Introduction to Quantum Information and Computation

LECTURE NOTES

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Lecture 1 - Keval Jain

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Postulates

1.1 Complete description of a physical system is given by its state represented by $|\psi\rangle \in \mathcal{H}$, where $\mathcal{H}$ is the Hilbert Space on complex numbers.

Additionally quantum states are normalised, i.e.

$$\langle\psi| |\psi\rangle = 1$$

1.2 **Observables** such as energy, spin are given by **Hermetian Operators** which take only real eigenvalues.

1.2.1 Let $d = dim\ \mathcal{H}$. Then we can denote any element of $\mathcal{H}$ as a linear combination of no more than d elements. Now, denote the set $\{\alpha_1, ..., \alpha_d\}$ as an orthonormal basis for $\mathcal{H}$. If $|x\rangle \in \mathcal{H}$, we can say that

$$|x\rangle = \sum_{i=1}^{d} c_i \cdot |\alpha_i\rangle$$

Since $\langle x| |x\rangle = 1$, we can conclude that

$$\sum_{i=1}^{d} |c_i|^2 = 1$$

A linear operator $\hat{O} : \mathcal{H}_1 \to \mathcal{H}_2$ is a hermitian operator if

$$\hat{O} = \hat{O}^\dagger$$

where $\hat{A}^\dagger$ is the hermitian adjoint of the linear operator $\hat{A}$ if it satisfies

$$\langle x, \hat{A}y\rangle \ = \ \langle \hat{A}^\dagger x, y\rangle$$

for any valid vectors x and y.

1.3 **Measurement** $\hat{M}$ corresponding to an observable $\hat{O}$ for any state $|\psi\rangle$, operates as follows:

$$\hat{M} |\psi\rangle \to |a_i\rangle$$

with outcome a_i, where $|\psi\rangle$ is a quantum state. We can write the observable $\hat{O}$ as follows:

$$\hat{O} = \sum_{i=1}^{d} a_i \cdot |a_i\rangle \langle a_i|$$

where $a_i, ..., a_d$ are the eigenvalues of $\hat{O}$ and $|a_i\rangle, ..., |a_d\rangle$ are the corresponding eigenvectors.

An observation:

$$\hat{O} |a_j\rangle = \sum_{i=1}^{d} a_i |a_i\rangle \langle a_i| \cdot |a_j\rangle \qquad = \sum_{i=1}^{d} a_i |a_i\rangle \cdot \delta_{ij} = a_i |a_i\rangle \tag{1}$$

$\hat{M} |a_i\rangle \to |a_i\rangle$ (guaranteed)

1.4 Evolution of quantum states is given by unitary transformation.

$U : \mathcal{H} \to \mathcal{H}$ is a unitary transformation if

$$U^\dagger U = U U^\dagger = \mathbb{I}$$

Lecture 3 - Romica Raisinghani

Operators

Suppose the current state is $|\phi\rangle$, then an operator can change its state to $|\phi\rangle'$ such that:

$$|\phi'\rangle = A\,|\phi\rangle$$

or

$$\langle\varphi'| = \langle\phi|\,A^\dagger$$

In order to check whether $|\phi\rangle'$ is a valid state, check whether:

$$\langle\phi'|\,\phi'\rangle = 1$$

Also,

$$\langle\phi|\,AA^\dagger\,|\phi\rangle = 1$$
$$\langle\phi|\,I\,|\phi\rangle = 1$$
$$\langle\phi|\,\phi\rangle = 1$$

Tensor Product

Suppose that H1 and H2 are two Hilbert spaces of dimension N1 and N2.

We can put these two Hilbert spaces together to construct a larger Hilbert space. We denote this larger space by H and use the tensor product operation symbol $\otimes$.So we write:

$$\mathrm{H} = \mathrm{H1} \otimes \mathrm{H2}$$

The dimension of the larger Hilbert space is the product of the dimensions of H1 and H2

$$\dim(H\ ) = N1N2$$

Let $|\phi\rangle\ \in$ H1 and $|\lambda\rangle \in$ H2 be two vectors that belong to the Hilbert spaces used to construct H. We can construct a vector $|\psi\rangle \in$ H using the tensor product in the following way:

$$|\psi\rangle = |\phi\rangle\ \otimes |\lambda\rangle$$

An example on how tensor product works on matrices is shown below:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes A = \begin{pmatrix} aA & bA \\ cA & dA \end{pmatrix}$$

