Problem 1

First, we have to find the equations of state from dynamics of the system:

$$\begin{cases} F_m = c\dfrac{I^2}{1-y} \\ m\ddot{y} = -mg - f_v\dot{y} + F_m \\ V = RI + L\dot{I} \end{cases} \rightarrow \begin{cases} x_1 = y \\ x_2 = \dot{y} \\ x_3 = I \\ u = V \end{cases} \rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{1}{m}(-mg - f_v x_2 + \dfrac{c x_3^2}{1-x_1}) \\ \dot{x}_3 = \dfrac{u - Rx_3}{L} \end{cases}$$

Now, based on the parameters given in Table 1 from the project, we simplify the equations of states. (Consider that the parameter a is considered 5)

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -9.8 - 0.1904x_2 + 2.8571\dfrac{x_3^2}{1-x_1} \\ \dot{x}_3 = -\,250x_3 + 50u \\ \\ y = x_1 \end{cases}$$

The stationary points of the system are those in which $\dot{x} = 0$. Based on this information, we now begin to find those points.

$$\begin{cases} \dot{x}_1 = 0 \rightarrow x_2 = 0 \\ \dot{x}_2 = 0 \rightarrow -9.8 + 2.8571\dfrac{x_3^2}{1-x_1} = 0 \qquad\qquad * \\ y_d = \ 0.35 = x_1 \overset{*}{\rightarrow} \ x_3 = 1.4931 \\ \\ \dot{x}_3 = 0 \rightarrow -\,250x_3 + 50u = 0 \rightarrow u = 7.4655 \end{cases}$$

Now that we have found the stationary point, we use the Jacobian matrix to linearize the system of equations.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 15.0757 & -0.1904 & 13.1260 \\ 0 & 0 & -250 \end{bmatrix} \rightarrow \dot{X} = AX + \begin{bmatrix} 0 \\ 0 \\ 50 \end{bmatrix} u$$

Now we have our linearized system as follow:

$$\begin{cases} \dot{X} = AX + \begin{bmatrix} 0 \\ 0 \\ 50 \end{bmatrix} u \\ y = [1 \quad 0 \quad 0]X \end{cases}$$

To calculate the transfer function of the system we use the following equation:

$$H(s) = C(sI - A)^{-1}B + D = 50 \times \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s & -1 & 0 \\ -15.0757 & s + 0.1904 & -13.1260 \\ 0 & 0 & s + 250 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= 50 \times \frac{\begin{vmatrix} -1 & 0 \\ s + 0.1904 & -13.1260 \end{vmatrix}}{\begin{vmatrix} s & -1 & 0 \\ -15.0757 & s + 0.1904 & -13.1260 \\ 0 & 0 & s + 250 \end{vmatrix}}$$

$$= \frac{656.3}{s^3 + 250.1904s^2 + 32.5243s - 3768.9250}$$

$$= \frac{656.3}{(s + 250)(s + 3.9791)(s - 3.7887)}$$

Problem 2

We choose our desired closed-loop poles as follows:

$p_1^* = -5 + j5, \qquad p_2^* = -5 - j5, \qquad p_3^* = ?$

We also design the controller as:

$$G_C(s) = K\frac{(s - 3.7887)(s + 3.9791)}{s(s - a)}$$

This controller has a pole at zero to ensure zero error in step response. It also replaces the open-loop pole -3.9791 with a variable a, so that we can choose two conjugate poles for our system with a desired real part.

We should find p_3^*, K, and a, such that for $s \in \{p_1^*, p_2^*, p_3^*\}$:

$$1 + G_C(s)H(s) = 0$$

We have:

$$1 + G_C(s)H(s) = 1 + K\frac{656.3}{s(s - a)(s + 250)} = 0$$

$$\Rightarrow \frac{s(s - a)(s + 250)}{656.3} = -K$$

$$\Rightarrow \angle s + \angle(s - a) + \angle(s + 250) - \angle 656.3 = \pi + 2k\pi$$

By letting $a = -10.2083$, this criterion is met for $s \in \{p_1^*, p_2^*\}$. This gives, using $1 + G_C(s)H(s) = 0$:

$$K = 19.0620$$

These parameters give p_3^* as -250.2084.

