Integrals over ellipses defined in the $r - \phi$ plane

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Vortices in protoplanetary disks appear as banana-shaped object in the disk. However, in the $r - \phi$ plane, they have an elliptical shape. The vortices can be identified by closed lines of constant vortensity, $\varpi = \frac{(\nabla\times\vec{v})_z}{\Sigma}$, where v is the velocity of the gas and Σ the surface density. To account for radial variation of Σ and the Keplerian velocity profile of the disk, it is advantageous to normalize the vortensity by the background vortensity, $\varpi_0 = \frac{(\nabla\times\vec{v}_{\mathrm{Kepler}}(r))_z}{\Sigma_0(r)}$ of the Keplerian flow divided by a background surface density profile. Meaningful values of the resulting quantity $\frac{\varpi}{\varpi_0}$ can be expected to be in the range -1 to 1 for retrograde vortices.

Having identified a suitable ellipse in the $r - \phi$ plane, we can then define a vortex as the material enclosed by the ellipse.

These ellispes are described by their center coordinates, r_0, ϕ_0, and their extents $2h_r$ and $2h_\phi$. Thus the ellipse equation takes the form

$$\left(\frac{r - r_0}{h_r}\right)^2 + \left(\frac{\phi - \phi_0}{h_\phi}\right)^2 = 1 \tag{1}$$

Without loss of generality, we can set $\phi_0 = 0$ by rotating the frame of reference.

Note that the ϕ direction does not carry a unit of length. The corresponding length is the distance on the arc at radius r. This needs to be accounted for during integration of quantities over the ellipse.

The area of the ellipse in the $r - \phi$ plane does not carry physical meaning. The physically relevant quantity is the area enclosed by the corresponding region in the disk.

In the r - s plane, where s is the length on an arc along the azimuthal direction, the ellipse is stretched for $r > r_0$ and pinched for $r < r_0$. See the center panel of Fig. 1 for a sketch of this geometry. The bounding lines indicated with $\phi = \pi$ and $\phi = -\pi$ identify the same line in the right panel and represent a periodic boundary.

To calculate the area in the physically meaningful cartesian plane, we need to integrate in polar coordinates with the appropriate area measure, where r and ϕ must fulfill Eq. (1). The shape of the ellipse is not symmetric in cartesian coordinates, as can be seen in Fig. 1 in the center and right panels. This is done by integrating from $r_0 - h_r$ to $r_0 + h_r$ in radial direction.

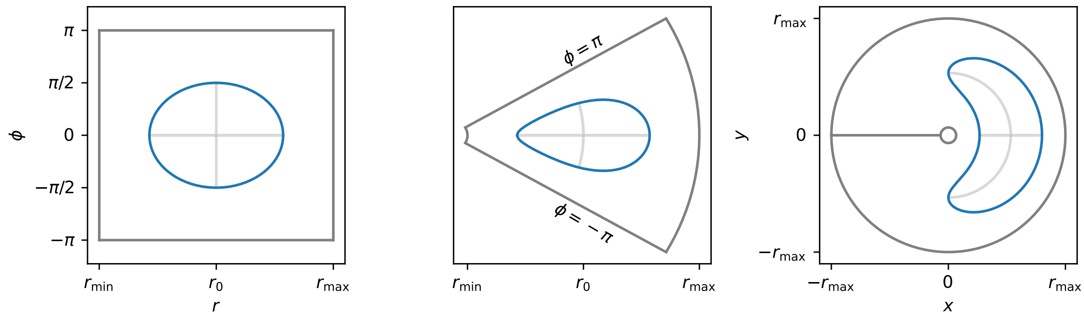

Figure 1: Illustration of the shape of a vortex which is defined in the $r-\phi$ plane. The panels show, from left to right, the shape of the vortex in the $r-\phi$ plane, in a geometry where the disk is folded into a wedge shape, and in cartesian coordinates, as it would be observed in a face-on disk. The light gray lines indicate constant r and constant ϕ of the ellipse in the $r-\phi$ plane, and the darker gray lines indicated the boundaries of the disk.

The azimuthal integration domain is $[-\alpha(r), \alpha(r)]$, where $\alpha(r) = h_\phi \sqrt{1-\left(\frac{r-r_0}{h_r}\right)^2}$ is defined by Eq. (1).

$$A = \int_{Ellipse} \mathrm{d}A = \int_{r_0-h_r}^{r_0+h_r} \int_{-\alpha(r)}^{\alpha(r)} \mathrm{d}\phi \, r \, \mathrm{d}r \tag{2}$$

$$= 2h_\phi \int_{r_0-h_r}^{r_0+h_r} \mathrm{d}r \, r \sqrt{1-\left(\frac{r-r_0}{h_r}\right)^2} \tag{3}$$

$$= 2h_r^2 h_\phi \int_{-1}^{1} \mathrm{d}x \left(x + \frac{r_0}{h_r}\right) \sqrt{1-x^2} = 2h_\phi h_r^2 \left(I_1 + \frac{r}{h_r} I_2\right) \tag{4}$$

The substitution $r - r_0 = h_r x$ was performed and the integral was split in two parts. The second integral disappears because the integrant is anti-symmetric, $I_2 = \int_{-1}^{1} \mathrm{d}x \, x \sqrt{1-x^2} = 0$. The remaining integral can be identified as the area of a half-circle, $I_1 = \int_{-1}^{1} \mathrm{d}x \sqrt{1-x^2} = \frac{\pi}{2}$.

