CLASSICAL LINEAR MULTISTEP FORMULAS

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1. Introduction

Differential equations are among the most important mathematical tools used in producing models in the physical sciences, biological sciences, and engineering. In this paper, we consider multistep methods in which the computation of the numerical solution y_{n+1} uses the solution values at several previous nodes.

The differential equation we consider is of the form

$$y' = f(t, y) \tag{1.1}$$

where y is an unknown function from $\mathbb{R}$ to $\mathbb{R}^d$ that is being sought and f is a given function of two variables $(t, y) \in \mathbb{R} \times \mathbb{R}^d$ with $d \geq 1$.
Thereafter, we always consider that f satisfies the Cauchy-Lipschitz conditions. In other words, f is uniformly Lipschitz continuous in y and continuous in t, i.e. the functions $f(t, \cdot) : y \mapsto f(t, y)$ and $f(\cdot, y) : y \mapsto f(t, y)$ are continuous for all $t \in \mathbb{R}$ and for all $y \in \mathbb{R}^d$ respectively and there exists C non-negative such that

$$\forall t \in \mathbb{R},\, \forall y, \widetilde{y} \in \mathbb{R}^d,\, \|f(t, y) - f(t, \widetilde{y})\| \leq C\, \|y - \widetilde{y}\|.$$

Besides, we will work on the following Cauchy problem:

$$y' = f(t, y), \quad y(t_0) = y_0 \in \mathbb{R}^d. \tag{1.2}$$

A such problem possesses a unique solution according to the Cauchy-Lipschitz theorem that we remind just below.

Theorem 1.1 (Cauchy-Lipschitz theorem)**.** *Consider the initial value problem* (1.2)*. Supppose that f is uniformly Lipschitz continuous in y and continuous in t. Then, for some value $\varepsilon > 0$,* (1.2) *has a unique solution on the interval* $[t_0 - \varepsilon, t_0 + \varepsilon]$.

Let $h > 0$. We define $t_n = t_0 + nh$ as the grid points for the time variable.
We reformulate the differential equation (1.1) by integrating it over the interval $[t_n,\, t_{n+1}]$, obtaining

$$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} y'(t)dt = \int_{t_n}^{x_{t+1}} f(t, y(t))dt.$$

We will develop numerical methods (the explicit and implicit Adams methods) to compute the solution y by approximating the integral in the following result:

$$y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(t, y(t))dt. \tag{1.3}$$

To simplify the notation, we will now use $y_n, y_{n-1}, \ldots, y_{n-k+1}$ for the numerical approximation to the exact solution $y(t_n), y(t_{n-1}), \ldots, y(t_{n-k+1})$.

Since we want to construct the numerical approximation to $y(t_{n+1})$ based on the previous k steps, we will suppose that $y_n, y_{n-1}, \dots, y_{n-k+1}$ are known and y_{n+1} will be determined in terms of them.

2. Explicit Adams Methods

Since in (1.3), we have no idea about $f(t, y(t))$, the main question is how to approximate $f(t, y(t))$ by a known function (or a function we know the form, precisely speaking).
Since the approximations $y_n, y_{n-1}, \cdots, y_{n-k+1}$ are known, the approximation values of $f(t, y(t))$ are also available:

$$f_i := f(t_i, y_i) \quad \text{for} \quad i = n-k+1, \dots, n.$$

In the explicit Adams method, we replace $f(t, y(t))$ by the interpolation polynomial p through k points $\{(t_i, f_i) \mid i = n-k+1, \dots, n\}$ and we replace (1.3) by its approximate version:

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} p(t)dt. \tag{2.1}$$

2.1. Computation of the interpolation polynomial.

After giving the definitions of an interpolation polynomial and the divided difference, we will give the form of p as described before.

Definition 2.1 (Interpolation polynomial)**.** Given a set of $n+1$ data points $(t_i, y_i) \in [t_0, T] \times \mathbb{R}^d$ where no two t_i are the same, a polynomial $p : \mathbb{R} \to \mathbb{R}^d$ is said to interpolate the data if $p(t_j) = y_j$ for each $j \in \{0, 1, \dots, n\}$.

Definition 2.2 (Divided difference)**.** The divided difference $f[t_0, \dots, t_n]$ on $n+1$ points $t_0, t_1, \dots, t_n$ of a function f is defines by

$$f[t_0] = f(t_0), \quad f[t_0, \dots, t_n] = \frac{f[t_0, \dots, t_{n-1}] - f[t_1, \dots, t_n]}{t_0 - t_n}.$$

We want to calculate p under the Newton basis. To do that, we firs recall the interpolation polynomial of Newton basis.

Lemma 2.3. *Suppose $\{(t_i, f_i) \mid i = 0, \dots, n\}$ are the interpolation points and then the interpolation polynomial under the Newton basis is in the form*

$$p(t) = f[t_0] + \sum_{k=1}^{n} f[t_0, \cdots, t_k](t - t_0) \cdots (t - t_{k-1}) \tag{2.2}$$

where $f[t_0, \dots, t_k]$ denotes the divided difference.

Proof. Let us prove the Newton's Interpolation Formula.
Let p be the interpolation polynomial with $deg(p) = n$. So, p is given by the following formula:

$$p(t) = a_0 + a_1(t - t_0) + a_2(t - t_0)(t - t_1) + \cdots + a_k(t - t_0) \cdots (t - t_{k-1}) + \cdots + a_n(t - t_0) \cdots (t - t_{n-1}).$$

for some constants $a_0, \dots, a_n$ to be determined using the fact that $p(t_i) = f_i$ for $i = 0, 1, \dots, n$.
For $i = 0$, we get for $t = t_0$: $p(t_0) = f_0$. That gives us $a_0 = y_0$.
Next,

$$p(t_1) = f_1 \Rightarrow f_1 = a_0 + (t_1 - t_0)a_1 \Rightarrow a_1 = \frac{f_1 - f_0}{h} = f[t_1, t_0].$$

For $i = 2$, $f_2 = a_0 + a_1(t_2 - t_0) + a_2(t_2 - t_0)(t_2 - t_1)$, or equivalently

$$2h^2 a_2 = f_2 - f_0 - 2hf[t_0, t_1] = f_2 - 2f_1 + f_0 \quad or \quad a_2 = f[t_0, t_1, t_2].$$

Now, using induction, we obtain

$$a_k = f[t_0, \dots, t_k]; \quad k = 0, \dots, n.$$

Then we have

$$p(t) = f_n + \sum_{j=1}^{k-1} f[t_n, \cdots, t_{n-j}](t - t_n) \cdots (t - t_{n-j+1}).$$

□

Thereafter, we introduce the notations

$$\nabla^0 f_n := f_n, \quad \nabla^{j+1} f_n := \nabla^j f_n - \nabla^j f_{n-1} \quad j \in \mathbb{N}$$

and

$$f[x_n] := f_n = \nabla^0 f_n.$$

We now can give the formula of p.

