

# 1 Main

For an irreducible second-order polynomial over the  $\mathbb{Z}$ -plane:

$$\begin{aligned}
 f(z) &= \frac{1}{2|\rho|} \left[ 1 - 2 \cos(\angle \rho) |\rho| z^{-1} + |\rho|^2 z^{-2} \right], \quad \rho \in \mathbb{C} \\
 \Rightarrow |f(\omega)|^2 &= \left[ \cos(\omega) - \Gamma \right] \left[ \cos(\omega) - \Gamma^* \right], \quad \Gamma \in \mathbb{C} \\
 \Rightarrow \eta &= |\Gamma|^2 + \sqrt{|\Gamma|^4 - 2 \cos(2\angle \Gamma) |\Gamma|^2 + 1} \\
 \Rightarrow |\rho| &= \sqrt{\eta - \sqrt{\eta^2 - 1}} \\
 \Rightarrow \angle \rho &= \arctan \left( \frac{\tan(\angle \rho)}{\tanh(|\rho|)} \right) + \pi * \mu(\angle \Gamma^\circ - 90^\circ)
 \end{aligned}$$

Similarly, for a first-order polynomial over the  $\mathbb{Z}$ -plane:

$$\begin{aligned}
 f(z) &= \frac{1}{\sqrt{2|\rho|}} [1 - \rho z^{-1}] \\
 \Rightarrow |f(\omega)|^2 &= -\operatorname{sgn}(\Gamma) \left[ \cos(\omega) - \Gamma \right] \\
 \Rightarrow |\rho| &= |\Gamma| - \sqrt{|\Gamma|^2 - 1} \\
 \Rightarrow \operatorname{sgn}(\rho) &= \operatorname{sgn}(\Gamma)
 \end{aligned}$$

Therefore, for a frequency magnitude consisting of  $N$  real positive roots,  $M$  real negative roots, and  $K$  complex conjugate pair roots, the following squared spectral magnitude  $|f(\omega)|^2$  has the corresponding z-transform:

$$\begin{aligned}
 |f(\omega)|^2 &= \left( \prod_{n=0}^{N-1} -[\cos(\omega) - \Gamma_n] \right) \left( \prod_{m=0}^{M-1} [\cos(\omega) - \Gamma_m] \right) \left( \prod_{k=0}^{K-1} [\cos(\omega) - \Gamma_k] [\cos(\omega) - \Gamma_k^*] \right) \\
 &= \left( \prod_{n=0}^{N-1} -1 \right) \left( \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} \prod_{k=0}^{K-1} [\cos(\omega) - \Gamma_n] [\cos(\omega) - \Gamma_m] [\cos(\omega) - \Gamma_k] [\cos(\omega) - \Gamma_k^*] \right) \\
 &= (-1)^N \left( \prod_{l=0}^{L-1} [\cos(\omega) - \Gamma_l] \right), \quad L \equiv N + M + K \\
 &= \sum_{l=0}^L a_l T_l(\cos(\omega))
 \end{aligned}$$

$$\begin{aligned}
\Rightarrow f(z) &= \sum_{l=0}^{N+M+K-1} c[l] z^{-l}, \quad \text{where } c[l] = f_0 \circledast f_1 \circledast \dots \circledast f_{N+M+K-2} \circledast f_{N+M+K-1} \\
&= \left( \bigcircledast_{n=0}^{N-1} f_n(z) \right) \circledast \left( \bigcircledast_{m=0}^{M-1} f_m(z) \right) \circledast \left( \bigcircledast_{k=0}^{K-1} f_k(z) \right)
\end{aligned}$$

## Appendix A: First-Order Spectral Magnitude Design

For an first-order polynomial over the  $\mathbb{Z}$ -plane:

$$\begin{aligned}
 f(z) &= 1 - \rho z^{-1} , \quad \rho \in \mathbb{R} , \quad |\rho| < 1 \\
 \Rightarrow |f(\omega)|^2 &= 1 + |\rho|^2 - 2\rho \cos(\omega) \\
 &= 2|\rho| \left( \frac{|\rho| + |\rho|^{-1}}{2} \right) - 2\rho \cos(\omega) \\
 &= 2|\rho| \cosh \left( \ln [|\rho|] \right) - 2\rho \cos(\omega) \\
 &= 2 \left[ |\rho| \cosh \left( \ln [|\rho|] \right) - \rho \cos(\omega) \right] \\
 &= 2 \left[ \text{sgn}^2(\rho) |\rho| \cosh \left( \ln [|\rho|] \right) - \rho \cos(\omega) \right] \\
 &= 2 \left[ \text{sgn}(\rho) \rho \cosh \left( \ln [|\rho|] \right) - \rho \cos(\omega) \right] \\
 &= 2\rho \left[ \text{sgn}(\rho) \cosh \left( \ln [|\rho|] \right) - \cos(\omega) \right] \\
 &= 2\rho \left[ \Gamma - \cos(\omega) \right] , \quad \Gamma \equiv \text{sgn}(\rho) \cosh \left( \ln [|\rho|] \right) , \\
 |\Gamma| &= \cosh \left( \ln [|\rho|] \right) , \\
 \text{sgn}(\Gamma) &= \text{sgn}(\rho)
 \end{aligned}$$

Focusing on  $|\Gamma|$ , we can derive the following equations:

$$\begin{aligned}
 \Gamma &= \text{sgn}(\rho) \cosh \left( \ln [|\rho|] \right) \\
 \Rightarrow |\rho| &= |\Gamma| \pm \sqrt{|\Gamma|^2 - 1} \\
 &= |\Gamma| - \sqrt{|\Gamma|^2 - 1}
 \end{aligned}$$

This yields the following equations:

$$\begin{aligned}
f(z) &= 1 - \rho z^{-1} \\
\Rightarrow \left| f(\omega) \right|^2 &= 2\rho \left[ \Gamma - \cos(\omega) \right] \\
&= 2 \operatorname{sgn}(\rho) |\rho| \left[ \Gamma - \cos(\omega) \right] \\
&= 2 \operatorname{sgn}(\Gamma) |\rho| \left[ \Gamma - \cos(\omega) \right] \\
&= -2 |\rho| \operatorname{sgn}(\Gamma) \left[ \cos(\omega) - \Gamma \right]
\end{aligned}$$

With a normalized spectral magnitude, we then get the final equations:

$$\begin{aligned}
f(z) &= \frac{1}{\sqrt{2|\rho|}} [1 - \rho z^{-1}] \\
\Rightarrow \left| f(\omega) \right|^2 &= -\operatorname{sgn}(\Gamma) \left[ \cos(\omega) - \Gamma \right] \\
\Rightarrow |\rho| &= |\Gamma| - \sqrt{|\Gamma|^2 - 1} \\
\Rightarrow \operatorname{sgn}(\rho) &= \operatorname{sgn}(\Gamma)
\end{aligned}$$

## Appendix B: Irreducible Second-Order Spectral Magnitude Design

For an irreducible second-order polynomial over the  $\mathbb{Z}$ -plane:

$$\begin{aligned}
 f(z) &= (1 - \rho z^{-1})(1 - \rho^* z^{-1}) , \quad \rho \in \mathbb{C} , \quad |\rho| < 1 \\
 &= 1 - 2 \cos(\angle \rho) |\rho| z^{-1} + |\rho|^2 z^{-2} \\
 \Rightarrow |f(\omega)|^2 &= K^2 \left[ \cos(\omega) - \Gamma \right] \left[ \cos(\omega) - \Gamma^* \right] , \quad \Gamma \in \mathbb{C} \\
 &\text{where } K \equiv 2|\rho|
 \end{aligned}$$

$$\Gamma_R \equiv \cos(\angle \rho) \cosh\left(\ln[|\rho|]\right)$$

$$\Gamma_I \equiv \sin(\angle \rho) \sinh\left(\ln[|\rho|]\right)$$

Effectively, the squared spectral magnitude can be expressed as an irreducible polynomial (over  $\cos(\omega)$  instead of  $x$ ) with  $\Gamma$  and its complex conjugate acting as the roots for this polynomial.

