Chapter 1

1.1 Introduction to digital signals and systems, Properties and operations on digital signals.
1.2 Classification of signals, system, LTI system
1.3 Convolution in time domain (linear & circular), Correlation.
Self-Learning Topic: Correlation (Circular)

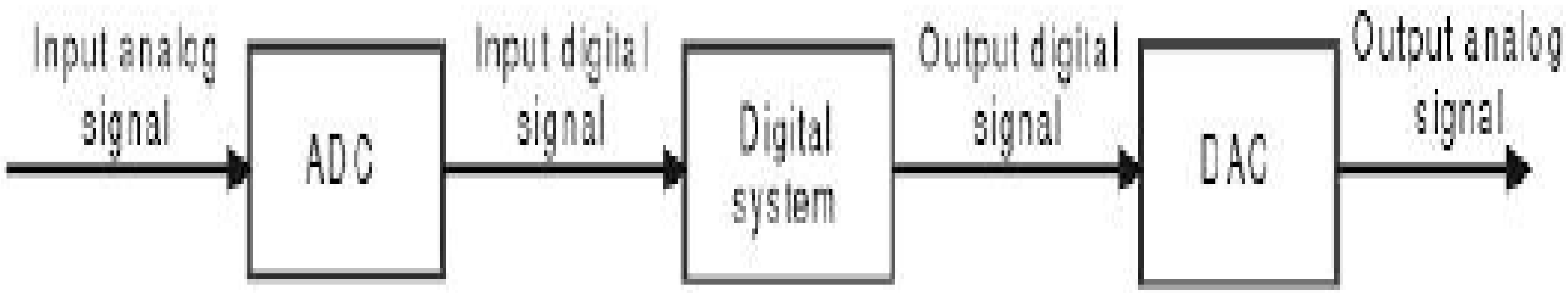


Fig 1.1 : Basic components of a DSP system.

Signal

- A signal is a pattern of variation that carry information.
- Signals are represented mathematically as a function of one or more independent variable.
- A picture is brightness as a function of two spatial variables, x and y.
- In this course signals involving a single independent variable, generally refer to as a time, t are considered. Although it may not represent time in specific application.
- A signal is a real-valued or scalar-valued function of an independent variable t.

Example of signals

- Electrical signals like voltages, current and EM field in circuit
- Acoustic signals like audio or speech signals (analog or digital)
- Video signals like intensity variation in an image
- Biological signal like sequence of bases in gene
- Noise which will be treated as unwanted signal

Classification of Signal

- Continuous-time and discrete-time signal
- Analog and digital signal (time and amplitude)

(1) Continuous-time signal：

(2) Discrete-time signal： Discrete variable→Continuous amplitude

Time-domain discrete signals

(3) Analog Signal: Continuous variable→Continuous amplitude

Speech, Television, Time-domain continuous signals

(4) Digital Signal： Discrete variables→Discrete amplitude

Quantized discrete-time signals

Signal Processing

Representation, transformation and manipulation of signals and the information they contain.

Signal operation include:

(1) Transform, filter, inspection, spectrum analysis;

(2) Modulation and coding;

(3) Analog Signal Processing;

(4) Digital Signal Processing.

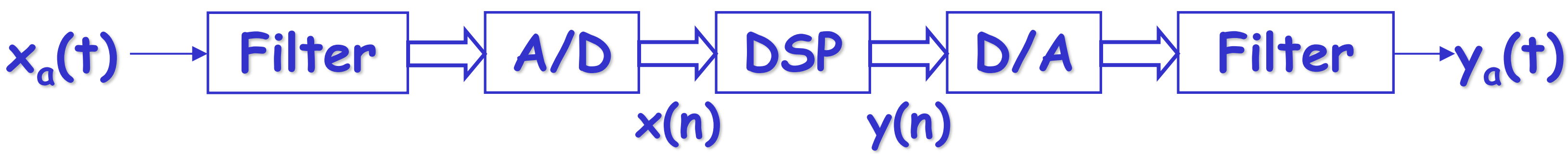
Computer, Semiconduct and Information Science

→1960's-1970's

Processing of analog signal with digital methods

(1) Digitalized process for analog signals

(2) Digital processing method



Feature of Digital System

Advantages

(1) High accuracy: Floating point-8,16,32,64 bits

(2) High reliability: VLSI (analog: drift, calibration)

(3) Flexible: DSP, Software, FPGA, VHDL

(4) Easy to integrate

(5) Deal with high dimensional signals

(6) Low costs: reusable, reconfigurable

(7) Data logging

(8) Adaptive capability

Objective of Digital Signal Processing

Digital Signals

Manipulation → **Digital filter**

Measurement

- **Spectrum analysis**

Digital Signals

- **Frequency division**
- **Disturbance attenuation**

(1) Selective of A/D → Signal representation - Sampling

(2) Manipulation and transform → feature extraction and analysis

(3) Noise process → Digital filter

Discrete Signal and Discrete Time Signal

The ***discrete signal*** is a function of a discrete independent variable. The independent variable is divided into uniform intervals and each interval is represented by an integer. The letter "n" is used to denote the independent variable. The discrete or digital signal is denoted by x(n).

Example :

$x(n) = \{ 2, 4, -1, 3, 3, 4 \}$

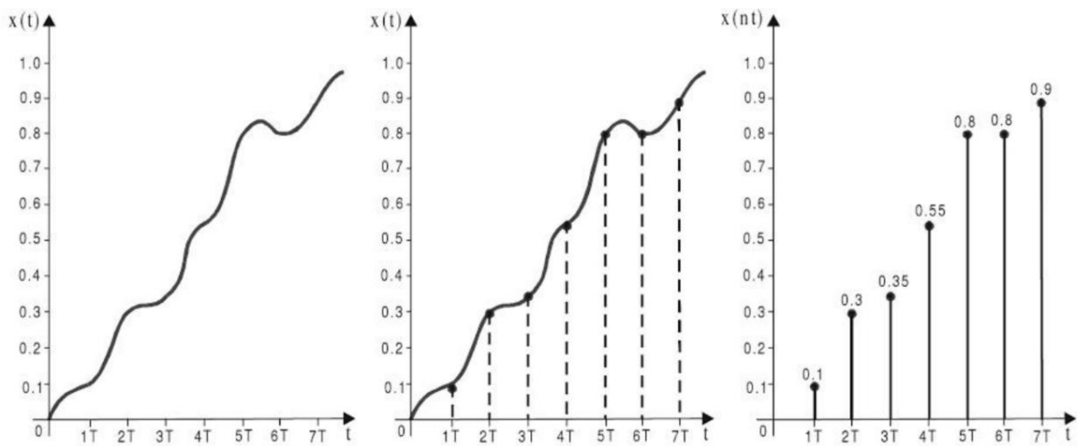
Here the discrete signal x(n) is defined for, $n = 0, 1, 2, 3, 4, 5$

\ $x(0) = 2$; $x(1) = 4$; $x(2) = -1$; $x(3) = 3$; $x(4) = 3$; $x(5) = 4$.

When the independent variable is time t, the discrete signal is called ***discrete time signal***. In discrete time signal, the time is divided uniformly using the relation t = nT, where T is the sampling time period. (The sampling time period is the inverse of sampling frequency). The discrete time signal is denoted by x(n) or x(nT).

Digital Signal

The ***digital signal*** is same as discrete signal except that the magnitude of the signal is quantized. The magnitude of the signal can take one of the values in a set of quantized values. Here quantization is necessary to represent the signal in binary codes.



When $t = 0$;	$x(t) = 0$	When $t = 4T$;	$x(t) = 0.55$
When $t = 1T$;	$x(t) = 0.1$	When $t = 5T$;	$x(t) = 0.8$
When $t = 2T$;	$x(t) = 0.3$	When $t = 6T$;	$x(t) = 0.8$
When $t = 3T$;	$x(t) = 0.35$	When $t = 7T$;	$x(t) = 0.9$

$$x(n) = \{\ 0,\ 0.1,\ 0.3,\ 0.35,\ 0.55,\ 0.8,\ 0.8,\ 0.9\ \}$$

Here the discrete signal $x(n)$ is defined for, $n = 0, 1, 2, 3, 4, 5, 6, 7$

Digital Signal

Quantization level (R = Range = 1)	Binary code	Range represented by quantization level for quantization by truncation
$0 \times \frac{R}{2^3} = 0 \times \frac{1}{8} = 0$	000	$0.000 \le x(n) < 0.125 \Rightarrow 0.000$
$1 \times \frac{R}{2^3} = 1 \times \frac{1}{8} = 0.125$	001	$0.125 \le x(n) < 0.250 \Rightarrow 0.125$
$2 \times \frac{R}{2^3} = 2 \times \frac{1}{8} = 0.25$	010	$0.250 \le x(n) < 0.375 \Rightarrow 0.250$
$3 \times \frac{R}{2^3} = 3 \times \frac{1}{8} = 0.375$	011	$0.375 \le x(n) < 0.500 \Rightarrow 0.375$
$4 \times \frac{R}{2^3} = 4 \times \frac{1}{8} = 0.5$	100	$0.500 \le x(n) < 0.625 \Rightarrow 0.500$
$5 \times \frac{R}{2^3} = 5 \times \frac{1}{8} = 0.625$	101	$0.625 \le x(n) < 0.75 \Rightarrow 0.625$
$6 \times \frac{R}{2^3} = 6 \times \frac{1}{8} = 0.75$	110	$0.750 \le x(n) < 0.875 \Rightarrow 0.750$
$7 \times \frac{R}{2^3} = 7 \times \frac{1}{8} = 0.875$	111	$0.875 \le x(n) \le 1.000 \Rightarrow 0.875$

Let, $x_q(n)$ = Quantized discrete time signal.

$x_c(n)$ = Quantized and coded discrete time signal.

Now, $x_q(n) = \{ 0, 0, 0.25, 0.25, 0.5, 0.75, 0.75, 0.875 \}$

$x_c(n) = \{ 000, 000, 010, 010, 100, 110, 110, 111 \}$

The quantized and coded discrete time signal $x_c(n)$ is called digital signal.

Representation of Discrete Time Signals

- Functional Representation

$$
\begin{aligned}
x(n) &= -0.5 &;\quad n &= -2 \\
&= 1.0 &;\quad n &= -1 \\
&= -1.0 &;\quad n &= 0 \\
&= 0.6 &;\quad n &= 1 \\
&= 1.2 &;\quad n &= 2 \\
&= 1.5 &;\quad n &= 3 \\
&= 0 &;\quad &\text{other } n
\end{aligned}
$$

Representation of Discrete Time Signals

- Graphical Representation

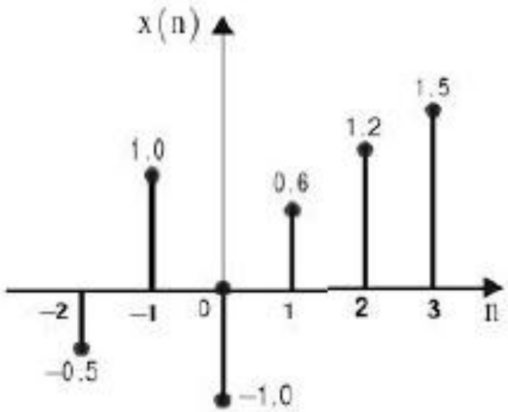


Fig 2.2 : Graphical representation of a discrete time signal.

Representation of Discrete Time Signals

- Tabular Representation

n	....	–2	-1	0	1	2	3	....
x(n)	....	–0.5	1.0	–1.0	0.6	1.2	1.5	....