Expectation value of an Operator

The expectation value of an operator is the mean or average value of that operator with respect to a given quantum state.
If a quantum state $|\psi\rangle$ is prepared many times, and we measure a given operator A each time, then the average of the measurement results is the expectation value of operator A which is given by:

$$\langle A\rangle = \langle\psi|A|\psi\rangle$$

Density Operator

1. A quantum state is represented by a density operator on a Hilbert space H.

* Here, a density operator maps elements in one Hilbert space to elements in same Hilbert space, that is:

$$\rho : \mathrm{H} \to \mathrm{H}$$

A density operator has the following three properties:

1. **Positive Semi-definite** : Eigenvalues are non-negative

$$\rho \geq 0$$

or,

$$\langle u|\rho|u\rangle \geq 0 \text{ for any state vector } |u\rangle$$

2. **Hermitian**: The density operator is Hermitian

$$\rho = \rho^\dagger$$

3. **Trace is 1**: Trace of density operator is 1

$$\mathrm{Tr}[\rho] = 1$$

Also, any Hermitian operator will have spectral decomposition:

$$\rho = \sum_i p_i \, |i\rangle \langle i|$$

where $|i\rangle \in$ H and $\{|i\rangle\}$ forms orthonormal basis.

Proving that the trace of density operator is 1:

$$\mathrm{Tr}[\textstyle\sum_i p_i \, |i\rangle \langle i|] = \sum_i p_i \, \mathrm{Tr}[|i\rangle \langle i|]$$
$$= \textstyle\sum_i \mathrm{Tr}[\langle i|i\rangle] \text{ (Cyclicity of Trace)}$$
$$= \textstyle\sum_i p_i = 1 \text{ (sum of probabilities equals 1)}$$

*** If the density operators for two quantum states is same, then the states are same**

For a pure state $|\psi\rangle$, the density operator is represented by $|\psi\rangle\langle\psi|$ and for a mixed state, the density operator is represented by $\sum_i p_i \; |\psi_i\rangle\langle\psi_i|$ where p_i are probability of the pure state $|\psi_i\rangle$.

For a pure state, we can say that,

$$\mathrm{Tr}[\rho^2] = 1$$

whereas for mixed state,

$$\mathrm{Tr}[\rho^2] < 1$$

Observables

In quantum theory, dynamical variables like position, momentum, angular momen- tum, and energy are called observables. This is because observables are things we measure in order to characterize the quantum state of a particle.

It turns out that an important postulate of quantum theory is that **there is an operator that corresponds to each physical observable**.

Expectation value of an observable $\hat{O}$ for a quantum state ρ is given by $\mathrm{Tr}[\hat{O}\ \rho]$

If $|rho$ is a pure state ($\rho := |\psi\rangle\langle\psi|$), then:

$$\mathrm{Tr}[\hat{O}\ |\psi\rangle|\psi\rangle] = \langle\psi|\hat{O}|\psi\rangle$$

If ρ is a mixed state, then:

$$\langle\hat{O}\rangle = \sum_i p_i \langle\psi_i|\hat{O}|\psi_i\rangle$$

Lecture 4 - Sriteja Reddy Pashya

Lecture 5 - Kyrylo Shyvam Kumar

Lecture 5

POVMs

Measurements can be represented by Positive Operator-valued Measure (POVM) denoted as $\{\Lambda^x\}$, wher x are possible outputs. The measure has positive semi-definite operators on a Hilbert space.

Alternatively, $\Lambda^x \geq 0$ (where Λ^x are eigenvalues)

Also, $\sum_{0}^{n-1} \Lambda^x = \mathbb{I}$ (where Λ^x are all matrices belonging to set of POVM)

Prob(x) $= Tr[\Lambda^x \rho]$, for a state ρ denotes the probability of outcome x, when $\{\Lambda^x\}$ is performed.

Projective Measurements

Projective measurement $\{\mathbb{P}_i\}$ are operators that satisfy:

- $\mathbb{P}_x = \mathbb{P}_x^2$
- $\mathbb{P}_i \mathbb{P}_j = \delta_{ij} \mathbb{P}_i$
- $\sum_{0}^{n-1} \mathbb{P}_i = \mathbb{I}$

Also, $\sum Tr[\Lambda^x \rho] = Tr[\sum \Lambda^x \rho] = Tr[\sum (\Lambda^x)\rho] = Tr[\mathbb{I}\rho] = 1$

$*$ where ρ is density operator

In other words, $\sum Prob(x) = 1$, which is known to be true.

This show that $Tr[\Lambda^x \rho]$ is perfectly valid value of probability.

Functions of Operators

Following holds for normal matrices, $\mathrm{A}^\dagger \mathrm{A} = \mathrm{A}\mathrm{A}^\dagger$:

- $f(A) = \sum f(a_i) \left|i\right\rangle \left\langle i\right|$,

where by Spectral Decomposition $A = \sum a_i \left|i\right\rangle \left\langle i\right|$.