Lemma 2.4. *Suppose* $\{(t_i, f_i) \mid i = 0, \dots, n\}$ *are the interpolation points. Then we have:*

$$\forall s, h \in \mathbb{R}, \quad p(t_n + sh) = \sum_{j=0}^{k-1} (-1)^j \binom{-s}{j} \nabla^j f_n \tag{2.3}$$

where

$$\binom{-s}{j} = \frac{-s(-s-1)(-s-2)\cdots(-s-j+1)}{j!} = \frac{s(s+1)(s+2)\cdots(s+j-1)}{j!}$$

Proof. We compute:

$$f[t_n, t_{n-1}] = \frac{f[t_n] - f[t_{n-1}]}{t_n - t_{n-1}} = \frac{f_n - f_{n-1}}{t_n - t_{n-1}} = \frac{\nabla^1 f_n}{h}$$

$$f[t_n, t_{n-1}, t_{n-2}] = \frac{f[t_n, t_{n-1}] - f[t_{n-1}, t_{n-2}]}{t_n - t_{n-2}} = \frac{1}{2h}\left(\frac{\nabla^1 f_n}{h} - \frac{\nabla^1 f_{n-1}}{h}\right) = \frac{\nabla^2 f_n}{2h^2}.$$

So by an induction argument, we get:

$$f[t_n, t_{n-1}, \cdots, t_{n-j}] = \frac{f[t_n, t_{n-1}, \cdots, t_{n-j+1}] - f[t_{n-1}, t_{n-1}, \cdots, t_{n-j}]}{t_n - t_{n-j}} = \frac{\nabla^j f_n}{j! h^j}.$$

When $t = t_n + sh$, the term of order j in the polynomial p is

$$\begin{aligned}
\frac{\nabla^j f_n}{j! h^j}(t_n + sh - t_n) \cdots (t_n + sh - t_{n-j+1}) &= \frac{\nabla^j f_n}{j! h^j} sh(s+1)h(s+2)h \cdots (s+j-1)h \\
&= \frac{\nabla^j f_n}{j! h^j} (-1)^j (-s)h(-s-1)h(-s-2)h \cdots (-s-(j-1))h \\
&= \frac{\nabla^j f_n}{j! h^j} \frac{(-s)! h^j}{(-s-j)!} (-1)^j \\
&= (-1)^j \binom{-s}{j} \nabla^j f_n.
\end{aligned}$$

Then we get the interpolation polynomial

$$p(t_n + sh) = \sum_{j=0}^{k-1} (-1)^j \binom{-s}{j} \nabla^j f_n \tag{2.4}$$

as claimed. □

2.2. **Estimation of the interpolation error.**
Here we expose a theorem that estimates the error between a function and its interpolation polynomial at a given point. This estimation will be useful for the study of the properties of the Adams methods.

Theorem 2.5. *Let f be a C^{n+1} real-valued function on $[a,b]$ and p its interpolation polynomial on $n+1$ points $x_0,\dots,x_n \in [a,b]$. Then, we have:*

$$\forall x \in [a,b], \exists \alpha \in [a,b], f(x) - p(x) = \frac{f^{(n+1)}(\alpha)}{(n+1)!}(x-x_0)\dots(x-x_n). \tag{2.5}$$

Proof. We give the proof for y going from $\mathbb{R}$ to $\mathbb{R}$. If there exists j in $0,\dots,n$ such that $x = x_j$, the conclusion is obvious.
Else, we note $q(x) = (x-x_0)\dots(x-x_n)$ and

$$W(t) = f(t) - p(t) - \frac{q(t)}{q(x)}(f(x) - p(x)).$$

W is C^{n+1} like f and $W(t) = 0$ for $t = x, x_0, \dots, x_n$ so W has at least $n+2$ zeros. By using Rolle's theorem by iteration, we get that $W^{(n+1)}$ has at least one zero in $[a,b]$ there is α in $[a,b]$ such that $W^{(n+1)}(\alpha) = 0$, which gives (2.5) . □

Lemma 2.6. *Here we note $p(X,h)$ the interpolation polynomial at the interpolation points $0,\dots,(q-1)h$. Due to the previous theorem, if f is C^q on $[0,T]$ and real-valued , we have:*

$$\forall t \in [(q-1)h, qh], |f(t) - p(t,h)| = O(h^q). \tag{2.6}$$

Proof. If f is C^q on $[0,T]$ there is $M > 0$ such that $|f^{(q)}(x)| \le M$ for $x \in [0,T]$. Using (2.5), we get:

$$\forall t \in [(q-1)h, qh], |f(t) - p(t,h)| \le M\frac{(hq)^q}{q!}.$$

which gives (2.6) . □

2.3. **Explicit Adams method formula.**

The explicit Adams method is given by

$$\begin{aligned}
\forall n \in \mathbb{N} \setminus \{0,\dots,k-1\}, \quad y_{n+1} - y_n &= \int_{t_n}^{t_{n+1}} p(t)dt \\
&= h\int_0^1 p(t_n + sh)ds \\
&= h\int_0^1 \sum_{j=0}^{k-1} (-1)^j \binom{-s}{j} \nabla^j f_n ds \\
&= h\sum_{j=0}^{k-1} \nabla^j f_n (-1)^j \int_0^1 \binom{-s}{j} ds.
\end{aligned}$$

Denote

$$\forall j \in \mathbb{N}, \quad \gamma_j := (-1)^j \int_0^1 \binom{-s}{j} ds. \tag{2.7}$$

We get

$$y_{n+1} - y_n = h \sum_{j=0}^{k-1} \gamma_j \nabla^j f_n \tag{2.8}$$

where we recall the notation $f_n = f(t_n, y_n)$ for all $n \in \mathbb{N}$.
So the updated vector y_{n+1} is computed by knowing $y_n, f_n, \dots, f_{n+1-k}$.

The first 4 Explicit Adams methods are given by the following formulas:

$$\begin{aligned}
y_{n+1} &= y_n + hf_n \\
y_{n+1} &= y_n + h\left(\tfrac{3}{2}f_n - \tfrac{1}{2}f_{n-1}\right) \\
y_{n+1} &= y_n + h\left(\tfrac{23}{12}f_n - \tfrac{16}{12}f_{n-1} + \tfrac{5}{12}f_{n-2}\right) \\
y_{n+1} &= y_n + h\left(\tfrac{55}{24}f_n - \tfrac{59}{24}f_{n-1} + \tfrac{37}{24}f_{n-2} - \tfrac{9}{24}f_{n-3}\right)
\end{aligned}$$

Notice that the first formula coincides with the explicit Euler method.