Representation of Discrete Time Signals

- Sequence Representation

An infinite duration discrete time signal with the time origin, n = 0, indicated by the symbol – is represented as,

x(n) = { ..... – 0.5, 1.0, –1.0, 0.6, 1.2, 1.5, ... }

An infinite duration discrete time signal that satisfies the condition x(n) = 0 for n < 0 is represented as,

x(n) = { –1.0, 0.6, 1.2, 1.5, ... }   or   x(n) = {–1.0, 0.6, 1.2, 1.5, ... }

A finite duration discrete time signal with the time origin, n = 0, indicated by the symbol – is represented as,

x(n) = { – 0.5, 1.0, –1.0, 0.6, 1.2, 1.5 }

A finite duration discrete time signal that satisfies the condition x(n) = 0 for n < 0 is represented as,

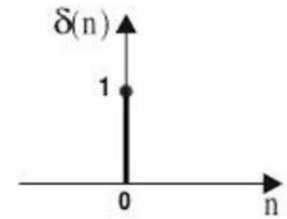
x(n) = { –1.0, –0.6, 1.2, 1.5 }   or   x(n) = { –1.0, 0.6, 1.2, 1.5}

Standard Discrete Time Signal

1. Digital impulse signal or unit sample sequence

Impulse signal, $\delta(n) = 1 \;\; ; \; n = 0$

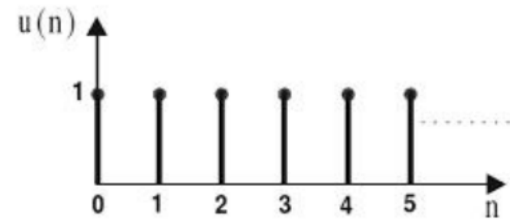
$= 0 \;\; ; \; n \neq 0$



2. Unit step signal

Unit step signal, $u(n) = 1 \; ; \; n \geq 0$

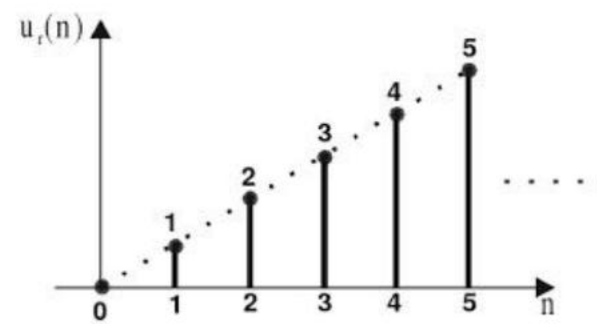
$= 0 \; ; \; n < 0$



3. Ramp signal

Ramp signal, $u_r(n) = n \;\; ; \; n \geq 0$

$= 0 \;\; ; \; n < 0$



Standard Discrete Time Signal

4. Exponential signal

Exponential signal, $g(n) = a^n \; ; n \geq 0$

$= 0 \; ; n < 0$

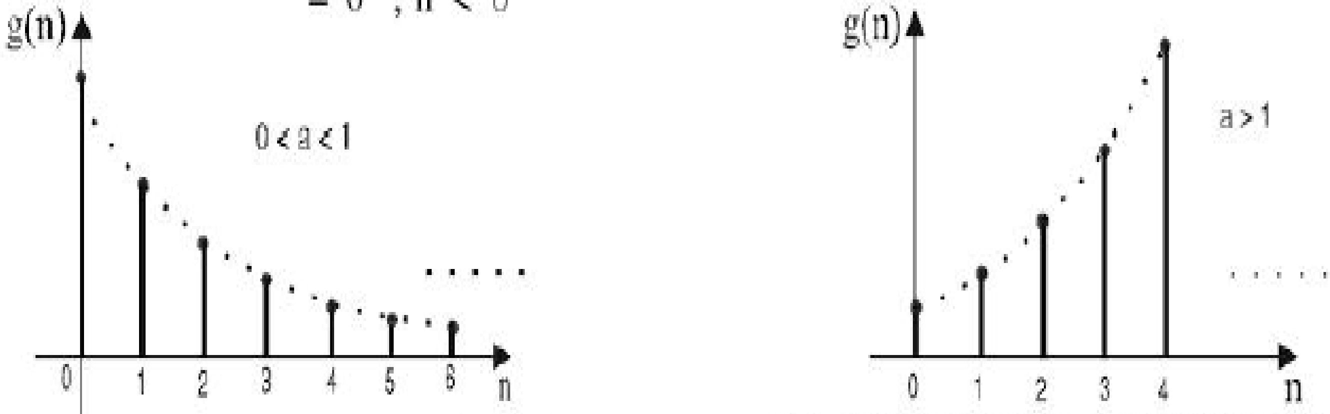


Fig 2.6a : Decreasing exponential signal.

Fig 2.6b : Increasing exponential signal.

Fig 2.6 : Exponential signal.

Standard Discrete Time Signal

5. Discrete time sinusoidal signal

The discrete time sinusoidal signal may be expressed as,

$$x(n) = A\cos(\omega_0 n + \theta) \text{ ; for n in the range } -¥ < n < +¥$$

$$x(n) = A\sin(\omega_0 n + \theta) \text{ ; for n in the range } -¥ < n < +¥$$

where, w_0 = Frequency in radians/sample ; q = Phase in radians

$$f_0 = \frac{\omega_0}{2\pi} = \text{Frequency in cycles/sample}$$

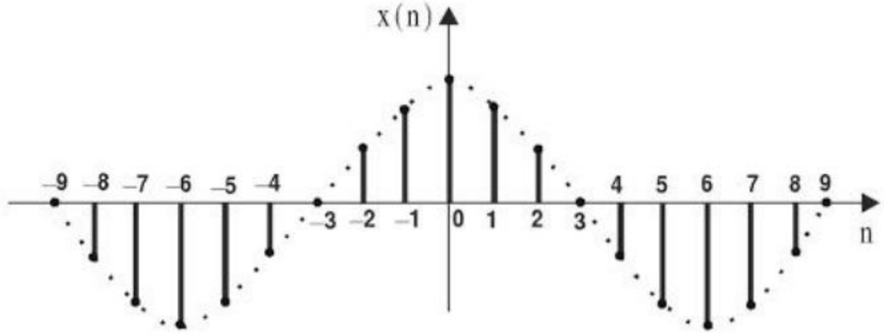


Fig 2.7a : *Discrete time sinusoidal signal represented by equation $x(n) = A\cos(\omega_0 n)$.*

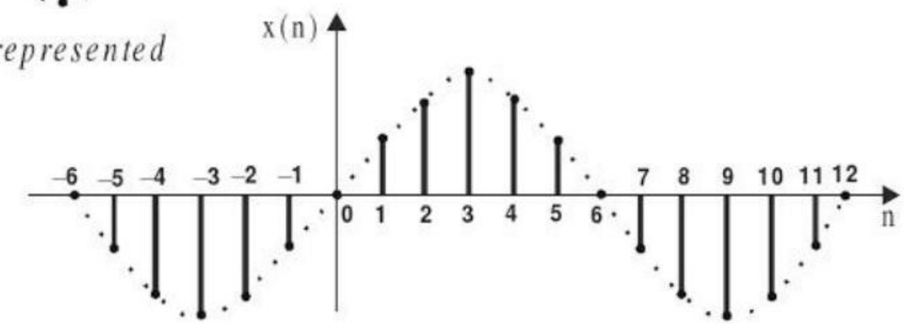


Fig 2.7b : *Discrete time sinusoidal signal represented by equation $x(n) = A\sin(\omega_0 n)$.*

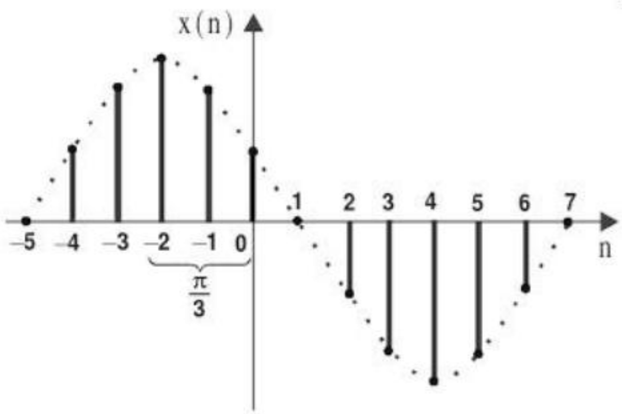


Fig 2.7c : *Discrete time sinusoidal signal represented by equation,*

$$x(n) = A\cos\left(\frac{\pi}{6}n + \frac{\pi}{3}\right);\ \omega_0 = \frac{\pi}{6}\ ;\ \theta = \frac{\pi}{3}$$

Standard Discrete Time Signal

6. Discrete time complex exponential signal

The discrete time complex exponential signal is defined as,

$$x(n) = a^n e^{j(\omega_0 n+\theta)} = a^n [\cos(w_0 n + q) + j \sin(w_0 n + q)]$$

$$= a^n \cos(w_0 n + q) + j a^n \sin(w_0 n + q) = x_r(n) + j x_i(n)$$

where, $x_r(n)$ = Real part of $x(n) = a^n \cos(w_0 n + q)$

$x_i(n)$ = Imaginary part of $x(n) = a^n \sin(w_0 n + q)$

The real part of x(n) will give an exponentially increasing cosinusoid sequence for a > 1 and exponentially decreasing cosinusoid sequence for 0 < a < 1.

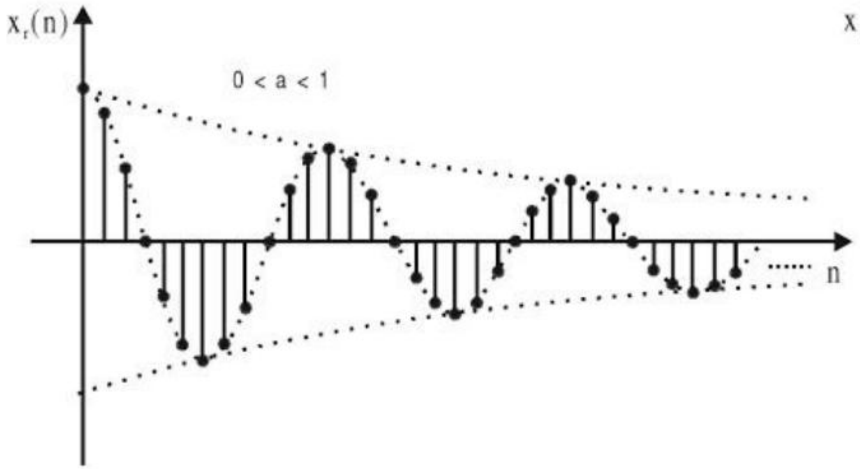


Fig 2.8a : The discrete time sequence represented by the equation, $x_r(n) = a^n \cos \omega_0 n$ for $0 < a < 1$.

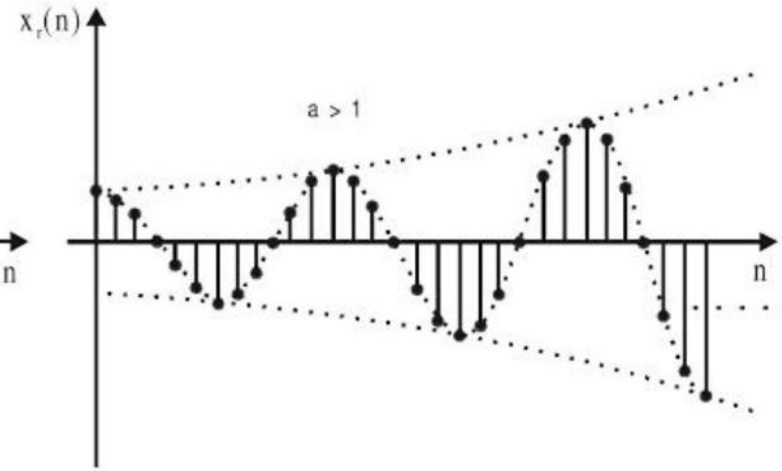


Fig 2.8b : The discrete time sequence represented by the equation, $x_r(n) = a^n \cos \omega_0 n$ for $a > 1$.

