(-x)$

3. This `recurrence formula`, derived from Equation 2, can be used in constructing the Bessel coefficients

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \tag{5}$$

4. For small values of x

$$J_n(x) \approx \frac{x^n}{2^n n!} \tag{6}$$

5. When $x \in \mathbb{R}$ and fixed, $J_n(x) \longrightarrow 0$ as $n \longrightarrow \infty$

6. $\sum_{n=-\infty}^{\infty} J_n^2(x) = 1$ for all x

3 Numerical Program for calculating and plotting the values of Bessel Function

```
import numpy as np
import matplotlib.pyplot as plt

def bessel_function(x, n):
    def integrand(t):
        return np.cos(n * t - x * np.sin(t))

    a = 0
    b = np.pi
    N = 1000  # Number of intervals for Simpson's rule

    h = (b - a) / N
    integral = integrand(a) + integrand(b)  # Initializing integral with endpoint values

    # Applying Simpson's rule
    for i in range(1, N, 2):
        integral += 4 * integrand(a + i * h)
    for i in range(2, N-1, 2):
        integral += 2 * integrand(a + i * h)

    integral *= h / 3

    return integral / np.pi


x_values = np.linspace(0, 20, 400)
```

```
for order in range(4):
    bessel_values = [bessel_function(x, order) for x in x_values]
    plt.plot(x_values, bessel_values, label=f"J_{order}(x)")

plt.title("Bessel Functions of Orders 0 to 3")
plt.xlabel("x")
plt.ylabel("J_n(x)")
plt.legend()
plt.grid(True)
plt.savefig("bessel_functions_plot.png", dpi=600, bbox_inches="tight")
plt.show()
```

4 Methodology

The methodology of the above simulation project code involves several key steps:

1. **Defining the Bessel Function:** The Bessel function of the first kind of order n, denoted as $J_n(x)$, is computed using Simpson's rule for numerical integration. The Bessel function is defined as the result of integrating a cosine function over the interval [0, π].

2. **Integration using Simpson's Rule**: Simpson's rule is a numerical integration method that approximates the integral of a function by subdividing the interval into small segments and approximating the area under the curve within each segment using quadratic interpolation. The integral of the function is then approximated as the sum of these areas.

3. **Looping through Orders:** The code loops through each order of the Bessel function from 0 to 3. For each order, it computes the corresponding Bessel function values for a range of x values.

4. **Plotting the Bessel Functions:** Finally, the code plots the computed Bessel functions for each order over the specified range of x values. Each Bessel function is plotted with a different color and labeled accordingly to distinguish between them.

5. **Visualization and Analysis**: The resulting plot visually represents the behavior of Bessel functions for orders 0 to 3. This allows for the analysis of their oscillatory behavior, zeros, and other characteristics as the order increases.