$$\begin{aligned}
 |\Gamma|^2 &= \Gamma_R^2 + \Gamma_I^2 \\
 &= \cos^2(\angle \rho) \cosh^2\left(\ln[|\rho|]\right) + \sin^2(\angle \rho) \sinh^2\left(\ln[|\rho|]\right) \\
 &= \cosh^2\left(\ln[|\rho|]\right) - \sin^2(\angle \rho) \\
 &= \left(\frac{1}{2}\right) \left( \cosh\left(\ln[|\rho|^2]\right) + \cos(2\angle \rho) \right) \\
 \Rightarrow 2|\Gamma| &= \cosh\left(\ln[|\rho|^2]\right) + \cos(2\angle \rho)
 \end{aligned}$$

The equation above is one of two equations needed to express  $\Gamma$  in terms of  $\rho$ . The second equation is derived down below:

$$\begin{aligned}
 \cosh^2\left(\ln[|\rho|]\right) - \sinh^2\left(\ln[|\rho|]\right) &= 1 \\
 \Rightarrow \frac{\Gamma_R^2}{\cos^2(\angle \rho)} - \frac{\Gamma_I^2}{\sin^2(\angle \rho)} &= 1 \\
 \Rightarrow \sin^2(\angle \rho) \Gamma_R^2 - \cos^2(\angle \rho) \Gamma_I^2 &= \cos^2(\angle \rho) \sin^2(\angle \rho) \\
 \Rightarrow \Gamma_R^2 - |\Gamma|^2 \cos^2(\angle \rho) &= \cos^2(\angle \rho) - \cos^4(\angle \rho)
 \end{aligned}$$

$$\begin{aligned}
&\Rightarrow \Gamma_R^2 - |\Gamma|^2 \cos^2(\angle \rho) = \cos^2(\angle \rho) - \cos^4(\angle \rho) \\
&\Rightarrow \left[ \cos^2(\angle \rho) - \left( \frac{|\Gamma|^2 + 1}{2} \right) \right]^2 = \left( \frac{|\Gamma|^2 + 1}{2} \right)^2 - \Gamma_R^2 \\
&\Rightarrow \left[ \cos(2\angle \rho) - |\Gamma|^2 \right]^2 = |\Gamma|^4 - 2 \cos(2\angle \Gamma) |\Gamma|^2 + 1 \\
&\Rightarrow \left[ \cos(2\angle \rho) - |\Gamma|^2 \right]^2 = \left[ |\Gamma|^2 - e^{2j\angle \Gamma} \right] \left[ |\Gamma|^2 - e^{-2j\angle \Gamma} \right] \\
&\Rightarrow \cos(2\angle \rho) = |\Gamma|^2 \pm \sqrt{\left[ |\Gamma|^2 - e^{2j\angle \Gamma} \right] \left[ |\Gamma|^2 - e^{-2j\angle \Gamma} \right]}
\end{aligned}$$

From the two different given solutions, the numerically smaller solution is chosen:

$$\Rightarrow \cos(2\angle \rho) = |\Gamma|^2 - \sqrt{\left[ |\Gamma|^2 - e^{2j\angle \Gamma} \right] \left[ |\Gamma|^2 - e^{-2j\angle \Gamma} \right]}$$

This second equation can then be plugged into the first equation to yield  $|\rho|$  as a function of  $|\Gamma|$  and  $\angle \Gamma$ :

$$\begin{aligned}
&\Rightarrow 2|\Gamma|^2 = \cosh\left(\ln[|\rho|^2]\right) + |\Gamma|^2 - \sqrt{\left[ |\Gamma|^2 - e^{2j\angle \Gamma} \right] \left[ |\Gamma|^2 - e^{-2j\angle \Gamma} \right]} \\
&\Rightarrow \eta = \cosh\left(\ln[|\rho|^2]\right) = |\Gamma|^2 + \sqrt{\left[ |\Gamma|^2 - e^{2j\angle \Gamma} \right] \left[ |\Gamma|^2 - e^{-2j\angle \Gamma} \right]} \\
&\Rightarrow |\rho|^2 = \eta \pm \sqrt{\eta^2 - 1} \\
&\Rightarrow |\rho|^2 = \eta + \sqrt{\eta^2 - 1}, \quad \eta - \sqrt{\eta^2 - 1} \\
&\Rightarrow |\rho|^2 = \eta + \sqrt{\eta^2 - 1}, \quad \frac{1}{\eta + \sqrt{\eta^2 - 1}}
\end{aligned}$$