Fig 2.8 : Real part of complex exponential signal.

The imaginary part of x(n) will give rise to an exponentially increasing sinusoid sequence for a > 1 and exponentially decreasing sinusoid sequence for 0 < a < 1.

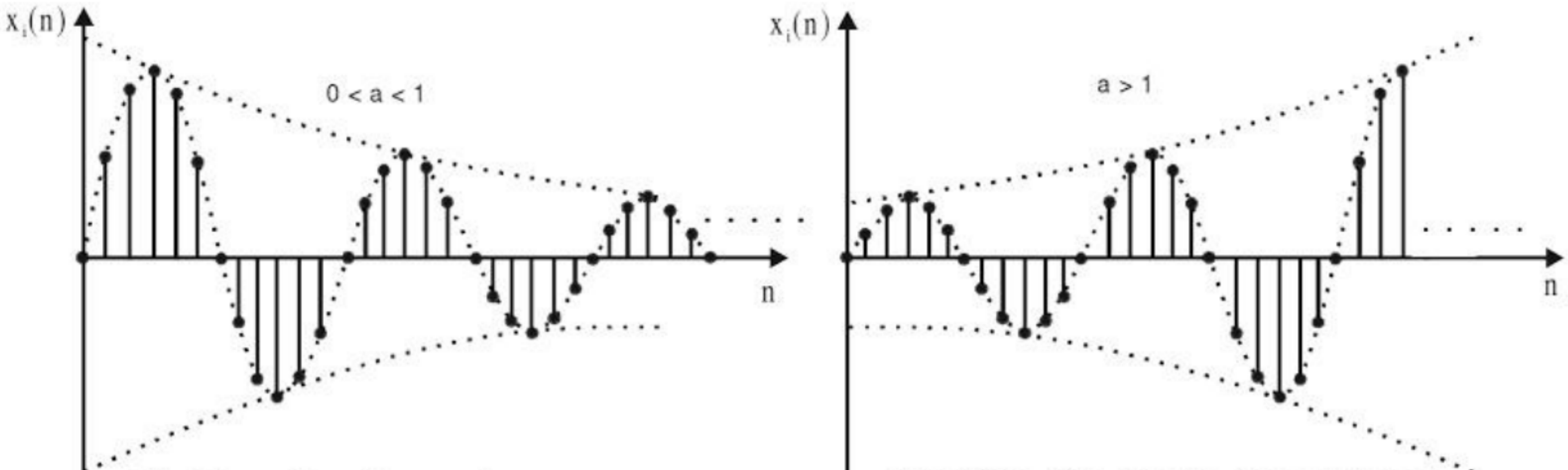


Fig 2.9a : *The discrete time sequence represented by the equation,* $x_i(n) = a^n \sin \omega_0 n$ *for* $0 < a < 1$.

Fig 2.9b : *The discrete time sequence represented by the equation,* $x_i(n) = a^n \sin \omega_0 n$ *for* $a > 1$.

Fig 2.9 : *Imaginary part of complex exponential signal.*

Sampling theorem

- A continuous time signal can be represented in its samples and can be recovered back when sampling frequency f_s is greater than or equal to the twice the highest frequency component of message signal (fm).

$$f_s \geq 2 f_m.$$

Relationship between analog and digital signal by sampling

The time interval between successive samples is called ***sampling time***

The inverse of sampling period is called ***sampling frequency*** (or sampling rate), and it is denoted by F_s.

Let, $x_a(t)$ = Analog / Continuous time signal.

$x(n)$ = Discrete time signal obtained by sampling $x_a(t)$.

Mathematically, the relation between $x(n)$ and $x_a(t)$ can be expressed as,

$$x(n) = x_a(t)\Big|_{t=nT} = x_a(nT) = x_a\left(\frac{n}{F_s}\right);$$ for n in the range -¥ < n < ¥

where, T = Sampling period or interval in seconds

$$F_s = \frac{1}{T}$$ = Sampling rate or sampling frequency in hertz

Classification of Discrete Time Signals

1. Deterministic and nondeterministic signals
2. Periodic and aperiodic signals
3. Symmetric and antisymmetric signals
4. Energy and power signals
5. Causal and noncausal signals

Deterministic and Random Signal

Signals specified by mathematical equations are deterministic signals

Eg: Ramp , Unit Step, exponential, sinusoidal

Signals whose characteristics are random in nature are random signals

Eg: Noise

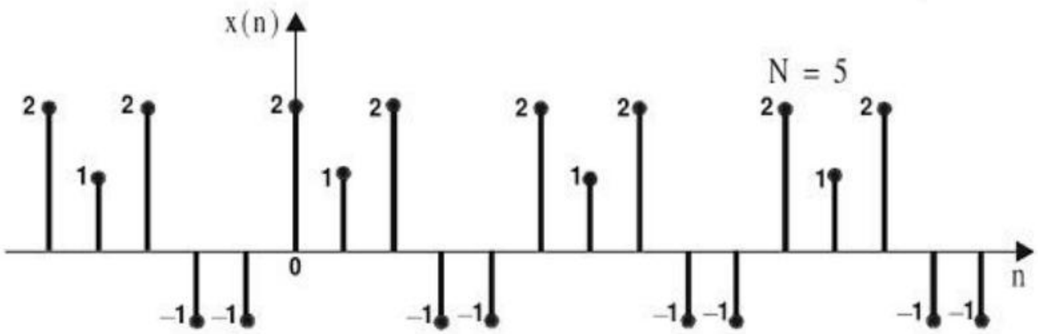
Periodic and Aperiodic Signals

When a discrete time signal x(n), satisfies the condition $\mathbf{x(n+N)=x(n)}$ for integer values of N, then the discrete time signal x(n) is called ***periodic signal***. Here N is the number of samples of a period.

$$\boxed{\text{i.e, if, } \mathbf{x(n+N)=x(n)}, \text{ for all n, then x(n) is periodic.}}$$

The smallest value of N for which the above equation is true is called ***fundamental period***. If there is no value of N that satisfies the above equation, then x(n) is called ***aperiodic*** or ***nonperiodic*** signal.

When N is the fundamental period, the periodic signals will also satisfy the condition $x(n+kN)=x(n)$, where k is an integer. The periodic signals are power signals. The discrete time sinusoidal and complex exponential signals are periodic signals when their fundamental frequency, f_0 is a rational number.



$$\mathbf{x(n)} = \{\ldots\ldots.2, 1, 2, -1, -1, \underset{\uparrow}{2}, 1, 2, -1, -1, 2, 1, 2, -1, -1, \ldots\ldots.\}$$

Fig 2.10 : *Periodic discrete time signal.*

When a discrete time signal is a sum or product of two periodic signals with fundamental periods N_1 and N_2, then the discrete time signal will be periodic with period given by LCM of N_1 and N_2.

$\sin^2 \frac{\pi}{4} n$

$\Downarrow$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\downarrow$

Here $\omega = \frac{\pi}{4}$.

$\sin^2 \frac{\pi}{4} n = \frac{1-\cos 2(\pi/4) n}{2}$.

$\omega = \frac{\pi}{4} = 2\pi f$

so $f = \frac{\omega}{2\pi} = \frac{\cancel{\pi}}{4} \times \frac{1}{2\cancel{\pi}}$

$f = \frac{1}{8} = \frac{K}{N}$ rational value.

$\rightarrow$ So as $\frac{\omega_0}{2\pi}$ is rational value which $\frac{1}{8}$ signal $\sin^2 \frac{4\pi}{4} n$ is periodic signal.

$\rightarrow$ Number of samples will be

$\boxed{N = 8}$

1) $x[n] = \sin^2 \frac{\pi}{4} n$

→ Trigo formula $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

→ $\omega = 2\pi f$

→ fundamental frequency $= \frac{\omega}{2\pi}$

$\frac{\omega}{2\pi}$ should be an ~~integer~~ (rational) value for signal to be periodic

$\frac{\omega}{2\pi}$ is equivalent to $\frac{M}{N}$ → no. of ~~full~~ cycles of s/g

N → no. of samples.

Note:- Identify periodicity of s/g

$\boxed{\omega}$

↓ $\omega = 2\pi f$

$\boxed{f}$

↓

$\boxed{N}$ - time period.

3) $x(n) = \sin\left(\frac{n}{2} - \frac{\pi}{2}\right)$

compare with $\sin(\omega n \pm \theta)$

$$\omega = \frac{1}{2}$$

$$\sin\left(\frac{n}{2} - \frac{\pi}{2}\right) \approx \sin(\omega n - \theta)$$

$$f = \frac{\omega}{2\pi} = \frac{1}{2} \times \frac{1}{2\pi} = \frac{1}{4\pi}$$ not rational number

not periodic.

7) $\sin\frac{2\pi}{3}n + \cos\frac{\pi}{2}n$

$\omega_1 = \frac{2\pi}{3}$ $\omega_2 = \frac{\pi}{2}$.

$f_1 = \frac{\omega_1}{2\pi} = \frac{\cancel{2\pi}}{3} \times \frac{1}{\cancel{2\pi}} = \frac{1}{3}$.

$f_2 = \frac{\omega_2}{2\pi} = \frac{\cancel{\pi}}{2} \times \frac{1}{2\cancel{\pi}} = \frac{1}{4}$.

$N_1 = 3$ $N_2 = 4$

LCM of N_1 & N_2 = 3×4

$\boxed{N = 12 \text{ samples}}$

8) $x(n) = e^{j\frac{\pi}{2}n}$

Compare to $e^{j\omega n}$.

$\omega = \frac{\pi}{2}$

$f = \frac{\omega}{2\pi} = \frac{\pi}{2} \times \frac{1}{2\pi}$

$\omega = 2\pi f$

$f = \frac{\omega}{2\pi}$

$f = \frac{1}{4} - \frac{M}{N}$

periodic $N = 4$.

9) $x[n] = e^{j2n}$

Compare $e^{j\omega n}$.

$\omega = 2$.

$f = \frac{\omega}{2\pi} = \frac{2}{2\pi} = \frac{1}{\pi}$

Non-periodic

Given that, $\mathbf{x(n) = \cos\left(\frac{5\pi}{9}n + 1\right)}$

Let N and M be two integers.

Now, $x(n+N) = \cos\left(\frac{5\pi}{9}(n+N) + 1\right) = \cos\left(\frac{5\pi n}{9} + 1 + \frac{5\pi}{9}N\right)$

Since, $\cos(q + 2pM) = \cos q$, for periodicity $\frac{5\pi}{9}N$ should be integral multiple of 2π.

Let, $\frac{5\pi}{9}N = M \times 2\pi$

$$\therefore \; N = M \times 2\pi \times \frac{9}{5\pi} = \frac{18M}{5}$$

Here N is an integer if, M = 5, 10, 15, 20, ...

Let, $\quad M = 5 \; ; \quad \backslash \quad N = 18$

When $N = 18$; $x(n+N) = \cos\left(\frac{5\pi n}{9} + 1 + \frac{5\pi}{9} \times 18\right) = \cos\left(\frac{5\pi n}{9} + 1 + 10\pi\right) = \cos\left(\frac{5\pi n}{9} + 1\right) = x(n)$

Hence x(n) is periodic with fundamental period of 18 samples.

Given that, $x(n) = \sin\left(\frac{n}{9} - \pi\right)$

Let N and M be two integers.

Now, $x(n+N) = \sin\left(\frac{n+N}{9} - \pi\right) = \sin\left(\frac{n}{9} + \frac{N}{9} - \pi\right) = \sin\left(\frac{n}{9} - \pi + \frac{N}{9}\right)$

Since, $\sin(q + 2pM) = \sin q$, for periodicity $\frac{N}{9}$ should be equal to integral multiple of 2p.

