ROMICA RAISINGHANI
2021101053

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Introduction to Quantum Information & Computation

ASSIGNMENT - 1

Q1) $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

a) To find the eigenvalues, we need to solve the characteristic eq^n for λ,

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda(\lambda^2 - 1) - (-\lambda) = 0$$

$$\Rightarrow -\lambda^3 + \lambda + \lambda = 0$$

$$\Rightarrow \lambda(\lambda^2 - 2) = 0$$

$$\Rightarrow \lambda = 0, \sqrt{2}, -\sqrt{2}$$

Eigenvalues of A are 0, $\sqrt{2}$ and $-\sqrt{2}$

To find the eigenvectors ~~for~~ $\lambda = 0$, we need to find the null space for $(A - \lambda I)x = 0$

For $\lambda = 0$,

$$Ax = 0$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow x_2 = 0$$

$$x_1 + x_3 = 0$$

By above eq^{n}s, the eigenvector is $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Normalizing the eigenvector, we get $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

For $\lambda = \sqrt{2}$,

$$(A - \sqrt{2}I)X = 0$$

$$\begin{bmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$-\sqrt{2}x_1 + x_2 = 0$$
$$x_1 - \sqrt{2}x_2 + x_3 = 0$$
$$x_2 - \sqrt{2}x_3 = 0$$

Solving the above eq^{n}s, the eigenvector is : $\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$

Normalizing the eigenvector, we get : $\frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$

For $\lambda = -\sqrt{2}$

$$(A + \sqrt{2}I)X = 0$$

$$\begin{bmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\sqrt{2}x_1 + x_2 = 0$$
$$x_1 + \sqrt{2}x_2 + x_3 = 0$$
$$x_2 + \sqrt{2}x_3 = 0$$

Solving the above eq^{n}s, the eigenvector is : $\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$

Normalizing the eigenvector, we get : $\frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$

The normalized eigenvectors are $1/\sqrt{2}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $1/2\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$ and $1/2\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$ corresponding to the eigenvalues 0, $\sqrt{2}$ and $-\sqrt{2}$ respectively.

b) $|a_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}$; $|a_2\rangle = \frac{1}{2}\begin{bmatrix}1\\\sqrt{2}\\1\end{bmatrix}$; $|a_3\rangle = \frac{1}{2}\begin{bmatrix}1\\-\sqrt{2}\\1\end{bmatrix}$

Now, $\langle a_1| = \frac{1}{\sqrt{2}}[1\ 0\ -1]$; $\langle a_2| = \frac{1}{2}[1\ \sqrt{2}\ 1]$; $\langle a_3| = \frac{1}{2}[1\ -\sqrt{2}\ 1]$

$$\langle a_1|a_2\rangle = \frac{1}{\sqrt{2}}[1\ 0\ -1]\frac{1}{2}\begin{bmatrix}1\\\sqrt{2}\\1\end{bmatrix} = \frac{1}{2\sqrt{2}}(1+0-1) = 0$$

$$\langle a_2|a_3\rangle = \frac{1}{2}[1\ \sqrt{2}\ 1]\frac{1}{2}\begin{bmatrix}1\\-\sqrt{2}\\1\end{bmatrix} = \frac{1}{4}(1-2+1) = 0$$

$$\langle a_3|a_1\rangle = \frac{1}{2}[1\ -\sqrt{2}\ 1]\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix} = \frac{1}{2\sqrt{2}}(1+0-1) = 0$$

Also, $$\langle a_1|a_1\rangle = \frac{1}{\sqrt{2}}[1\ 0\ -1]\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix} = \frac{1}{2}(1+0+1) = 1$$

$$\langle a_2|a_2\rangle = \frac{1}{2}[1\ \sqrt{2}\ 1]\frac{1}{2}\begin{bmatrix}1\\\sqrt{2}\\1\end{bmatrix} = \frac{1}{4}[1+2+1] = 1$$

$$\langle a_3|a_3\rangle = \frac{1}{2}[1\ -\sqrt{2}\ 1]\frac{1}{2}\begin{bmatrix}1\\-\sqrt{2}\\1\end{bmatrix} = \frac{1}{4}[1+2+1] = 1$$