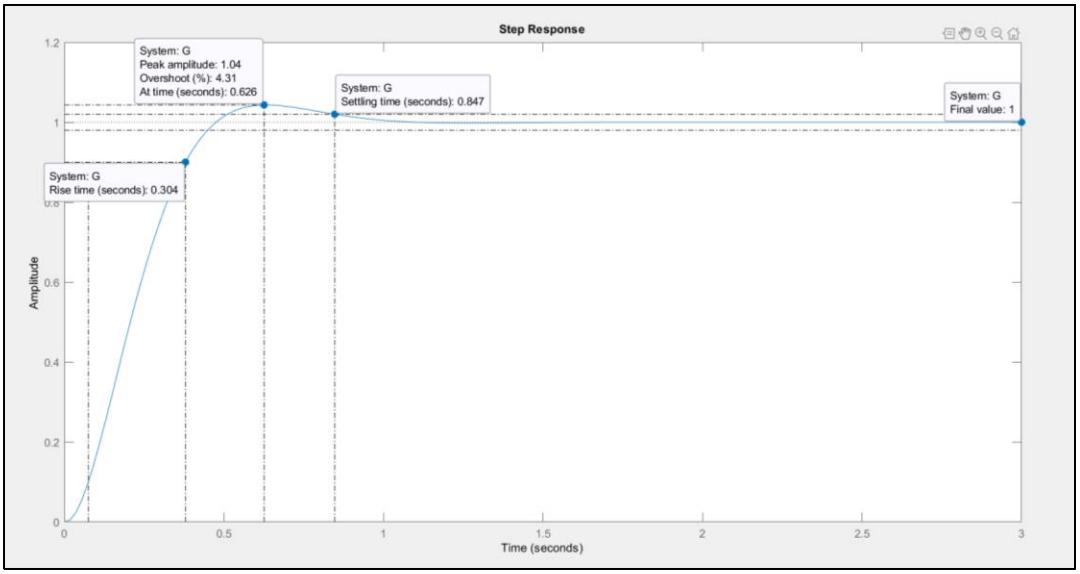
Therefore:

$$\boxed{G_C(s) = 19.0620\frac{(s - 3.7887)(s + 3.9791)}{s(s + 10.2083)}}$$

$$G(s) = \frac{G_C(s)H(s)}{1 + G_C(s)\,H(s)} = \frac{\dfrac{12510.3906}{s(s + 10.2083)(s + 250)}}{1 + \dfrac{12510.3906}{s(s + 10.2083)(s + 250)}}$$

$$= \boxed{\frac{12510.3906}{s^3 + 260.2083s^2 + 2552.075s + 12510.3906}}$$

The step response, according to MATLAB:



$$p_{1,2}^{*} = -5 \pm 5j = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}\ (p_3^{*}\ is\ an\ insignificant\ open-loop\ pole)$$

$$\Rightarrow \begin{cases} \zeta = \dfrac{1}{\sqrt{2}} \\ \omega_n = 5\sqrt{2} \\ \omega_d = \omega_n\sqrt{1-\zeta^2} = 5 \end{cases}$$

$$\begin{cases} t_p = 0.626 \\ \dfrac{\pi}{\omega_d} = 0.628 \end{cases}, \qquad \begin{cases} O.S. = 4.31\% \\ \exp\left(-\dfrac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) = 0.0432 = 4.32\% \end{cases}, \qquad \begin{cases} t_s = 0.847 \\ \dfrac{4}{\zeta\omega_n} = 0.800 \end{cases}$$

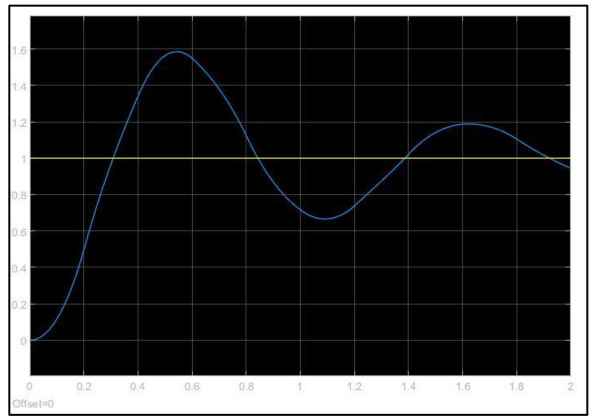
Problem 3

We use the Simulink model `vrmaglev.slx`, and `P3.m` for defining constants. The blue graph shows the output, and the yellow one is for the step input.

$$\begin{cases} G_{C,DM,i}(z) = \overline{K}_i \dfrac{(z - e^{3.7887T_i})(z - e^{-3.9791T_i})}{(z-1)(z - e^{-10.2083T_i})} \\ G_{C,DM,i}(-1) = G_C(\infty) = 19.0620 \end{cases}$$