Let, $\frac{N}{9} = M \times 2\pi$

$\backslash \quad N = 18\,pM$

Here N cannot be an integer for any integer value of M and so x(n) will not be periodic.

Symmetric(EVEN) and antisymmetric(ODD) signals

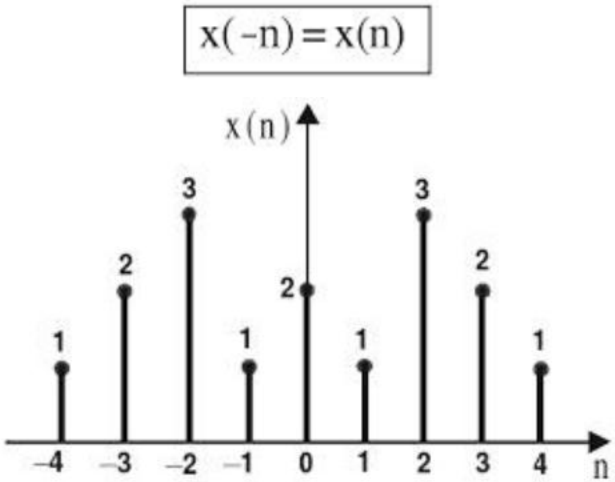
The discrete time signals may exhibit symmetry or antisymmetry with respect to n = 0. When a discrete time signal exhibits symmetry with respect to n = 0 then it is called an ***even signal.*** Therefore, the even signal satisfies the condition,

$$\boxed{x(-n) = x(n)}$$

When a discrete time signal exhibits antisymmetry with respect to n = 0, then it is called an ***odd signal***. Therefore the odd signal satisfies the condition,

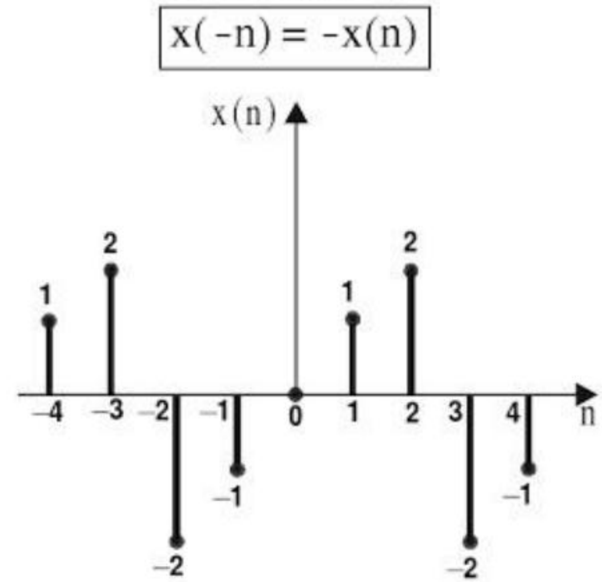
$$\boxed{x(-n) = -x(n)}$$

Symmetric(EVEN) and antisymmetric(ODD) signals



$$\mathbf{x(n)} = \{1, 2, 3, 1, \underset{\uparrow}{2}, 1, 3, 2, 1\}$$

Fig 2.11a : *Symmetric (or even) signal.*



$$\mathbf{x(n)} = \{1, 2, -2, -1, \underset{\uparrow}{0}, 1, 2, -2, -1\}$$

Fig 2.11b : *Antisymmetric (or odd) signal.*

Fig 2.11 : *Symmetric and antisymmetric discrete time signal.*

Symmetric(EVEN) and antisymmetric(ODD) signals

A discrete time signal x(n) which is neither even nor odd can be expressed as a sum of even and odd signal.

Let, $x(n) = x_e(n) + x_0(n)$

where, $x_e(n) =$ Even part of $x(n)$

$x_0(n) =$ Odd part of $x(n)$

Note : If x(n) is even then its odd part will be zero. If x(n) is odd then its even part will be zero.

$$\text{Even part, } x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

$$\text{Odd part, } x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

Symmetric(EVEN) and antisymmetric(ODD) signals

$$\text{Even part, } x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

$$\text{Odd part, } x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

Symmetric(EVEN) and antisymmetric(ODD) signals

Determine the even and odd parts of the signals.

$$x(n) = 3^n$$

Even part, $x_e(n) = \frac{1}{2}[x(n) + x(-n)]$

Odd part, $x_o(n) = \frac{1}{2}[x(n) - x(-n)]$

Solution

a) Given that, $x(n) = 3^n$

$$\therefore \quad x(-n) = 3^{-n}$$

Even part, $x_e(n) = \frac{1}{2}[x(n) + x(-n)] = \frac{1}{2}[3^n + 3^{-n}]$

Odd part, $x_o(n) = \frac{1}{2}[x(n) - x(-n)] = \frac{1}{2}[3^n - 3^{-n}]$

Symmetric(EVEN) and antisymmetric(ODD) signals

Determine the even and odd parts of the signals.

$$\mathbf{x(n) = 3\, e^{j\frac{\pi}{5}n}}$$

Even part, $x_e(n) = \frac{1}{2}[x(n) + x(-n)]$

Odd part, $x_o(n) = \frac{1}{2}[x(n) - x(-n)]$

Even part, $x_e(n) = \frac{1}{2}[x(n) + x(-n)]$

$$= \frac{1}{2}\left[3\cos\frac{\pi}{5}n + j3\sin\frac{\pi}{5}n + 3\cos\frac{\pi}{5}n - j3\sin\frac{\pi}{5}n\right] = \frac{1}{2}\left[6\cos\frac{\pi}{5}n\right] = 3\cos\frac{\pi}{5}n$$

Odd part, $x_o(n) = \frac{1}{2}[x(n) - x(-n)]$

$$= \frac{1}{2}\left[3\cos\frac{\pi}{5}n + j3\sin\frac{\pi}{5}n - 3\cos\frac{\pi}{5}n + j3\sin\frac{\pi}{5}n\right]$$

$$= \frac{1}{2}\left[j6\sin\frac{\pi}{5}n\right] = j3\sin\frac{\pi}{5}n$$

Energy and Power Signals

The ***energy*** E of a discrete time signal x(n) is defined as,

$$\boxed{\text{Energy, } E = \sum_{n=-\infty}^{\infty} |x(n)|^2}$$

The energy of a signal may be finite or infinite, and can be applied to complex valued and real valued signals.

If energy E of a discrete time signal is finite and nonzero, then the discrete time signal is called an ***energy signal.*** The exponential signals are examples of energy signals.

Energy and power signals

The average ***power*** of a discrete time signal x(n) is defined as,

$$\boxed{\text{Power, } P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2} \qquad \text{.....(2.12)}$$

If power P of a discrete time signal is finite and nonzero, then the discrete time signal is called a ***power signal***. The periodic signals are examples of power signals.

For energy signals, the energy will be finite and average power will be zero. For power signals the average power is finite and energy will be infinite.

$$\boxed{\begin{array}{l} \backslash \quad \text{For energy signal, } 0 < E < \infty \text{ and } P = 0 \\ \\ \quad\ \ \text{For power signal, } 0 < P < \infty \text{ and } E = \infty \end{array}}$$

Determine whether the following signals are energy or power signals.

$$x(n) = \left(\frac{1}{4}\right)^n u(n)$$

Solution

a) Given that, $x(n) = \left(\frac{1}{4}\right)^n u(n)$

Here, $x(n) = \left(\frac{1}{4}\right)^n u(n)$ for all n.

$$\therefore \; x(n) = \left(\frac{1}{4}\right)^n = 0.25^n \quad ; \quad n \geq 0$$

Infinite geometric series sum formula. $\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$

$$\text{Energy, } E = \sum_{n=-\infty}^{+\infty} |x(n)|^2 = \sum_{n=0}^{\infty} \left|(0.25)^n\right|^2 = \sum_{n=0}^{\infty} (0.25^2)^n$$

$$= \sum_{n=0}^{\infty} (0.0625)^n = \frac{1}{1-0.0625} = 1.067 \text{ joules}$$

$$\text{Power, } P = \underset{N\to\infty}{\mathrm{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x(n)|^2 = \underset{N\to\infty}{\mathrm{Lt}} \frac{1}{2N+1} \sum_{n=0}^{N} \left|(0.25)^n\right|^2$$

$$= \underset{N\to\infty}{\mathrm{Lt}} \frac{1}{2N+1} \sum_{n=0}^{N} (0.25^2)^n = \underset{N\to\infty}{\mathrm{Lt}} \frac{1}{2N+1} \sum_{n=0}^{N} (0.0625)^n$$

Finite geometric series sum formula.

$$\sum_{n=0}^{N} C^n = \frac{C^{N+1}-1}{C-1}$$

$$= \underset{N\to\infty}{\mathrm{Lt}} \frac{1}{2N+1} \frac{(0.0625)^{N+1}-1}{0.0625-1}$$

$$= \frac{1}{\infty} \times \frac{0.0625^{\infty}-1}{0.0625-1} = 0$$

Here E is finite and P is zero and so x(n) is an energy signal.

Determine whether the following signals are energy or power signals.

$x(n) = u(n)$

Given that, $x(n) = u(n)$

$$E = \sum_{n=-\infty}^{+\infty} |x(n)|^2 = \sum_{n=0}^{+\infty} (u(n))^2$$

$$= \sum_{n=0}^{+\infty} u(n) = 1+1+1 \,\ldots\ldots\ldots \infty = \infty$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} u(n) = \underset{N\to\infty}{Lt} \frac{1}{2N+1}\left(\underbrace{1+1+1+\ldots\ldots\ldots+1}_{N+1 \text{ terms}}\right)$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1}(N+1) = \underset{N\to\infty}{Lt} \frac{N\left(1+\frac{1}{N}\right)}{N\left(2+\frac{1}{N}\right)} = \frac{1+\frac{1}{\infty}}{2+\frac{1}{\infty}} = \frac{1+0}{2+0} = \frac{1}{2} \text{ watts}$$

Since P is finite and E is infinite, x(n) is a power signal.

Causal, Noncausal, Anticausal Signals

- Causal Signal

$$x(n) = 0 \text{ for } n < 0.$$

- Non Causal if defined for n<=0 or for both n>0 and n<=0
- If system is non causal then:

$$x(n) \neq 0 \text{ for } n < 0.$$

- A non-causal signal can be converted into causal by multiplying it by unit step signal.
- When a non causal signal is defined for only n<0 then it is called anticausal signal.

Examples of Causal and Noncausal Signals

$$x(n) = \{\underset{\uparrow}{1}, -1, 2, -2, 3, -3\}$$

$$x(n) = \{\underset{\uparrow}{2}, 2, 3, 3, \ldots\ldots\ldots\ldots\}$$

Causal signals

$$x(n) = \{1, -1, 2, -2, 3, \underset{\uparrow}{-3}\}$$

$$x(n) = \{\ldots\ldots\ldots\ldots, 2, 2, 3, \underset{\uparrow}{3}\}$$

Anticausal signals

$$x(n) = \{2, 3, 4, \underset{\uparrow}{5}, 4, 3, 2\}$$

$$x(n) = \{\ldots\ldots, 2, 3, 4, \underset{\uparrow}{5}, 4, 3, 2, \ldots\ldots\}$$

Noncausal signals

Mathematical operations on Discrete Time Signals

1. Scaling : Amplitude scaling and time scaling
2. Folding
3. Shifting : Right shift (or advance) and left shift (or delay)
4. Addition
5. Multiplication

1. Scaling

Amplitude Scaling (or Scalar Multiplication)

Amplitude scaling of a discrete time signal by a constant A is accomplished by multiplying the value of every signal sample by the constant A.

