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# Assignment 3

Q1) $|\psi_k\rangle = \cos\frac{2\pi k}{5}|0\rangle + \sin\frac{2\pi k}{5}|1\rangle$

$\langle\psi_k| = \cos\frac{2\pi k}{5}\langle 0| + \sin\frac{2\pi k}{5}\langle 1|$

$|\psi_k\rangle\langle\psi_k| = \left(\cos\frac{2\pi k}{5}|0\rangle + \sin\frac{2\pi k}{5}|1\rangle\right)\left(\cos\frac{2\pi k}{5}\langle 0| + \sin\frac{2\pi k}{5}\langle 1|\right)$

$\Rightarrow |\psi_k\rangle\langle\psi_k| = \cos^2\frac{2\pi k}{5}|0\rangle\langle 0| + \sin\frac{2\pi k}{5}\cos\frac{2\pi k}{5}|1\rangle\langle 0| + \sin\frac{2\pi k}{5}\cos\frac{2\pi k}{5}|0\rangle\langle 1| + \sin^2\frac{2\pi k}{5}|1\rangle\langle 1|$

Let $E = |\psi_k\rangle\langle\psi_k\rangle$

Now, let us find the matrix corresponding to $|\psi_k\rangle\langle\psi_k|$

$$E_k = \begin{bmatrix} \langle 0|E|0\rangle & \langle 0|E|1\rangle \\ \langle 1|E|0\rangle & \langle 1|E|1\rangle \end{bmatrix}$$

$$\Rightarrow E_k = \begin{bmatrix} \cos^2\frac{2\pi k}{5} & \sin\frac{2\pi k}{5}\cos\frac{2\pi k}{5} \\ \sin\frac{2\pi k}{5}\cos\frac{2\pi k}{5} & \sin^2 2\pi k/5 \end{bmatrix}$$

Hence the set of operators,

$\left\{\frac{2}{5}|\psi_k\rangle\langle\psi_k|\right\}_{k=0}^{4}$ is given by $\left\{\frac{2}{5}E_k\right\}_{k=0}^{4}$

To check whether the above set of operators is a POVM or not, we need to check the following conditions:

1) The eigenvalues corresponding to each operator must be non-negative $\Rightarrow$ POSITIVE SEMIDEFINITE

$$A = \frac{2}{5}\begin{bmatrix} \cos^2\frac{2\pi k}{5} & \sin\frac{2\pi k}{5}\cos\frac{2\pi k}{5} \\ \sin\frac{2\pi k}{5}\cos\frac{2\pi k}{5} & \sin^2\frac{2\pi k}{5} \end{bmatrix}$$

$\det(A - \lambda I) = 0$

$$\left(\frac{2}{5}\cos^2\frac{2\pi k}{5} - \lambda\right)\left(\frac{2}{5}\sin^2\frac{2\pi k}{5} - \lambda\right) - \left(\sin^2\frac{2\pi k}{5}\cos^2\frac{2\pi k}{5}\right) = 0$$

$$\frac{4}{25}\cos^2\frac{2\pi k}{5}\sin^2\frac{2\pi k}{5} - \lambda\times\frac{2}{5} + \lambda^2 - \sin^2\frac{2\pi k}{5}\cos^2\frac{2\pi k}{5} = 0$$

$$\Rightarrow \lambda^2 - \frac{2\lambda}{5} - \frac{21}{25}\sin^2\frac{2\pi k}{5}\cos^2\frac{2\pi k}{5} = 0$$

$$\Rightarrow \lambda = \frac{2/5 \pm \sqrt{4/25 - 84/25 \sin^2\frac{2\pi k}{5}\cos^2\frac{2\pi k}{5}}}{2}$$

$$\Rightarrow \lambda = \frac{1}{5} + \frac{1}{5}\sqrt{\underbrace{1 - 21\sin^2\frac{2\pi k}{5}\cos^2\frac{2\pi k}{5}}_{+\text{ve fraction always}}}$$

Hence $\lambda \geq 0$

2) completeness: sum of all operators must equal the identity operator.

$$\sum_{k=0}^{4} \frac{2}{5} E_k = \sum_{k=0}^{4} \frac{2}{5}\begin{bmatrix} \cos^2\frac{2\pi k}{5} & \sin\frac{2\pi k}{5}\cos\frac{2\pi k}{5} \\ \sin\frac{2\pi k}{5}\cos\frac{2\pi k}{5} & \sin^2\frac{2\pi k}{5} \end{bmatrix}$$