Again, between the two given solutions, the numerically smaller solution is given:

$$\begin{aligned}
&\Rightarrow |\rho|^2 = \eta - \sqrt{\eta^2 - 1} \\
&\Rightarrow |\rho| = \sqrt{\eta - \sqrt{\eta^2 - 1}}, \quad \eta = |\Gamma|^2 + \sqrt{|\Gamma|^4 - 2 \cos(2\angle \Gamma) |\Gamma|^2 + 1}
\end{aligned}$$

Similarly,  $\angle \rho$  can be calculated as a function of  $|\Gamma|$  and  $\angle \Gamma$ :

$$\angle \rho = \arctan \left( \frac{\tan (\angle \rho)}{\tanh (|\rho|)} \right)$$

Just as a warning, the angle from the equation above has to be wrapped since it'll produce a negative  $\angle \rho$  for any  $\angle \Gamma^\circ > 90^\circ$  (i.e.,  $\Gamma_R < 0$ ).

$$\angle \rho = \arctan \left( \frac{\tan (\angle \rho)}{\tanh (|\rho|)} \right) + \pi * \mu (\angle \Gamma^\circ - 90^\circ)$$

Finally, the squared spectral magnitude and corresponding z-transform coefficients are normalized to yield the final equations down below:

$$f(z) = \frac{1}{2|\rho|} \left[ 1 - 2 \cos (\angle \rho) |\rho| z^{-1} + |\rho|^2 z^{-2} \right], \quad \rho \in \mathbb{C}$$

$$\Rightarrow |f(\omega)|^2 = [\cos (\omega) - \Gamma] [\cos (\omega) - \Gamma^*], \quad \Gamma \in \mathbb{C}$$

$$\text{where } \Gamma_R \equiv \cos (\angle \rho) \cosh \left( \ln [|\rho|] \right),$$

$$\Gamma_I \equiv \sin (\angle \rho) \sinh \left( \ln [|\rho|] \right)$$

$$\Rightarrow \eta = |\Gamma|^2 + \sqrt{|\Gamma|^4 - 2 \cos (2 \angle \Gamma) |\Gamma|^2 + 1}$$

$$\Rightarrow |\rho| = \sqrt{\eta - \sqrt{\eta^2 - 1}}$$

$$\Rightarrow \angle \rho = \arctan \left( \frac{\tan (\angle \rho)}{\tanh (|\rho|)} \right) + \pi * \mu (\angle \Gamma^\circ - 90^\circ)$$

## Appendix C: Spectral Magnitude of Irreducible Second-Order DTFT

$$f(z) = (1 - \rho z^{-1})(1 - \rho^* z^{-1}), \quad \rho \in \mathbb{C}, \quad |\rho| < 1$$

$$= 1 - 2|\rho| \cos(\angle \rho) z^{-1} + |\rho|^2 z^{-2}$$

$$= \vec{c}^T [z^{-n}]^N, \quad \vec{c} \equiv \begin{bmatrix} |\rho|^2 \\ -2|\rho| \cos(\angle \rho) \\ 1 \end{bmatrix}$$

$$\Rightarrow R_{cc}[n] = \begin{cases} |\rho|^2, & n = 2 \\ -2|\rho|(|\rho|^2 + 1) \cos(\angle \rho), & n = 1 \\ 1 + 4|\rho| \cos^2(\angle \rho) + |\rho|^4, & n = 0 \end{cases}$$

$$\Rightarrow |f(\omega)|^2 = R_{cc}[0] + 2 \sum_{n=1}^{N-1} R_{cc}[n] T_n(\cos(\omega))$$

$$= R_{cc}[0] + 2 \begin{bmatrix} 2R_{cc}[2] \\ R_{cc}[1] \\ -R_{cc}[2] \end{bmatrix}^T \begin{bmatrix} \cos^2(\omega) \\ \cos(\omega) \\ 1 \end{bmatrix}$$

$$= 4R_{cc}[2] \cos^2(\omega) + 2R_{cc}[1] \cos(\omega) + R_{cc}[0] - 2R_{cc}[2]$$

$$= 4R_{cc}[2] \left( \cos(\omega) + \frac{R_{cc}[1]}{4R_{cc}[2]} \right)^2 + R_{cc}[0] - 2R_{cc}[2] - \frac{R_{cc}^2[1]}{4R_{cc}[2]}$$