Example :

Let y(n) be amplitude scaled signal of x(n), then y(n) = A x(n)

Let, x(n) = 10	; n = 0	and A = 0.2,	When n = 0	; y(0) = A x(0) = 0.2 ´ 10	= 2.0
= 16	; n = 1		When n = 1	; y(1) = A x(1) = 0.2 ´ 16	= 3.2
= 20	; n = 2		When n = 2	; y(2) = A x(2) = 0.2 ´ 20	= 4.0

1. Scaling

Time Scaling (or Downsampling and Upsampling)

There are two ways of time scaling a discrete time signal. They are downsampling and upsampling.

In a signal x(n), if n is replaced by Dn, where D is an integer, then it is called ***downsampling***.

In a signal x(n), if n is replaced by $\frac{n}{I}$, where I is an integer, then it is called ***upsampling***.

Example :

If $x(n) = b^n$; $n \geq 0$; $0 < b < 1$, then

$x_1(n) = x(2n)$ will be a down sampled version of $x(n)$ and

$x_2(n) = x\left(\frac{n}{2}\right)$ will be an up sampled version on $x(n)$.

When $n = 0$; $x_1(0) = x(0) = b^0$

When $n = 1$; $x_1(1) = x(2) = b^2$

When $n = 2$; $x_1(2) = x(4) = b^4$ and so on.

When $n = 0$; $x_2(0) = x\left(\frac{0}{2}\right) = x(0) = b^0$

When $n = 1$; $x_2(1) = x\left(\frac{1}{2}\right) = 0$

When $n = 2$; $x_2(2) = x\left(\frac{2}{2}\right) = x(1) = b^1$

When $n = 3$; $x_2(3) = x\left(\frac{3}{2}\right) = 0$ and so on.

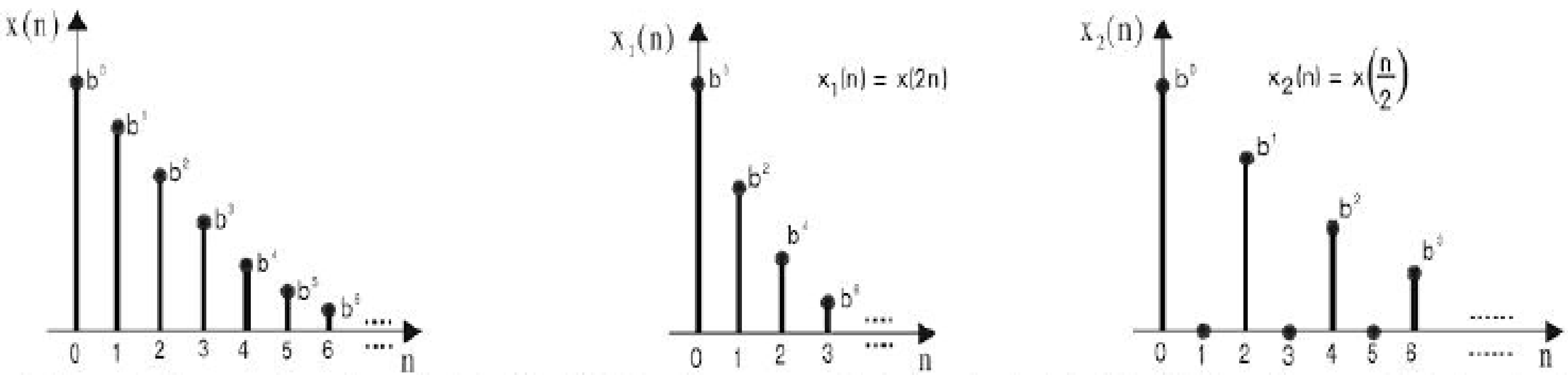


Fig 2.12a : A discrete time signal x(n). *Fig 2.12b : Down sampled signal of x(n).* *Fig 2.12c : Up sampled signal x(n).*

Fig 2.12 : A discrete time signal and its time scaled version.

2. Folding

The ***folding*** of a discrete time signal x(n) is performed by changing the sign of the time base n in x(n). The folding operation produces a signal x(–n) which is a mirror image of the signal x(n) with respect to time origin n = 0.

Example :

Let x(n) = 0.8n ; –2 £ n £ 2. Now the folded signal, $x_1(n)$ = x(–n) = –0.8n ; –2 £ n £ 2

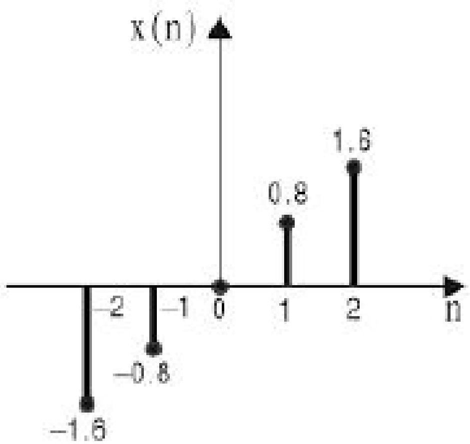


***Fig 2.13a** : A discrete time signal x(n).*

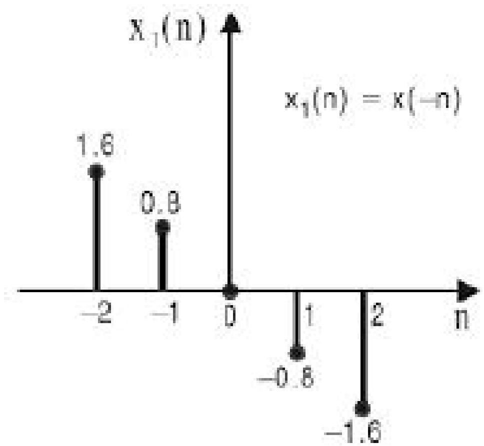


***Fig 2.13b** : Folded signal of x(n).*

***Fig 2.13** : A discrete time signal and its folded version.*

3.Time Shifting of Discrete Time Signals

- X(n) signal will be shifted by variable m i.e. x(n)=x(n-m)
- m is positive integer signal is delayed by m units of time.
- Delay results in shifting each sample of x(n) to right.
- m is negative integer signal is advanced by m units of time.
- Advance results in shifting each sample of x(n) to left.

Let, $x(n) = 3 \;;\; n = 2$

$= 2 \;;\; n = 3$

$= 1 \;;\; n = 4$

$= 0 \;;\;$ for other n

Let, $x_1(n) = x(n-2)$, where $x_1(n)$ is delayed signal of x(n)

When $n = 4 \;;\; x_1(4) = x(4-2) = x(2) = 3$

When $n = 5 \;;\; x_1(5) = x(5-2) = x(3) = 2$

When $n = 6 \;;\; x_1(6) = x(6-2) = x(4) = 1$

The sample x(2) is available at n = 2 in the original sequence x(n), but the same sample is available at n = 4 in $x_1(n)$. Similarly every sample of x(n) is delayed by two sampling times.

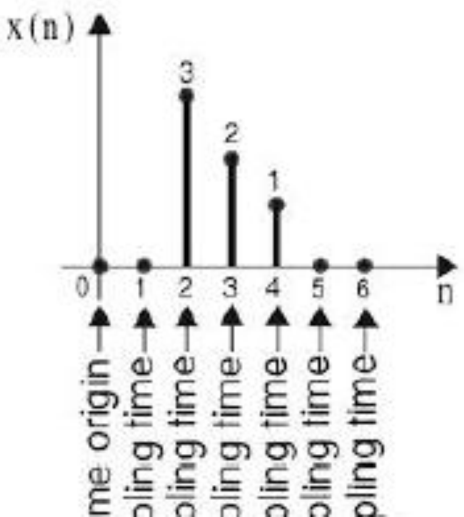
Let, $x_2(n) = x(n+2)$, where $x_2(n)$ is an advanced signal of x(n)

When $n = 0 \;;\; x_2(0) = x(0+2) = x(2) = 3$

When $n = 1 \;;\; x_2(1) = x(1+2) = x(3) = 2$

When $n = 2 \;;\; x_2(2) = x(2+2) = x(4) = 1$

The sample x(2) is available at n = 2 in the original sequence x(n), but the same sample is available at n = 0 in $x_2(n)$. Similarly every sample of x(n) is advanced by two sampling times. Hence the signal $x_2(n)$ is an advanced version of x(n).



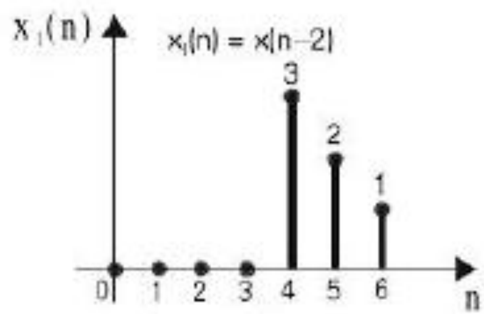


Fig 2.14b : Delayed signal of x(n).

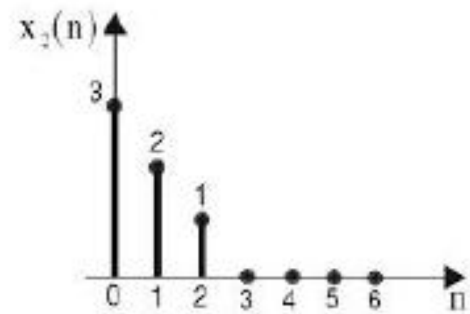


Fig 2.14c : Advanced signal of x(n).

4. Addition of Discrete Time Signal

The ***addition*** of two discrete time signals is performed on a sample-by-sample basis.

The sum of two signals $x_1(n)$ and $x_2(n)$ is a signal $y(n)$, whose value at any instant is equal to the sum of the samples of these two signals at that instant.

i.e., $y(n) = x_1(n) + x_2(n)$; $-$¥ $< n <$ ¥ .

Example :

Let, $x_1(n) = \{2, 2, -1\}$ and $x_2(n) = \{-1, 1, 2\}$

When $n = 0$; $y(0) = x_1(0) + x_2(0) = 2 + (-1) = 1$

When $n = 1$; $y(1) = x_1(1) + x_2(1) = 2 + 1 = 3$

When $n = 2$; $y(2) = x_1(2) + x_2(2) = -1 + 2 = 1$

\ $y(n) = x_1(n) + x_2(n) = \{1, 3, 1\}$

Example :

Let, $x_1(n) = \{2, 2, -1\}$ and $x_2(n) = \{-1, 1, 2\}$

When $n = 0$; $y(0) = x_1(0) + x_2(0) = 2 + (-1) = 1$

When $n = 1$; $y(1) = x_1(1) + x_2(1) = 2 + 1 = 3$

When $n = 2$; $y(2) = x_1(2) + x_2(2) = -1 + 2 = 1$

$\therefore \quad y(n) = x_1(n) + x_2(n) = \{1, 3, 1\}$

5.Multiplication of Discrete time signals

The ***multiplication*** of two discrete time signals is performed on a sample-by-sample basis.The product of two signals $x_1(n)$ and $x_2(n)$ is a signal $y(n)$, whose value at any instant is equal to the product of the samples of these two signals at that instant. The product is also called ***modulation***.