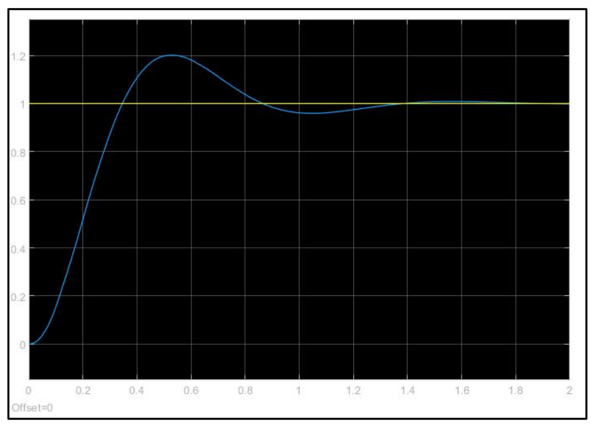
$T_1 = 0.2$

$$G_{C,DM,1}(z) = 9.4722 \frac{(z - 2.1334)(z - 0.4512)}{(z-1)(z - 0.1298)}$$



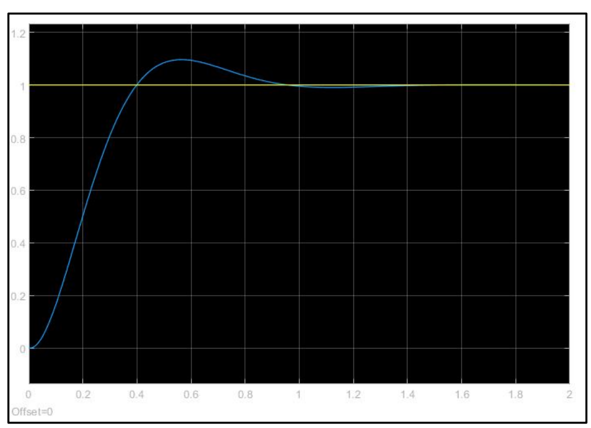
$T_2 = 0.1$

$$G_{C,DM,2}(z) = 12.6073 \frac{(z - 1.4606)(z - 0.6717)}{(z-1)(z - 0.3603)}$$



$T_3 = 0.05$

$$G_{C,DM,3}(z) = 15.1811 \frac{(z - 1.2086)(z - 0.8196)}{(z-1)(z - 0.6002)}$$



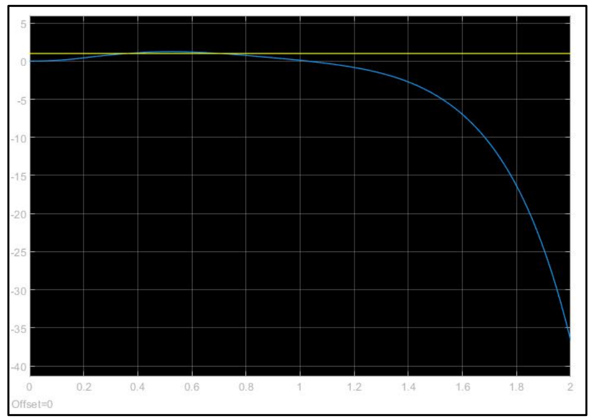
$$G_{C,DB,i}(z) = G_C\left(\frac{2}{T_i}\frac{z-1}{z+1}\right) = 19.0620\frac{\big(2(z-1) - 3.7887T_i(z+1)\big)\big(2(z-1) + 3.9791T_i(z+1)\big)}{2(z-1)\big(2(z-1) + 10.2083T_i(z+1)\big)}$$

$$\Rightarrow \begin{cases} G_{C,DB,i}(z) = 19.0620A_i\dfrac{(z+C_{i1})(z+C_{i2})}{(z-1)(z+C_{i3})} \\ A_i = \dfrac{(2-3.7887T_i)(2+3.9791T_i)}{2(2+10.2083T_i)} \\ C_{i1} = \dfrac{-2-3.7887T_i}{2-3.7887T_i}, \qquad C_{i2} = \dfrac{-2+3.9791T_i}{2+3.9791T_i}, \qquad C_{i3} = \dfrac{-2+10.2083T_i}{2+10.2083T_i} \end{cases}$$

The BLT-based designs suffer from numerical instability, as seen in the following figures.

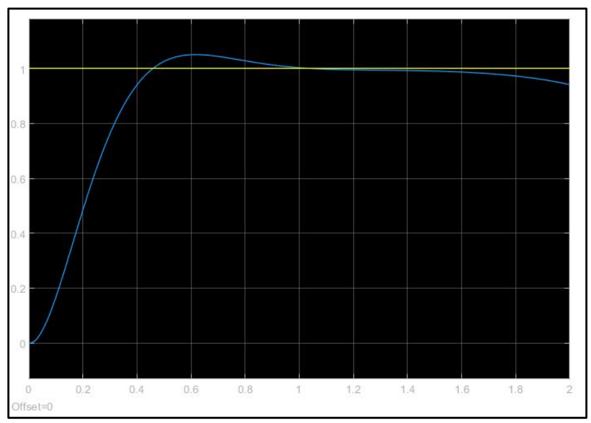
$T_1 = 0.2$

$$G_{C,DB,1}(z) = 8.1903\frac{(z-2.2199)(z-0.4307)}{(z-1)(z+0.0103)}$$



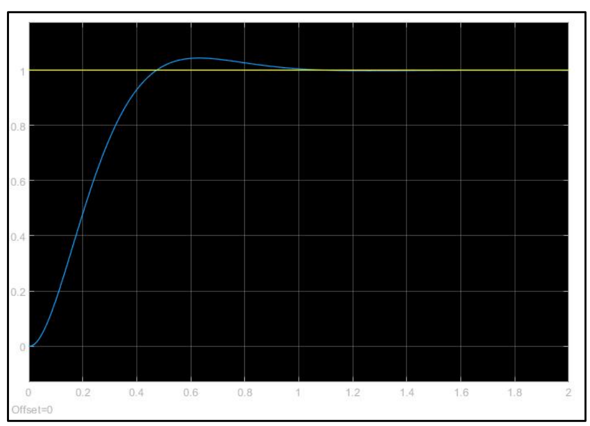
$T_2 = 0.01$

$$G_{C,DB,2}(z) = 18.1467\frac{(z-1.0386)(z-0.9610)}{(z-1)(z-0.9029)}$$



$T_3 = 0.001$

$$G_{C,DB,3}(z) = 18.9669\frac{(z-1.0038)(z-0.9960)}{(z-1)(z-0.9898)}$$



We set $T = 0.001$ from now on, and measure the transient and steady state properties of the systems. With this new T, we have:

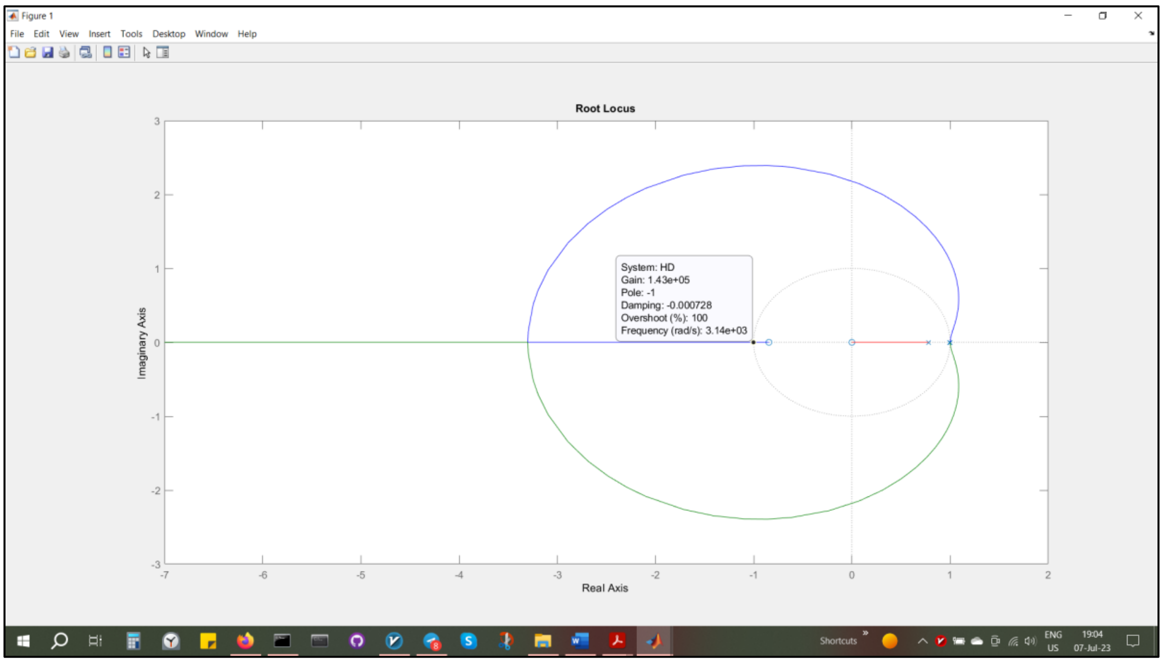
Type	MZT	BLT	Continuous
Formula	$G_{C,DM}(z) = 18.9669\frac{(z-1.0038)(z-0.9960)}{(z-1)(z-0.9898)}$	$G_{C,DB}(z) = 18.9669\frac{(z-1.0038)(z-0.9960)}{(z-1)(z-0.9898)}$	-
Screenshot			-
t_p	$0.640\ [s]$	$0.640\ [s]$	$0.628\ [s]$
$O.S.$	4.37%	4.37%	4.32%
t_s	$0.935\ [s]$	$0.935\ [s]$	$0.800\ [s]$
Numerical instability time	$> 5\ [s]$	$2.750\ [s]$	-

Problem 4

$$H(s) = \frac{656.3}{(s+250)(s+3.9791)(s-3.7887)} = \frac{0.0105}{s+250} + \frac{-0.3434}{s+3.9791} + \frac{0.3329}{s-3.7887}$$

$$\Rightarrow H_D(z) = \frac{0.0105}{1-0.7788z^{-1}} - \frac{0.3434}{1-0.9960z^{-1}} + \frac{0.3329}{1-1.0038z^{-1}}$$

Root locus: stability in $k \geq 1.43 \times 10^5$



Bode plot: Bandwidth = 2.5027 $[rad/s]$

Problem 7

For designing the dead-beat controller for our digital system first we need to find the z-transform of our continues system from the transfer function which was calculated in part one:

$$G(s) = \frac{656.3}{(s+250)(s+3.9791)(s-3.7887)} \rightarrow \mathbb{Z}\{G(s)\} = \mathbb{Z}\left\{\frac{0.0105}{(s+250)}\right\} +$$

$$\mathbb{Z}\left\{\frac{-0.3434}{(s+3.9791)}\right\} + \mathbb{Z}\left\{\frac{0.3329}{(s-3.7887)}\right\} = \frac{0.0105}{1-e^{-0.25}z^{-1}} - \frac{0.3434}{1-e^{-0.0039}z^{-1}} +$$

$$\frac{0.3329}{1-e^{0.0037}z^{-1}} \rightarrow \boxed{G(z) = \frac{0.0002481z^2 + 0.00032z}{z^3 - 2.7786z^2 + 2.5572z - 0.7786}}$$

