1 Main

For an irreducible second-order polynomial over the $\mathbb{Z}$-plane:

$$f(z) = \frac{1}{2|\rho|}\left[1 - 2\cos\left(\angle\rho\right)\left|\rho\right|z^{-1} + \left|\rho\right|^2 z^{-2}\right], \quad \rho \in \mathbb{C}$$

$$\Rightarrow \left|f\left(\omega\right)\right|^2 = \left[\cos\left(\omega\right) - \Gamma\right]\left[\cos\left(\omega\right) - \Gamma^*\right], \quad \Gamma \in \mathbb{C}$$

$$\Rightarrow \eta = \left|\Gamma\right|^2 + \sqrt{\left|\Gamma\right|^4 - 2\cos\left(2\angle\Gamma\right)|\Gamma|^2 + 1}$$

$$\Rightarrow \left|\rho\right| = \sqrt{\eta - \sqrt{\eta^2 - 1}}$$

$$\Rightarrow \angle\rho = \arctan\left(\frac{\tan\left(\angle\rho\right)}{\tanh\left(\left|\rho\right|\right)}\right) + \pi * \mu\Big(\angle\Gamma^\circ - 90^\circ\Big)$$

Similarly, for a first-order polynomial over the $\mathbb{Z}$-plane:

$$f(z) = \frac{1}{\sqrt{2|\rho|}}\left[1 - \rho z^{-1}\right]$$

$$\Rightarrow \left|f\left(\omega\right)\right|^2 = -\operatorname{sgn}\left(\Gamma\right)\left[\cos\left(\omega\right) - \Gamma\right]$$

$$\Rightarrow |\rho| = \left|\Gamma\right| - \sqrt{\left|\Gamma\right|^2 - 1}$$

$$\Rightarrow \operatorname{sgn}\left(\rho\right) = \operatorname{sgn}\left(\Gamma\right)$$

Therefore, for a frequency magnitude consisting of N real positive roots, M real negative roots, and K complex conjugate pair roots, the following squared spectral magnitude $\left|f\left(\omega\right)\right|^2$ has the corresponding z-transform:

$$\left|f\left(\omega\right)\right|^2 = \left(\prod_{n=0}^{N-1} -\left[\cos\left(\omega\right) - \Gamma_n\right]\right)\left(\prod_{m=0}^{M-1}\left[\cos\left(\omega\right) - \Gamma_m\right]\right)\left(\prod_{k=0}^{K-1}\left[\cos\left(\omega\right) - \Gamma_k\right]\left[\cos\left(\omega\right) - \Gamma_k^*\right]\right)$$

$$= \left(\prod_{n=0}^{N-1} -1\right)\left(\prod_{n=0}^{N-1}\prod_{m=0}^{M-1}\prod_{k=0}^{K-1}\left[\cos\left(\omega\right) - \Gamma_n\right]\left[\cos\left(\omega\right) - \Gamma_m\right]\left[\cos\left(\omega\right) - \Gamma_k\right]\left[\cos\left(\omega\right) - \Gamma_k^*\right]\right)$$

$$= \Big(-1\Big)^N\left(\prod_{l=0}^{L-1}\left[\cos\left(\omega\right) - \Gamma_l\right]\right), \quad L \equiv N + M + K$$

$$= \sum_{l=0}^{L} a_l T_l\Big(\cos\left(\omega\right)\Big)$$

$$\Rightarrow f(z) = \sum_{l=0}^{N+M+K-1} c[l] z^{-l} \ , \quad \text{where } c[l] = f_0 \circledast f_1 \circledast \ldots \circledast f_{N+M+K-2} \circledast f_{N+M+K-1}$$

$$= \left(\mathop{\circledast}_{n=0}^{N-1} f_n(z) \right) \circledast \left(\mathop{\circledast}_{m=0}^{M-1} f_m(z) \right) \circledast \left(\mathop{\circledast}_{k=0}^{K-1} f_k(z) \right)$$