2.4. **Recurrence relation for γ_i.**
Consider the power series

$$G(t) = \sum_{j=0}^{+\infty} \gamma_j t^j.$$

We will prove that it converges when $t \in (-1, 1)$ by using the following lemma:

Lemma 2.7. *Let $\alpha \in \mathbb{R}$. Then for all $t \in (-1, 1)$,*

$$\sum_{j=0}^{+\infty} \binom{\alpha}{j} t^j = (1+t)^\alpha.$$

Proof. Let $\alpha \in \mathbb{R}$. For $t \in (-1, 1)$, we set $A(t) = (1+t)^\alpha$. The function A is differentiable for $t \in (-1, 1)$ and we have $A'(t) = \alpha(1+t)^{\alpha-1}$. Therefore A is solution of the following Cauchy problem

$$(1+t)A'(t) - \alpha A(t) = 0, \quad A(0) = 1. \tag{2.9}$$

Since $t \mapsto -\frac{\alpha}{1+t}$ is a continuous function on $(-1, 1)$, we have that A is the unique solution of the Cauchy problem (2.9) (according to the local Cauchy-Lipschitz theorem).
Let $\sum a_n t^n$ a power series of radius of convergence $R_\alpha > 0$. For $t \in (-R_\alpha, R_\alpha)$, we set $B(x) =$

$\sum_{n=0}^{+\infty} a_n t^n$. Then, for all $t \in (-R_\alpha, R_\alpha)$,

$$\begin{aligned}(1+t)B'(t) - \alpha B(t) &= (1+t)\sum_{n=1}^{+\infty} n a_n t^{n-1} - \alpha \sum_{n=0}^{+\infty} a_n t^n \\ &= \sum_{n=1}^{+\infty} n a_n t^{n-1} + \sum_{n=0}^{+\infty} n a_n t^n - \alpha a_n t^n \\ &= \sum_{n=0}^{+\infty} (n+1) a_{n+1} t^n + \sum_{n=0}^{+\infty} (n-\alpha) a_n t^n \\ &= \sum_{n=0}^{+\infty} \left((n+1)a_{n+1} + (n-\alpha)a_n\right) t^n.\end{aligned}$$

Using the uniqueness of the coefficient of a power series,

$$\begin{aligned}& g(0) = 1 \quad \text{and} \quad \forall t \in (-R_\alpha, R_\alpha),\, (1+t)B'(t) - \alpha B(t) = 0 \\ \Leftrightarrow\ & a_0 = 1 \quad \text{and} \quad \forall n \in \mathbb{N},\, (n+1)a_{n+1} + (n-\alpha)a_n = 0 \\ \Leftrightarrow\ & a_0 = 1 \quad \text{and} \quad \forall n \in \mathbb{N}^*,\, a_n = \frac{(\alpha - (n-1))}{n} a_{n-1} \\ \Leftrightarrow\ & \forall n \in \mathbb{N},\, a_n = \binom{\alpha}{n}.\end{aligned}$$

Let determine R_α. If $\alpha \in \mathbb{N}$, then $(a_n)_{n\in\mathbb{N}}$ is null for $n \geq \alpha + 1$. In this case, $R_\alpha = +\infty$. We get the well-known formula $(1+t)^\alpha = \sum_{n=0}^{\alpha} \binom{\alpha}{n} t^n$, for all $t \in \mathbb{R}$:it is the binomial theorem.
We now suppose than $\alpha \notin \mathbb{N}$, then $(a_n)_{n\in\mathbb{N}}$ is not null and for $n \geq \alpha + 1$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\alpha(\alpha-1)\dots(\alpha-(n-1))}{\alpha(\alpha-1)\dots(\alpha-(n-2))}\right| \frac{n!}{(n+1)!} = \frac{|\alpha-(n-1)|}{n+1} = \frac{n-\alpha-1}{n+1}.$$

This expression tends to one when n goes to $+\infty$ and according to the ratio test, $R_\alpha = 1$.
The function B is defined on $(-1,1)$, differentiable on $(-1,1)$ and it is a solution of the previous Cauchy problem. Then by uniqueness of the solution, $A = B$, i.e.

$$\forall t \in (-1,1), \quad (1+t)^\alpha = \sum_{n=0}^{+\infty} \binom{\alpha}{n} t^n.$$

□

We replace γ_j in G as described in (2.7) and we get

$$G(t) = \sum_{j=0}^{+\infty} (-t)^j \int_0^1 \binom{-s}{j} ds.$$

We can apply the Fubini–Tonelli theorem here since we have

$$(-1)^j \binom{-s}{j} = \frac{s(s+1)\cdots(s+j-1)}{j!} \geq 0, \text{ for } s \in [0,1] \text{ and } j \in \mathbb{N}$$

and therefore

$$\begin{aligned}\int_0^1 \sum_{j=0}^{+\infty} |(-t)^j (\begin{array}{c} -s \\ j \end{array})| ds &= \int_0^1 \sum_{j=0}^{+\infty} (-1)^j (\begin{array}{c} -s \\ j \end{array}) |t|^j ds \\ &= \int_0^1 (1-|t|)^{-s} ds \\ &= -\frac{|t|}{(1-|t|)\ln(1-|t|)} < \infty \text{ for } \forall t \in (-1,1).\end{aligned}$$

Then we have

$$\sum_{j=0}^{+\infty} (-t)^j \int_0^1 \binom{-s}{j} ds = \int_0^1 \sum_{j=0}^{+\infty} (-t)^j \binom{-s}{j} ds.$$

Therefore, by using the lemma 2.7,

$$G(t) = \int_0^1 \sum_{j=0}^{+\infty} (-t)^j \binom{-s}{j} ds = \int_0^1 (1-t)^{-s} ds = -\left[\frac{(1-t)^{-s}}{\ln(1-t)}\right]_0^1 = -\frac{t}{(1-t)\ln(1-t)}.$$

Then we have

$$-\frac{\ln(1-t)}{t} G(t) = \frac{1}{1-t}$$

for $t \in (-1,1)$, we have

$$-\frac{\ln(1-t)}{t} = -\frac{1}{t}(-t - \frac{t^2}{2} - \frac{t^3}{3} - \cdots) = (1 + \frac{t}{2} + \frac{t^2}{3} + \cdots)$$

and

$$\frac{1}{1-t} = (1 + t + t^2 + \cdots).$$

Therefore we use the multiplication for the power series

$$(1 + \frac{1}{2}t + \frac{1}{3}t^2 + \ldots)(\gamma_0 + \gamma_1 t + \gamma_2 t^2 + \ldots) = (1 + t + t^2 + \ldots)$$

Consider the coefficients of t^j we obtain the induction formula:

$$\gamma_0 = 1 \quad ; \quad \gamma_j + \frac{\gamma_{j-1}}{2} + \frac{\gamma_{j-2}}{3} + \cdots + \frac{\gamma_0}{j+1} = 1 \tag{2.10}$$

for $j \in \mathbb{N}$.

3. Implicit Adams Methods

The formula (2.8) is obtained by integrating the interpolation polynomial (2.4) from t_n to t_{n+1}, i.e., outside the interpolation interval (t_{n-k+1}, t_n). It is well known that an interpolation polynomial is usually a rather poor approximation outside this interval (Even in the interval near the edge: Runge's Phenomenon). Adams therefore replaced (2.4) by interpolation polynomial which uses in addition the point (t_{n+1}, f_{n+1}) which is still unknown. Use the previous method, we can easily calculate

$$p^*(t_n + sh) = p^*(t_{n+1} + (s-1)h) = \sum_{j=0}^{k} (-1)^j (\begin{array}{c} -s+1 \\ j \end{array}) \nabla^j f_{n+1}. \tag{3.1}$$

Insert this into the (1.3) we obtain

$$y_{n+1} = y_n + h \sum_{j=0}^{k} \gamma_j^* \nabla^j f_{n+1} \tag{3.2}$$

where

$$\gamma_j^* = (-1)^j \int_0^1 \binom{-s+1}{j} ds. \tag{3.3}$$

We remark that $\gamma_j^* = \gamma_j - \gamma_{j-1}$ for all $j \in \mathbb{N}$.
Also we can use the same method to achieve the recurrence relation of γ_j^*:

$$\gamma_0^* = 1 \quad ; \quad \gamma_m^* + \frac{\gamma_{j-1}^*}{2} + \frac{\gamma_{j-2}^*}{3} + \cdots + \frac{\gamma_0^*}{j+1} = 0 \tag{3.4}$$

for $j \in \mathbb{N}$.