1- As we can see, G(z) starts with z^{-1}, so f(z) should also start with z^{-1}. We consider f(z) as follow (because of condition 2,3 and 4, we need to consider 3 terms for f(z) and write it till the term z^{-3}):

$$f(z) = f_1 z^{-1} + f_2 z^{-2} + f_3 z^{-3}$$

2- f(z) should contain the unstable zeros of G(z):

$$f(z) = (z+1.2898)M(z) \rightarrow f_1 z^{-1} + f_2 z^{-2} + f_3 z^{-3} =$$

$$(z+1.2898)(m_2 z^{-2} + m_3 z^{-3}) \rightarrow f_1 = m_2, f_2 = 1.2898m_2 + m_3,$$

$$f_3 = 1.2898m_3 \rightarrow \boxed{f_2 = 1.2898f_1 + \frac{1}{1.2898}f_3}$$

3- 1 - f(z) should contain unstable poles of G(z):

$$1 - f(z) = (z-1.0037)Q(z) \rightarrow 1 - f_1 z^{-1} - f_2 z^{-2} - f_3 z^{-3}$$

$$= (z-1.0037)(z^{-1} + q_2 z^{-2} + q_3 z^{-3}) \rightarrow f_1 = 1.0037 - q_2,$$

$$f_2 = 1.0037q_2 - q_3,\ f_3 = 1.0037q_3 \rightarrow \boxed{f_2 = 1.0037(1.0037 - f_1) - \frac{f_3}{1.0037}}$$

4- 1 – f(z) should contain $(1 - z^{-1})$:

$$1 - f(z) = (1 - z^{-1})N(z) \rightarrow 1 - f_1 z^{-1} - f_2 z^{-2} - f_3 z^{-3} =$$

$$= (1 - z^{-1})(1 + n_1 z^{-1} + n_2 z^{-2}) \to f_1 = 1 - n_1, f_2 = n_1 - n_2, f_3 = n_2 \to$$

$$\boxed{f_2 = 1 - f_1 - f_3}$$

So, we have 3 equations and 3 parameters:

$$\begin{cases} f_2 = 1.2898 f_1 + \dfrac{1}{1.2898} f_3 \\ f_2 = 1.0037(1.0037 - f_1) - \dfrac{f_3}{1.0037} \\ f_2 = 1 - f_1 - f_3 \end{cases} \to \boxed{\begin{cases} f_1 = 1.1224 \\ f_2 = 0.7620 \\ f_3 = -0.8844 \end{cases}}$$

$$\boxed{\to\ f(z) = 1.1224 z^{-1} + 0.7620 z^{-2} - 0.8844 z^{-3}}$$

5- $N_G(z)U(z) \mathrel{?=} F(z)D_G(z)R(z)$

$(0.0002481z^2 + 0.00032z)(u_0 + u_1 z^{-1} + u_2 z^{-2} + \cdots) \mathrel{?=} (1.1224z^{-1} + 0.7620z^{-2} - 0.8844z^{-3})(1 - z^{-1})(z^3 - 2.7786z^2 + 2.5572z - 0.7786)$ ...

