Parcial 1 Señales y Sistemas

1. La distancia media entre dos señales periódicas $x_1(t) \in \mathbb{R}, \mathbb{C}$ y $x_2(t) \in \mathbb{R}, \mathbb{C}$; se puede expresar a partir de la potencia media de la diferencia entre ellas:

$$d^2(x_1, x_2) = \bar{P}_{x_1 - x_2} = \lim_{T \to \infty} \frac{1}{T} \int_T |x_1(x) - x_2(t)|^2 d$$

Sea $x_1(t)$ y $x_2(t)$ dos señales definidas como:

$$x_1(t) = A e^{-Jn w_0 t}$$

$$x_2(t) = B e^{Jm w_0 t}$$

Con $w_0 = \frac{2\pi}{T}$; $T, A, B \in \mathbb{R}^+$ y $n, m \in \mathbb{Z}$.

Determine la distancia entre las dos señales

Hallamos $|x_1(t) - x_2(t)|^2$

$$|x_1(t) - x_2(t)|^2 = (x_1(t) - x_2(t))(x_1(t) - x_2(t))$$

$$|x_1(t) - x_2(t)|^2 = |A e^{-Jn w_0 t} - B e^{Jm w_0 t}|^2$$

$$|x_1(t) - x_2(t)|^2 = (A e^{-Jn w_0 t} - B e^{Jm w_0 t})(A e^{Jn w_0 t} - B e^{-Jm w_0 t})$$

$$= A^2 - AB e^{-Jn w_0 t} e^{-Jm w_0 t} - AB e^{Jm w_0 t} e^{Jn w_0 t} + B^2$$

$$= A^2 + B^2 - AB[e^{-J(n+m) w_0 t} + e^{J(n+m) w_0 t}]$$

$$= A^2 + B^2 - 2AB \cos((n+m) w_0 t)$$

Potencia media

$$d^2(x_1, x_2) = \lim_{T \to \infty} \frac{1}{T} \int_T [A^2 + B^2 - 2AB \cos((n+m) w_0 t)]$$

Para señales periódicas, podemos calcular sobre un periodo

$$d^2(x_1, x_2) = \frac{1}{T} \int_0^T [A^2 + B^2 - 2AB \cos((n+m) w_0 t)]$$

$$d^2(x_1, x_2) = \frac{1}{T} \left[\int_0^T (A^2 + B^2) dt - 2AB \int_0^T \cos((n+m) w_0 t) dt \right]$$

Evaluamos las integrales

1) $\int_0^T (A^2+B^2)\,dt = (A^2+B^2)T$

2) $\int_0^T \cos((n+m)\omega_0 t)\,dt = \int_0^T \cos\left((n+m)\frac{2\pi}{T}t\right) dt$

Caso 1 $n+m \neq 0$

$$d^2(x_1,x_2) = \frac{1}{T}\left[(A^2+B^2)T - 0\right] = A^2+B^2$$

Caso 2 $n+m = 0$

$$d^2(x_1,x_2) = \frac{1}{T}\left[(A^2+B^2)T - 2AB\cdot T\right] = A^2+B^2-AB =$$

$$= (A-B)^2$$

$$d(x_1,x_2) = \begin{cases} \sqrt{A^2+B^2} & si\ n+m \neq 0 \\ |A-B| & si\ n+m = 0 \end{cases}$$

2. Encuentre la señal en tiempo discreto al utilizar un conversor análogo digital con frecuencia de muestreo de 5kHz y 4 bits de capacidad de representación, aplicado a la señal continua:

$$x(t) = 3\cos(1000\pi t) + 5\sin(3000\pi t) + 10\cos(11000\pi t)$$

$\omega_1 = 1000$ rad/s $\quad T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{1000\pi} \quad f_1 = \frac{1}{T_1} = \frac{1000\pi}{2\pi} = 500$ Hz

$\omega_2 = 3000$ rad/s $\quad T_2 = \frac{2\pi}{\omega_2} = \frac{2\pi}{3000\pi} \quad f_2 = \frac{1}{T_2} = \frac{3000\pi}{2\pi} = 1500$ Hz

