Mathematics for Decisions

Basics of Linear Programming

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October 15, 2021

Overview

Linear Programming

A *Linear Programming* (LP) problem is a mathematical programming problem of the form:

$$
\begin{aligned}
\text{min } \textit{or} \text{ max} \quad & c^T x \\
\text{s.t.} \quad & Ax \leq b \\
& x \geq 0
\end{aligned}
$$

where $x \in \mathbb{R}^n$ are the decision variables, $b \in \mathbb{R}^m$ is the vector of known values, $A \in \mathbb{R}^{m \times n}$ and $c \in \mathbb{R}^n$ is the vector of the coefficients in the objective function.

Canonical and standard form

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Canonical

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Standard

The two formulations are equivalent but the conversion from one form to the other may change the number of constraints and variables. Rules to follow:

- Conversion from "min" to "max", by changing the sign of c^T
- Constraints conversion from "$\leq$" to "$=$", by introducing **slack variables**: $\mathbf{a}_i^T\mathbf{x} \geq b_i \rightarrow \mathbf{a}_i^T\mathbf{x} + s_i = b_i$, with $s_i \geq 0$;
- Free variables: if x_i free, then $x_i = x_i^+ - x_i^-$, with $x_i^+, x_i^- \geq 0$;

Integer, Mixed Integer, and Binary Linear Programming

- Integer LP (ILP): when all variables assume integer values;
- Mixed Integer LP (MILP): when some variables are integer and other continuous;
- Binary LP (0-1 LP): when all variables can only assume 0 or 1 as values.

Geometry of Linear Programming

- **(Convex) Polyhedron**: intersection of a finite number of affine half-spaces and hyperplanes.
- **Feasible region**: set of feasible solutions $\mathbf{x} \in \mathbb{R}^n$ that satisfy all linear inequalities $\rightarrow$ It's a polyhedron.
- **Polytope**: a bounded polyhedron.
- **Vertex** or **extreme point**: a point $\mathbf{x}$ of a polyhedron P that cannot be expressed as a strict convex combination of other two points of the polyhedron, i.e., there exist no $\mathbf{y}, \mathbf{z} \in P, \mathbf{y} \neq \mathbf{z}$ and $\lambda \in (0,1)$ such that $\mathbf{x} = \lambda\mathbf{y} + (1-\lambda)\mathbf{z}$.
- Each polyhedron has a **finite number of vertices**.
- **Minkowski-Weyl Theorem**: every point of a polytope P can be obtained as a convex combination of its vertices $\rightarrow$ If the feasible region of an LP is a bounded polytope, then there exists at least one optimal vertex of P.

Vertices and basic solutions

- The optimal solution of an LP is a vertex: we can start from one vertex arbitrarily and iterate through the vertices, moving to an adjacent one, until the optimal is found.
- **Basis of A**: a collection of m linearly independent columns of A
- **Basic and non-basic variables**: $\mathbf{x}_B$ and $\mathbf{x}_N$

$$A\mathbf{x} = \mathbf{b} \text{ can be written as } B\mathbf{x}_B + N\mathbf{x}_N = \mathbf{b}$$

- When $\mathbf{x}_N = 0$, $\mathbf{x}_B = B^{-1}\mathbf{b}$ is the **basic solution** associated to the basis B. It is:
  - *feasible*, if $B^{-1}\mathbf{b} \geq 0$;
  - *degenerate*, if $B^{-1}\mathbf{b}$ has one or more zero components.
- A point $\mathbf{x}$ of the polyhedron $P := \{\mathbf{x} \geq 0 : A\mathbf{x} = \mathbf{b}\}$ is a **vertex** iff $\mathbf{x}$ is a basic feasible solution of $A\mathbf{x} = \mathbf{b}$.

The Simplex method

How to solve an LP?

- Enumerate all possible vertices, i.e., all the basic solutions to the problem $\rightarrow$ Number of vertices $= \binom{n}{m} = \frac{n!}{m!(n-m)!}$
- Improving this procedure:
  - Verify the optimality of the current solution;
  - Find a way to move from a basic feasible solution to another adjacent with a better value of the objective function.
- **Tableau form**: `https://www.youtube.com/watch?v=XK26I9eoS18` and `https://www.hec.ca/en/cams/help/topics/The_steps_of_the_simplex_algorithm.pdf`

Particular cases

- Loop between entering and exiting variables
- Empty feasible region
- Unlimited solution
- Multiple optimal solutions

The graphical method (I) – Exam 31/07/2017, ex. 7

Consider the following LP problem:

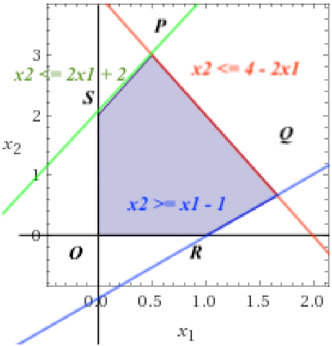
$$
\begin{aligned}
\max 3x_1 + 2x_2 & \\
2x_1 + x_2 & \leq 4 \\
-2x_1 + x_2 & \leq 2 \\
x_1 - x_2 & \leq 1 \\
x_1, x_2 \geq 0. &
\end{aligned}
$$

1. Solve it with the graphical method, specifying the objective function and the variables values at optimum.

   Note: the problem is presented in its canonical form (not the standard one).