$$= 4|\rho|^2 \left( \cos(\omega) - \cos(\angle \rho) \cosh(\ln[|\rho|]) \right)^2 + R_{cc}[0] - 2R_{cc}[2] - \frac{R_{cc}^2[1]}{4R_{cc}[2]}$$

$$= 4|\rho|^2 \left( \cos(\omega) - \cos(\angle \rho) \cosh(\ln[|\rho|]) \right)^2 + 4|\rho|^2 \sin^2(\angle \rho) \sinh^2(\ln[|\rho|])$$

$$= K \left[ \left( \cos(\omega) - \Gamma_R \right)^2 + \Gamma_I^2 \right]$$

$$= K \left[ \cos(\omega) - \Gamma \right] \left[ \cos(\omega) - \Gamma^* \right]$$



## Appendix D: Spectral Magnitude of General DTFT

$$\begin{aligned}
f(z) &= \sum_{n=0}^{N-1} c_n z^{-n} \\
\Rightarrow f(\omega) &= \sum_{n=0}^{N-1} c_n e^{-jn\omega} \\
&= \vec{c}^T [q[n]]^N, \quad \vec{c} \equiv \begin{bmatrix} c_{N-1} \\ c_{N-2} \\ c_{N-3} \\ \vdots \\ c_2 \\ c_1 \\ c_0 \end{bmatrix}, \quad q[n] = e^{-jn\omega} \\
\Rightarrow |f(\omega)|^2 &= f(\omega) f^*(\omega) \\
&= \left( \vec{c}^T [q[n]]^N \right) \left( \vec{c}^T [q[n]]^N \right)^H \\
&= \vec{c}^T [q[n]]^N {}_N[q^*[n]] \vec{c} \\
&= \vec{c}^T \left( I_N + \sum_{n=1}^{N-1} \left( e^{-jn\omega} R_{N,n} + e^{jn\omega} R_{N,n}^T \right) \right) \vec{c} \\
&= \vec{c}^T \vec{c} + \sum_{n=1}^{N-1} \left( e^{-jn\omega} \vec{c}^T R_{N,n} \vec{c} + e^{jn\omega} \vec{c}^T R_{N,n}^T \vec{c} \right) \\
&= ||\vec{c}||^2 + \sum_{n=1}^{N-1} \left( e^{-jn\omega} \vec{c}^T R_{N,n} \vec{c} + e^{jn\omega} \vec{c}^T R_{N,n} \vec{c} \right) \\
&= ||\vec{c}||^2 + \sum_{n=1}^{N-1} \vec{c}^T R_{N,n} \vec{c} \left( e^{-jn\omega} + e^{jn\omega} \right) \\
&= ||\vec{c}||^2 + \sum_{n=1}^{N-1} R_{cc}[n] \left( 2 \cos(n\omega) \right) \\
&= R_{cc}[0] + 2 \sum_{n=1}^{N-1} R_{cc}[n] T_n(\cos(\omega))
\end{aligned}$$

## Appendix E: Complex Multi-Dimensional Matrix Recurrence Relations

Consider the complex causal sequence  $q[n]$  and its implicit scalar recurrence relation:

$$\begin{aligned} q[n] &= e^{-jn\omega} \mu[n] \\ &= \delta[n] + e^{-j\omega} q[n-1] \end{aligned}$$

The sequence generated by  $e^{-jn\omega}$  can then be decomposed into three individual components:

$$\begin{aligned} e^{-jn\omega} &= e^{-jn\omega} \left( \mu[-(n+1)] + \mu[n] \right) \\ &= e^{-jn\omega} \mu[-(n+1)] + e^{-jn\omega} \mu[n] \\ &= e^{j\omega} q^*[-(n+1)] + q[n] \\ &= e^{j\omega} q^*[-(n+1)] + \delta[n] + e^{-j\omega} q[n-1] \\ &= \delta[n] + e^{-j\omega} q[n-1] + e^{j\omega} q^*[-(n+1)] \end{aligned}$$