The first 4 Adams implicit formula is given here:

$$\begin{aligned}
y_{n+1} &= y_n + h f_{n+1} \\
y_{n+1} &= y_n + h\left(\tfrac{1}{2} f_{n+1} + \tfrac{1}{2} f_n\right) \\
y_{n+1} &= y_n + h\left(\tfrac{5}{12} f_{n+1} + \tfrac{8}{12} f_n - \tfrac{1}{12} f_{n-1}\right) \\
y_{n+1} &= y_n + h\left(\tfrac{9}{24} f_{n+1} + \tfrac{19}{24} f_n - \tfrac{5}{24} f_{n-1} + \tfrac{1}{24} f_{n-2}\right)
\end{aligned}$$

Notice that the first two formulas are the implicit Euler method and the trapezoidal rule, respectively.

4. Numerical Experiment

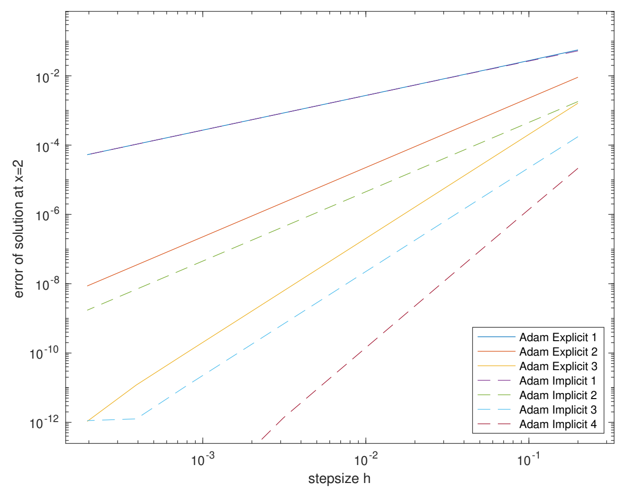
We consider the following Cauchy problem

$$\begin{cases} y'(t) = -y(t) - 3t, t \in [0, 2] \\ y(0) = 1 \end{cases}$$

whose analytical solution is $y(t) = -2e^{-t} - 3t + 3$. We have applied the above explicit and implicit Adams methods. We calculated the round-off error on the point $t = 2$, which is $|y_n - y(2)|$ where y is the analytical solution and y_n is the approximated solution at $x = 2$. We have also plotted the relation between the error and the step size in logarithmic scale (Figure:1). For different Adams method, we need several start values to iterate (e.g. we need the values on the first two points for the second formula of the explicit Adams method). In our numerical experiments the missing start values were replaced by the analytical solutions. The numerical experiments shows that:

- The implicit methods generally achieve better result than the explicit ones for formulas of the same number.
- There is linear relation between the step size and the round-off error in the logarithmic scale. The result is expected if we refer to 8.2 which is the final result of our project. The slope of the corresponding lines is called the convergence order which we will explore later.

FIGURE 1. round-off errors with different methods and step sizes



5. Local error of a multistep method

Let us consider the general difference equation

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f(t_{n+j}, y_{n+j}) = h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{5.1}$$

where α_j and β_j are real parameters and h denotes the step size.
Besides, we suppose that

$$\alpha_k \neq 0, \quad |\alpha_0| + |\beta_0| > 0. \tag{5.2}$$

Let us define some notions.

Definition 5.1 (Local error)**.** The local error of the multistep method (5.1) is given by

$$y(t_k) - y_k$$

where $y(t)$ is the exact solution of $y' = f(t, y), y(t_0) = y_0$ and y_k is the numerical solution obtained from (5.1) by using the exact starting values $y_i = y(t_i)$ for $i = 0, 1, \ldots, k-1$.

Definition 5.2 (Consistency error)**.** The consistency error of the multistep method (5.1) is given by

$$\epsilon(t, y, h) = \sum_{j=0}^{k} \alpha_j y(t + jh) - h \sum_{j=0}^{k} \beta_j y'(t + jh) \tag{5.3}$$

where y is a real function of class $\mathcal{C}^1$.

6. Order of the Adams methods

To characterize the behavior and efficiency of Adams methods, it is a good idea to look at its order.

6.1. General definition of the order for multisteps methods.

Definition 6.1 (Order)**.** A multistep method is said to be of order p if the consistency error is zero (identically) when y is a polynomial of degree less than p.
A method is said to be consistent when it is of order one or higher.

Definition 6.2 (Generating polynomials)**.** We define the generating polynomials of the multistep method (5.1) by these formulas:

$$\begin{array}{rcl} \rho(\zeta) & = & \alpha_k\zeta^k + \alpha_{k-1}\zeta^{k-1} + \cdots + \alpha_0 \\ \sigma(\zeta) & = & \beta_k\zeta^k + \beta_{k-1}\zeta^{k-1} + \cdots + \beta_0 \end{array} \tag{6.1}$$

The hypothesis that must be verified to conclude that a method is of a given order may be difficult to check. However, there exists a theorem that gives equivalent characterizations of the order of a multistep method that will be easier to verify and apply to the Adams methods.

6.2. Characterization of the order of linear multistep methods.

Theorem 6.3. *The following propositions are equivalent:*

(i) *The multistep method* (5.1) *is of order* p*;*
(ii) *For all function* y *of class* $\mathcal{C}^{p+1}(\mathbb{R})$ *with values in* $\mathbb{R}$*,*

$$\epsilon(t,y,h) = O(h^{p+1}) \quad \text{when } h \text{ goes to } 0;$$

(iii) *We have the following relations:*

$$\sum_{j=0}^{k}\alpha_j = 0, \quad \text{and} \quad \sum_{j=0}^{k} j^\ell\alpha_j = \ell\sum_{j=0}^{k} j^{\ell-1}\beta_j \quad \text{for } \ell = 1,\ldots,p; \tag{6.2}$$

(iv) *Let* ρ *and* σ *be the generating polynomials as described in* (6.1)*. Then,*

$$\rho(e^h) - h\sigma(e^h) = O(h^{p+1}) \qquad \text{when } h \text{ goes to } 0.$$

6.3. Proof of the theorem.

First of all, let expose a preliminary result that will be useful for the entire proof.
Suppose y is $C^{p+1}(\mathbb{R})$ which is real valued. By applying Taylor-Young to the functions y and y' and inserting into (5.3) we get:

$$\begin{aligned} \epsilon(t,y,h) &= \sum_{j=0}^{k}\left(\alpha_j\sum_{i=0}^{p}\frac{y^{(i)}(t)}{i!}(jh)^i - h\beta_j\sum_{i=0}^{p}\frac{y^{(i+1)}(t)}{i!}(jh)^i\right) + O(h^{p+1}) \\ &= y(t)\sum_{j=0}^{k}\alpha_j + \sum_{i=1}^{p}\frac{y^{(i)}(t)}{i!}h^i\left(\sum_{j=0}^{k}\alpha_j j^i - i\sum_{j=0}^{k}\beta_i j^{i-1}\right) + O(h^{p+1}) \end{aligned} \tag{6.3}$$

Proof. Let us prove the theorem by using several implications.