Hence, $$\langle a_i|a_k\rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$\Rightarrow$ $\langle a_j|a_k\rangle = \delta_{jk}$ where δ_{jk} is entry of identity matrix

Also,
$$|a_1\rangle\langle a_1| = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}_{3\times1}\frac{1}{\sqrt{2}}[1\ 0\ -1]_{1\times3} = \frac{1}{2}\begin{bmatrix}1&0&-1\\0&0&0\\-1&0&1\end{bmatrix}$$

$$|a_2\rangle\langle a_2| = \frac{1}{2}\begin{bmatrix}1\\\sqrt{2}\\1\end{bmatrix}\frac{1}{2}[1\ \sqrt{2}\ 1] = \frac{1}{4}\begin{bmatrix}1&\sqrt{2}&1\\\sqrt{2}&2&\sqrt{2}\\1&\sqrt{2}&1\end{bmatrix}$$

$$|a_3\rangle\langle a_3| = \frac{1}{2}\begin{bmatrix}1\\-\sqrt{2}\\1\end{bmatrix}\frac{1}{2}[1\ -\sqrt{2}\ 1] = \frac{1}{4}\begin{bmatrix}1&-\sqrt{2}&1\\-\sqrt{2}&2&-\sqrt{2}\\1&-\sqrt{2}&1\end{bmatrix}$$

$$\sum_{j=1}^{3}|a_j\rangle\langle a_j| = |a_1\rangle\langle a_1| + |a_2\rangle\langle a_2| + |a_3\rangle\langle a_3|$$
$$= \frac{1}{2}\begin{bmatrix}1&0&-1\\0&0&0\\-1&0&1\end{bmatrix} + \frac{1}{4}\begin{bmatrix}1&\sqrt{2}&1\\\sqrt{2}&2&\sqrt{2}\\1&\sqrt{2}&1\end{bmatrix} + \frac{1}{4}\begin{bmatrix}1&-\sqrt{2}&1\\-\sqrt{2}&2&-\sqrt{2}\\1&-\sqrt{2}&1\end{bmatrix}$$
$$= \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = I$$

Hence $|a_1\rangle, |a_2\rangle, |a_3\rangle$ forms orthonormal & complete basis

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c) $|b_1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$; $|b_2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$; $|b_3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Also,
$|a_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$; $|a_2\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$; $|a_3\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$

Suppose U is the transformation matrix for the transformation from basis $\{|a\rangle\}$ to basis $\{|b\rangle\}$

Let there be a matrix X whose columns are $\{|a_i\rangle\}$ and a matrix Y whose columns are $\{|b_i\rangle\}$

Then,
$$X = UY$$
$$\Rightarrow XY^{-1} = UYY^{-1}$$
$$\Rightarrow U = XY^{-1}$$

where
$$X = \begin{bmatrix} 1/\sqrt{2} & 1/2 & 1/2 \\ 0 & 1/2\sqrt{2} & -1/2\sqrt{2} \\ -1/\sqrt{2} & 1/2 & 1/2 \end{bmatrix} \quad Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{cof}(Y) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{adj}(Y) = [\text{cof}(Y)]^T = \text{cof}(Y)$$

$$Y^{-1} = \frac{\text{adj}(Y)}{|Y|} = Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence, $U = XY^{-1}$
$$\Rightarrow U = XI$$
$$\Rightarrow U = X = \begin{bmatrix} 1/\sqrt{2} & 1/2 & 1/2 \\ 0 & 1/2\sqrt{2} & -1/2\sqrt{2} \\ -1/\sqrt{2} & 1/2 & 1/2 \end{bmatrix}$$