Change in State

For an initial state ρ,

$$\rho \xrightarrow{\{\mathbb{P}_i\}} i$$

$*$ where i is outcome

The post-measurement state is:

$$\frac{(\mathbb{P}_i \rho \mathbb{P}_i)}{Tr[\mathbb{P}_i \rho \mathbb{P}_i]}$$

$where\ Tr[\mathbb{P}_i \rho \mathbb{P}_i] = Tr[\mathbb{P}\mathbb{P}_i \rho] = Tr[\mathbb{P}_i \rho]$ (by cyclicity of trace)

So, $Prob(i) = Tr[\mathbb{P}\mathbb{P}_i \rho] = Tr[\mathbb{P}_i \rho]$

Also, for some changed states, ρ_i and ρ_j we have:

- $\rho_i \rho_j = \delta_{ij} \rho_i$
- $Tr[\rho_i \rho_j] = \delta_{ij}$

In other words, performing a measurement again will lead to the same outcome with $probability = 1$

Examples of projective measurement:

- $\{\, |0\rangle \langle 0| + |1\rangle \langle 1| \,, |2\rangle \langle 2| \,\}$ for dimenension 2.

But $\{1/2\, |0\rangle \langle 0| + |1\rangle \langle 1| \,, 1/2\, |0\rangle \langle 0| + |2\rangle \langle 2| \,\}$ is not a projection.

It is just a POVM as:

- Summation of all values $= \sum \Lambda_x = \mathbb{I}$.
- Eigenvalues are > 0. If we take Λ_x as Spectral Decomposition of subspace, the cofficients of orthogonal bases will be eigenvalues, which in this case are positive.

Transformation/Evolution

Quantum channel — Anything that transforms the quantum state.

They are completely positive and trace preserving. A completely positive map is mapping that sends positive elements to positive elements. This is required as density operator was positive semi-definite.

$$N_{A\to B}: \quad B\big(H_A\big) \to B\big(H_B\big)$$

where B is set of operators.

In the giving mapping, dimension of Hilbert space can change (both increase and decrease).

Lecture 6 - Jai Bhatnagar

Quantum Channel

$$\mathcal{N}_{A\to B} = \beta(\mathcal{H}_A) \to \beta(\mathcal{H}_B)$$

where,

$\mathcal{N}$ represents a Quantum Channel.

$\mathcal{N}$ is a super operator, i.e. it acts on other operators.

β is a bounded, trace class operators, i.e. the trace is finite.

Therefore, $\mathcal{N}_{A\to B}$ is trace-preserving.

$\mathcal{N}_{A\to B}$ is also a completely positive operator i.e. if:

$$X \geq 0$$

then :

$$\implies \mathcal{N}(X) \geq 0$$

where, $A \geq 0$ means eigenvalues of A are all positive.

Tensor Product of Hilbert Spaces

$$\mathcal{H}_A \otimes \mathcal{H}_B$$

$$= \{|i\rangle_A\}_i \otimes \{|j\rangle_B\}_j$$

$$= \{|i\rangle_A \otimes |j\rangle_B\}_{i,j}$$

where,

$|i\rangle_A$ is the standard basis of $\mathcal{H}_A$.

$|j\rangle_B$ is the standard basis of $\mathcal{H}_B$.

Composite Systems

1) Product State

If :

$$\rho_{AB} = \rho_A \otimes \rho_B$$

$\implies$ A and B are in a product state.

$\implies$ A and B are mutually independent.

2) Separable State

$$\rho_{AB} = \sum_x p_x(\rho_A^x \otimes \rho_B^x)$$

If such a decomposition exists, A and B are in a separable state.

Finding such a decomposition is a NP-hard problem.

3) Entangled State

If a decomposition does not exist, A and B are in a entangled state.

$$\Phi_{AB} = \frac{1}{d}\sum_{i,j} |i\rangle_A \otimes |i\rangle_B \langle j|_A \otimes \rangle j|_B$$

where,

$$d = min\{dim(A), dim(B)\}$$

$$\Phi = |\psi\rangle\langle\psi|$$

where,

$$|\psi\rangle = \sum \frac{1}{\sqrt{d}} |i\rangle_A \otimes |i\rangle_B$$

$\implies$ Φ is a special entangled state, checking for this is enough to satisfy the conditions for a valid quantum channel :

$$Tr[\rho_A] = Tr[\mathcal{N}_{A\to C}(\Phi_A)]$$

$$id_B \otimes \mathcal{N}_{A\to C}(\Phi_{AB}) \geq 0$$

Kraus Operators

$$\mathcal{N}_{A\to C}(\mathcal{X}) = \sum_i K_i(\mathcal{X})K_i^\dagger$$

such that,

$$\sum_i K_i^\dagger K_i = \mathcal{I}$$

$\{K_i\}_i \to$ Kraus operators

This is a necessary and sufficient condition for a completely positive super-operator.