(ii) $\Rightarrow$ (iv) We consider the function $y = \exp$ which is $\mathcal{C}^\infty$. Then we have

$$\epsilon(0,y,h) = \sum_{j=1}^{k}\alpha_j e^{0+jh} - h\sum_{j=0}^{k}\beta_j e^{0+jh} = \rho(e^h) - h\sigma(e^h).$$

The result is immediate.

(i) $\Rightarrow$ (iii) Let suppose that the multistep method is of order p. By definition, for all polynomial y with $deg(y) \leq p$, the consistency error is identically null. In particular for $y : x \mapsto x^\ell$ where $\ell = 0, 1, \ldots, p$, we have for all $x \in \mathbb{R}$:

$$0 = \epsilon(t, y, h) = \sum_{j=1}^{k} \alpha_j (t + jh)^\ell - h\ell \sum_{j=0}^{k} \beta_j (t + jh)^{\ell-1}$$

Thus, for $\ell = 0$, we get:

$$0 = \sum_{j=1}^{k} \alpha_j (t + jh)^0 = \sum_{j=1}^{k} \alpha_j.$$

For $1 \leq \ell \leq p$, with $t = 0$, we get:

$$0 = \sum_{j=1}^{k} \alpha_j (jh)^\ell - h\ell \sum_{j=0}^{k} \beta_j (jh)^{\ell-1} = h^\ell \left(\sum_{j=1}^{k} \alpha_j j^\ell - \ell \sum_{j=0}^{k} \beta_j j^{\ell-1} \right).$$

Therefore,

$$\sum_{j=0}^{k} j^\ell \alpha_j = \ell \sum_{j=0}^{k} j^{\ell-1} \beta_j$$

(iii) $\Rightarrow$ (i) Let y be a polynomial function of degree inferior of p. Then, y is equal to its Taylor expansion of order p. Using (6.3), we have

$$\epsilon(t, y, h) = y(t) \sum_{j=0}^{k} \alpha_j + \sum_{i=1}^{p} \frac{y^{(i)}(t)}{i!} h^i (\sum_{j=0}^{k} \alpha_j j^i - i \sum_{j=0}^{k} \beta_i j^{i-1})$$

which is a polynomial function of the h variable.
Then, using (6.2), we get that the coefficients of the corresponding polynomial are equal to zero and we get that the consistency error is identically null.

(iv) $\Rightarrow$ (iii) By applying (6.2) to $y = exp$ at $t = 0$ we get:

$$\begin{aligned} \rho(e^t) - t\sigma(e^t) &= \sum_{j=0}^{k} \alpha_j + \sum_{i=1}^{p} \frac{t^i}{i!} (\sum_{j=0}^{k} \alpha_j j^i - i \sum_{j=0}^{k} \beta_i j^{k-1}) + O(t^{p+1}) \\ &= P(t) + O(t^{p+1}) \end{aligned}$$

Where P is a polynomial of degree less than p. If (iv) is verified, we get that this expression is equal to $O(t^{p+1})$ which implies that $P(t) = O(t^{p+1})$. As the degree of P is $< p + 1$, this implies that the coefficients of P are equal to 0, which gives the relations (6.2).

(i) $\Rightarrow$ (ii) Let y be a C^{p+1} function. With (6.3) calculus, we know that the consistency error of y is equal to the sum of the consistency error of the Taylor polynomial of y of degree p and the correction term $O(h^{p+1})$. By supposing that (i) is verified, we know that the first term in this sum is equal to zero. We then conclude that the consistency error of y is equal to $O(h^{p+1})$.

□

Remark 6.1. A multistep method is consistent when it is of order 1, which can written in the form

$$\rho(1) = 0, \quad \rho'(1) = \sigma(1). \tag{6.4}$$

It is called the consistency condition.

6.4. **Application to the Adams methods.**
With the previous theorem, the order of the Adams methods is easy to find. According to this theorem, to find the order of a linear multistep method, we only need to calculate $\rho(e^h) - h\sigma(e^h)$ which is by definition the consistency error for the exponential function. Considering the following Adams method with $k+1$ steps:

$$y_{n+1} - y_n = h \sum_{j=0}^{k} \gamma_j \nabla^j f_n \tag{6.5}$$

we get $\rho(e^h) = e^{hk} - e^{h(k-1)}$. We consider the interpolation polynomial of the exponential function at the points of interpolation : $0, \dots, (k-1)h$, that we note $p(X, h)$. By construction of the Adams method, we get that:

$$h\sigma(e^h) = h \sum_{j=0}^{k} \beta_j e^{jh} = \int_{(k-1)h}^{kh} p(t, h) dt \tag{6.6}$$

Using lemma 2.6, we get that the error between the exponential function and its interpolation polynomial is:

$$\forall t \in [(k-1)h, kh], e^t - p(t, h) = O(h^k)$$

By integrating this equality between $(k-1)h$ and kh we get:

$$e^{kh} - e^{(k-1)h} - \int_{(k-1)h}^{kh} P(s, h) dh = O(h^{k+1})$$

Which gives:

$$\rho(e^h) - h\sigma(e^h) = O(h^{k+1})$$

By using theorem 6.3, we conclude that the order of explicit Adams methods of $k+1$ steps as defined in (6.5) is k.

7. Stability of a multistep method

However, it is observed that high order and a small local error do not guarantee a useful multistep method. Actually it may be extremely unstable and we get the totally wrong approximations. One numerical experiment here is the highest 2-step method. From the previous results of the order we can show that it is of order 3 with the form:

$$y_{n+2} + 4y_{n+1} - 5y_n = h(4f_{n+1} + 2f_n). \tag{7.1}$$

We apply the method to the Cauchy problem

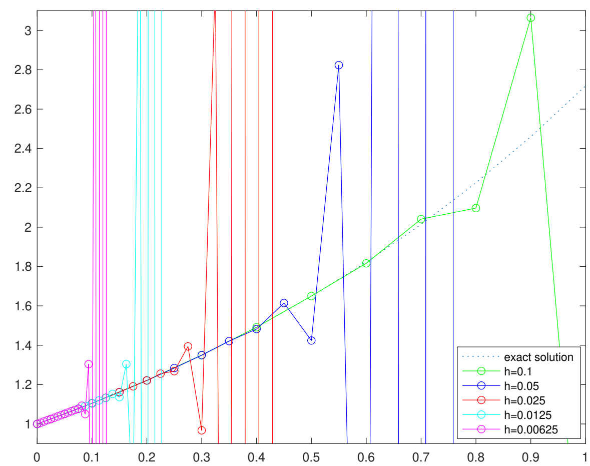
$$y' = y, y(0) = 1. \tag{7.2}$$

to achieve the linear difference relation

$$y_{n+2} + 4(1-h)y_{n+1} - (5+2h)y_n = 0. \tag{7.3}$$

The graph shows that the numerical solutions are very bad and become worse when the step size decreases (Notice that the analytical solution is $y(t) = e^t$).