Example :

Let, $x_1(n) = \{ 2, 2, -1 \}$ and $x_2(n) = \{ -1, 1, 2 \}$

When $n = 0$; $y(0) = x_1(0) \times x_2(0) = 2 \times (-1) = -2$

When $n = 1$; $y(1) = x_1(1) \times x_2(1) = 2 \times 1 = 2$

When $n = 2$; $y(2) = x_1(2) \times x_2(2) = -1 \times 2 = -2$

$\therefore$ $y(n) = x_1(n) \times x_2(n) = \{-2, 2, -2\}$

Example :

Let, $x_1(n) = \{\ 2,\ 2,\ -1\ \}$ and $x_2(n) = \{\ -1,\ 1,\ 2\ \}$

When $n = 0$; $y(0) = x_1(0)\ \text{´}\ x_2(0) = 2\ \text{´}\ (-1) = -2$

When $n = 1$; $y(1) = x_1(1)\ \text{´}\ x_2(1) = 2\ \text{´}\ 1 \quad = \quad 2$

When $n = 2$; $y(2) = x_1(2)\ \text{´}\ x_2(2) = -1\ \text{´}\ 2 \quad = -2$

$\backslash\ \ y(n) = x_1(n)\ \text{´}\ x_2(n) = \{-2,\ 2,\ -2\}$

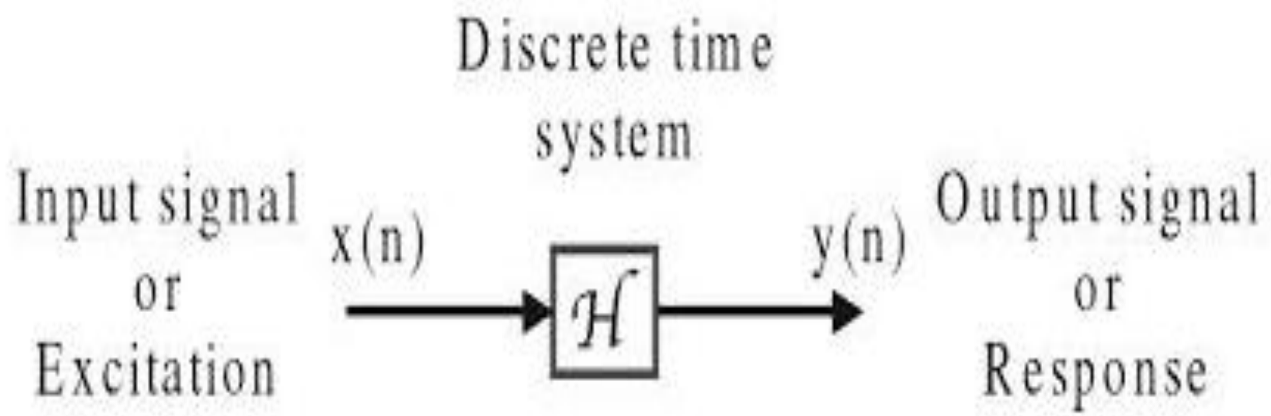
Discrete Time System

- Discrete time system is device or algorithm which accepts input in discrete signal form to produce output or response in discrete time signal form.

Discrete Time System

Response, $y(n) = \mathcal{H}\{x(n)\}$

where, $\mathcal{H}$ denotes the transformation (also called an operator).



LTI (Linear Time Invariant) System

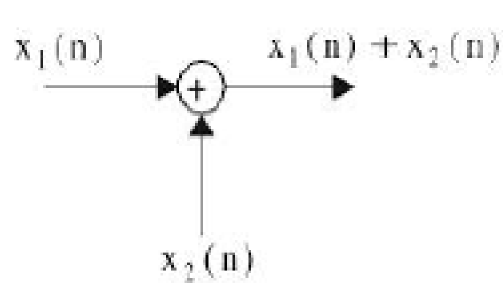
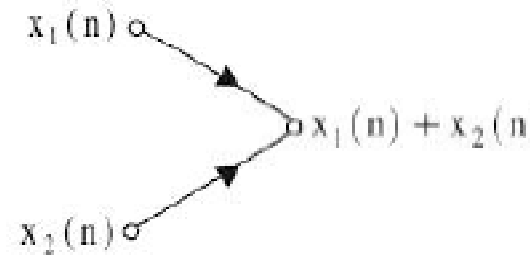
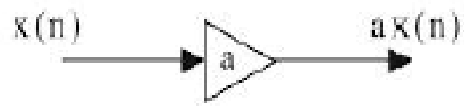
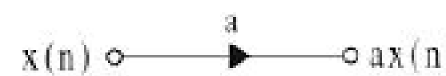
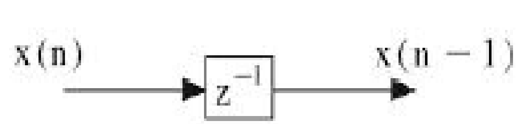
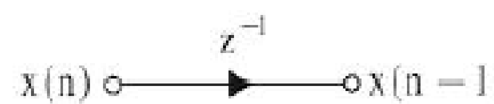
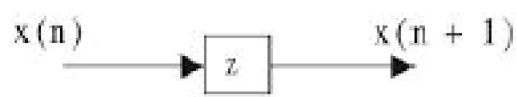
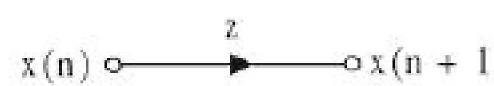
- A discrete time system is linear if it obeys principle of superposition and time invariant if input output relationship does not change with time.

- When system satisfies property of linearity and time invariance it is called as LTI system.

Block Diagram and Signal Flow graph Representation of DT System

- **Basic elements of Representation are:**

1) Adder
2) Constant Multiplier – Constant scaling factor
3) Unit Delay Element – one sample unit to delay
4) Unit Advance Element - one sample unit to advance

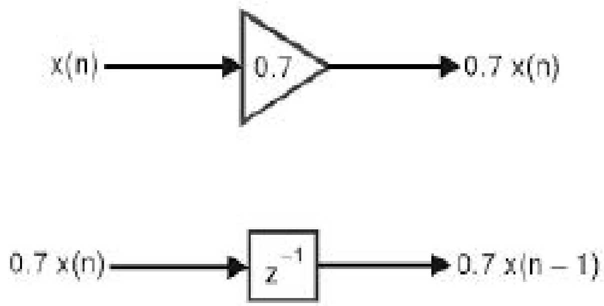
Element	Block diagram representation	Signal flow graph representation
Adder		
Constant multiplier		
Unit delay element		
Unit advance element		

Block Diagram and Signal Flow graph Representation of DT System

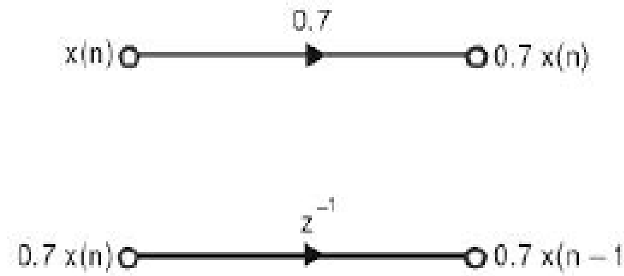
Construct the block diagram and signal flow graph of the discrete time systems whose input-output relations are described by the following difference equations.

a) $y(n) = 0.7\,x(n) + 0.7\,x(n-1)$

Block diagram representation



Signal flow graph representation



The input to the system is x(n) and the output of the system is y(n). The above elements are connected as shown below to get the output y(n).

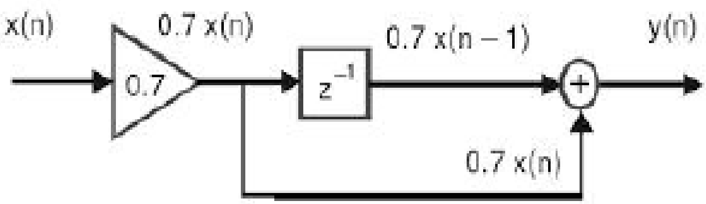


Fig 1 : Block diagram of the system
$y(n) = 0.7\ x(n) + 0.7\ x(n-1).$

Fig 2 : Signal flow graph of the system
$y(n) = 0.7\ x(n) + 0.7\ x(n-1).$

Classification of Discrete Time Systems

The discrete time systems are classified based on their characteristics. Some of the classifications of discrete time systems are,

1. Static and dynamic systems
2. Time invariant and time variant systems
3. Linear and nonlinear systems
4. Causal and noncausal systems
5. Stable and unstable systems
6. FIR and IIR systems
7. Recursive and nonrecursive systems

Static system and Dynamic System

- Static System – Output depends on only current input samples but not on past or future samples – no memory

- Dynamic System – Opposite of static

1) $y(n) = x(n)$

$n=0 \rightarrow y(0) = x(0)$ system is static

Here o/p $y(0)$ value is only dependent

current i/p $x(0)$ no memory

present time $n=0$.

2) $y(t) = 2x^2(t)$

Here if $t = (-2)$

$y(-2) = 2x^2(-2)$

Here also present time $t = -2$ ~~s~~

o/p $y(t)$ is dependent on present

$x(t)$ not dependent on any past

or future values

System is static.

3) $y(n) = x(n) + x(n-1)$

Here at time $n = 0$ or $n = 1$

$n = 1$

$y(1) = x(1) + x(1-1)$

$y(1) = x(1) + x(0)$.

Here current o/p $y(1)$ is dependent on time equal to $x(1)$ & $x(0)$

$x(1)$ is present i/p while $x(0)$ is past i/p to maintain this i/p memory is required so the system is dynamic.

4) $y(t) = x(t) + x(t+3)$

$t = -1$

$y(-1) = x(-1) + x(-1+3)$

$y(-1) = x(-1) + x(2)$

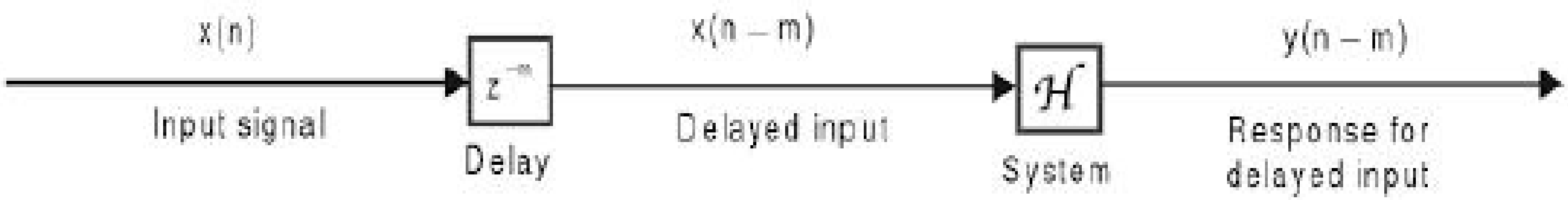
Here also o/p $y(-1)$ is dependent on future & current values i.e (-1) & (2) so the system is dynamic

Time Invariant and Time Variant System

- A system is said to be time invariant if its input output characteristics do not change with time.

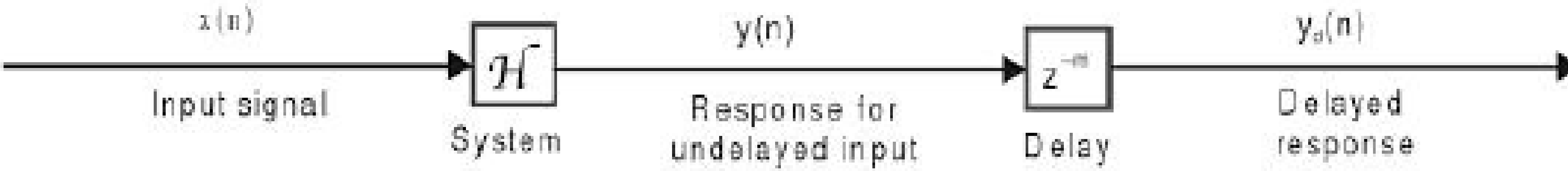
Procedure to test for time Invariance

1. Delay the input signal by m units of time and determine the response of the system for this delayed input signal. Let this response be $y(n - m)$.



Procedure to test for time Invariance

2. Delay the response of the system for undelayed input by m units of time. Let this delayed response be $y_d(n)$.



Procedure to test for time Invariance

3. Check whether $y(n-m) = y_d(n)$. If they are equal then the system is time invariant. Otherwise the system is ***time variant***.