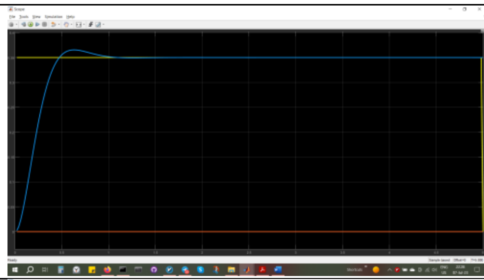
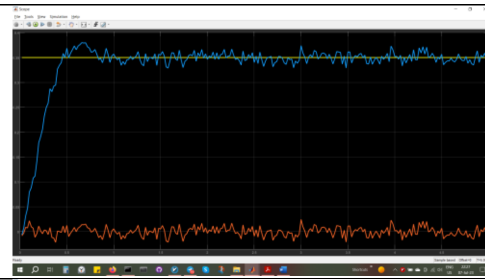
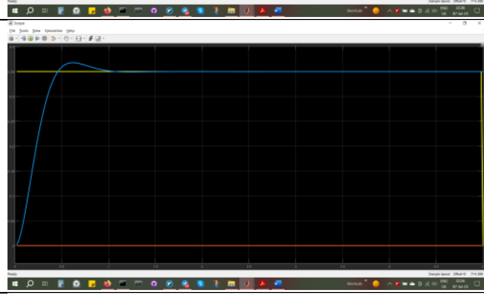
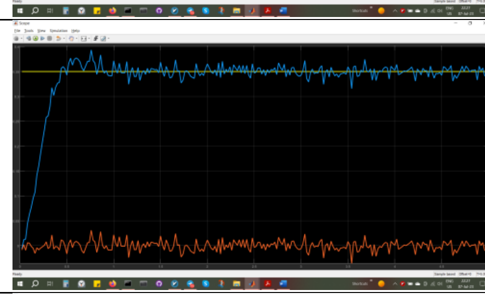
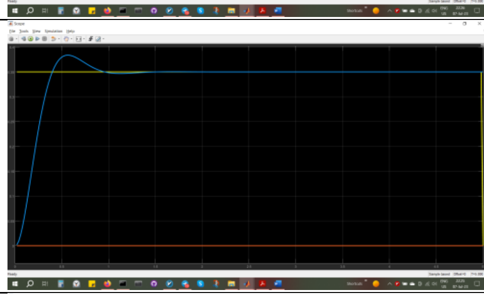
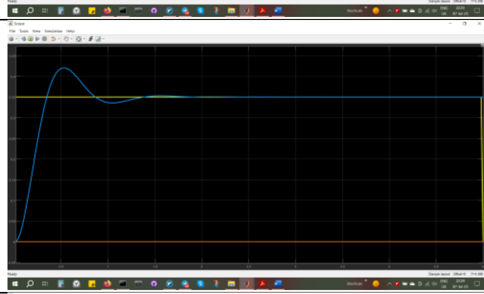
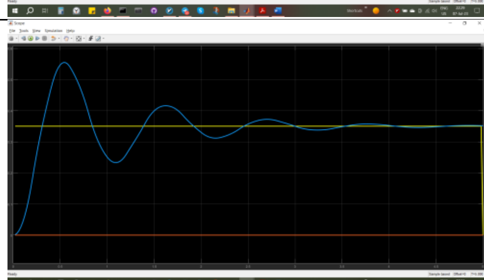
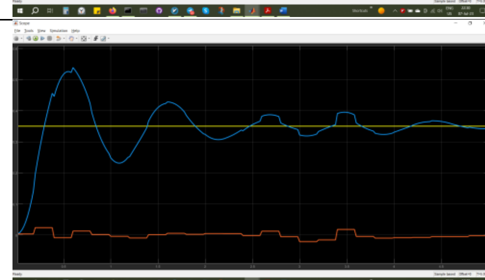
So now we can have our dead-beat controller:

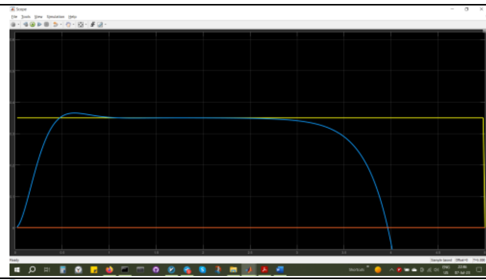
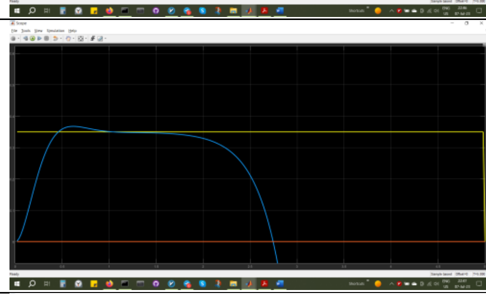
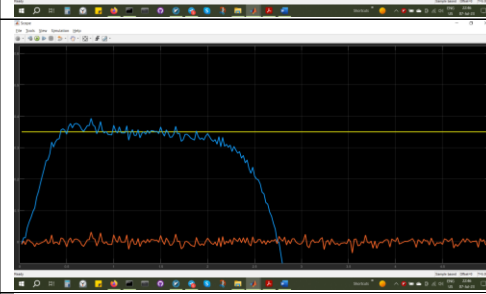
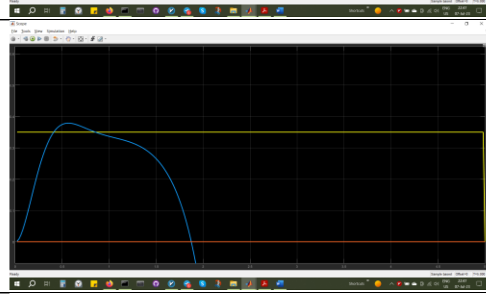
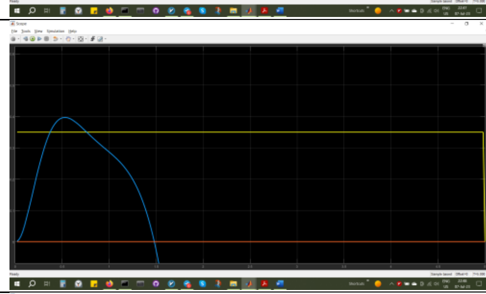
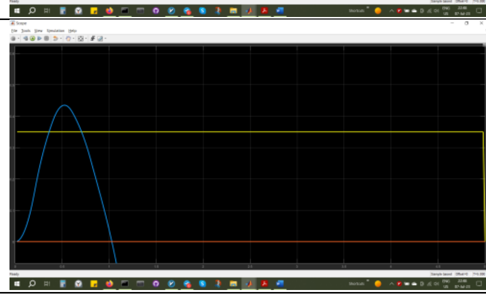
$$G_D(z) = \frac{f(z)}{\big(1 - f(z)\big)G(z)} \to G_D(z) = \frac{(1.1224z^{-1} + 0.7620z^{-2} - 0.8844z^{-3})(z^3 - 2.7786z^2 + 2.5572z - 0.7786)}{(1 - (1.1224z^{-1} + 0.7620z^{-2} - 0.8844z^{-3}))(0.0002481z^2 + 0.00032z)}$$

$$\boxed{\to G_D(z) = \frac{1.122\,z^5 - 2.357\,z^4 - 0.1314\,z^3 + 3.532\,z^2 - 2.855\,z + 0.6886}{0.00024\,z^5 + 0.00004\,z^4 - 0.00054\,z^3 - 0.00002\,z^2 + 0.000283\,z}}$$