The following outer product can then be expanded via this recurrence relation:

$$\begin{aligned} \vec{q}[n] &= [q[n]]^N \quad ( ) \\ \Rightarrow \vec{q}[n] \vec{q}^H[n] &= [q[n]]^N {}_N[q^*[n]] \\ &= {}_N[q[n] q^*[n]]^N \\ &= {}_N[q[n-m]]^N \\ &= {}_N[\delta[n-m] + e^{-j\omega} q[n-m-1] + e^{j\omega} q^*[-(n-m+1)]]^N \\ &= {}_N[\delta[n-m]]^N + (e^{-j\omega}) {}_N[q[n-m-1]]^N + (e^{j\omega}) {}_N[q^*[-(n-m+1)]]^N \\ &= I_N + e^{-j\omega} \begin{bmatrix} \vec{0} & {}_{N-1}[q[n-m]]^{N-1} \\ 0 & \vec{0}^T \end{bmatrix} + e^{j\omega} \begin{bmatrix} \vec{0}^T & 0 \\ {}_{N-1}[q^*[-(n-m)]]^{N-1} & \vec{0} \end{bmatrix} \\ &= I_N + e^{-j\omega} \begin{bmatrix} \vec{0} & {}_{N-1}[q[n-m]]^{N-1} \\ 0 & \vec{0}^T \end{bmatrix} + e^{j\omega} \begin{bmatrix} \vec{0} & {}_{N-1}[q[n-m]]^{N-1} \\ 0 & \vec{0}^T \end{bmatrix}^H \end{aligned}$$

$$\begin{aligned}
&= I_N + e^{-j\omega} \begin{bmatrix} \vec{0} & I_{N-1} + e^{-j\omega} {}_{N-1}[q[n-m-1]]^{N-1} \\ 0 & \vec{0}^T \end{bmatrix} \\
&\quad + e^{j\omega} \begin{bmatrix} \vec{0} & I_{N-1} + e^{-j\omega} {}_{N-1}[q[n-m-1]]^{N-1} \\ 0 & \vec{0}^T \end{bmatrix}^H \\
&= I_N + e^{-j\omega} \begin{bmatrix} \vec{0} & I_{N-1} \\ 0 & \vec{0}^T \end{bmatrix} + e^{-2j\omega} \begin{bmatrix} \vec{0} & {}_{N-1}[q[n-m-1]]^{N-1} \\ 0 & \vec{0}^T \end{bmatrix} \\
&\quad + e^{j\omega} \begin{bmatrix} \vec{0} & I_{N-1} \\ 0 & \vec{0}^T \end{bmatrix}^T + e^{2j\omega} \begin{bmatrix} \vec{0} & {}_{N-1}[q[n-m-1]]^{N-1} \\ 0 & \vec{0}^T \end{bmatrix}^H \\
&= I_N + e^{-j\omega} \begin{bmatrix} 0_{N-1,1} & I_{N-1} \\ 0_{1,1} & 0_{1,N-1} \end{bmatrix} + e^{-2j\omega} \begin{bmatrix} 0_{N-2,2} & {}_{N-2}[q[n-m]]^{N-2} \\ 0_{2,2} & 0_{2,N-2} \end{bmatrix} \\
&\quad + e^{j\omega} \begin{bmatrix} 0_{N-1,1} & I_{N-1} \\ 0_{1,1} & 0_{1,N-1} \end{bmatrix}^T + e^{2j\omega} \begin{bmatrix} 0_{N-2,2} & {}_{N-2}[q[n-m]]^{N-2} \\ 0_{2,2} & 0_{2,N-2} \end{bmatrix}^H \\
&= I_N + e^{-j\omega} \begin{bmatrix} 0_{N-1,1} & I_{N-1} \\ 0_{1,1} & 0_{1,N-1} \end{bmatrix} + e^{-2j\omega} \begin{bmatrix} 0_{N-2,2} & I_{N-2} \\ 0_{2,2} & 0_{2,N-2} \end{bmatrix} + e^{-3j\omega} \begin{bmatrix} 0_{N-3,2} & {}_{N-3}[q[n-m]]^{N-3} \\ 0_{3,2} & 0_{3,N-3} \end{bmatrix} \\
&\quad + e^{j\omega} \begin{bmatrix} 0_{N-1,1} & I_{N-1} \\ 0_{1,1} & 0_{1,N-1} \end{bmatrix}^T + e^{2j\omega} \begin{bmatrix} 0_{N-2,2} & I_{N-2} \\ 0_{2,2} & 0_{2,N-2} \end{bmatrix}^T + e^{3j\omega} \begin{bmatrix} 0_{N-3,3} & {}_{N-3}[q[n-m]]^{N-3} \\ 0_{3,3} & 0_{3,N-3} \end{bmatrix}^H \\
&= I_N + \sum_{n=1}^{N-1} \left( e^{-jn\omega} \begin{bmatrix} 0_{N-n,n} & I_n \\ 0_{n,n} & 0_{n,N-n} \end{bmatrix} + e^{jn\omega} \begin{bmatrix} 0_{N-n,n} & I_n \\ 0_{n,n} & 0_{n,N-n} \end{bmatrix}^T \right) \\
&= I_N + \sum_{n=1}^{N-1} \left( e^{-jn\omega} R_{N,n} + e^{jn\omega} R_{N,n}^T \right)
\end{aligned}$$