Lecture 7 - Hardik Sharma

Quantum Channels(Recap)

We have looked at quantum channels in the previous class and now we have the following results/conditions regarding them :

- $\mathcal{N}_{A\to B} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$
- Quantum channels are *trace preserving*, that is $Tr[\mathcal{S}_A] = Tr[\mathcal{S}_B]$. Where $\mathcal{S}_A, \mathcal{S}_B$ are the representation of the initial and final states in the Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$.
- Quantum channels are *completely positive*, which means that any extended system, of which the state is just a part, is still mapped to a valid state.
- The last property can be condensed into the equation : $\mathcal{N}_{A\to B}(\Phi_{RA}) \geq 0$. where Φ_{RA} is a special state known as the *maximal entangled state*.
- $\Phi_{RA} = |\phi\rangle\langle\phi|_{RA}$, where $|\phi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_R |i\rangle_A$. With $d = \min\{|R|, |A|\}$.
- In case, dimension of R is greater than A, then we choose arbitary d basis vectors for R.

Note: Technically we should check for all such R's, but mathematically it turns out that checking for the maximal entangled state is sufficient.

Schmidt Decomposition and Choi State

Schmidt Decomposition is represented as :

$$|\psi\rangle_{RA} = \sum_{i=0}^{d-1} \sqrt{p_i} |\varphi_i\rangle_R |\psi_i\rangle_A$$

Notice the extra $\sqrt{p_i}$ term in addition to general form, when all such p_i's are equal, we get the *maximal entangled state*'s ket component.

If the quantum channel $\mathcal{N}_{A\to B}$ is a valid quantum channel then $\mathcal{N}_{A\to B}(\Phi_{AB})$ is called the **choi state**.

Partial Trace

Assume we have a system composed of two states $\{\rho_A, \omega_E\}$. Further supppose, we don't have access to the ω_E part. Let a unitary transform $U_{A\to B}$ act on the system of states and result in the system $\{\rho_B, \omega_{E'}\}$.

Such systems can often arise in real life, since we may not always have access to a certain portion of the system. We have the relation :

$$\mathcal{N}_{A\to B}(\rho_A) = Tr_{E'}\left[U_{AE\to BE'}\Big(\rho_A \otimes \omega_E\Big)U^{\dagger}_{AE\to BE'}\right]$$

Since we have an unitary operator operating from system AE to BE', we have the important realtion $|AE| = |BE'|$. Notice how the above relation does not have any terms associated with E', that is the part about which we have no information.

This is called *partial trace.*

In general, to eliminate the contribution of a part B in a system AB, we have the following relation :

$$Tr_B[X_{AB}] = \sum_i \langle i|_B X_{AB}|i\rangle_B$$

Where $|i\rangle_B$ represents any orthonormal basis for B. Also, we can represent X_{AB} as :

$$X_{AB} = \sum \alpha_{\{i,j,k,l\}}|i\rangle\langle j|_A \otimes |k\rangle\langle l|_B$$

Suppose we have a bipartite state :

$$\Phi_{AB} = \frac{1}{2}\Big((|00\rangle\langle 00|)_{AB} + (|00\rangle\langle 11|)_{AB} + (|11\rangle\langle 00|)_{AB} + (|11\rangle\langle 11|)_{AB}\Big)$$

Note that $|00\rangle_{AB} = |0\rangle_A \otimes |0\rangle_B, \langle 00| = \langle 0|_A \otimes \langle 0|_B$

Now, if calculate the partial trace of the state with respect to B, we get :

$$Tr_B[\Phi_{AB}] = \frac{1}{2}(|0\rangle_A\langle 0|_A + |1\rangle_A\langle 1|_A)$$

Selective Operation

$$(\mathcal{M}_A \otimes \mathcal{N}_B)(\rho_{AB})$$

This can be interpreted as first the operator $\mathcal{M}$ acts on A, unaffecting B, then $\mathcal{N}_B$ acts on B leaving A unchanged. This operator is commutative, so the process can be thought of in any order.

Based on the above defintion, we have the following relation that establishes the relation between Quantum Channels, partial trace and selective operation.

$$Tr_A[\mathcal{M}_A \otimes \mathcal{N}_B] = \mathcal{N}_B(\rho_B), |\rho_B = Tr_A[\rho_{AB}]$$