Now, $\sum_{k=0}^{4} \cos^2\frac{2\pi k}{5}$

$= 1 + \cos^2\frac{2\pi}{5} + \cos^2\frac{4\pi}{5} + \cos^2\frac{6\pi}{5} + \cos^2\frac{8\pi}{5}$

$= 1 + 2\left[\cos^2\frac{2\pi}{5} + \cos^2\frac{\pi}{5}\right]$

$= 1 + \frac{3}{2}$

$= \frac{5}{2}$

Similarly $\sum_{k=0}^{4} \sin^2\frac{2\pi k}{5} = \frac{5}{2}$

$\sum_{k=0}^{4} \sin\frac{2\pi k}{5}\cos\frac{2\pi k}{5} = \frac{1}{2}\sum_{k=0}^{4} \sin\frac{4\pi k}{5}$

$= \frac{1}{2}\left[\sin\left(\frac{4\pi}{5} + \frac{3}{\cancel{2}}\frac{\cancel{2}\pi}{5}\right)\right] \sin\left(\frac{3 \times \cancel{2}\pi}{\cancel{2} \cdot 5}\right)$

$\sin\frac{2\pi}{5 \times 2}$

$= 0$

Hence $\sum_{k=0}^{4} \frac{2}{5} E_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

3) Hermitian : $E^{+} = E$

$$E^{+} = \begin{bmatrix} \cos^2 2\pi k/5 & i \sin\frac{2\pi k}{5}\cos\frac{2\pi k}{5} \\ -i \sin\frac{2\pi k}{5}\cos\frac{2\pi k}{5} & \sin^2 2\pi k/5 \end{bmatrix}^{T}$$

$$= \begin{bmatrix} \cos^2 2\pi k/5 & -i \sin\frac{2\pi k}{5}\cos\frac{2\pi k}{5} \\ i \sin\frac{2\pi k}{5}\cos\frac{2\pi k}{5} & \sin^2\frac{2\pi k}{5} \end{bmatrix} = E$$

Since, all these conditions hold true,
The set of operators $\left\{ \frac{2}{5} |\psi_k\rangle\langle\psi_k| \right\}_{k=0}^{4}$ is a POVM.

Now, let us check whether $\left\{ \frac{2}{5} |\psi_k\rangle\langle\psi_k| \right\}_{k=0}^{4}$ is a projective measurement or not.

$$A_k = \frac{2}{5} E_k$$

Firstly, $A_k^2 = A_k$ for it to be projective measurement

$$A_k^2 = \frac{4}{5} \begin{bmatrix} \cos^2 2\pi k/5 & -i \sin\frac{2\pi k}{5}\cos\frac{2\pi k}{5} \\ i \sin\frac{2\pi k}{5}\cos\frac{2\pi k}{5} & \sin^2 2\pi k/5 \end{bmatrix} \times \begin{bmatrix} \cos^2 2\pi k/5 & -i \sin\frac{2\pi k}{5}\cos 2\pi k/5 \\ i \sin\frac{2\pi k}{5}\cos\frac{2\pi k}{5} & \sin^2\frac{2\pi k}{5} \end{bmatrix}$$

$$= \frac{4}{5} \begin{bmatrix} \cos^4\frac{2\pi k}{5} + \sin^2\frac{2\pi k}{5}\cos^2\frac{2\pi k}{5} & -i \sin\frac{2\pi k}{5}\cos^3\frac{2\pi k}{5} - i \sin^3\frac{2\pi k}{5}\cos\frac{2\pi k}{5} \\ i \sin\frac{2\pi k}{5}\cos^3\frac{2\pi k}{5} + i \sin^3\frac{2\pi k}{5}\cos\frac{2\pi k}{5} & \sin^4\frac{2\pi k}{5} + \sin^2\frac{2\pi k}{5}\cos^2\frac{2\pi k}{5} \end{bmatrix}$$

$$\neq A_k$$

Hence, the set of operators $\left\{ \frac{2}{5} |\psi_k\rangle\langle\psi_k| \right\}_{k=0}^{4}$ is not a projective measurement and we need not check the rest of the conditions

2) For the Walsh-Hadamard Transformation:

$$|0\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|1\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

i) The matrix representation of a linear transformation can be found by applying the transformation to the standard basis and expressing the result as a linear combination of standard basis vectors.