FIGURE 2. Numerical solutions of the unstable method



In the following part, we are going to introduce the stability for the multistep method and develop a beautiful theorem to guarantee the stability for the multistep method.

We first define the notion of stability for linear multistep methods.

Definition 7.1. A multistep method is said to be stable if there exists $h^\star > 0$ such that for all $T > 0$ there exists $C > 0$ which has the following properties: for every h in $]0, h^\star]$, for all sequences y_n and $\widetilde{y}_n$ which verifie:

$$\begin{aligned} \textstyle\sum_{j=0}^{k} \alpha_j y_{n+j} &= h \textstyle\sum_{j=0}^{k} \beta_j f_{n+j}, \\ \textstyle\sum_{j=0}^{k} \alpha_j \widetilde{y}_{n+j} &= h \textstyle\sum_{j=0}^{k} \beta_j \widetilde{f}_{n+j} + \varepsilon_n \end{aligned} \tag{7.4}$$

the following estimation is verified for $k \leq n \leq T/h$:

$$|y_n - \widetilde{y}_n| \leq C(\sum_{j=0}^{k-1} |y_j - \tilde{y_j}| + \sum_{j=0}^{n-k} |\varepsilon_j|) \tag{7.5}$$

If a multistep method is stable, it means that the error on each step can be controlled by the errors on the previous steps, which is of great importance when we do the numerical experiments on the computers.

Before giving the main theorem of this section, we first give some lemmas and properties which are useful for the theorem proof.

Lemma 7.2. *Let $A \in M_n(\mathbb{K})$ (where $\mathbb{K}$ is $\mathbb{C}$ or $\mathbb{R}$) such that its spectral radius less than one, i.e. $\rho(A) < 1$. Then we can construct from any norm M, an other norm N such that its induced operator norm $\|\cdot\|_N$ satisfies*

$$\|A\|_N < 1.$$

Proof. We first set

$$\forall x \in \mathbb{K}^n, \quad N(x) = \sum_{j\in\mathbb{N}} M(A^j x).$$

This is well-defined thanks to $\rho(A) < 1$. Indeed, there exists $C > 0$ and $r > 0$ such that

$$\forall k \in \mathbb{N}, \quad M(A^k) \leq Cr^k.$$

Thus, this series is convergent. Besides, N is a norm thanks to the properties of M and the series. For $N(x) = 1$, let us compute $N(Ax)$:

$$N(Ax) = \sum_{j=0}^{\infty} M(A^{j+1}x) = \sum_{j=1}^{\infty} M(A^j x) = 1 - M(x).$$

Then, we get

$$\|A\|_N = \max_{N(x)=1} N(Ax) = \max_{N(x)=1} \left(1 - M(x)\right).$$

The norm M attains a minimum on the compact set $\{x \mid N(x) = 1\}$. Therefore, $\|A\|_N < 1$. □

Lemma 7.3. *Let A be a complex matrix in $M_d(\mathbb{C})$. The following propositions are equivalent:*

i) *there exists a norm on $\mathbb{C}^d$ such that for the corresponding norm on $M_d(\mathbb{C})$, A verifies:*

$$\|A\|_N \leq 1.$$

ii) *in $M_d(\mathbb{C})$, the powers of A are uniformly bounded.*

iii) *the eigenvalues of A are of modulus at most equal to 1 and the algebraic multiplicity of eigenvalue with modulus equal to 1 is equal to their geometric multiplicity.*

Then A is said to be a stable matrix if A satisfies one of the above propositions.

Proof. Let us prove this theorem by using different implications.

i) $\Rightarrow$ *ii)* Suppose that there exists a norm on $\mathbb{C}^d$ such that for the corresponding norm on $M_d(\mathbb{C})$, A verifies: $\|A\|_N \leq 1$. Then for all $i \in \mathbb{N}$,

$$\left\|A^i\right\|_N \leq \|A\|_N^i \leq 1.$$

ii) $\Rightarrow$ *iii)* Suppose that the powers of A are uniformly bounded. The Jordan normal form of A are

$$A = P^{-1}AP$$

where P is an invertible matrix and J such that

$$J = \begin{pmatrix} J(\lambda_1, m_1) & 0 & \cdots & 0 \\ 0 & J(\lambda_2, m_2) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & J(\lambda_k, m_k) \end{pmatrix}.$$

The Jordan's blocks are given by

$$J(\lambda, m) = \lambda I_m + N_m, \quad N_m = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Since, the powers of A are uniformly bounded by a constant $C_i > 0$ where $i \in \mathbb{N}$ is the power of A , the power of J are also uniformly bounded by the same constant C_i. Indeed, for all $i \in \mathbb{N}$, $J^i = \left(PAP^{-1}\right)^i = PA^iP^{-1}$, then

$$\|J^i\| \leq \|P\| \cdot \|A^i\| \cdot \|P^{-1}\| \leq \|P\| \cdot \|A\|^i \cdot \|P\|^{-1} \leq C_i.$$

Nevertheless,

$$J^i = \begin{pmatrix} J(\lambda_1, m_1)^i & 0 & \cdots & 0 \\ 0 & J(\lambda_2, m_2)^i & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & J(\lambda_k, m_k)^i \end{pmatrix} \quad \text{with} \quad J(\lambda, m)^i = \lambda^i I_m + \sum_{j=1}^{m-1} \binom{i}{j} \lambda^{i-j} N_m^j.$$

Suppose that there exists one eigenvalue $\widetilde{\lambda}$ of modulus 1 with a Jordan's block of dimension at least 2. Then the coefficient of $N_{\widetilde{m}}$ in $J(\widetilde{\lambda}, \widetilde{m})^i$ is $i(\widetilde{\lambda})^{i-1}$ which is not bounded (*contradiction*). Then the algebraic multiplicity of eigenvalue with modulus equal to 1 is equal to their geometric multiplicity.

iii) $\Rightarrow$ *i*) Suppose that *iii*) is verified.
Let $\mathbb{C}^d$ be the direct sum of two linear space $V_{<1}$ and V_1 such that $V_{<1}$ is the direct sum of the eigenspace of A associated with the eigenvalue of modulus strictly lower than 1 and V_1 is the direct sum of the eigenspace of A associated with the eigenvalue of modulus 1. We denot $P_{<1}$ and P_1 the projection on $V_{<1}$ and V_1.
Let $\nu_1, \ldots, \nu_\ell$ be a basis of V_1 where ν_i is an eigenvector of A. Then we can write $x = \sum_{i=1}^{\ell} \xi_i \nu_i$. Thus, the following expression is a norm on V_1

$$N_1(x) = \sum_{i=1}^{\ell} |\xi_i|.$$

Since the spectral radius is strictly lower than 1 on $V_{<1}$, we have thanks to (7.2) that there exists a norm $N_{<1}$ on $V_{<1}$ such that the restriction of A to this space is of norm strictly lower than one.
Therefore, we can define a new norm N such that for all $x \in \mathbb{C}$,

$$N(x) = N_1\left(P_1(x)\right) + N_{<1}\left(P_{<1}(x)\right) \leq 1.$$

□

We expose here an easy lemma that will be useful for characterizing the stability of a given linear multistep method.