Q2) $|\psi\rangle = |\phi_1\rangle + |\phi_2\rangle$ and $|\chi\rangle = |\phi_1\rangle - |\phi_2\rangle$ → given

To show: $\langle\psi|\psi\rangle + \langle\chi|\chi\rangle = 2\langle\phi_1|\phi_1\rangle + 2\langle\phi_2|\phi_2\rangle$

$$\langle\psi|\psi\rangle = (\langle\phi_1| + \langle\phi_2|)(|\phi_1\rangle + |\phi_2\rangle)$$
$$= (\langle\phi_1|\phi_1\rangle + \langle\phi_1|\phi_2\rangle + \langle\phi_2|\phi_1\rangle + \langle\phi_2|\phi_2\rangle)$$

$$\langle\chi|\chi\rangle = (\langle\phi_1| - \langle\phi_2|)(|\phi_1\rangle - |\phi_2\rangle)$$
$$= (\langle\phi_1|\phi_1\rangle - \langle\phi_1|\phi_2\rangle - \langle\phi_2|\phi_1\rangle + \langle\phi_2|\phi_2\rangle$$

$$\Rightarrow \langle\psi|\psi\rangle + \langle\chi|\chi\rangle = (\langle\phi_1|\phi_1\rangle + \langle\phi_1|\phi_2\rangle + \langle\phi_2|\phi_1\rangle + \langle\phi_2|\phi_2\rangle)$$
$$+$$
$$(\langle\phi_1|\phi_1\rangle - \langle\phi_1|\phi_2\rangle - \langle\phi_2|\phi_1\rangle + \langle\phi_2|\phi_2\rangle$$

$$\Rightarrow \langle\psi|\psi\rangle + \langle\chi|\chi\rangle = 2(\langle\phi_1|\phi_1\rangle) + 2(\langle\phi_2|\phi_2\rangle)$$

Since, $|\phi_1\rangle$ and $\langle\phi_2\rangle$ are not orthonormal,
Hence, $\langle\phi_1|\phi_1\rangle \neq 1$ and $\langle\phi_2|\phi_2\rangle \neq 1$

$$\Rightarrow \langle\psi|\psi\rangle + \langle\chi|\chi\rangle = 2\langle\phi_1|\phi_1\rangle + 2\langle\phi_2|\phi_2\rangle$$

Hence, proved.

Q3) $|\psi\rangle = \frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{5}}|\phi_2\rangle + \frac{1}{\sqrt{10}}|\phi_3\rangle$

Since, $|\phi_1\rangle$, $|\phi_2\rangle$ and $|\phi_3\rangle$ are orthonormal eigenstates, hence:

$$\langle\phi_1|\phi_1\rangle = \langle\phi_2|\phi_2\rangle = \langle\phi_3|\phi_3\rangle = 1 \quad -(1)$$

and,

$$\langle\phi_1|\phi_2\rangle = \langle\phi_2|\phi_3\rangle = \langle\phi_3|\phi_1\rangle = 0 \quad -(2)$$

Also, for operator $\hat{B}$, $\hat{B}|\phi_n\rangle = n^2|\phi_n\rangle$ - (3)

a) $$\langle\psi|\psi\rangle = \left(\frac{1}{\sqrt{2}}\langle\phi_1| + \frac{1}{\sqrt{5}}\langle\phi_2| + \frac{1}{\sqrt{10}}\langle\phi_3|\right)\left(\frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{5}}|\phi_2\rangle + \frac{1}{\sqrt{10}}|\phi_3\rangle\right)$$

$$= \frac{1}{2}\langle\phi_1|\phi_1\rangle + \frac{1}{5}\langle\phi_2|\phi_2\rangle + \frac{1}{10}\langle\phi_3|\phi_3\rangle$$