$\omega_3 = 11000$ rad/s $\quad T_3 = \frac{2\pi}{\omega_3} = \frac{2\pi}{11000\pi} \quad f_3 = \frac{1}{T_3} = \frac{11000\pi}{2\pi} = 5500$ Hz

Hallamos frecuencia de muestreo

$$f_s \geq 2\cdot f_{max} \Rightarrow f_s \geq 2(5500) \Rightarrow f_s \geq 11000$$

Como $fs = 5000\,Hz$ no cumple con el teorema de Nyquiest

Discretizamos la señal

para ir de $x(t)$ a $x[n]$ usamos

$t = nTs = \frac{n}{Fs}$ $\qquad Fs = 5000\,Hz$

$$x\left(\frac{n}{Fs}\right) = 3\cos\left(1000\pi \frac{n}{Fs}\right) + 5\,\text{sen}\left(3000\pi \frac{n}{Fs}\right) + 10\cos\left(11000\pi \frac{n}{Fs}\right)$$

$$x\left(\frac{n}{Fs}\right) = 3\cos\left(\frac{1000\pi}{5000} n\right) + 5\,\text{sen}\left(\frac{3000\pi}{5000} n\right) + 10\cos\left(\frac{11000\pi}{5000} n\right)$$

$$x\left(\frac{n}{Fs}\right) = 3\cos\left(\frac{\pi}{5} n\right) + 5\,\text{sen}\left(\frac{3\pi}{5} n\right) + 10\cos\left(\frac{11\pi}{5} n\right)$$

$10\cos\left(\frac{11}{5}\pi n\right)$ no está en $[-\pi, \pi]$

Para que esté en el intervalo restamos 2π

$$\Omega ori = \Omega copia - 2\pi$$

$$= \frac{11\pi}{5} - 2\pi = \frac{\pi}{5}$$

Entonces

$$x\left(\frac{n}{Fs}\right) = 3\cos\left(\frac{\pi}{5} n\right) + 5\,\text{sen}\left(\frac{3\pi}{5} n\right) + 10\cos\left(\frac{\pi}{5} n\right)$$

$$x\left(\frac{n}{Fs}\right) = 13\cos\left(\frac{\pi}{5} n\right) + 5\,\text{sen}\left(\frac{3\pi}{5} n\right)$$

Esta será la señal que obtendrá en tiempo discreto el conversor análogo digital.

3. Sea $x''(t)$ la segunda derivada de la señal $x(t)$, donde $t \in [t_i, t_f]$. Demuestre que los coeficientes de la serie exponencial de Fourier se pueden calcular según:

$$C_n = \frac{1}{(t_i - t_f) n^2 w_0^2} \int_{t_i}^{t_f} x''(t) e^{-Jnw_0 t} dt; \quad n \in \mathbb{Z}.$$

¿Cómo se pueden calcular los coeficientes a_n y b_n desde $x''(t)$ en la serie trigonométrica de Fourier?

$$C_n = \frac{1}{T} \int_{t_i}^{t_f} x(t) e^{-Jnw_0 t} dt \quad \therefore x(t) = \sum_n C_n e^{Jnw_0 t}$$

$$x'(t) = \frac{d}{dt} x(t) = \frac{d}{dt} \left\{ \sum_n C_n e^{Jnw_0 t} \right\} = \sum_n C_n e^{Jnw_0 t} Jnw_0$$

$$x''(t) = \frac{d}{dt} \left\{ \sum_n C_n e^{Jnw_0 t} (Jnw_0) \right\} = \sum_n C_n e^{Jnw_0 t} (Jnw_0)^2$$

$$\tilde{C}_n = \frac{(x''(t) \cdot e^{Jnw_0 t})}{\| e^{Jnw_0 t} \|^2} = \int_{t_i}^{t_f} \frac{x''(t) e^{-Jnw_0 t}}{T} dt \quad \text{cuando } T = t_f - t_i$$