Lemma 7.4. *Let A be a complex matrix that is diagonal in blocks. Then this matrix is stable if and only if its blocks are stable.*

Proof. Let A be a complex matrix that is diagonal in blocks. Suppose A has n different blocks that we note A_i for i in $1, \ldots, n$. Then we have that the characteristic polynomial $\chi(A)$ of A is the product of the characteristic polynomials $\chi(A_i)$ of the blocks A_i for i in $1, \ldots, n$. For this reason, the set of the roots of $\chi(A)$ is exactly the set of the roots of the polynomials $\chi(A_i)$ for i in $1, \ldots, n$. The eigenvalues of a matrix are exactly the roots of its characteristic polynomial so we deduce that the set of the eigenvalues of A is exactly the set of the eigenvalues of the blocks A_i for i in $1, \ldots, n$, with the same multiplicities. With the characterization of stability in terms of eigenvalues given by, we get that the stability of a matrix can be characterized just by the properties of its eigenvalues. We conclude that A is stable if and only if all its blocks are stable.

□

We also present the discrete Gronwall's Lemma which we will use later.

Lemma 7.5. *Let Λ and h be two positive numbers and let $(a_j)_{j\geq 0}$ and $(b_j)_{j\geq 0}$ be two non-negative sequences which satisfy for all $j \geq 0$:*

$$a_{j+1} \leq (1+\Lambda h)a_j + b_j \tag{7.6}$$

Then we have

$$a_j \leq e^{\Lambda j h}a_0 + \sum_{k=0}^{j-1} b_k e^{\Lambda(j-k-1)h}. \tag{7.7}$$

Proof. Let $\alpha_j = a_j e^{-\Lambda j h}$, then we have that

$$\begin{aligned}
\alpha_{j+1} &\leq e^{-\Lambda(j+1)h}(1+\Lambda h)a_j + e^{-\Lambda(j+1)h}b_j \\
&\leq \alpha_j e^{-\Lambda h}(1+\Lambda h) + e^{-\Lambda(j+1)h}b_j \\
&\leq \alpha_j + e^{-\Lambda(j+1)h}b_j
\end{aligned}$$

The last inequality is due to the fact that

$$e^x \geq 1+x, \forall x \geq 0.$$

Then we have

$$\alpha_j - \alpha_0 \leq \sum_{k=1}^{j} e^{-\Lambda k h} b_k$$

Changing α_j back to a_j, we obtain the inequality 7.7. □

We can now give the main result of this section. It connects the polynomial ρ to the stability (or unstability) of a multistep method.

Theorem 7.6. *Let f be a continuous map from $\mathbb{R}\times\mathbb{R}^d$ to $\mathbb{R}^d$ which is $L-$lipschitzian with respect to its state variable. Suppose that ρ satisfies the root condition, i.e. :*

i) *The roots of ρ lie or within the unit circle;*
ii) *The roots on the unit circle are simple.*

Then, the method 5.1 is stable.
Reciprocally, if a multistep method is stable then the polynomial ρ satisfies the root condition.

Proof. Suppose that ρ satisfies the root condition. Let us take two sequences $(Y_n)_n$ and $(\widetilde{Y}_n)_n$ such that:

$$\begin{array}{rcl}
\sum_{j=0}^{k}\alpha_j y_{n+j} & = & h\sum_{j=0}^{k}\beta_j f_{n+j}, \\
\sum_{j=0}^{k}\alpha_j \widetilde{y}_{n+j} & = & h\sum_{j=0}^{k}\beta_j \widetilde{f}_{n+j} + \varepsilon_n
\end{array} \tag{7.8}$$

Let define

$$\alpha_j^{'} = \frac{\alpha_j}{\alpha_k} \quad \text{and} \quad \beta_j^{'} = \frac{\beta_j}{\alpha_k}.$$

These are well-defined thanks to the hypothesis (5.2). Let A be the companion matrix of ρ

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0^{'} & -\alpha_1^{'} & -\alpha_2^{'} & \cdots & -\alpha_{k-1}^{'} \end{pmatrix}$$

Since, ρ satisfies the root condition, A is stable.
Let define B a $kd-$square matrix such that

$$B = \begin{pmatrix} 0 & I_d & 0 & \cdots & 0 \\ 0 & 0 & I_d & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_d \\ -\alpha_0^{'} I_d & -\alpha_1^{'} I_d & -\alpha_2^{'} I_d & \cdots & -\alpha_{k-1}^{'} I_d \end{pmatrix}$$

By reorganizing the canonical basis of $\mathbb{R}^{kd}$, we can prove that B is similar to a diagonal block matrix, with A as the constant block. Then, by using the lemma (7.4), we have that B is stable.
Let $|\cdot|$ be a norm on $\mathbb{R}^{kd}$ such that the operator norm of B is lower than 1.
We now set u_n, $\widetilde{u}_n \in \mathbb{R}^{kd}$

$$u_n = \begin{pmatrix} y_n \\ \vdots \\ y_{n+k-1} \end{pmatrix} \quad \widetilde{u}_n = \begin{pmatrix} y_n \\ \vdots \\ \widetilde{y}_{n+k-1} \end{pmatrix}$$

and the functions ϕ and ψ from $\mathbb{R} \times \mathbb{R}^{kd}$ to $\mathbb{R}^{kd}$ such that

$$\forall\, x = \begin{pmatrix} x_1 \\ \vdots \\ x_{k-1} \\ x_k \end{pmatrix} \in \mathbb{R}^k, \quad \phi(t,x) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \beta_k^{'} f(t+kh, x_k) \end{pmatrix} \quad \text{and} \quad \psi(t,x) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sum_{j=0}^{q-1} \beta_j^{'} f(t+jh, x_j) \end{pmatrix}$$

Since f is $L-$lipschitzian with respect to its state variable, the functions ϕ and ψ are also lipschitzian with respect to their state variable. We denote L_ϕ and L_ψ their Lipschitz constants.
We introduce $\eta_n = (0, \ldots, 0, \varepsilon_n)^t \in \mathbb{R}^{kd}$. Then (7.8) can be written as the following

$$\begin{aligned} u_{n+1} &= Bu_n + h\phi(t_n, u_{n+1}) + h\psi(t_n, u_n), \\ \widetilde{u}_{n+1} &= B\widetilde{u}_n + h\phi(t_n, \widetilde{u}_{n+1}) + h\psi(t_n, \widetilde{u}_n) + \eta_n. \end{aligned}$$