(by eq^n (2))

$$\Rightarrow \langle\phi|\psi\rangle = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} \quad \text{(by eq}^n\text{ (1))}$$

$$= \frac{5+2+1}{10}$$

$$\Rightarrow \langle\psi|\psi\rangle = \frac{4}{5}$$

Since $\langle\psi|\psi\rangle \neq 1$, hence $|\psi\rangle$ is not normalized.

b) Expectation value $\langle\hat{B}\rangle = \frac{\langle\psi|\hat{B}|\psi\rangle}{\langle\psi|\psi\rangle}$

$$\langle\psi|\hat{B}|\psi\rangle = (\langle\psi|)(\hat{B}|\psi\rangle)$$

$$= \left(\frac{1}{\sqrt{2}}\langle\phi_1| + \frac{1}{\sqrt{5}}\langle\phi_2| + \frac{1}{\sqrt{10}}\langle\phi_3|\right)\hat{B}\left(\frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{5}}|\phi_2\rangle + \frac{1}{\sqrt{10}}|\phi_3\rangle\right)$$

$$= \left(\frac{1}{\sqrt{2}}\langle\phi_1| + \frac{1}{\sqrt{5}}\langle\phi_2| + \frac{1}{\sqrt{10}}\langle\phi_3|\right)\left(\frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{4}{\sqrt{5}}|\phi_2\rangle + \frac{9}{\sqrt{10}}|\phi_3\rangle\right)$$

(by eq^n (3))

$$\Rightarrow \langle \psi | \hat{B} | \psi \rangle = \frac{1}{2} + \frac{4}{5} + \frac{9}{10} \quad (\text{by eq}^n \; 1 \;\&\; \text{eq}^n \; 2)$$

$$= \frac{5+8+9}{10}$$

$$= \frac{22}{10}$$

expectation value of $\hat{B}$ for the state $|\psi\rangle$ is

$$\langle \hat{B} \rangle = \frac{\langle \psi | \hat{B} | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{5}{4} \; \frac{22}{10} = \frac{11}{4}$$

c) By eq^n 3,

$$\hat{B} | \phi_n \rangle = n^2 | \phi_n \rangle$$

$$\hat{B} \hat{B} | \phi_n \rangle = n^2 \hat{B} | \phi_n \rangle$$

$$(\hat{B})^2 | \phi_n \rangle = n^2 (n^2 | \phi_n \rangle) \quad (\text{by eq}^n \; 3)$$

$$\Rightarrow \hat{B}^2 | \phi_n \rangle = n^4 | \phi_n \rangle \quad - \; 4$$

$$\langle \psi | \hat{B}^2 | \psi \rangle = \left(\frac{1}{\sqrt{2}} \langle \phi_1 | + \frac{1}{\sqrt{5}} \langle \phi_2 | + \frac{1}{\sqrt{10}} \langle \phi_3 | \right) \hat{B}^2 \left(\frac{1}{\sqrt{2}} | \phi_1 \rangle + \frac{1}{\sqrt{5}} | \phi_2 \rangle + \frac{1}{\sqrt{10}} | \phi_3 \rangle \right)$$

$$= \left(\frac{1}{\sqrt{2}} \langle \phi_1 | + \frac{1}{\sqrt{5}} \langle \phi_2 | + \frac{1}{\sqrt{10}} \langle \phi_3 | \right) \left(\frac{1}{\sqrt{2}} | \phi_1 \rangle + \frac{16}{\sqrt{5}} | \phi_2 \rangle + \frac{81}{\sqrt{10}} | \phi_3 \rangle \right)$$

$$(\text{by eq}^n \; 4)$$

$$\Rightarrow \langle \psi | \hat{B}^2 | \psi \rangle = \frac{1}{2} + \frac{16}{5} + \frac{81}{10} \quad (\text{by eq}^n \; 1 \;\&\; \text{eq}^n \; 2)$$

$$= \frac{5+32+81}{10} = \frac{118}{10} = \frac{59}{5}$$

expectation value of $\hat{B}^2$ for the state $|\psi\rangle$ is

$$\langle \hat{B}^2 \rangle = \frac{\langle \psi | \hat{B}^2 | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{5}{4} \; \frac{59}{5} = \frac{59}{4}$$

24) Given: A Hermitian operator $\hat{A}$ such that:

$$\hat{A}^{+} = \hat{A}$$

To prove: All eigenvalues are real and the eigenvectors corresponding to different eigenvalues are orthogonal.