The graphical method (II)

We represent the three constraints in the plane (x_1, x_2) obtaining the feasible region through their intersection:



The graphical method (III)

- The feasible region has five vertices:
  - O $= x_1 \geq 0 \cap x_2 >= 0$;
  - P $= cons_1 \cap cons_2$;
  - Q $= cons_1 \cap cons_3$;
  - R $= cons_3 \cap x_1 \geq 0$;
  - S $= cons_2 \cap x_2 \geq 0$.
- The objective function can be seen as a family of straight lines moving towards the direction for maximizing the function;
- We consider the gradient (3, 2).

The graphical method (IV)

- The optimal solution is given by point $P = (\frac{1}{2}, 3)$, the last vertex reached by the family of straight lines;
- Here, the objective function is $\frac{15}{2}$.

Vertices, variables and basic solutions, non-basic variables, ...

- Given a system of linear constraints defined over n variables, a **solution** is a point $x \in \mathbb{R}^n$ that satisfies all the constraints;
- We write the problem in the standard form, introducing the three slack variables s_1, s_2 e s_3:

$$
\begin{array}{ll}
\max & 3x_1 + 2x_2 \\
\text{s.t.} & 2x_1 + x_2 + s_1 = 4 \\
& -2x_1 + x_2 + s2 = 2 \\
& x_1 - x_2 + s_3 = 1 \\
& x_1, x_2 \geq 0 \\
& s_1, s_2, s_3 \geq 0
\end{array}
$$

Associated bases

2. Determine the bases associated to the vertices of the feasible region;

We can rewrite $Ax = b$ as $(B|N) \cdot (x_B|x_N)^T = b$ where:

- B = basic matrix ($m \times m$, composed of m columns of A)
- N = non-basic matrix
- x_B = basic variables
- x_N = non-basic variables

Simplex

3. Specify the sequence of the bases visited by the Simplex method to reach the optimal solution (choose x_1 as the first entering variable);

Reduced costs

4. Determine the values of the reduced costs related to the basic solutions associated to the following vertices, expressed as intersections of straight lines in $\mathbb{R}^2$:
   - $cons_1 \cap cons_2$;
   - $cons_1 \cap cons_3$.

Opposite direction of the gradient vector

5. Verify that the opposite direction of the gradient vector can be expressed as a nonnegative linear combination of the gradients for **active** constraints **only** in the optimal vertex (keep in mind that, since it is a maximization problem, constraints have to be expressed with $\leq$; e.g., $x_1 \geq 0$ has to be rewritten as $-x_1 \leq 0$).

Duality in Linear Programming

Any **primal** LP in maximization form is associated to a **dual** LP in minimization form:

Primal Problem	Dual Problem
opt=max	opt=min
Constraint i :	Variable i :
<= form	y_i >= 0
= form	y_i urs
Variable j:	Constraint j:
x_j >= 0	>= form
x_j urs	= form

Duality theorems

Given the primal problem $P : \max \mathbf{c}^T\mathbf{x}$ s.t. $A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0$ and the dual problem $D : \min \mathbf{b}^T\mathbf{u}$ s.t. $A^T\mathbf{u} \geq \mathbf{c}, \mathbf{u} \geq 0$:

- The dual of the dual problem D is the primal P.
- **Weak duality**: $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{u}$.
- **Strong duality**: P has a finite optimal solution iff D has it too and the value of the two objective functions is the same $\rightarrow \mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{u}$.

Relationships between Primal and Dual

		DUAL		
		FINITE OPTIMAL SOLUTION	UNBOUNDED	INFEASIBLE
PRIMAL	FINITE OPTIMAL SOLUTION	YES	NO	NO
	UNBOUNDED	NO	NO	YES
	INFEASIBLE	NO	YES	YES

Optimality conditions

Two vectors $\bar{\mathbf{x}} \in \mathbb{R}^n$ and $\bar{\mathbf{u}} \in \mathbb{R}^m$ are optimal for the primal problem P and the dual problem D, respectively, iff the following optimality conditions hold:

1. $A\bar{\mathbf{x}} \geq \mathbf{b}, \bar{\mathbf{x}} \geq 0$ (primal feasibility);
2. $\mathbf{c}^T \geq \bar{\mathbf{u}}^T A, \bar{\mathbf{u}} \geq 0$ (dual feasibility);
3. $\bar{\mathbf{u}}^T (A\bar{\mathbf{x}} - b) = 0$ (complementary slackness);
4. $(\mathbf{c}^T - \bar{\mathbf{u}}^T A)\bar{\mathbf{x}} = 0$ (complementary slackness).

