

2024-03-26: problemi di decisione, di ottimizzazione, di search/ costruzione + i linguaggi di si e no (dimostrazione della buona congettura per il problema pirellone)

Si comincia a comporre il presente documento contestualmente e collaborativamente all'URL:

<https://etherpad.wikimedia.org/p/DiscreteOptimization2024-03-26>

Gli incontri avvengono sia in presenza che nella stanza Zoom:

<https://univr.zoom.us/j/83514538731>

E, quando disponibile, la loro registrazione è nel folder:

<https://univr.cloud.panopto.eu/Panopto/Pages/Sessions/List.aspx?folderID=c27d5938-629e-48c0-a238-b12a00b86b38>

Incontro 2024-03-26

Good Conjecture:

Every matrix is solvable unless it contains one of the 8 forbidden 2x2 matrices (those with an odd number of 1's).

What do we mean by submatrix?

Strong conjecture: a 2x2 submatrix of a matrix is obtained by:

1. removing all rows except two consecutive ones, and
2. removing all columns except two consecutive ones.

Weak conjecture: a 2x2 submatrix of a matrix is obtained by:

1. removing all rows except just two of them, and
2. removing all columns except just two of them.

Assume we start to believe the conjecture is true.

Then we go all-in about proving it.

We know we must do it by induction.

We could then seek for an algorithm (its efficiency is not important, it is not on our agenda right now) for the following computational problema:

INPUT: a matrix with no 2x2 bad submatrix

TASK: compute and return a solution

We could also go for an algorithm for the following computational problema:

INPUT: a matrix M

OUTPUT: either a 2x2 bad submatrix of M or a solution for M

$\max f(x)$

with x in X

SEARCH FORM:

INPUT: X, f

OUTPUT: an x in X maximizing f

OPTIMIZATION FORM:

INPUT: X, f

OUTPUT: I'm curious, tell me what is the maximum I could achieve in terms of f

DECISION FORM:

INPUT: X, f, b

OUTPUT: Yes if the optimum value is at least b , No otherwise.

Linear programming is the mathematical programming problem where:

1. x is a vector of variables
2. f is a linear function
3. X is described in terms of linear constraints
(i.e., $3x_1 + 2x_2 \leq 5$ or $2x_2 - x_3 \geq 7$)

Note: strict inequalities are not allowed in order to guarantee the existence of an optimal solution when the problem is feasible and bounded)

max $3x$
 $x < 2$ not allowed

Every linear program can be cast in standard form:

max $x_1 + x_2$	max $c'x$
$3x_1 + 2x_2 \leq 5$	$Ax \leq b$
$2x_1 + 3x_2 \leq 5$	$x \geq 0$
$x_1 \geq 0, x_2 \geq 0$	

What are the elements that make the form standard:

1. see it as a maximization rather than as a minimization

What if my problem was:

min $2x_1$	I regard it as: -max $-2x_1$
$x_1 + x_2 \geq 3$	$x_1 + x_2 \geq 3$

In this way I can reduce minimization problems to maximization problems

2. no constraints are of the form $=$

What if my problem was:

max $2x_1$	I regard it as:
$x_1 + x_2 = 3$	max $2x_1$
	$x_1 + x_2 \leq 3$
	$x_1 + x_2 \geq 3$

3. no constraints are of the form \geq (we are only allowed \leq except for the non-negativity constraints of the variables)

What if my problem was:

max $2x_1$	I regard it as:
$x_1 + x_2 \geq 3$	max $2x_1$
	$-x_1 - x_2 \leq -3$

And

max $2x_1$	I regard it as:
$x_1 + x_2 = 3$	max $2x_1$
	$x_1 + x_2 \leq 3$
	$-x_1 - x_2 \leq -3$

4. all variables should be subject to non-negativity constraints, i.e., $x \geq 0$.

What if my problem was:

max $2x_1$	after considering $x'_2 := -x_2$, I regard it as:
$x_1 + x_2 \leq 3$	max $2x_1$
$x_1 \geq 0, x_2 \leq 0$	$x_1 - x'_2 \leq 3$
	$x_1 \geq 0, x'_2 \geq 0$

What if my problem was:

max $2x_1$
 $x_1 + x_2 \leq 3$
 $x_1 \geq 0, x_2$ free in sign

I introduce $x_2^+ := \max\{0; x_2\}$
 $x_2^- := \max\{0; -x_2\}$

Not that they:

1. are both non-negative
2. together they encode x_2 without any loss of information

after this the problem:

$$\max 2x_1 + 3x_2$$

$$x_1 + x_2 \leq 3$$

$$x_1 \geq 0, x_2 \text{ free in sign}$$

becomes:

$$\max 2x_1 + 3x_2^+ - 3x_2^-$$

$$x_1 + x_2^+ - x_2^- \leq 3$$

$$x_1 \geq 0, x_2^+ \geq 0, x_2^- \geq 0$$

given that $x_2 = x_2^+ - x_2^-$

So, we can assume that the problem is in standard form.

$\max x_1 + x_2$	$\max c'x$
$3x_1 + 2x_2 \leq 5$	$Ax \leq b$
$2x_1 + 3x_2 \leq 5$	$x \geq 0$
$x_1 \geq 0, x_2 \geq 0$	

Yes-language:

Merlin tells King Arthur: you know guy, you can gain at least 2!

King Arthur: really? Whow! But ... I do not believe you. Please, convince me!

Merlin: $(x_1, x_2) = (1, 1)$

King Arthur homework:

$$f = (1) + (1) = 2 \text{ (would be nice)}$$

$$3(1) + 2(1) = 5 \leq 5 \text{ ok}$$

$$2(1) + 3(1) = 5 \leq 5 \text{ ok} \quad x \geq 0$$

$$x_1 \geq 0, x_2 \geq 0$$