Proof: $\hat{A}^{+} = \hat{A}$ — (1) (given)

Let λ be the eigenvalue corresponding to the eigenvector x such that:

$$Ax = \lambda x \quad -(2)$$

Taking the conjugate transpose on both sides of the above eqn, we get:

$$(Ax)^{+} = (\lambda x)^{+}$$

$$\Rightarrow x^{+} A^{+} = \lambda^{+} x^{+}$$

$$\Rightarrow x^{+} A = \lambda^{+} x^{+} \quad (\text{by eq}^n\ (1))$$

Multiplying by eigenvector x on the right side of the above eqn on both sides,

$$x^{+} A x = \lambda^{+} x^{+} x$$

$$\Rightarrow x^{+} \lambda x = \lambda^{+} x^{+} x \quad (\text{by eq}^n\ (2))$$

$$\Rightarrow \lambda x^{+} x = \lambda^{+} x^{+} x$$

$$\Rightarrow (\lambda - \lambda^{+}) x^{+} x = 0$$

Either $\lambda - \lambda^{+} = 0$ or $x^{+} x = 0$

But since x is a non-zero vector,

Hence $x^{+} x \neq 0$

$$\Rightarrow \lambda - \lambda^{+} = 0$$

$$\Rightarrow \lambda = \lambda^{+}$$

This shows that λ is real.

Hence the eigenvalues of a Hermitian operator are real.

Now,

suppose λ_1 and λ_2 are two distinct eigenvalues of the Hermitian operator having v_1 and v_2 as eigenvectors respectively.

$$\Rightarrow \quad Av_1 = \lambda_1 v_1 \quad -(3)$$

and

$$Av_2 = \lambda_2 v_2 \quad -(4)$$

Now, the inner product,

$$\langle v_1, Av_2\rangle = v_2^+ A^+ v_1 = v_2^+ A v_1 \quad (\text{by eq}^n (1))$$

$$= v_2^+ \lambda_1 v_1 \quad (\text{by eq}^n (3))$$

$$= \lambda_1 v_2^+ v_1$$

$$\Rightarrow \langle v_1, Av_2\rangle = \lambda_1 \langle v_1, v_2\rangle \quad -(5)$$

Also,

$$\langle v_1, Av_2\rangle = \langle v_1, \lambda_2 v_2\rangle \quad (\text{by eq}^n (4))$$

$$= \lambda_2^+ v_2^+ v_1$$

$$\langle v_1, Av_2\rangle = \lambda_2^+ \langle v_1, v_2\rangle \quad -(6)$$

By eq^n (5) and eq^n (6)

$$\lambda_1 \langle v_1, v_2\rangle = \lambda_2^+ \langle v_1, v_2\rangle$$

$$\Rightarrow \lambda_1 \langle v_1, v_2\rangle = \lambda_2 \langle v_1, v_2\rangle \quad \{\text{since the eigenvalues are real, hence } \lambda^+ = \lambda\}$$

$$\Rightarrow (\lambda_1 - \lambda_2)\langle v_1, v_2\rangle = 0$$

Either $\lambda_1 - \lambda_2 = 0$ or $\langle v_1, v_2\rangle = 0$

But $\lambda_1 \neq \lambda_2$ as they are distinct eigenvalues.

Hence $\langle v_1, v_2\rangle = 0$

$\Rightarrow$ v_1 and v_2 are orthogonal eigenvectors corresponding to eigenvalues λ_1 and λ_2 respectively.

Hence, proved.