Therefore,

$$\begin{aligned} |u_{n+1} - \widetilde{u}_{n+1}| &\leq |B||u_n - \widetilde{u}_n| + h\,|\phi(t_n, u_{n+1}) - \phi(t_n, \widetilde{u}_{n+1})| + h\,|\psi(t_n, u_n) - \psi(t_n, \widetilde{u}_n)| + |\eta_n| \\ &\leq |B||u_n - \widetilde{u}_n| + hL_\phi |u_{n+1} - \widetilde{u}_{n+1}| + hL_\psi |u_n - \widetilde{u}_n| + |\varepsilon_n| \\ &\leq |u_n - \widetilde{u}_n| + hL_\phi |u_{n+1} - \widetilde{u}_{n+1}| + hL_\psi |u_n - \widetilde{u}_n| + |\varepsilon_n|. \end{aligned}$$

We obtain

$$(1 - hL_\phi)|u_{n+1} - \widetilde{u}_{n+1}| \leq |\varepsilon_n| + (1 + hL_\psi)|u_n - \widetilde{u}_n|$$

Select $h^\star$ such that for all $h \in]0, h^\star[$, we have $1 - hL_\phi \geq \frac{1}{2}$, then

$$\begin{aligned} |u_{n+1} - \widetilde{u}_{n+1}| &\leq \frac{1}{(1-hL_\phi)}|\varepsilon_n| + \frac{(1+hL_\psi)}{(1-hL_\phi)}|u_n - \widetilde{u}_n| \\ &\leq 2|\varepsilon| + \left(1 + h\frac{L_\phi + Ł_\psi}{1-hL_\phi}\right)|u_n - \widetilde{u}_n| \\ &\leq 2|\varepsilon| + (1 + 2h(L_\phi + Ł_\psi))\,|u_n - \widetilde{u}_n|. \end{aligned}$$

Bu using Gronwall's lemma, we get

$$\forall n \leq 0, \quad |u_n - \widetilde{u}_n| \leq e^{2nh(L_\phi + L_\psi)}|u_0 - \widetilde{u}_0| + 2\sum_{i=0}^{n-1}|\varepsilon_i|e^{2h(n-1-i)}.$$

Sine $2e^{2nh(L_\phi+L_\psi)} \leq 2e^{2T(L_\phi+L_\psi)} =: C$, we have the following result

$$\forall n \leq 0, \quad |u_n - \widetilde{u}_n| \leq C\left(|u_0 - \widetilde{u}_0| + \sum_{i=0}^{n-1}|\varepsilon_i|\right) = C\left(\sum_{i=0}^{k-1}|y_i - \widetilde{y}_i| + \sum_{i=0}^{n-1}|\varepsilon_i|\right).$$

Therefore,

$$|y_n - \widetilde{y}_n| \leq |u_{n-k} - \widetilde{u}_{n-k}| \leq C\left(\sum_{i=0}^{k-1}|y_i - \widetilde{y}_i| + \sum_{i=0}^{n-1}|\varepsilon_i|\right)$$

which prove that the multistep method is stable.

Reciprocally, we define two reel sequences u_n and $\widetilde{u}_n$ and we choose $f = 0$:

$$\forall n \geq 0,\ u_n = 0; \quad \forall i \geq 0, \sum_{j=0}^{k}\widetilde{u}^i_{n+j} = 0.$$

If we choose $\widetilde{u}^i_j = \delta_{ij}$ for all $i, j = 1, \ldots, k-1$, then there exists C_i such that for all h small enough,

$$\forall n \in \left\{0, \ldots, \frac{T}{h}\right\}, \quad \left|\widetilde{u}^i_n\right| \leq C_i.$$

Thus, if A is the companion matrix of ρ,

$$\left|A^n\left(\widetilde{u}^i_j\right)_{0\leq j\leq k-1}\right|$$

is bounded independently of i and n. Therefore, A^n is bounded independently of n.
We conclude with the lemma (7.3).

$\square$

7.1. **Application to the Adams methods.** The stability of the Adams method of $k+1$ steps defined in (6.5) is given by the calculation of ρ. We have $\rho(X) = X^k - X^{k-1}$. The roots of ρ are 0 and 1 and the root of modulus 1 is simple. Hence, the hypothesis on the roots of ρ in 7.6 is verified. We conclude that the Adams method of $k+1$ steps defined in (6.5) is stable.

8. Convergence of linear multistep methods: definition and main results

In this section, we define and characterize the notion of convergence for a linear multistep method. We expose the main result of this project which is that a linear multistep method that is both stable and consistent is then convergent.

Definition 8.1 (Convergence of a linear multistep method)**.** The method defined in (5.1) is said to be convergent of order p if and only if the following assumption is verified:

$$\exists C > 0, \max_{0 \leq nh \leq T} |y_n - y(nh)| \leq Ch^p \tag{8.1}$$

Theorem 8.2. *Let f be a C^p function, lipschitzian in regard to its second argument. Suppose that the method defined in* (5.1) *is stable and of order $p \geq 1$. The initial value y_0 is arbitrary in R^d and $T > 0$; y is the solution to* (1.2)*. If there is $\alpha > 0$ such that for h small enough:*

$$\sum_{j=0}^{k-1} |y_n - y(jh)| \leq \alpha h^p, \tag{8.2}$$

then the method is convergent of order p.

Proof. We define $N = \left\lfloor \dfrac{T}{h} \right\rfloor$ As y is a C^1 solution of (1.2) and as f is C^p, we have that y is C^{p+1}. We define $\epsilon_n = \epsilon(t_n, y, h)$ and $\tilde{y_n} = y(nh)$. With theorem 6.3, the hypothesis on the order of the method gives that there exists $C > 0$ such that :

$$|\epsilon_n| \leq Ch^{p+1}.$$

Since the method is stable, we get $C' > 0$ such that the following inequality is verified:

$$|y_n - \tilde{y_n}| \leq C'(\sum_{j=0}^{k-1} |y_j - \tilde{y_j}| + \sum_{j=0}^{n-k} |\epsilon_j|). \tag{8.3}$$

Thanks to the estimation (8.2), we get that the first term is inferior or equal to Ch^p. With the bound of ϵ_n given by the hypothesis on the order, we get $\beta > 0$ such that:

$$\sum_{j=0}^{n-k} |\epsilon_j| \leq (n-k+1)Ch^{p+1} \leq NCh^{p+1} \leq \beta h^p$$

Combining the estimations of both members of the sum in (8.3) we get:

$$\exists C > 0, \forall n \in \{n \in N | 0 \leq nh \leq T\}, |y_n - y(nh)| \leq Ch^p$$

This estimation gives the convergence of the method.

□

8.1. **Convergence of the Adams methods.** Thanks to the precedent results on the Adams methods, we know that the Adams method of $k+1$ step defined in (6.5) is both stable and consistent. By 8.2, we get that this method is convergent.

9. Conclusion

In this project, we've studied important properties of the linear multistep methods by characterizing crucial notions that are consistency order, stability and convergence. The most important result is that as for linear one-step methods, a linear multistep method which is both consistent and stable is then convergent. By applying these results to the Adams methods, we have a concrete representation of the possible applications of these results with a quite simple method.

References

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