Hence the matrix representation for this transformation is given by:

$$M = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

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ii) $|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. $|-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$|+\rangle = \frac{(|0\rangle + |1\rangle)}{\sqrt{2}}$ — (1) $|-\rangle = \frac{1}{\sqrt{2}}[|0\rangle - |1\rangle]$ — (2)

$$|+\rangle \xrightarrow{H} \frac{|0\rangle + |1\rangle}{\sqrt{2}} \xrightarrow{H} \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)$$

$$= \frac{1}{2}(2|0\rangle)$$

$$= |0\rangle$$

$$|-\rangle \xrightarrow{H} \frac{|0\rangle - |1\rangle}{\sqrt{2}} \xrightarrow{H} \frac{1}{\sqrt{2}}\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}} - \frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$= \frac{1}{2} \times 2|1\rangle$$

$$= |1\rangle$$

Also by eq$^n$ (1) and eq$^n$ (2)

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}} \quad \text{and} \quad |1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

Hence,

$$|+\rangle \xrightarrow{H} |0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$$

$$|-\rangle \xrightarrow{H} |1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

Hence the matrix representation of H in the basis $\{|+\rangle, |-\rangle\}$ is given by:

$$M = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

iii) As found in part 1, the matrix representation of H in the basis $\{|0\rangle, |1\rangle\}$ is given by :

$$M = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\text{cof}(M) = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -1 \\ -1 & +1 \end{bmatrix} = \text{adj}(M)$$

$$|M| = \left(\frac{1}{\sqrt{2}}\right)^2 (-1-1) = \cancel{\sqrt{2}} = -1$$

$$M^{-1} = \frac{\text{adj}(M)}{|M|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = M$$

Hence, inverse of Walsh-Hadamard Transform is the Transform itself. This means that applying the Transform twice will return the original state.

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Q3) $|\psi\rangle = \frac{1}{6}\begin{bmatrix}1\\0\\4\end{bmatrix}$

Observables A and B:

$$A = \frac{1}{\sqrt{2}}\begin{bmatrix}2 & 0 & 0\\0 & 1 & i\\0 & -i & 1\end{bmatrix} \qquad B = \begin{bmatrix}1 & 0 & 0\\0 & 0 & -i\\0 & i & 0\end{bmatrix}$$

i) The probability of obtaining a value 0 for A can be obtained by multiplying the state $|\psi\rangle$ with the eigenvector corresponding to eigenvalue 0 of A and then taking the square of the magnitude of the result.

characteristic eq$^n$ of A:

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix}\sqrt{2}-\lambda & 0 & 0\\0 & \frac{1}{\sqrt{2}}-\lambda & i/\sqrt{2}\\0 & -i/\sqrt{2} & 1/\sqrt{2}-\lambda\end{vmatrix}=0$$

$$(\sqrt{2}-\lambda)\left(\left(\frac{1}{\sqrt{2}}-\lambda\right)^2-\frac{1}{2}\right)=0$$

$\lambda=\sqrt{2}$ $\qquad \frac{1}{\sqrt{2}}-\lambda = \pm\frac{1}{\sqrt{2}}$

$$\lambda = 0,\ \sqrt{2}$$

For eigenvector corresponding to eigenvalue 0, we find the null space of $Ax = \cancel{0}\lambda x$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}2 & 0 & 0\\0 & 1 & i\\0 & -i & 1\end{bmatrix}x = 0$$

$\Rightarrow$ $2x_1 = 0$

$x_2 + ix_3 = 0$

$-ix_2 + x_3 = 0$

Hence, $x = \begin{bmatrix}0\\-i\\1\end{bmatrix}$

$$P(0) = |\langle x|\psi\rangle|^2 = \frac{1}{2\times 17}\left|\begin{bmatrix} 0 & -i & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}\right|^2$$

$$= \frac{1}{2\times 17}\,16$$

$$= \frac{8}{17}$$

ii) Now since 0 is obtained as a result of measurement A, the state of the system is : $|a\rangle = \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$

characteristic eq$^n$ of B :

$$\det(B - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & -i \\ 0 & i & -\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[\lambda^2 - 1] = 0$$

$$\lambda = 1, -1$$

for $\lambda = 1$, $\quad B x = \lambda x$

$$\Rightarrow B x = x$$

$$\Rightarrow (B - I) x = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -i \\ 0 & i & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$x_1 = 0 \qquad -x_2 - i x_3 = 0 \Rightarrow x_2/x_3 = -i$

$\qquad\qquad i x_2 - x_3 = 0 \Rightarrow x_2/x_3 = 1/i = -i$

Hence, $\quad x = \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$

$$\Rightarrow P(1) = |\langle a|x\rangle|^2 = \left|\frac{1}{2\times 2}[0 \;\; i \;\; 1]\begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}\right|^2$$

$$= \frac{1}{4} \; 2^2$$

$$= 1$$