

## 2024-04-02: i linguaggi di si e no della Programmazione Lineare (PL)

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### Incontro 2024-04-02

The other time we saw:

1. what is a linear program
2. what is a linear program in standard form

```
max c'x
Ax <= b
```

3. la NO language, that is, since we are working with an (optimization/search problem), an (upper bound/certificate for it) on the optimal objective function value)

For the following example problem:

```
max x1 + x2
3x1 + 2x2 <= 5
2x1 + 3x2 <= 5
x1, x2 >= 0
```

We saw a feasible solution (Yes certificate, lower bound) of value 2:

```
x1=1,x2=1
```

It was easy for King Arthur to check that this solution was feasible and so we have the following lower bound of 2 on the optimal objective function value:

```
x1 + x2 = (1)+(1) = 2
```

We also saw a way to certificate that 2 was the optimal value.

```
(1/5) 3x1 + 2x2 <= 5 +
(1/5) 2x1 + 3x2 <= 5
-----
1x1 + 1x2 <= 2
```

We already observed that Linear Programming is in NP (easy to certify any possible valid lower-bound).

good conjecture (any conjecture such that, if it were true, then it would also be easy to certify any possible valid upper-bound):

it is enough to combine (with suitable multipliers) the constraints that are tight (=satisfied at equality) in the optimal solution.

what does it mean combine:

choose a multiplier for each such constraint and produce the linear combination of them via these multipliers.

```
(1/5) 3x1 + 2x2 <= 5 +    --> 3x1 + 2x2 <= 5 +
(-1/5) 2x1 + 3x2 <= 5    --> -2x1 - 3x2 >= -5
-----
```

$1x_1 + 1x_2 \leq 2$                       ???  
 VALID

The multiplier should be non-negative

Definition: a conic combination is any linear combination where all multipliers are non-negative.

Now that we have a suitable good conjecture, it might be a suitable plan to (prove it/obtain an algorithm that finds the optimum and builds the certificates)

Still one thing should disturb us.

Consider the problem:

$\max x_1 + x_2 + x_3$   
 $x_1 \leq 1$   
 $x_2 \leq 1$   
 $x_3 \leq 1$   
 $x_1, x_2, x_3 \geq 0$

Lower bounds:

0 yes-cert:  $(x_1=0, x_2=0, x_3=0)$   
 1 yes-cert:  $(x_1=1, x_2=0, x_3=0)$   
 3 yes-cert:  $(x_1=1, x_2=1, x_3=1)$

Upper bounds:

(\*1)  $x_1 \leq 1$   
 (\*1)  $x_2 \leq 1$   
 (\*1)  $x_3 \leq 1$

-----

$x_1 + x_2 + x_3 \leq 3$

Great! Our nice good conjecture is still alive!

$\max x_1 - x_2 + x_3$   
 $x_1 \leq 1$   
 $x_2 \leq 1$   
 $x_3 \leq 1$   
 $x_1, x_2, x_3 \geq 0$

Lower bounds:

0 yes-cert:  $(x_1=0, x_2=0, x_3=0)$   
 2 yes-cert:  $(x_1=1, x_2=0, x_3=1)$

Upper bounds:

(\*1)  $x_1 \leq 1$   
 (\*1)  $x_3 \leq 1$   
 (\*-1)  $x_2 \geq 0$

we should use a non-negative constraint as well, but of course we should give it a non-positive multiplier.

We do not like this asymmetry, this is the standard way to solve it:

Consider the canonic form of the problem which makes all distances from a constraint a non-negative variable:

$\max x_1 - x_2 + x_3$   
 $x_1 \leq 1 \quad \rightarrow \quad s_1 := 1 - x_1$   
 $x_2 \leq 1 \quad \rightarrow \quad s_2 := 1 - x_2$   
 $x_3 \leq 1 \quad \rightarrow \quad s_3 := 1 - x_3$   
 $x_1, x_2, x_3 \geq 0 \quad \quad \quad x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$

Thus, this is a problem in canonic for:

$$\begin{aligned} \max \quad & x_1 - x_2 + x_3 \\ \text{s.t.} \quad & s_1 = 1 - x_1 \\ & s_2 = 1 - x_2 \\ & s_3 = 1 - x_3 \\ & x_1, x_2, x_3, s_1, s_2, s_3 \geq 0 \end{aligned}$$

Lower bounds:

$$2 \text{ yes-cert: } (x_1=1, x_2=0, x_3=1, s_1=0, s_2=1, s_3=0)$$

Upper bounds:

$$\begin{aligned} (*-1) \quad & s_1 \geq 0 \\ (*-1) \quad & s_3 \geq 0 \\ (*-1) \quad & x_2 \geq 0 \\ & -s_1 - x_2 - s_3 \leq 0 \end{aligned} \quad \text{should be true for every feasible solution}$$

what does this truth imply for the objective function value?

$$\text{objective function value} = x_1 - x_2 + x_3 = (1-s_1) - x_2 + (1-s_3) = 2 - s_1 - x_2 - s_3 \leq 2$$

The simplex algorithm:

$$\begin{aligned} \max \quad & 4x_1 + x_2 + 3x_3 \\ \text{s.t.} \quad & x_1 + 4x_2 \leq 1 \\ & 3x_1 - x_2 + x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & z = 4x_1 + x_2 + 3x_3 \\ w_1 \quad & = 1 - x_1 - 4x_2 \\ w_2 \quad & = 3 - 3x_1 + x_2 - x_3 \\ & x_1, x_2, x_3, w_1, w_2 \geq 0 \end{aligned}$$

basic solution  $x_1=0, x_2=0, x_3=0$  (hence  $w_1, w_2=1, 3$ ) is feasible if all the known terms (the  $b$  vector of problem in standard form  $\max \{c'x: Ax \leq b, x \geq 0\}$ ) are non negative, hence it certifies that the optimal objective function value is at least 0.

The tight constraints for this basic solution are  $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$ .

Can we obtain a certificate of optimality combining these constraints or can we build a new basic feasible solution with better objective function value?

Yes, since  $x_3$  appears with coefficient  $3 > 0$  in the current objective function expression, then we have good reasons to lift it away from its current value of 0. The more we push it, the more we gain.

However, there will be of course a limit: the only problem is that we loose feasibility. The only way we might loose feasibility is because a basic variable gets negative. We stop at the first variable becoming 0. In this way, not only we exploit till the very end the direction of improvement that we have found, but we also get a new tight constraints (the non-negative constraint of  $x_3$  will not be any longer tight) as needed to express a NO-certificate.

Now we want to produce the next step rewriting, where the non-basic variable are  $x_1, x_2, w_2$  and  $w_1, x_3$  are the basic one.

The procedure for obtaining this rewriting is called pivoting.

$$\begin{aligned} \max \quad & z = 4x_1 + x_2 + 3x_3 \\ w_1 \quad & = 1 - x_1 - 4x_2 \\ w_2 \quad & = 3 - 3x_1 + x_2 - x_3 \\ & x_1, x_2, x_3, w_1, w_2 \geq 0 \end{aligned}$$