5 Results

On calculating the Bessel Values and plotting them against x Figure 1

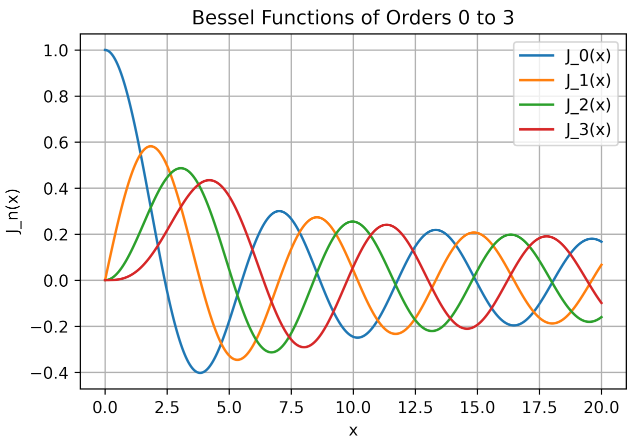

Figure 1: Bessel Functions of First Kind of order upto 3

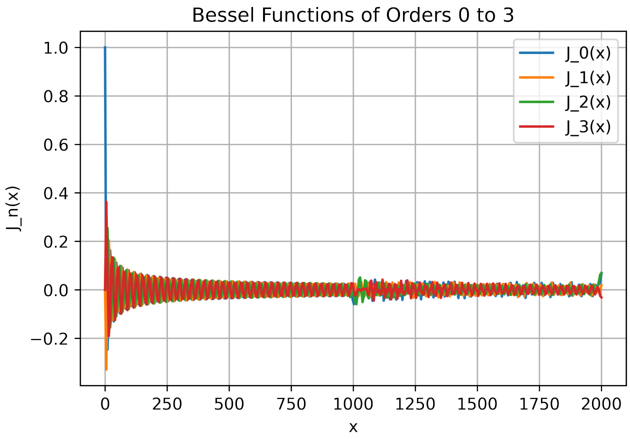

Figure 2: Bessel Functions at higher values of x

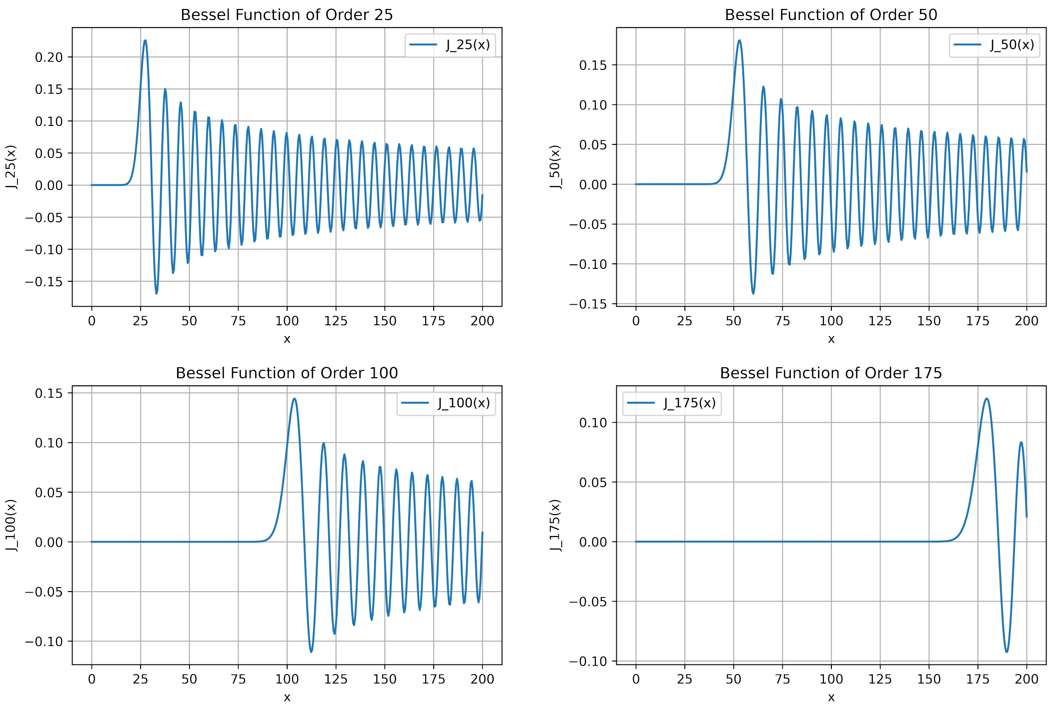

Figure 3: Bessel Functions of different higher orders n=25, 50, 100,175

6 Discussion

The plotted results showcase the behavior of Bessel functions of orders 0 to 3 over a specified range of x values from 0 to 20. Several key observations can be made from the plot:

1. **Oscillatory Nature:** Bessel functions exhibit oscillatory behavior, with the number of oscillations increasing with the order of the function. As the order increases, the oscillations become more rapid and closely spaced.

2. **Amplitude Variation:** The amplitude of the Bessel functions varies with both the order of the function and the value of x. For $n = 0$, at $x = 0$, the Bessel functions have a maximum amplitude that decreases as the order increases. For larger values of x, the amplitude of the functions oscillates around zero, with the amplitude decreasing as the order increases. Also, for higher value of n Figure 3, the function takes maximum amplitude at close to n.

3. **Zero Crossings:** Bessel functions have multiple zero crossings, where the function crosses the x-axis. The number of zero crossings increases with the order of the function. These zero crossings are characteristic of Bessel functions and play a significant role in various applications in communication systems, such as in

carrier recovery, symbol time recovery, channel equalization, bit synchronization etc.

4. **Asymptotic Behavior**: At large values of x, the Bessel functions exhibit asymptotic behavior, approaching zero as x tends towards infinity Figure 2. This behavior is more pronounced for higher-order Bessel functions, indicating that they decay more rapidly as x increases.

Overall, the plotted results provide valuable insights into the characteristics of Bessel functions and their behavior across different orders and x values. These observations are essential for understanding the properties of Bessel functions and their applications in various fields, including engineering.

7 Bessel Function in Frequency Modulation

I would like to determine the spectrum of a single-tone FM signal Equation 7

$$s(t) = A_c \cos\left[2\pi f_c t + \beta \sin\left(2\pi f_m t\right)\right] \tag{7}$$

We may simplify this by using complex representation of band-pass signal and assuming the carrier frequency f_c is large enough compared to the BW of FM Signal and we can write the above equation as:

$$s(t) = Re[A_c \exp\left(j2\pi f_c t + j\beta \sin\left(2\pi f_m t\right)\right)] = Re[\tilde{s}(t) \exp\left(j2\pi f_c t\right)] \tag{8}$$

Where $\tilde{s}(t)$ is the complex envelope of FM Signal $s(t)$, defined by

$$\tilde{s}(t) = A_c \exp\left[j\beta \sin\left(2\pi f_m t\right)\right] \tag{9}$$

Now let's expand Equation 9 in the form of complex fourier series,

$$\tilde{s}(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left[j\beta \sin\left(2\pi f_m t\right)\right] \tag{10}$$

where the complex fourier coefficients c_n is defined by,

$$c_n = f_m A_c \int_{-1/2f_m}^{1/2f_m} \exp\left(j\beta \sin\left(2\pi f_m t\right) - j2\pi n f_m t\right) dt \tag{11}$$

Now from Equation 4, we can rewrite as

$$c_n = \frac{A_c}{2\pi} \int_{-\pi}^{\pi} \exp\left[j(\beta \sin x - nx)\right] dx = A_c J_n(\beta) \tag{12}$$

Where $x = 2\pi f_m t$ and $J_n(\beta)$ is the *nth order Bessel function of the first kind*.

Then, we can substitute Equation 12 in Equation 9 and that in Equation 8

$$s(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos\left[2\pi(f_c + n f_m)t\right] \tag{13}$$

here β is the modulation index which and on taking Fourier Transform on both sides of Equation 13, we get

$$S(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta)[\delta(f - f_c - n f_m) + \delta(f + f_c + n f_m)] \tag{14}$$

References

- *Communication Systems by Simon Haykins, 4th Edition.*
- *Numerical Methods by Physics by Alejandro Garcia.*
- *Wikipedia.*