Appendix A: First-Order Spectral Magnitude Design

For an first-order polynomial over the $\mathbb{Z}$-plane:

$$f(z) = 1 - \rho z^{-1} \ , \quad \rho \in \mathbb{R} \ , \quad \left|\rho\right| < 1$$

$$\Rightarrow \left|f\left(\omega\right)\right|^2 = 1 + \left|\rho\right|^2 - 2\rho\cos\left(\omega\right)$$

$$= 2\left|\rho\right|\left(\frac{\left|\rho\right| + \left|\rho\right|^{-1}}{2}\right) - 2\rho\cos\left(\omega\right)$$

$$= 2\left|\rho\right|\cosh\Big(\ln\left[\left|\rho\right|\right]\Big) - 2\rho\cos\left(\omega\right)$$

$$= 2\left[\left|\rho\right|\cosh\Big(\ln\left[\left|\rho\right|\right]\Big) - \rho\cos\left(\omega\right)\right]$$

$$= 2\left[\operatorname{sgn}^2\left(\rho\right)\left|\rho\right|\cosh\Big(\ln\left[\left|\rho\right|\right]\Big) - \rho\cos\left(\omega\right)\right]$$

$$= 2\left[\operatorname{sgn}\left(\rho\right)\rho\cosh\Big(\ln\left[\left|\rho\right|\right]\Big) - \rho\cos\left(\omega\right)\right]$$

$$= 2\rho\left[\operatorname{sgn}\left(\rho\right)\cosh\Big(\ln\left[\left|\rho\right|\right]\Big) - \cos\left(\omega\right)\right]$$

$$= 2\rho\Big[\Gamma - \cos\left(\omega\right)\Big] \ , \quad \Gamma \equiv \operatorname{sgn}\left(\rho\right)\cosh\Big(\ln\left[\left|\rho\right|\right]\Big) \ ,$$

$$\left|\Gamma\right| = \cosh\Big(\ln\left[\left|\rho\right|\right]\Big) \ ,$$

$$\operatorname{sgn}\left(\Gamma\right) = \operatorname{sgn}\left(\rho\right)$$

Focusing on $\left|\Gamma\right|$, we can derive the following equations:

$$\Gamma = \operatorname{sgn}\left(\rho\right)\cosh\Big(\ln\left[\left|\rho\right|\right]\Big)$$

$$\Rightarrow \left|\rho\right| = \left|\Gamma\right| \pm \sqrt{\left|\Gamma\right|^2 - 1}$$

$$= \left|\Gamma\right| - \sqrt{\left|\Gamma\right|^2 - 1}$$

This yields the following equations:

$$
\begin{aligned}
f(z) &= 1 - \rho z^{-1} \\
\Rightarrow \left|f(\omega)\right|^2 &= 2\rho\left[\Gamma - \cos\left(\omega\right)\right] \\
&= 2\,\mathrm{sgn}\left(\rho\right)\left|\rho\right|\left[\Gamma - \cos\left(\omega\right)\right] \\
&= 2\,\mathrm{sgn}\left(\Gamma\right)\left|\rho\right|\left[\Gamma - \cos\left(\omega\right)\right] \\
&= -2\left|\rho\right|\mathrm{sgn}\left(\Gamma\right)\left[\cos\left(\omega\right) - \Gamma\right]
\end{aligned}
$$

With a normalized spectral magnitude, we then get the final equations:

$$
\begin{aligned}
f(z) &= \frac{1}{\sqrt{2\left|\rho\right|}}\left[1 - \rho z^{-1}\right] \\
\Rightarrow \left|f(\omega)\right|^2 &= -\,\mathrm{sgn}\left(\Gamma\right)\left[\cos\left(\omega\right) - \Gamma\right] \\
\Rightarrow \left|\rho\right| &= \left|\Gamma\right| - \sqrt{\left|\Gamma\right|^2 - 1} \\
\Rightarrow \mathrm{sgn}\left(\rho\right) &= \mathrm{sgn}\left(\Gamma\right)
\end{aligned}
$$

Appendix B: Irreducible Second-Order Spectral Magnitude Design

For an irreducible second-order polynomial over the $\mathbb{Z}$-plane:

$$f(z) = \left(1 - \rho z^{-1}\right)\left(1 - \rho^* z^{-1}\right) \ , \quad \rho \in \mathbb{C} \ , \quad \left|\rho\right| < 1$$

$$= 1 - 2\cos\left(\angle\rho\right)\left|\rho\right| z^{-1} + \left|\rho\right|^2 z^{-2}$$

$$\Rightarrow \left|f\left(\omega\right)\right|^2 = K^2\Big[\cos\left(\omega\right) - \Gamma\Big]\Big[\cos\left(\omega\right) - \Gamma^*\Big] \ , \quad \Gamma \in \mathbb{C}$$

$$\text{where } K \equiv 2\left|\rho\right|$$

$$\Gamma_R \equiv \cos\left(\angle\rho\right)\cosh\Big(\ln\left[\left|\rho\right|\right]\Big)$$

$$\Gamma_I \equiv \sin\left(\angle\rho\right)\sinh\Big(\ln\left[\left|\rho\right|\right]\Big)$$