If, $y(n-m) = y_d(n)$, then the system is time invariant

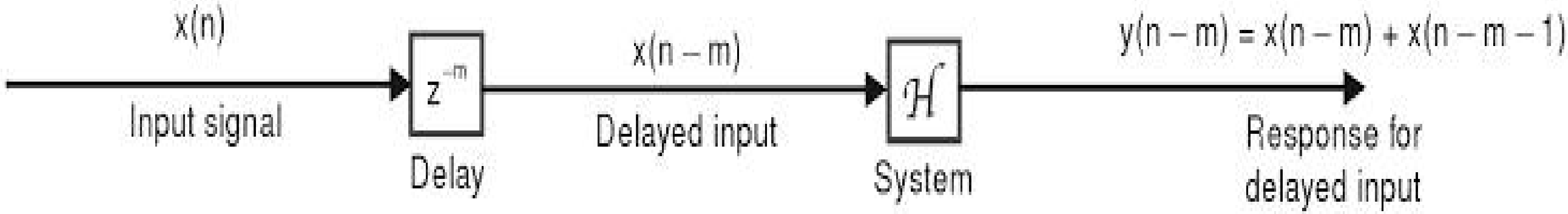
Test the following system for time variance

$$y(n) = x(n) + x(n-1)$$

Given that, $y(n) = x(n) + x(n-1)$

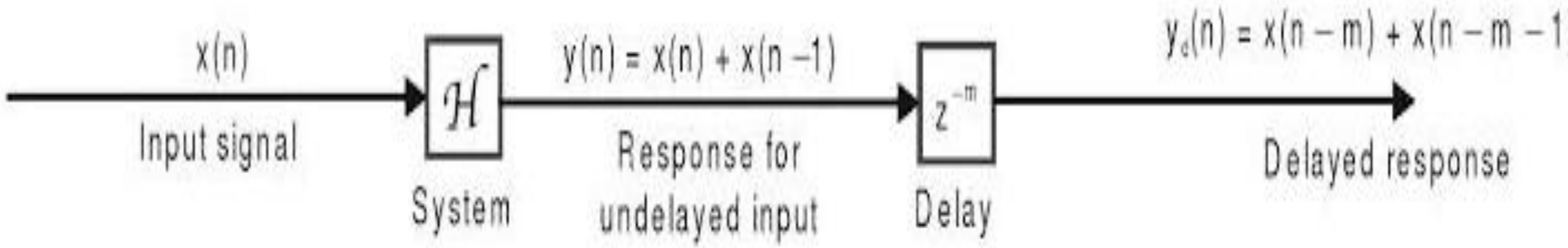
<u>Test 1 : Response for delayed input</u>

Let, $y(n-m)$ = Response for delayed input.



Test 2 : Delayed response

Let, $y_d(n)$ = Delayed response.



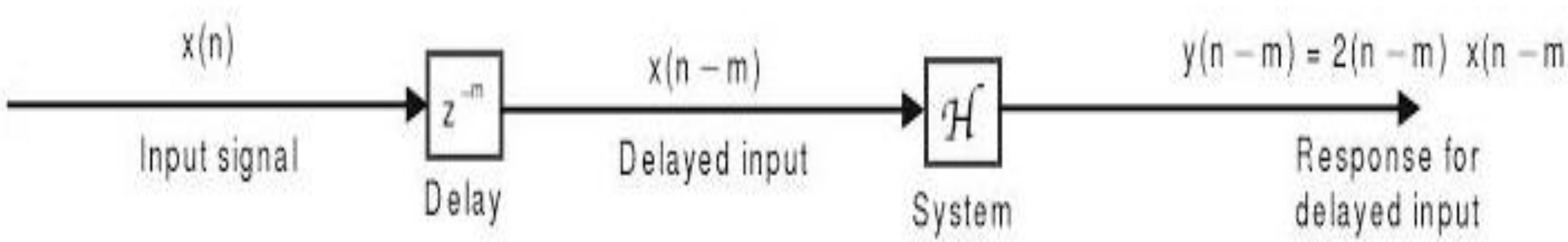
Conclusion : Here, $y(n-m) = y_d(n)$, therefore the system is time invariant.

Test the following system for time variance

y(n) = 2n x(n)

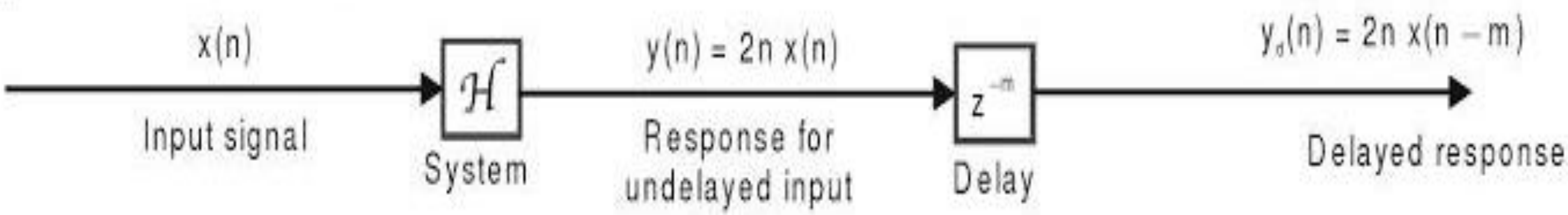
Test 1 : Response for delayed input

Let, $y(n-m)$ = Response for delayed input.



Test 2 : Delayed response

Let, $y_d(n)$ = Delayed response.



Conclusion : Here, $y(n-m)$ ¹ $y_d(n)$, therefore the system is time variant.

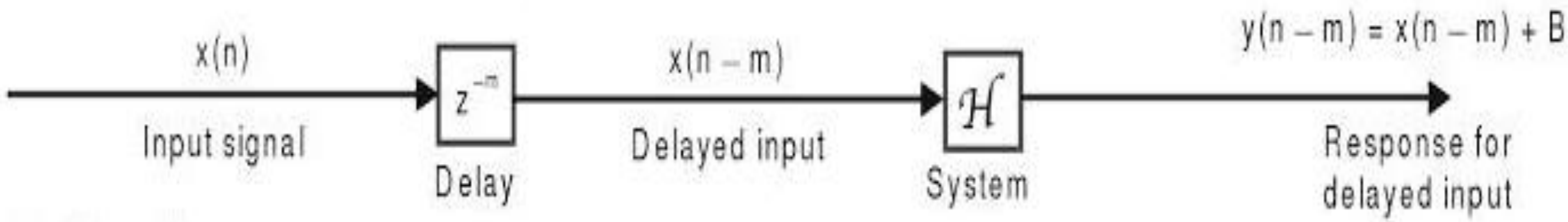
Test the following system for time variance

$$y(n) = x(n) + B$$

Given that, $y(n) = x(n) + B$

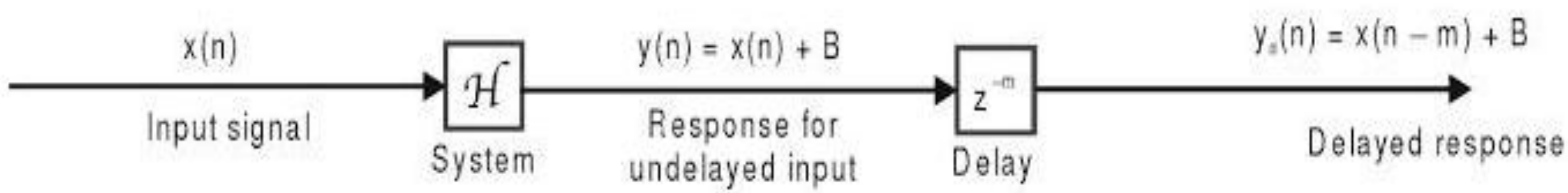
<u>Test 1 : Response for delayed input</u>

Let, $y(n-m)$ = Response for delayed input.



<u>Test 2 : Delayed response</u>

Let, $y_d(n)$ = Delayed response.



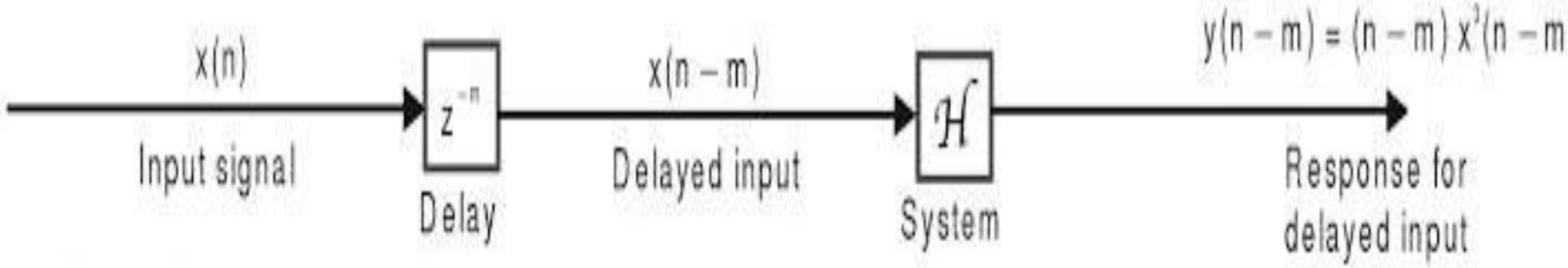
<u>Conclusion</u> : Here, $y(n-m) = y_d(n)$, therefore the system is time invariant.

Test the following system for time variance

$y(n) = n\,x^3(n)$

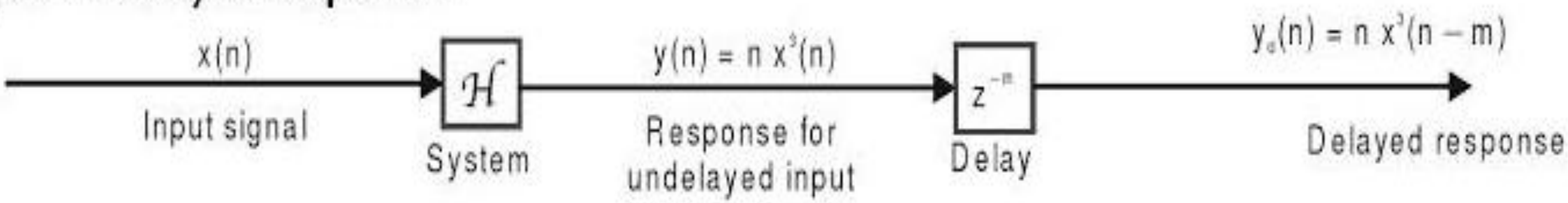
<u>Test 1 : Response for delayed input</u>

Let, $y(n-m)$ = Response for delayed input.



<u>Test 2 : Delayed response</u>

Let, $y_d(n)$ = Delayed response.



<u>Conclusion</u> : Here, $y(n-m) \neq y_d(n)$, therefore the system is time variant.