## Appendix F: Matrix Representation of Autocorrelation Sequence

For a given causal sequence  $x[n]$ , the Autocorrelation is defined as  $R_{xx}[n]$ :

$$\begin{aligned}
 R_{xx}[l] &= \sum_{n=0}^{\infty} s[n]s[n+l] \quad , \quad s[n] \equiv x[n](\mu[n] - \mu[n-N]) \\
 &= \sum_{n=0}^{N-(l+1)} x[n]x[n+l] \\
 &= {}_N[x[n]] \begin{bmatrix} \vec{0} \\ [x[n]]_{n=l}^{n=N-1} \end{bmatrix} \\
 &= \left( I_N [x[n]]^N \right)^T \left( \begin{bmatrix} 0_{l,N-l} & 0_{l,l} \\ I_{N-l} & 0_{N-l,l} \end{bmatrix} [x[n]]^N \right) \\
 &= {}_N[x[n]] I_N \begin{bmatrix} 0_{l,N-l} & 0_{l,l} \\ I_{N-l} & 0_{N-l,l} \end{bmatrix} [x[n]]^N \\
 &= {}_N[x[n]] \begin{bmatrix} 0_{l,N-l} & 0_{l,l} \\ I_{N-l} & 0_{N-l,l} \end{bmatrix} [x[n]]^N \\
 &= {}_N[x[n]] \begin{bmatrix} 0_{l,N-l} & I_{N-l} \\ 0_{l,l} & 0_{N-l,l} \end{bmatrix} [x[n]]^N \\
 &= {}_N[x[n]] R_{N,l} [x[n]]^N \quad , \quad R_{N,l} \equiv \begin{bmatrix} 0_{l,N-l} & I_{N-l} \\ 0_{l,l} & 0_{N-l,l} \end{bmatrix}
 \end{aligned}$$

## Appendix G: Custom Notation for Vector/Matrix-generating Sequences

Some of the recurrence relations in Appendix A rely on the mathematics of vector/matrix-generating sequences. However, there seems to be no standard notation for any kind of vector/matrix generated from such sequences. This Appendix contains some of my own custom notation which I created for expressing these vectors/matrices in a concise manner.

Take a generic sequence  $f[n]$ . The column vector  $\vec{v}$  (populated by applying  $f[n]$  element-wise to each row in the vector as a function of the row) can be denoted below as such:

$$\vec{v} = [f[n]]^N = [f[n]]_{n=0}^{n=N-1} = \begin{bmatrix} f[N-1] \\ f[N-2] \\ f[N-3] \\ \vdots \\ f[2] \\ f[1] \\ f[0] \end{bmatrix}$$

Accordingly, the row vector  $\vec{v}^T$  can be denoted below as such:

$$\begin{aligned} \vec{v}^T &= {}_N[f[n]] = {}_{n=N-1}[f[n]]_{n=0} \\ &= [f[N-1] \quad f[N-2] \quad f[N-3] \quad \dots \quad f[2] \quad f[1] \quad f[0]] \end{aligned}$$