Effectively, the squared spectral magnitude can be expressed as an irreducible polynomial (over $\cos\left(\omega\right)$ instead of x) with Γ and its complex conjugate acting as the roots for this polynomial.

$$\left|\Gamma\right|^2 = \Gamma_R^2 + \Gamma_I^2$$

$$= \cos^2\left(\angle\rho\right)\cosh^2\Big(\ln\left[\left|\rho\right|\right]\Big) + \sin^2\left(\angle\rho\right)\sinh^2\Big(\ln\left[\left|\rho\right|\right]\Big)$$

$$= \cosh^2\Big(\ln\left[\left|\rho\right|\right]\Big) - \sin^2\left(\angle\rho\right)$$

$$= \left(\frac{1}{2}\right)\Bigg(\cosh\Big(\ln\left[\left|\rho\right|^2\right]\Big) + \cos\left(2\angle\rho\right)\Bigg)$$

$$\Rightarrow 2\left|\Gamma\right| = \cosh\Big(\ln\left[\left|\rho\right|^2\right]\Big) + \cos\left(2\angle\rho\right)$$

The equation above is one of two equations needed to express Γ in terms of ρ. The second equation is derived down below:

$$\cosh^2\Big(\ln\left[\left|\rho\right|\right]\Big) - \sinh^2\Big(\ln\left[\left|\rho\right|\right]\Big) = 1$$

$$\Rightarrow \frac{\Gamma_R^2}{\cos^2\left(\angle\rho\right)} - \frac{\Gamma_I^2}{\sin^2\left(\angle\rho\right)} = 1$$

$$\Rightarrow \sin^2\left(\angle\rho\right)\Gamma_R^2 - \cos^2\left(\angle\rho\right)\Gamma_I^2 = \cos^2\left(\angle\rho\right)\sin^2\left(\angle\rho\right)$$

$$\Rightarrow \Gamma_R^2 - \left|\Gamma\right|^2\cos^2\left(\angle\rho\right) = \cos^2\left(\angle\rho\right) - \cos^4\left(\angle\rho\right)$$

$$\Rightarrow \Gamma_R^2 - \left|\Gamma\right|^2 \cos^2\left(\angle\rho\right) = \cos^2\left(\angle\rho\right) - \cos^4\left(\angle\rho\right)$$

$$\Rightarrow \left[\cos^2\left(\angle\rho\right) - \left(\frac{\left|\Gamma\right|^2 + 1}{2}\right)\right]^2 = \left(\frac{\left|\Gamma\right|^2 + 1}{2}\right)^2 - \Gamma_R^2$$

$$\Rightarrow \left[\cos\left(2\angle\rho\right) - \left|\Gamma\right|^2\right]^2 = \left|\Gamma\right|^4 - 2\cos\left(2\angle\Gamma\right)|\Gamma|^2 + 1$$

$$\Rightarrow \left[\cos\left(2\angle\rho\right) - \left|\Gamma\right|^2\right]^2 = \left[\left|\Gamma\right|^2 - e^{2j\angle\Gamma}\right]\left[\left|\Gamma\right|^2 - e^{-2j\angle\Gamma}\right]$$

$$\Rightarrow \cos\left(2\angle\rho\right) = \left|\Gamma\right|^2 \pm \sqrt{\left[\left|\Gamma\right|^2 - e^{2j\angle\Gamma}\right]\left[\left|\Gamma\right|^2 - e^{-2j\angle\Gamma}\right]}$$

From the two different given solutions, the numerically smaller solution is chosen:

$$\Rightarrow \cos\left(2\angle\rho\right) = \left|\Gamma\right|^2 - \sqrt{\left[\left|\Gamma\right|^2 - e^{2j\angle\Gamma}\right]\left[\left|\Gamma\right|^2 - e^{-2j\angle\Gamma}\right]}$$

This second equation can then be plugged into the first equation to yield $\left|\rho\right|$ as a function of $\left|\Gamma\right|$ and $\angle\Gamma$:

$$\Rightarrow 2\left|\Gamma\right|^2 = \cosh\left(\ln\left[\left|\rho\right|^2\right]\right) + \left|\Gamma\right|^2 - \sqrt{\left[\left|\Gamma\right|^2 - e^{2j\angle\Gamma}\right]\left[\left|\Gamma\right|^2 - e^{-2j\angle\Gamma}\right]}$$

$$\Rightarrow \eta = \cosh\left(\ln\left[\left|\rho\right|^2\right]\right) = \left|\Gamma\right|^2 + \sqrt{\left[\left|\Gamma\right|^2 - e^{2j\angle\Gamma}\right]\left[\left|\Gamma\right|^2 - e^{-2j\angle\Gamma}\right]}$$

$$\Rightarrow \left|\rho\right|^2 = \eta \pm \sqrt{\eta^2 - 1}$$

$$\Rightarrow \left|\rho\right|^2 = \eta + \sqrt{\eta^2 - 1}\ ,\ \eta - \sqrt{\eta^2 - 1}$$

$$\Rightarrow \left|\rho\right|^2 = \eta + \sqrt{\eta^2 - 1}\ ,\ \frac{1}{\eta + \sqrt{\eta^2 - 1}}$$