$$\tilde{C}_n = C_n (Jnw_0)^2 = \int_{t_i}^{t_f} \frac{x''(t) e^{-Jnw_0 t}}{T} dt$$

$$C_n = \frac{1}{(t_f - t_i)(Jnw_0)^2} \int_{t_i}^{t_f} x''(t) e^{-Jnw_0 t} dt$$

$$C_n = \frac{1}{(t_i - t_f) n^2 w_0^2} \int_{t_i}^{t_f} x''(t) e^{-Jnw_0 t} dt$$

$$x(t) = C_n + \sum_{n=1}^{N} a_n \cos(nw_0 t) + b_n \operatorname{sen}(nw_0 t)$$

$$x'(t) = \sum_{n=1}^{N} a_n (-nw_0) \operatorname{sen}(nw_0 t) + b_n (nw_0) \cos(nw_0 t)$$

$$x''(t) = \sum_{n=1}^{N} a_n (-nw_0)(nw_0) \cos(nw_0 t) + b_n (nw_0)(-nw_0) \operatorname{sen}(nw_0 t)$$

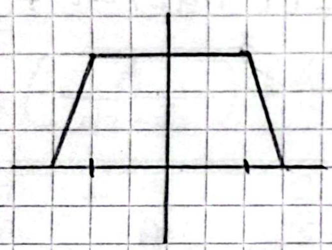
$$a_n(-n^2 w_0^2) = \frac{2}{T}\int_{ti}^{tf} x''(t)\cos(n w_0 t)\,dt$$

$$a_n = \frac{2}{-T n^2 w_0^2}\int_{ti}^{tf} x''(t)\cos(n w_0 t)\,dt$$

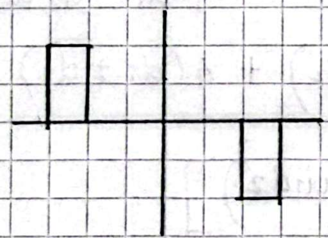
$$b_n = (-n^2 w_0^2) = \frac{2}{T}\int_{ti}^{tf} x''(t)\,\text{sen}(n w_0 t)\,dt$$

$$b_n = \frac{2}{-T n^2 w_0^2}\int_{ti}^{tf} x''(t)\,\text{sen}(n w_0 t)\,dt$$

4. Encuentre el espectro de Fourier, su parte real, Imaginaria, magnitud, fase y el error relativo para $n \in \{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5\}$, a partir de $x''(t)$ para la señal $x(t)$



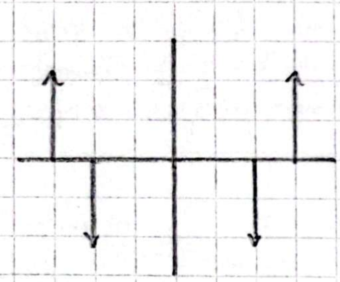
$$x''(t) - A\delta(t + d_2) - A\delta(b + d_1)$$
$$- A\delta(t - d_1) + A\delta(t - d_2)$$



$$C_n = \frac{1}{-t n^2 w_0^2}\int_{-T/2}^{T/2} x''(t) e^{-Jnw_0 t}\,dt$$

$$C_n = \frac{1}{T n^2 w_0^2}\int_{-T/2}^{T/2} A[\delta(t + d_2)]$$

$$- \delta(t + d_1) - \delta(t - d_1) + \delta(t - d_1)$$
$$\int e^{-Jnw_0 t}\,dt$$



$$C_n = \frac{-A}{Tn^2\omega_0^2}\left(e^{-jn\omega_0(-d_2)} + e^{-jn\omega_0(-d_1)} - e^{-jn\omega_0 d_1} + e^{-jn\omega_0 d_2}\right)$$

$$C_n = \frac{-A}{Tn^2\omega_0^2}\left(e^{jn\omega_0 d_2} + e^{-jn\omega_0 d_2} - \left(e^{jn\omega_0 d_1} + e^{-jn\omega_0 d_1}\right)\right)$$