The matrix  $M$  (populated by applying  $f[n, m]$  elementwise to each row  $n$  and column  $m$  in the matrix as a function of  $n$  and  $m$ ) can be denoted below as such:

$$\begin{aligned} M &= {}_N[f[n, m]]^N = {}_{m=N-1}[f[n, m]]_{n=0, m=0}^{n=N-1} \\ &= \begin{bmatrix} f[N-1, N-1] & f[N-1, N-2] & f[N-1, N-3] & \dots & f[N-1, 2] & f[N-1, 1] & f[N-1, 0] \\ f[N-2, N-1] & f[N-2, N-2] & f[N-2, N-3] & \dots & f[N-2, 2] & f[N-2, 1] & f[N-2, 0] \\ f[N-3, N-1] & f[N-3, N-2] & f[N-3, N-3] & \dots & f[N-3, 2] & f[N-3, 1] & f[N-3, 0] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ f[2, N-1] & f[2, N-2] & f[2, N-3] & \dots & f[2, 2] & f[2, 1] & f[2, 0] \\ f[1, N-1] & f[1, N-2] & f[1, N-3] & \dots & f[1, 2] & f[1, 1] & f[1, 0] \\ f[0, N-1] & f[0, N-2] & f[0, N-3] & \dots & f[0, 2] & f[0, 1] & f[0, 0] \end{bmatrix} \end{aligned}$$

Sometimes, some matrices will have portions systematically populated by zeroes. In such cases, we can have the original function ( $f[n, m]$ ) modified by another function as a sort of indicator function. For example, consider an upper triangular matrix  $U$  where  $f[n, m]$  is multiplied by a heaviside step function.  $U$  can be denoted as such below:

$$U = {}_N [g[n, m]]^N, \quad g(n, m) = f[n, m]\mu[n - m]$$

$$= \begin{bmatrix} f[N-1, N-1] & f[N-1, N-2] & f[N-1, N-3] & \dots & f[N-1, 2] & f[N-1, 1] & f[N-1, 0] \\ 0 & f[N-2, N-2] & f[N-2, N-3] & \dots & f[N-2, 2] & f[N-2, 1] & f[N-2, 0] \\ 0 & 0 & f[N-3, N-3] & \dots & f[N-3, 2] & f[N-3, 1] & f[N-3, 0] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & f[2, 2] & f[2, 1] & f[2, 0] \\ 0 & 0 & 0 & \dots & 0 & f[1, 1] & f[1, 0] \\ 0 & 0 & 0 & \dots & 0 & 0 & f[0, 0] \end{bmatrix}$$

The inner product of  $\vec{v}$  with itself can be denoted as such below:

$$\vec{v}^T \vec{v} = {}_N [f[n]] [f[n]]^N = {}_{n=N-1} [f[n]]_{n=0} [f[n]]_{n=0}^{n=N-1} = \sum_{n=0}^{n=N-1} f^2[n]$$

The outer product of  $\vec{v}$  with itself can be denoted as such below:

$$\vec{v} \vec{v}^T = [f[n]]^N {}_N [f[n]] = [f[n]]_{n=0}^{n=N-1} {}_{n=N-1} [f[n]]_{n=0} = {}_{m=N-1} [f[n]f[m]]_{n=0, m=0}^{n=N-1}$$

$$= \begin{bmatrix} f[N-1]f[N-1] & f[N-1]f[N-2] & f[N-1]f[N-3] & \dots & f[N-1]f[2] & f[N-1]f[1] & f[N-1]f[0] \\ f[N-2]f[N-1] & f[N-2]f[N-2] & f[N-2]f[N-3] & \dots & f[N-2]f[2] & f[N-2]f[1] & f[N-2]f[0] \\ f[N-3]f[N-1] & f[N-3]f[N-2] & f[N-3]f[N-3] & \dots & f[N-3]f[2] & f[N-3]f[1] & f[N-3]f[0] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ f[2]f[N-1] & f[2]f[N-2] & f[2]f[N-3] & \dots & f[2]f[2] & f[2]f[1] & f[2]f[0] \\ f[1]f[N-1] & f[1]f[N-2] & f[1]f[N-3] & \dots & f[1]f[2] & f[1]f[1] & f[1]f[0] \\ f[0]f[N-1] & f[0]f[N-2] & f[0]f[N-3] & \dots & f[0]f[2] & f[0]f[1] & f[0]f[0] \end{bmatrix}$$