Again, between the two given solutions, the numerically smaller solution is given:

$$\Rightarrow \left|\rho\right|^2 = \eta - \sqrt{\eta^2 - 1}$$

$$\Rightarrow \left|\rho\right| = \sqrt{\eta - \sqrt{\eta^2 - 1}}\ , \quad \eta = \left|\Gamma\right|^2 + \sqrt{\left|\Gamma\right|^4 - 2\cos\left(2\angle\Gamma\right)|\Gamma|^2 + 1}$$

Similarly, $\angle\rho$ can be calculated as a function of $\left|\Gamma\right|$ and $\angle\Gamma$:

$$\angle\rho = \arctan\left(\frac{\tan\left(\angle\rho\right)}{\tanh\left(\left|\rho\right|\right)}\right)$$

Just as a warning, the angle from the equation above has to be wrapped since it'll produce a negative $\angle\rho$ for any $\angle\Gamma^\circ > 90^\circ$ (i.e., $\Gamma_R < 0$).

$$\angle\rho = \arctan\left(\frac{\tan\left(\angle\rho\right)}{\tanh\left(\left|\rho\right|\right)}\right) + \pi * \mu\Big(\angle\Gamma^\circ - 90^\circ\Big)$$

Finally, the squared spectral magnitude and corresponding z-transform coefficients are normalized to yield the final equations down below:

$$f(z) = \frac{1}{2\left|\rho\right|}\left[1 - 2\cos\left(\angle\rho\right)\left|\rho\right|z^{-1} + \left|\rho\right|^2 z^{-2}\right], \quad \rho \in \mathbb{C}$$

$$\Rightarrow \left|f\left(\omega\right)\right|^2 = \Big[\cos\left(\omega\right) - \Gamma\Big]\Big[\cos\left(\omega\right) - \Gamma^*\Big], \quad \Gamma \in \mathbb{C}$$

$$\text{where } \Gamma_R \equiv \cos\left(\angle\rho\right)\cosh\Big(\ln\left[\left|\rho\right|\right]\Big),$$

$$\Gamma_I \equiv \sin\left(\angle\rho\right)\sinh\Big(\ln\left[\left|\rho\right|\right]\Big)$$

$$\Rightarrow \eta = \left|\Gamma\right|^2 + \sqrt{\left|\Gamma\right|^4 - 2\cos\left(2\angle\Gamma\right)|\Gamma|^2 + 1}$$

$$\Rightarrow \left|\rho\right| = \sqrt{\eta - \sqrt{\eta^2 - 1}}$$

$$\Rightarrow \angle\rho = \arctan\left(\frac{\tan\left(\angle\rho\right)}{\tanh\left(\left|\rho\right|\right)}\right) + \pi * \mu\Big(\angle\Gamma^\circ - 90^\circ\Big)$$

Appendix C: Spectral Magnitude of Irreducible Second-Order DTFT

$$f(z) = \left(1 - \rho z^{-1}\right)\left(1 - \rho^* z^{-1}\right) \ , \quad \rho \in \mathbb{C} \ , \quad \left|\rho\right| < 1$$

$$= 1 - 2\left|\rho\right| \cos\left(\angle\rho\right) z^{-1} + \left|\rho\right|^2 z^{-2}$$

$$= \vec{c}^T \left[z^{-n}\right]^N \quad , \quad \vec{c} \equiv \begin{bmatrix} \left|\rho\right|^2 \\ -2\left|\rho\right|\cos\left(\angle\rho\right) \\ 1 \end{bmatrix}$$

$$\Rightarrow R_{cc}[n] = \begin{cases} \left|\rho\right|^2 \ , & n = 2 \\ -2\left|\rho\right|\left(\left|\rho\right|^2 + 1\right)\cos\left(\angle\rho\right) \ , & n = 1 \\ 1 + 4\left|\rho\right|\cos^2\left(\angle\rho\right) + \left|\rho\right|^4 \ , & n = 0 \end{cases}$$

$$\Rightarrow \left|f(\omega)\right|^2 = R_{cc}[0] + 2\sum_{n=1}^{N-1} R_{cc}[n] T_n\left(\cos\left(\omega\right)\right)$$

