Romica Raisinghani
2021101053

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Assignment - 2

Q 1) Pauli Matrices are Hermitian Matrices

$$\Rightarrow \sigma_1 = (\sigma_1)^+$$

Also, $\sigma_1\sigma_2 = i\sigma_3$ — (1)

$\sigma_2\sigma_3 = i\sigma_1$ — (2)

$\sigma_2\sigma_1 = -i\sigma_3$ — (3)

and $\sigma_i^2 = I$ as they are unitary

so,

$$X = \begin{bmatrix} \sigma_0 & \sigma_1 \\ -i\sigma_2 & \sigma_3 \end{bmatrix}$$

$$\Rightarrow X^+ = \begin{bmatrix} \sigma_0^+ & -(\sigma_2)^+(i) \\ \sigma_1^+ & \sigma_3^+ \end{bmatrix} \quad \left\{ \text{since } (k\ X)^+ = k\ X^+ \right.$$

$$= \begin{bmatrix} \sigma_0 & i\sigma_2 \\ \sigma_1 & \sigma_3 \end{bmatrix}$$

Now, $$XX^+ = \begin{bmatrix} \sigma_0 & \sigma_1 \\ -i\sigma_2 & \sigma_3 \end{bmatrix} \begin{bmatrix} \sigma_0 & i\sigma_2 \\ \sigma_1 & \sigma_3 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_0^2 + \sigma_1^2 & i\sigma_0\sigma_2 + \sigma_1\sigma_3 \\ -i\sigma_2\sigma_0 + \sigma_3\sigma_1 & \sigma_2^2 + \sigma_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2I_2 & i\sigma_2 + \sigma_1\sigma_3 \\ -i\sigma_2 + \sigma_3\sigma_1 & 2I_2 \end{bmatrix}$$

From eqn (2),

$\sigma_2\sigma_3 = i\sigma_1$

$\Rightarrow \sigma_2\sigma_3\sigma_1 = iI_2$

$\Rightarrow \sigma_3\sigma_1 = i\sigma_2$

$\Rightarrow \sigma_1\sigma_3 = -i\sigma_2$

$$\Rightarrow \begin{bmatrix} 2I_2 & 0 \\ 0 & 2I_2 \end{bmatrix} = 2I_4$$

$\Rightarrow$ The given matrix is not unitary

Q2)-

By the properties of Bloch sphere, coordinates of points are:

$$(x, y, z) = (\sin\theta\cos\beta, \sin\theta\sin\beta, \cos\theta)$$

where $0 \le \theta \le \pi$ and $0 \le \beta \le 2\pi$

Its opposite is $(-x, -y, -z)$

$$\Rightarrow \theta' = \pi - \theta$$

$$\beta' = 180 + \beta$$

Now,

$$|\psi\rangle = \begin{pmatrix} \cos\theta/2\cos\alpha + i\sin\frac{\theta}{2}\sin\alpha \\ \sin\frac{\theta}{2}\cos(\alpha-\beta) + i\sin\frac{\theta}{2}\sin(\alpha+\beta) \end{pmatrix}$$

The opposite vector $|\psi'\rangle$

$$\langle\psi|\psi'\rangle = \begin{pmatrix} \cos\theta/2\cos\alpha + i\cos\frac{\theta}{2}\sin\alpha \\ \sin\frac{\theta}{2}\cos(\alpha-\beta) + \sin\frac{\theta}{2}\sin(\alpha+\beta) \end{pmatrix} \psi'^{+}$$

$$= \left(\cos\frac{\theta}{2}\cos\alpha + i\cos\frac{\theta}{2}\sin\alpha\right)\left(\sin\frac{\theta}{2}\cos\alpha - i\sin\frac{\theta}{2}\sin\alpha\right)$$

$$+$$

$$\sin\frac{\theta}{2}\left(\cos(\alpha-\beta) + \sin(\alpha+\beta)\right) \cdot \cos\frac{\beta}{2}\left(-\cos(\beta+\alpha) + i\sin(\alpha+\beta)\right)$$

$$= \sin\frac{\theta}{2}\cos\frac{\theta}{2}\left[(\cos^2\beta + \sin^2\beta) - (\cos^2(\alpha+\beta) + \sin^2(\alpha-\beta)\right]$$

$= \sin\theta_1 \cos\theta_2 (1-1)$

$= 0$

Hence, they are orthogonal.

Q3 - $\rho = \frac{1}{4}(1-\epsilon)I_4 + \epsilon(|0\rangle \otimes |0\rangle)(\langle 0| \otimes \langle 0|)$

Now,

$|0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

and,

$\langle 0| \otimes \langle 0| = [1\ 0\ 0\ 0]$

So,

$\rho = \frac{1}{4}(1-\epsilon)I_4 + \epsilon \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Now, ρ is a diagonal matrix

Its eigenvectors are given by: $\frac{5-\epsilon}{4}$ and $\frac{1-\epsilon}{4}$

Now

for $\epsilon \in [0,1]$, they are non-negative and hence positive semidefinite.

Trace $= 4\left(\frac{1-\epsilon}{4}\right) + \epsilon = 1.$

It is diagonal and real and hence Hermitian matrix

$\Rightarrow$ All three criteria are fulfilled and it is a density matrix

Q4) $\langle a| = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}^T$, $\langle \beta| = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}^T$

$$\langle \alpha| \otimes \langle \beta| = \begin{pmatrix} \cos\alpha\cos\theta \\ \cos\alpha\sin\theta \\ \sin\alpha\cos\theta \\ \sin\alpha\sin\theta \end{pmatrix}^T$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$$= \frac{1}{\sqrt{2}}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right)$$

$$\Rightarrow \quad p = \left(\frac{\cos\alpha\sin\theta - \sin\alpha\cos\theta}{\sqrt{2}} \right)^2$$

$$\Rightarrow \quad p(\alpha,\theta) = \frac{\sin^2(\theta-\alpha)}{2}$$