Finally, the area in the cartesian plane appears to be equivalent to the usual formula with the extent $a_s = r_0 h_\phi$,

$$A = \pi \, h_r \, h_\phi \, r_0 \tag{5}$$

A possible interpretation is that the stretching in the outer part is compensated by the pinching in the inner part. See the center panel of Fig. 1 for how the shape is affected by the stretching and pinching.

The definition for the vortex given above depends on the technicalities of how the lines of constant normalized vortensity are computed. To give a precise definition for the vortex which , we fit bell curves to the surface density, Σ, and the normalized vortensity, ϖ/ϖ_0. The function is composed of two bell curves in radial and azimuthal direction as

$$f(r,\phi) = c + a\exp\left(-\frac{(r-r_0)^2}{\sigma_r^2}\right)\exp\left(-\frac{(\phi-\phi_0)^2}{\sigma_\phi^2}\right). \tag{6}$$

Then the vortex can be defined as the region enclosed by an ellipse (in $r-\phi$) with the extent is defined by the full width at half maximum of the bell curve, i.e. $h_r = \sqrt{2\ln(2)}\,\sigma_r$ and $h_\phi = \sqrt{2\ln(2)}\,\sigma_\phi$, or by some multiple of σ_r and σ_ϕ. In general, any ratio $k = \frac{h_r}{\sigma_r} = \frac{h_\phi}{\sigma_\phi}$ can be chosen.

Again, the physical relevant integrals need to be performed in the cartesian plane with limits defined by the ellipse in the $r-\phi$ plane. To evaluate the integral, we again rotate into a reference system such that $\phi_0 = 0$. We will first split off the contribution from the constant part of f by reusing the result for the area.

$$F = \int_{Ellipse} f(r,\phi)\,\mathrm{d}A = \int_{r_0-h_r}^{r_0+h_r}\int_{-\alpha(r)}^{\alpha(r)} \mathrm{d}\phi\, r\,\mathrm{d}r\, f(r,\phi) = A\,c + J\,, \tag{7}$$

$$J = a\int_{r_0-h_r}^{r_0+h_r} r\,\mathrm{d}r\exp\left(-\frac{(r-r_0)^2}{2\sigma_r^2}\right)2\int_0^{\alpha(r)}\mathrm{d}\phi\exp\left(-\frac{\phi^2}{2\sigma_\phi^2}\right). \tag{8}$$

The ϕ integral can be evaluated using the error function, erf, and the substitution $y = \frac{\phi}{\sqrt{2}\sigma_\phi}$

$$\int_0^{\alpha(r)}\mathrm{d}\phi\exp\left(-\frac{\phi^2}{2\sigma_\phi^2}\right) = \sqrt{2}\sigma_\phi\int_0^{\frac{\alpha(r)}{\sqrt{2}\sigma_\phi}}\mathrm{d}y\exp(-y^2) \tag{9}$$

$$= \sqrt{\frac{\pi}{2}}\sigma_\phi \mathrm{erf}\left(\frac{h_\phi}{\sqrt{2}\sigma_\phi}\sqrt{1-\frac{(r-r_0)^2}{h_r^2}}\right). \tag{10}$$

Using this result and the substitution $r - r_0 = h_r x$,

$$J = \sqrt{2\pi}a\sigma_\phi\int_{r_0-h_r}^{r_0+h_r} r\mathrm{d}r\exp\left(-\frac{(r-r_0)^2}{2\sigma_r^2}\right)\mathrm{erf}\left(\frac{k}{\sqrt{2}}\sqrt{1-\frac{(r-r_0)^2}{h_r^2}}\right) \tag{11}$$

$$= \sqrt{2\pi}a\sigma_\phi\sigma_r k\int_{-1}^{1}\mathrm{d}x(r_0+h_r x)\exp\left(-\frac{k^2}{2}x^2\right)\mathrm{erf}\left(\frac{k}{\sqrt{2}}\sqrt{1-x^2}\right) \tag{12}$$

$$= \sqrt{2\pi}a\sigma_\phi\sigma_r r_0 k\int_{-1}^{1}\mathrm{d}x\exp\left(-\frac{k^2}{2}x^2\right)\mathrm{erf}\left(\frac{k}{\sqrt{2}}\sqrt{1-x^2}\right) \tag{13}$$

$$= \sqrt{2\pi}a\sigma_\phi\sigma_r r_0\,C_k\,. \tag{14}$$

The second term in the integrant of the second line, resulting from multiplication of $h_r x$ with the two functions, is anti-symmetric and thus its contribution to the integral vanishes.

At this point, the integral only depends on constants and can be integrated numerically to obtain $C_k = k \int_{-1}^{1} \mathrm{d}x \exp\left(-\frac{k^2}{2}x^2\right) \mathrm{erf}\left(\frac{k}{\sqrt{2}}\sqrt{1-x^2}\right)$.

Finally, the quantity integrated over the vortex region is

$$F = cA + \sqrt{2\pi} a \sigma_\phi \sigma_r r_0 k C_k \tag{15}$$

$$= \sigma_\phi \sigma_r r_0 \left(c\, k^2 \pi + a \sqrt{2\pi} C_k\right) \tag{16}$$

Interestingly, in the case of the double bell curve, the stretching and pinching effects again appear to cancel each other. An example for a quantity obtained in this way, the mass, $F = M$, of the vortex can be calculated from a fit to the surface density, $f_\Sigma(r, \phi)$.

Some computed values for $k = 1$, $k = \sqrt{2 \ln 2}$ (corresponding to a vortex extent of one FWHM), and $k = 2$ are $C_1 = 0.98628137356472$, $C_{2\ln(2)} = 1.25331413731550$, and $C_2 = 2.16739302711$.