$$= R_{cc}[0] + 2 \begin{bmatrix} 2R_{cc}[2] \\ R_{cc}[1] \\ -R_{cc}[2] \end{bmatrix}^T \begin{bmatrix} \cos^2\left(\omega\right) \\ \cos\left(\omega\right) \\ 1 \end{bmatrix}$$

$$= 4R_{cc}[2]\cos^2\left(\omega\right) + 2R_{cc}[1]\cos\left(\omega\right) + R_{cc}[0] - 2R_{cc}[2]$$

$$= 4R_{cc}[2]\left(\cos\left(\omega\right) + \frac{R_{cc}[1]}{4R_{cc}[2]}\right)^2 + R_{cc}[0] - 2R_{cc}[2] - \frac{R_{cc}^2[1]}{4R_{cc}[2]}$$

$$= 4\left|\rho\right|^2\left(\cos\left(\omega\right) - \cos\left(\angle\rho\right)\cosh\left(\ln\left[\left|\rho\right|\right]\right)\right)^2 + R_{cc}[0] - 2R_{cc}[2] - \frac{R_{cc}^2[1]}{4R_{cc}[2]}$$

$$= 4\left|\rho\right|^2\left(\cos\left(\omega\right) - \cos\left(\angle\rho\right)\cosh\left(\ln\left[\left|\rho\right|\right]\right)\right)^2 + 4\left|\rho\right|^2\sin^2\left(\angle\rho\right)\sinh^2\left(\ln\left[\left|\rho\right|\right]\right)$$

$$= K\left[\left(\cos\left(\omega\right) - \Gamma_R\right)^2 + \Gamma_I^2\right]$$

$$= K\left[\cos\left(\omega\right) - \Gamma\right]\left[\cos\left(\omega\right) - \Gamma^*\right]$$

Appendix D: Spectral Magnitude of General DTFT

$$f(z) = \sum_{n=0}^{N-1} c_n z^{-n}$$

$$\Rightarrow f\big(\omega\big) = \sum_{n=0}^{N-1} c_n e^{-jn\omega}$$

$$= \vec{c}^T \left[q[n]\right]^N \quad , \quad \vec{c} \equiv = \begin{bmatrix} c_{N-1} \\ c_{N-2} \\ c_{N-3} \\ \vdots \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} \quad , \quad q[n] = e^{-jn\omega}$$

$$\Rightarrow \left| f\big(\omega\big) \right|^2 = f\big(\omega\big) f^*\big(\omega\big)$$

$$= \left(\vec{c}^T \left[q[n]\right]^N \right) \left(\vec{c}^T \left[q[n]\right]^N \right)^H$$

$$= \vec{c}^T \left[q[n]\right]^N {}_N\left[q^*[n]\right] \vec{c}$$

$$= \vec{c}^T \left(I_N + \sum_{n=1}^{N-1} \left(e^{-jn\omega} R_{N,n} + e^{jn\omega} R_{N,n}^T \right) \right) \vec{c}$$

$$= \vec{c}^T \vec{c} + \sum_{n=1}^{N-1} \left(e^{-jn\omega} \vec{c}^T R_{N,n} \vec{c} + e^{jn\omega} \vec{c}^T R_{N,n}^T \vec{c} \right)$$

$$= \left|\left|\vec{c}\right|\right|^2 + \sum_{n=1}^{N-1} \left(e^{-jn\omega} \vec{c}^T R_{N,n} \vec{c} + e^{jn\omega} \vec{c}^T R_{N,n} \vec{c} \right)$$

$$= \left|\left|\vec{c}\right|\right|^2 + \sum_{n=1}^{N-1} \vec{c}^T R_{N,n} \vec{c} \Big(e^{-jn\omega} + e^{jn\omega} \Big)$$

$$= \left|\left|\vec{c}\right|\right|^2 + \sum_{n=1}^{N-1} R_{cc}[n] \Big(2\cos\big(n\omega\big) \Big)$$

$$= R_{cc}[0] + 2 \sum_{n=1}^{N-1} R_{cc}[n] T_n \Big(\cos\big(\omega\big) \Big)$$

Appendix E: Complex Multi-Dimensional Matrix Recurrence Relations

Consider the complex causal sequence $q[n]$ and its implicit scalar recurrence relation:

$$
\begin{aligned}
q[n] &= e^{-jn\omega}\mu[n] \\
&= \delta[n] + e^{-j\omega}q[n-1]
\end{aligned}
$$