Sensitivity analysis

Once we get the optimal solution, are we done?

- We could investigate how much the solution is **stable**, w.r.t. changing the parameter data;
- Do not forget that we are solving a **model** of the problem, not the problem itself! Thus, the less sensible is the solution, the more reliable is the model;
- **Sensitivity analysis**: study of perturbations of initial data whereby conditions:
  - $B^{-1}\mathbf{b} \geq 0$ (primal feasibility for $\bar{\mathbf{x}}$);
  - $\bar{\mathbf{c}}^T := \mathbf{c}^T - \mathbf{c}_B^T B^{-1} A \geq 0^T$ (dual feasibility for $\bar{\mathbf{u}}$, where $\bar{\mathbf{u}}^T := \mathbf{c}_B^T B^{-1}$).
- The basis B remains optimal (not the solution $\mathbf{x}$).
- We'll study three cases:
  - Changes in the right-hand sides;
  - Changes in the costs of non-basic variables;
  - Changes in the costs of basic variables.

Changes in the right-hand sides

We consider a change of $\Delta\mathbf{b}$:

- $B^{-1}(\mathbf{b} + \Delta\mathbf{b}) \geq 0$;
- $\bar{\mathbf{c}}^T := \mathbf{c}^T - \mathbf{c}_B^T B^{-1} A \geq 0^T$ (unchanged).

The basis B remain feasible and optimal iff:

$$B^{-1}\mathbf{b} \geq -B^{-1}\Delta\mathbf{b}$$

The optimal value changes from $\mathbf{c}_B^T B^{-1}\mathbf{b}$ to $\mathbf{c}_B^T B^{-1}(\mathbf{b} + \Delta\mathbf{b}) \rightarrow$ $\Delta z := (\mathbf{c}_B^T B^{-1})\Delta\mathbf{b} = \bar{\mathbf{u}}^T \Delta\mathbf{b}$

The dual variables $\bar{u}_i$, $i = 1, \ldots, m$, measure the **sensitivity** of the optimal value of the objective function w.r.t. changes Δb_i of the right-hand sides.

Changes in the costs of non-basic variables

Now we consider a change $\Delta\mathbf{c}_N^T$ and let $\mathbf{c}$ and $\widetilde{\mathbf{c}}$ be the reduced cost vectors before and after change $\Delta\mathbf{c}_N^T$.

- $B^{-1}\mathbf{b} \geq 0$ (unchanged);
- $\widetilde{\mathbf{c}}^T := [\widetilde{\mathbf{c}}_B^T, \widetilde{\mathbf{c}}_N^T] = [0^T, (\mathbf{c}_N^T + \Delta\mathbf{c}_N^T) - \mathbf{c}_B^T B^{-1} N] \geq 0^T$.

As before, we want B to remain optimal, and this happens iff:

$$\widetilde{\mathbf{c}}^T = \mathbf{c}_N^T - \mathbf{c}_B^T B^{-1} N + \Delta\mathbf{c}_N^T = \bar{\mathbf{c}}_N^T + \Delta\mathbf{c}_N^T \geq 0 \iff \Delta\mathbf{c}_N \geq -\bar{\mathbf{c}}_N.$$

We obtain $n - m$ inequalities, independent from each other:

$$\Delta c_j \geq -\bar{c}_j, \forall x_j \text{ non-basic}$$

The reduced cost $\bar{c}_j \geq 0$ can be interpreted as the maximum **decrease** in cost c_j under which B remains optimal.

Changes in the costs of basic variables

Finally, we consider a change $\Delta \mathbf{c}_B^T$ and, as before, let $\mathbf{c}$ and $\widetilde{\mathbf{c}}$ be the reduced cost vectors before and after change $\Delta \mathbf{c}_B^T$.

- $B^{-1}\mathbf{b} \geq 0$ (unchanged);
- $\widetilde{\mathbf{c}}^T := [\widetilde{\mathbf{c}}_B^T, \widetilde{\mathbf{c}}_N^T] = [0^T, \mathbf{c}_N^T - (\mathbf{c}_B^T + \Delta \mathbf{c}_B^T) B^{-1} N] \geq 0^T$.

B remains optimal iff:

$$\widetilde{\mathbf{c}}_N^T := \mathbf{c}_N^T - \mathbf{c}_B^T B^{-1} N - \Delta \mathbf{c}_B^T B^{-1} N \geq 0^T \iff \Delta \mathbf{c}_B^T B^{-1} N \leq \bar{\mathbf{c}}_N^T$$

We obtain a system which defines a polyhedron in $\mathbb{R}^m$, whose points correspond to the vectors $\Delta \mathbf{c}_B$ for which B does not change.

References

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