Problem 8

We examine each controller under noisy and noiseless conditions. We use a band-limited white noise with sampling time T and “Noise power” equal to $10^{-4}T$.

Controller	MZT	
Noise	No	Yes
$T = 0.001\ [s]$		
$T = 0.01\ [s]$		
$T = 0.05\ [s]$		
$T = 0.1\ [s]$		
$T = 0.2\ [s]$		

Controller	BLT	
Noise	No	Yes
$T = 0.001\ [s]$		
$T = 0.01\ [s]$		
$T = 0.05\ [s]$		
$T = 0.1\ [s]$		
$T = 0.2\ [s]$		

It can be seen that for increasing T (and keeping the noise amplitude fixed), the output signal is less sensitive to the implemented noise.

Problem 9

we know that if the continuous equation of system would be as follow:

$$\begin{cases}\dot{X} = AX + Bu \\ y = CX + Du\end{cases}$$

Then, the discrete equation of system is as follow:

$$\begin{cases} x[k+1] = e^{AT}x[k] + \int_0^T e^{A\lambda}Bd\lambda\, u[k] \\ y[k] = Cx[k] + Du[k] \end{cases}$$

So, the equations of discrete system can be calculated as follow:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 15.0757 & -0.1904 & 13.1260 \\ 0 & 0 & -250 \end{bmatrix};B = \begin{bmatrix} 0 \\ 0 \\ 50 \end{bmatrix}, C = [1 \quad 0 \quad 0], D = 0$$

$$\rightarrow e^{AT} = \sum_{k=0}^{\infty} \frac{(AT)^k}{k!} = L^{-1}\{(sI - A)^{-1}\} = L^{-1}\left\{\begin{bmatrix} s & -1 & 0 \\ -15.0757 & s + 0.1904 & -13.1260 \\ 0 & 0 & s + 250 \end{bmatrix}^{-1}\right\}$$

$$\rightarrow e^{AT} =$$

$$L^{-1}\left\{\begin{bmatrix} \frac{s+0.1904}{s^2+0.1904s-15.0757} & \frac{1}{s^2+0.1904s-15.0757} & \frac{13.126}{(s+250)(s^2+0.1904s-15.075)} \\ \frac{15.0757}{s^2+0.1904s-15.0757} & \frac{1}{\frac{s^2-15.0757}{s}+0.1904} & \frac{13.1260}{0.1904s+(\frac{s^3+250(s^2-15.0757)}{s}+32.5243} \\ 0 & 0 & \frac{1}{s+250} \end{bmatrix}\right\} = e^{AT} =$$

$$\begin{bmatrix} 0.4877e^{-3.9791T} + 0.5122e^{3.7887T} & 0.1287e^{3.7887T} - 0.1287e^{-3.9791T} & 0.0002e^{-250T} - 0.0068e^{-3.9791T} + 0.0066e^{3.7887T} \\ 1.9402e^{3.7887T} - 1.9402e^{-3.9791T} & 0.4877e^{3.7887T} + 0.5122e^{-3.9791T} & -0.0525e^{-250T} - 0.02738e^{-3.9791T} + 0.0252e^{3.7887T} \\ 0 & 0 & e^{-250T} \end{bmatrix}$$

$$\boxed{G = e^{AT} = \begin{bmatrix} 0.9998 & 0.0009 & 0 \\ 0.01474 & 0.9997 & -0.0428 \\ 0 & 0 & 0.7788 \end{bmatrix}}$$

$$\int_0^T e^{A\lambda}Bd\lambda = \int_0^{0.001} L^{-1}\{(sI - A)^{-1}\}d\lambda B =$$

$$\int_0^{0.001} \begin{bmatrix} 0.4877e^{-3.9791\lambda} + 0.5122e^{3.7887\lambda} & 0.1287e^{3.7887\lambda} - 0.1287e^{-3.9791\lambda} & 0.0002e^{-250\lambda} - 0.0068e^{-3.9791\lambda} + 0.0066e^{3.7887\lambda} \\ 1.9402e^{3.7887\lambda} - 1.9402e^{-3.9791\lambda} & 0.4877e^{3.7887\lambda} + 0.5122e^{-3.9791\lambda} & -0.0525e^{-250\lambda} - 0.02738e^{-3.9791\lambda} + 0.0252e^{3.7887\lambda} \\ 0 & 0 & e^{-250\lambda} \end{bmatrix} d\lambda B$$

$$= \begin{bmatrix} 0.0009 & 0 & 0 \\ 0 & 0.0009 & -0.00048 \\ 0 & 0 & 0.0008 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 50 \end{bmatrix} \rightarrow \boxed{H = \int_0^T e^{A\lambda} B d\lambda = \begin{bmatrix} 0 \\ -0.024 \\ 0.04 \end{bmatrix}}$$