The sequence generated by $e^{-jn\omega}$ can then be decomposed into three individual components:

$$
\begin{aligned}
e^{-jn\omega} &= e^{-jn\omega}\Big(\mu\big[-(n+1)\big] + \mu[n]\Big) \\
&= e^{-jn\omega}\mu\big[-(n+1)\big] + e^{-jn\omega}\mu[n] \\
&= e^{j\omega}q^*\big[-(n+1)\big] + q[n] \\
&= e^{j\omega}q^*\big[-(n+1)\big] + \delta[n] + e^{-j\omega}q[n-1] \\
&= \delta[n] + e^{-j\omega}q[n-1] + e^{j\omega}q^*\big[-(n+1)\big]
\end{aligned}
$$

The following outer product can then be expanded via this recurrence relation:

$$
\begin{aligned}
\vec{q}[n] &= \big[q[n]\big]^N \qquad () \\
\Rightarrow \vec{q}[n]\vec{q}^H[n] &= \big[q[n]\big]^N {}_N\big[q^*[n]\big] \\
&= {}_N\big[q[n]q^*[m]\big]^N \\
&= {}_N\big[q[n-m]\big]^N \\
&= {}_N\big[\delta[n-m] + e^{-j\omega}q[n-m-1] + e^{j\omega}q^*\big[-(n-m+1)\big]\big]^N \\
&= {}_N\big[\delta[n-m]\big]^N + \big(e^{-j\omega}\big)\,{}_N\big[q[n-m-1]\big]^N + \big(e^{j\omega}\big)\,{}_N\big[q^*\big[-(n-m+1)\big]\big]^N \\
&= I_N + e^{-j\omega}\begin{bmatrix} \vec{0} & {}_{N-1}\big[q[n-m]\big]^{N-1} \\ 0 & \vec{0}^T \end{bmatrix} + e^{j\omega}\begin{bmatrix} \vec{0}^T & 0 \\ {}_{N-1}\big[q^*\big[-(n-m)\big]\big]^{N-1} & \vec{0} \end{bmatrix} \\
&= I_N + e^{-j\omega}\begin{bmatrix} \vec{0} & {}_{N-1}\big[q[n-m]\big]^{N-1} \\ 0 & \vec{0}^T \end{bmatrix} + e^{j\omega}\begin{bmatrix} \vec{0} & {}_{N-1}\big[q[n-m]\big]^{N-1} \\ 0 & \vec{0}^T \end{bmatrix}^H
\end{aligned}
$$

$$
= I_N + e^{-j\omega} \begin{bmatrix} \vec{0} & I_{N-1} + e^{-j\omega} \, {}_{N-1}\big[q[n-m-1]\big]^{N-1} \\ 0 & \vec{0}^T \end{bmatrix}
$$

$$
+ e^{j\omega} \begin{bmatrix} \vec{0} & I_{N-1} + e^{-j\omega} \, {}_{N-1}\big[q[n-m-1]\big]^{N-1} \\ 0 & \vec{0}^T \end{bmatrix}^H
$$

$$
= I_N + e^{-j\omega} \begin{bmatrix} \vec{0} & I_{N-1} \\ 0 & \vec{0}^T \end{bmatrix} + e^{-2j\omega} \begin{bmatrix} \vec{0} & {}_{N-1}\big[q[n-m-1]\big]^{N-1} \\ 0 & \vec{0}^T \end{bmatrix}
$$

$$
+ e^{j\omega} \begin{bmatrix} \vec{0} & I_{N-1} \\ 0 & \vec{0}^T \end{bmatrix}^T + e^{2j\omega} \begin{bmatrix} \vec{0} & {}_{N-1}\big[q[n-m-1]\big]^{N-1} \\ 0 & \vec{0}^T \end{bmatrix}^H
$$

$$
= I_N + e^{-j\omega} \begin{bmatrix} 0_{N-1,1} & I_{N-1} \\ 0_{1,1} & 0_{1,N-1} \end{bmatrix} + e^{-2j\omega} \begin{bmatrix} 0_{N-2,2} & {}_{N-2}\big[q[n-m]\big]^{N-2} \\ 0_{2,2} & 0_{2,N-2} \end{bmatrix}
$$

$$
+ e^{j\omega} \begin{bmatrix} 0_{N-1,1} & I_{N-1} \\ 0_{1,1} & 0_{1,N-1} \end{bmatrix}^T + e^{2j\omega} \begin{bmatrix} 0_{N-2,2} & {}_{N-2}\big[q[n-m]\big]^{N-2} \\ 0_{2,2} & 0_{2,N-2} \end{bmatrix}^H
$$