$$C_n = \frac{-A}{Tn^2\omega_0^2}\left(2\cos(n\omega_0 d_2) - 2\cos(n\omega_0 d_1)\right)$$

$$C_n = \frac{-2A}{Tn^2 + \frac{4\pi^2}{T^2}}\left(\cos\left(n\frac{2\pi}{T}d_1\right) - \cos\left(n\frac{2\pi}{T}d_1\right)\right)$$

$$C_n = \frac{A}{\frac{2\pi n^2}{T}}\left(2\cos(n2\pi d_2)\right) - \cos\left(n\frac{2\pi}{T}d_1\right)\right)$$

$$C_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt = \frac{1}{T}\int_{-d_2}^{-d_1}\frac{A}{d_2-d_1}(t+d_2)\,dt + \frac{1}{T}\int_{-d_1}^{d_1} A\,dt +$$

$$+ \frac{1}{T}\int_{d_1}^{d_2}\frac{-A}{d_2-d_1}(t-d_2)\,dt$$

$$= \frac{1}{T}\left[\frac{A}{d_2-d_1}\left(\frac{t^2}{2}+d_2t\right)\Big|_{-d_2}^{-d_1} + At\Big|_{-d_1}^{d_1} - \frac{A}{d_2-d_1}\left(\frac{t^2}{2}+d_2t\right)\Big|_{d_1}^{d_2}\right.$$

$$= \frac{1}{T}\left[\frac{A}{d_2-d_1}\left(\frac{d_1^2}{2} - d_1d_2 - \frac{d_2^2}{2} + d_2d_2\right) + A(d_1+d_1)\right.$$

$$\left. - \frac{A}{d_2-d_1}\left(\frac{d_2^2}{2} - d_2d_2 - \frac{d_1^2}{2} + d_1d_2\right)\right]$$

$$= \frac{1}{T}\left[\frac{2A}{d_2-d_1}\left(\frac{d_1^2}{2} - d_1d_2 - \frac{d_2^2}{2} + d_2^2\right) + 2Ad_1\right.$$

Si $A = 1$ $d_1 = 1$ $d_2 = 2$ y $T = 2d_2 = 4$

$$C_n = \frac{-1}{\frac{2n^2n^2}{2}}\left(\cos\left(n\frac{2\pi}{4}\cdot 2\right) - \cos\left(n\frac{2\pi}{4}\cdot 1\right)\right)$$

$$= \frac{-1}{\frac{n^2n^2}{2}}\left(\cos(n\pi) - \cos\left(n\frac{\pi}{2}\right)\right)$$

$$C_0 = \frac{1}{4}\left[\frac{2\cdot 1}{1}\left(\frac{1}{2} - 2 - 2 + 4\right) + 2\cdot 1\cdot 1\right]$$

$$= \frac{1}{4}\left[2\left(\frac{1}{2}\right) + 2\right] = \frac{3}{4}$$

$$P_x = \frac{1}{T}\int_{-T/2}^{T/2} (x(t))^2 dt = \frac{2}{T}\int_{-T/2}^{0} (x(t))^2 dt$$

$$= \frac{2}{T}\int_{-d_2}^{-d_1} \left(\frac{A}{d_2 - d_1}\right)^2 (t + d_2)^2 dt + \frac{2}{T}\int_{-d_1}^{0} A^2 dt$$

$$\therefore \quad P_x = \frac{2}{T}\left(\frac{A}{d_2 d_1}\right)^2$$

$$(t^2 + 2 + d_2 + d_1^2)\Big|_{-d_2}^{-d_1} + \frac{2}{T} A^2 t\Big|_{-d_1}^{0}$$

$$P_x = \frac{2}{T}\left(\frac{A}{d_2 - d_1}\right)^2 (d_1^2 - 2d_2 d_1 + d_2^2 - d_2^2 + 2d_2^2 - d_2^2) + \frac{2}{T} A^2 (0 + d_1)$$

$$P_x = \frac{1}{6} + \frac{1}{2} = \frac{2}{3}$$