So, the discrete equation of states would be as follow:

$$\begin{cases} x[k+1] = \begin{bmatrix} 0.9998 & 0.0009 & 0 \\ 0.01474 & 0.9997 & -0.0428 \\ 0 & 0 & 0.7788 \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ -0.024 \\ 0.04 \end{bmatrix} u[k] \\ \\ y[k] = [1 \quad 0 \quad 0] x[k] \end{cases}$$

a. Controllability:

For checking controllability, we need to check if the controllability matrix which is as follow is full rank or not:

control matrix: $M = [H \quad GH \quad G^2H] = \begin{bmatrix} 0 & -0.00002 & 0 \\ -0.024 & -0.0257 & -0.027 \\ 0.04 & 0.0312 & 0.0243 \end{bmatrix}$

The rank of matrix M is 3 so this system is controllable.

b. Observability:

For checking observability, we need to check if the observability matrix which is as follow is full rank or not:

observe matrix: $N = \begin{bmatrix} C \\ CG \\ CG^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.9998 & 0.0009 & 0 \\ 0.9996 & 0.0018 & -0.00003 \end{bmatrix}$

The rank of matrix N, is 3 so this system is observable.

Problem 10

For designing the state feedback system, we need to find k that satisfy the following equations:

$$\begin{cases} \varphi(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = \lambda^3 - 2.7783\lambda^2 + 2.5567\lambda + 0.7784 \\ U = -Kx(k) \\ K = [-a_3 \quad -a_2 \quad -a_1]T^{-1} \\ T = MW \end{cases}$$

We know that M is the controllability matrix and W is defined as follow:

$$W = \begin{bmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2.5567 & -2.7783 & 1 \\ -2.7783 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \rightarrow$$

$$T = \begin{bmatrix} 0 & -0.00002 & 0 \\ -0.024 & -0.0257 & -0.027 \\ 0.04 & 0.0312 & 0.0243 \end{bmatrix} \begin{bmatrix} 2.5567 & -2.7783 & 1 \\ -2.7783 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \rightarrow$$

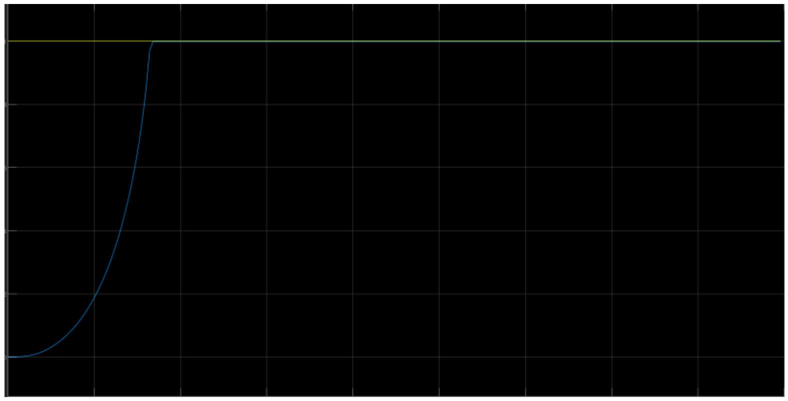
$$T = \begin{bmatrix} 0.00005 & -0.00002 & 0 \\ -0.0170 & 0.0410 & -0.0240 \\ 0.0400 & -0.0800 & 0.0400 \end{bmatrix} \rightarrow T^{-1} = \begin{bmatrix} 158390 & 490 & 290 \\ 158350 & 350 & 210 \\ 158320 & 200 & 150 \end{bmatrix} \rightarrow$$

$$K = [-0.7784 \quad -2.5567 \quad 2.7783\,] \begin{bmatrix} 158390 & 490 & 290 \\ 158350 & 350 & 210 \\ 158320 & 200 & 150 \end{bmatrix} \rightarrow$$

$$K = [-15644 \quad -384.2822 \quad -161.1118]$$

$$\boxed{u[k] = -[-15644 \quad -384.2822 \quad -161.1118]x[k]}$$

The implementation of this part is shown in part10.slx.



Problem 11

Now we consider that we do not have all the state available and we need to design an observer for our system:

$$\begin{cases} L = \varphi(G)N^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix} \\ K = L^T \\ u[k] = -Kx[k] \end{cases}$$

$$L = (G^3 - 2.7783G^2 + 2.5567G + 0.7784I)\begin{bmatrix}1 & 0 & 0\\0.9998 & 0.0009 & 0\\0.9996 & 0.0018 & -0.00003\end{bmatrix}^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix}$$
$$= \begin{bmatrix}0\\0.0037\\-51893\end{bmatrix}$$

Finally, now that we have designed our observer, we can control our system which is shown in part11.slx.