$$
= I_N + e^{-j\omega} \begin{bmatrix} 0_{N-1,1} & I_{N-1} \\ 0_{1,1} & 0_{1,N-1} \end{bmatrix} + e^{-2j\omega} \begin{bmatrix} 0_{N-2,2} & I_{N-2} \\ 0_{2,2} & 0_{2,N-2} \end{bmatrix} + e^{-3j\omega} \begin{bmatrix} 0_{N-3,2} & {}_{N-3}\big[q[n-m]\big]^{N-3} \\ 0_{3,2} & 0_{3,N-3} \end{bmatrix}
$$

$$
+ e^{j\omega} \begin{bmatrix} 0_{N-1,1} & I_{N-1} \\ 0_{1,1} & 0_{1,N-1} \end{bmatrix}^T + e^{2j\omega} \begin{bmatrix} 0_{N-2,2} & I_{N-2} \\ 0_{2,2} & 0_{2,N-2} \end{bmatrix}^T + e^{3j\omega} \begin{bmatrix} 0_{N-3,3} & {}_{N-3}\big[q[n-m]\big]^{N-3} \\ 0_{3,3} & 0_{3,N-3} \end{bmatrix}^H
$$

$$
= I_N + \sum_{n=1}^{N-1} \left(e^{-jn\omega} \begin{bmatrix} 0_{N-n,n} & I_n \\ 0_{n,n} & 0_{n,N-n} \end{bmatrix} + e^{jn\omega} \begin{bmatrix} 0_{N-n,n} & I_n \\ 0_{n,n} & 0_{n,N-n} \end{bmatrix}^T \right)
$$

$$
= I_N + \sum_{n=1}^{N-1} \left(e^{-jn\omega} R_{N,n} + e^{jn\omega} R_{N,n}^T \right)
$$

Appendix F: Matrix Representation of Autocorrelation Sequence

For a given causal sequence $x[n]$, the Autocorrelation is defined as $R_{xx}[n]$:

$$R_{xx}[l] = \sum_{n=0}^{\infty} s[n]s[n+l] \quad , \quad s[n] \equiv x[n]\Big(\mu[n] - \mu[n-N]\Big)$$

$$= \sum_{n=0}^{N-(l+1)} x[n]x[n+l]$$

$$= {}_N\big[x[n]\big] \begin{bmatrix} \vec{0} \\ \\ \big[x[n]\big]_{n=l}^{n=N-1} \end{bmatrix}$$

$$= \Big(I_N \, \big[x[n]\big]^N\Big)^T \left(\begin{bmatrix} 0_{l,N-l} & 0_{l,l} \\ I_{N-l} & 0_{N-l,l} \end{bmatrix} \big[x[n]\big]^N\right)$$

$$= {}_N\big[x[n]\big] I_N \begin{bmatrix} 0_{l,N-l} & 0_{l,l} \\ I_{N-l} & 0_{N-l,l} \end{bmatrix} \big[x[n]\big]^N$$

$$= {}_N\big[x[n]\big] \begin{bmatrix} 0_{l,N-l} & 0_{l,l} \\ I_{N-l} & 0_{N-l,l} \end{bmatrix} \big[x[n]\big]^N$$

$$= {}_N\big[x[n]\big] \begin{bmatrix} 0_{l,N-l} & I_{N-l} \\ 0_{l,l} & 0_{N-l,l} \end{bmatrix} \big[x[n]\big]^N$$

$$= {}_N\big[x[n]\big] R_{N,l} \big[x[n]\big]^N \quad , \quad R_{N,l} \equiv \begin{bmatrix} 0_{l,N-l} & I_{N-l} \\ 0_{l,l} & 0_{N-l,l} \end{bmatrix}$$

Appendix G: Custom Notation for Vector/Matrix-generating Sequences

Some of the recurrence relations in Appendix D rely on the mathematics of vector/matrix-generating sequences. However, there seems to be no standard notation for any kind of vector/matrix generated from such sequences. This Appendix contains some of my own custom notation which I created for expressing these vectors/matrixs in a concise manner.

Take a generic sequence $f[n]$. The column vector $\vec{v}$ (populated by applying $f[n]$ element-wise to each row in the vector as a function of the row) can be denoted below as such:

$$\vec{v} = \big[f[n]\big]^N = \big[f[n]\big]_{n=0}^{n=N-1} = \begin{bmatrix} f[N-1] \\ f[N-2] \\ f[N-3] \\ \vdots \\ f[2] \\ f[1] \\ f[0] \end{bmatrix}$$

Accordingly, the row vector $\vec{v}^T$ can be denoted below as such:

$$\vec{v}^T = {}_N\big[f[n]\big] = {}_{n=N-1}\big[f[n]\big]_{n=0}$$

$$= \begin{bmatrix} f[N-1] & f[N-2] & f[N-3] & \dots & f[2] & f[1] & f[0] \end{bmatrix}$$

The matrix M (populated by applying $f[n,m]$ elementwise to each row n and column m in the matrix as a function of n and m) can be denoted below as such:

$$M = {}_N\big[f[n,m]\big]^N = {}_{m=N-1}\big[f[n,m]\big]_{n=0,m=0}^{n=N-1}$$

$$= \begin{bmatrix} f[N-1,N-1] & f[N-1,N-2] & f[N-1,N-3] & \dots & f[N-1,2] & f[N-1,1] & f[N-1,0] \\ f[N-2,N-1] & f[N-2,N-2] & f[N-2,N-3] & \dots & f[N-2,2] & f[N-2,1] & f[N-2,0] \\ f[N-3,N-1] & f[N-3,N-2] & f[N-3,N-3] & \dots & f[N-3,2] & f[N-3,1] & f[N-3,0] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ f[2,N-1] & f[2,N-2] & f[2,N-3] & \dots & f[2,2] & f[2,1] & f[2,0] \\ f[1,N-1] & f[1,N-2] & f[1,N-3] & \dots & f[1,2] & f[1,1] & f[1,0] \\ f[0,N-1] & f[0,N-2] & f[0,N-3] & \dots & f[0,2] & f[0,1] & f[0,0] \end{bmatrix}$$

Sometimes, some matrices will have portions systematically populated by zeroes. In such cases, we can have the original function ($f[n,m]$) modified by another function as a sort of indicator function. For example, consider an upper triangular matrix U where $f[n,m]$ is multiplied by a heaviside step function. U can be denoted as such below:

$$U = {}_{N}\big[g[n,m]\big]^{N} \quad , \quad g(n,m) = f[n,m]\mu[n-m]$$

$$= \begin{bmatrix} f[N-1,N-1] & f[N-1,N-2] & f[N-1,N-3] & \dots & f[N-1,2] & f[N-1,1] & f[N-1,0] \\ 0 & f[N-2,N-2] & f[N-2,N-3] & \dots & f[N-2,2] & f[N-2,1] & f[N-2,0] \\ 0 & 0 & f[N-3,N-3] & \dots & f[N-3,2] & f[N-3,1] & f[N-3,0] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & f[2,2] & f[2,1] & f[2,0] \\ 0 & 0 & 0 & \dots & 0 & f[1,1] & f[1,0] \\ 0 & 0 & 0 & \dots & 0 & 0 & f[0,0] \end{bmatrix}$$

The inner product of $\vec{v}$ with itself can be denoted as such below:

$$\vec{v}^T\vec{v} = {}_{N}\big[f[n]\big]\,\big[f[n]\big]^{N} = {}_{n=N-1}\big[f[n]\big]_{n=0}\,\big[f[n]\big]_{n=0}^{n=N-1} = \sum_{n=0}^{n=N-1} f^2[n]$$

The outer product of $\vec{v}$ with itself can be denoted as such below:

$$\vec{v}\vec{v}^T = \big[f[n]\big]^{N}\,{}_{N}\big[f[n]\big] = \big[f[n]\big]_{n=0}^{n=N-1}\,{}_{n=N-1}\big[f[n]\big]_{n=0} = {}_{m=N-1}\big[f[n]f[m]\big]_{n=0,m=0}^{n=N-1}$$

$$= \begin{bmatrix} f[N-1]f[N-1] & f[N-1]f[N-2] & f[N-1]f[N-3] & \dots & f[N-1]f[2] & f[N-1]f[1] & f[N-1]f[0] \\ f[N-2]f[N-1] & f[N-2]f[N-2] & f[N-2]f[N-3] & \dots & f[N-2]f[2] & f[N-2]f[1] & f[N-2]f[0] \\ f[N-3]f[N-1] & f[N-3]f[N-2] & f[N-3]f[N-3] & \dots & f[N-3]f[2] & f[N-3]f[1] & f[N-3]f[0] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ f[2]f[N-1] & f[2]f[N-2] & f[2]f[N-3] & \dots & f[2]f[2] & f[2]f[1] & f[2]f[0] \\ f[1]f[N-1] & f[1]f[N-2] & f[1]f[N-3] & \dots & f[1]f[2] & f[1]f[1] & f[1]f[0] \\ f[0]f[N-1] & f[0]f[N-2] & f[0]f[N-3] & \dots & f[0]f[2] & f[0]f[1] & f[0]f[0] \end{bmatrix}$$