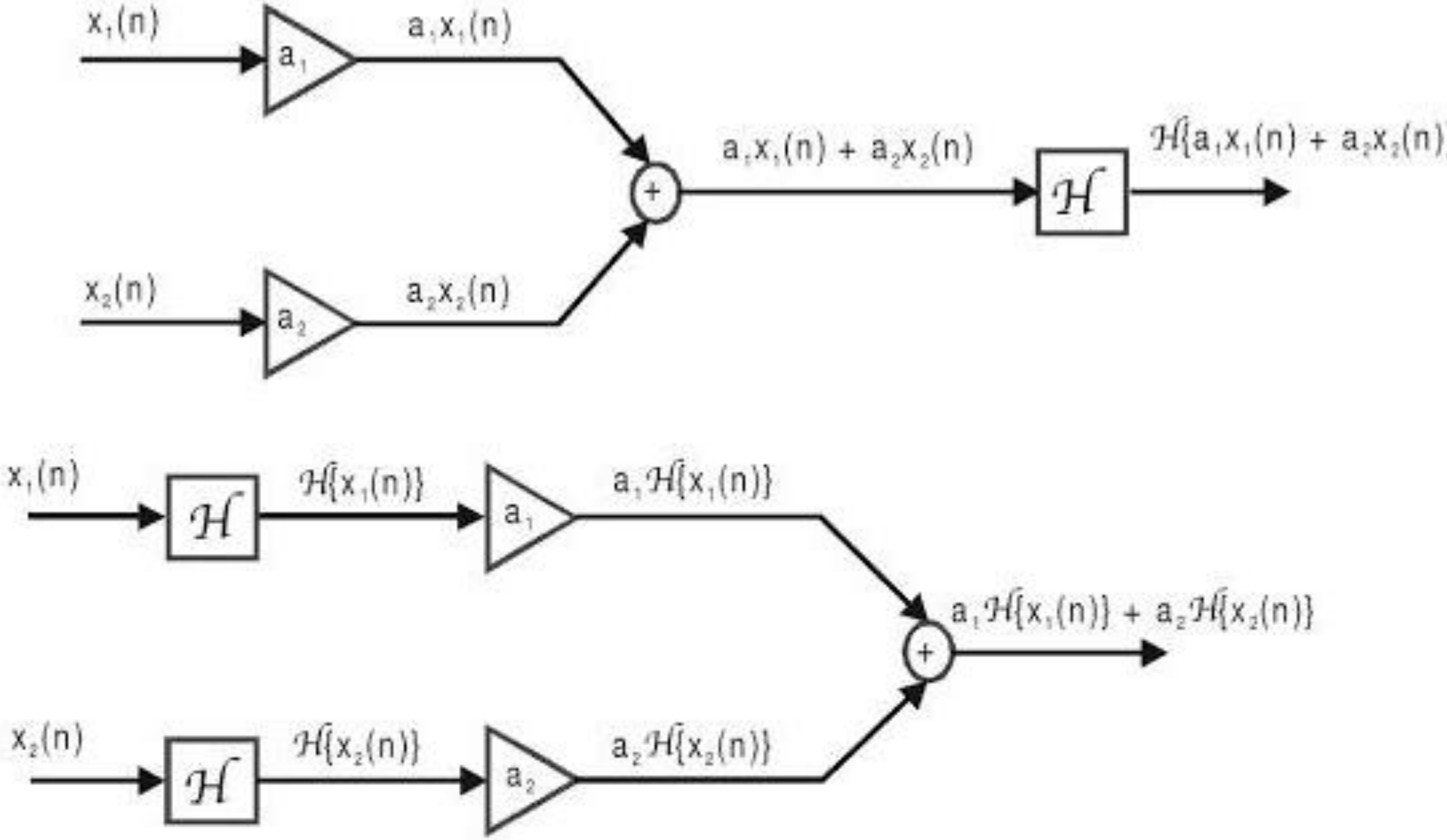
Linear and Non Linear System

A ***linear system*** is one that satisfies the superposition principle. The ***principle of superposition*** requires that the response of the system to a weighted sum of the signals is equal to the corresponding weighted sum of the responses of the system to each of the individual input signals.

Definition : A relaxed system $\mathcal{H}$ is ***linear*** if

$$\mathcal{H}\{a_1 x_1(n) + a_2 x_2(n)\} = a_1 \mathcal{H}\{x_1(n)\} + a_2 \mathcal{H}\{x_2(n)\}$$

Linear and Non Linear System(Check for Linearity)



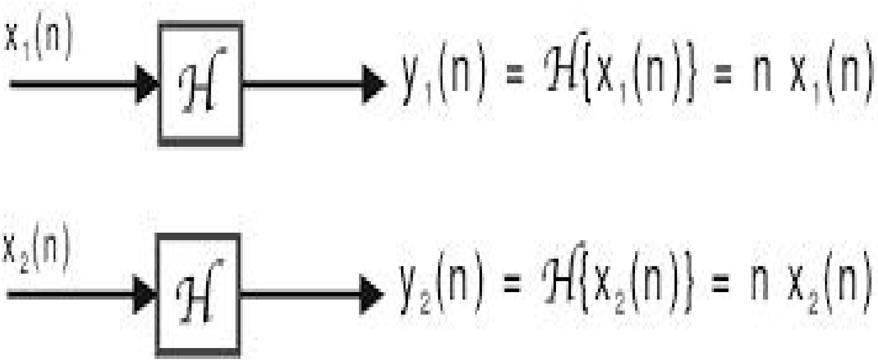
The system, $\mathcal{H}$ is linear if and only if, $\mathcal{H}\{a_1\, x_1(n) + a_2\, x_2(n)\} = a_1\, \mathcal{H}\{x_1(n)\} + a_2\, \mathcal{H}\{x_2(n)\}$

- Test the following system for linearity.

$$y(n) = n\,x(n)$$

Consider two signals, $x_1(n)$ and $x_2(n)$.

Let, $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.



\ $a_1 y_1(n) + a_2 y_2(n) = a_1\,n\,x_1(n) + a_2\,n\,x_2(n)$

Consider a linear combination of inputs, $a_1\, x_1(n) + a_2\, x_2(n) = x_3(n)$.

Let, $y_3(n)$ be the response for $x_3(n)$.

$$\backslash \quad y_3(n) = \mathcal{H}\{a_1\, x_1(n) + a_2\, x_2(n)\} = n[a_1\, x_1(n) + a_2\, x_2(n)] = a_1\, n\, x_1(n) + a_2\, n\, x_2(n)$$

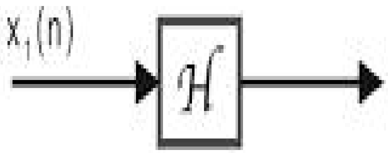
The condition to be satisfied for linearity is, $y_3(n) = a_1\, y_1(n) + a_2\, y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) = a_1\, y_1(n) + a_2\, y_2(n)$. Hence the system is linear.

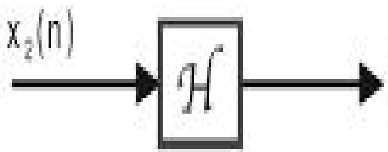
$$y(n) = B\,x(n) + C$$

Consider two signals, $x_1(n)$ and $x_2(n)$.

Let, $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.



$$y_1(n) = \mathcal{H}\{x_1(n)\} = B\,x_1(n) + C$$



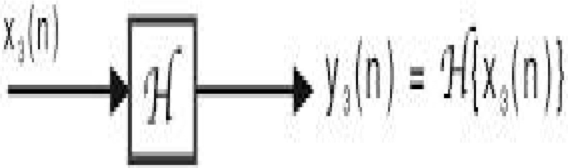
$$y_2(n) = \mathcal{H}\{x_2(n)\} = B\,x_2(n) + C$$

$$\backslash \quad a_1 y_1(n) + a_2 y_2(n) = a_1 [B\,x_1(n) + C] + a_2 [B\,x_2(n) + C]$$

$$= B\,a_1 x_1(n) + C\,a_1 + B\,a_2 x_2(n) + C\,a_2$$

Consider a linear combination of inputs, $a_1 x_1(n) + a_2 x_2(n) = x_3(n)$.

Let, $y_3(n)$ be the response for $x_3(n)$.



$$\backslash \quad y_3(n) = \mathcal{H}\{a_1 x_1(n) + a_2 x_2(n)\} = B\,[a_1 x_1(n) + a_2 x_2(n)] + C = B\,a_1 x_1(n) + B\,a_2 x_2(n) + C \qquad .....(2)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1 y_1(n) + a_2 y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n)$ ¹ $a_1 y_1(n) + a_2 y_2(n)$. Hence the system is nonlinear.

$$y(n) = e^{x(n)}$$

Consider two signals, $x_1(n)$ and $x_2(n)$.

Let, $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

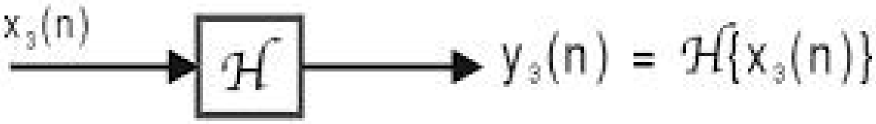
$$\xrightarrow{x_1(n)} \boxed{\mathcal{H}} \longrightarrow y_1(n) = \mathcal{H}\{x_1(n)\} = e^{x_1(n)}$$

$$\xrightarrow{x_2(n)} \boxed{\mathcal{H}} \longrightarrow y_2(n) = \mathcal{H}\{x_2(n)\} = e^{x_2(n)}$$

$$\therefore \; a_1 y_1(n) + a_2 y_2(n) = a_1 e^{x_1(n)} + a_2 e^{x_2(n)}$$

Consider a linear combination of inputs, $a_1 x_1(n) + a_2 x_2(n) = x_3(n)$.

Let, $y_3(n)$ be the response for $x_3(n)$.



$$\backslash \quad y_3(n) = \mathcal{H}\{a_1 x_1(n) + a_2 x_2(n)\} = e^{[a_1 x_1(n) + a_2 x_2(n)]} = e^{a_1 x_1(n)}\, e^{a_2 x_2(n)}$$

Hence the system is nonlinear.

$$y(n) = n\,x^2(n)$$

Consider two signals, $x_1(n)$ and $x_2(n)$.

Let, $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$\xrightarrow{x_1(n)} \boxed{\mathcal{H}} \longrightarrow y_1(n) = \mathcal{H}\{x_1(n)\} = n\,x_1^2(n)$$

$$\xrightarrow{x_2(n)} \boxed{\mathcal{H}} \longrightarrow y_2(n) = \mathcal{H}\{x_2(n)\} = n\,x_2^2(n)$$

$$\backslash \quad a_1\,y_1(n) + a_2\,y_2(n) = a_1\,n\,x_1^2(n) + a_2\,n\,x_2^2(n)$$

Consider a linear combination of inputs, $a_1\,x_1(n) + a_2\,x_2(n) = x_3(n)$.

Let, $y_3(n)$ be the response for $x_3(n)$.

$$\xrightarrow{x_3(n)} \boxed{\mathcal{H}} \longrightarrow y_3(n) = \mathcal{H}\{x_3(n)\}$$

$$\begin{aligned} \backslash \quad y_3(n) &= \mathcal{H}\{a_1\,x_1(n) + a_2\,x_2(n)\} = n[a_1\,x_1(n) + a_2\,x_2(n)]^2 \\ &= n\,a_1^2\,x_1^2(n) + n\,a_2^2\,x_2^2(n) + 2\,n\,a_1\,a_2\,x_1(n)\,x_2(n) \end{aligned}$$

Hence the system is nonlinear.

Causal and Non Causal Systems

Definition : A system is said to be ***causal*** if the output of the system at any time n depends only on the present input, past inputs and past outputs but does not depend on the future inputs and outputs.

Let, $x(n)$ = Present input and $y(n)$ = Present output

\ $x(n-1), x(n-2), \ldots\ldots$, are past inputs

$y(n-1), y(n-2), \ldots\ldots$, are past outputs

Causal and Non Causal Systems

Given that, $y(n) = x(n) - x(n-2)$

When $n = 0$, $y(0) = x(0) - x(-2)$ $\Rightarrow$ The response at $n = 0$, i.e., $y(0)$ depends on the present input $x(0)$ and past input $x(-2)$

When $n = 1$, $y(1) = x(1) - x(-1)$ $\Rightarrow$ The response at $n = 1$, i.e., $y(1)$ depends on the present input $x(1)$ and past input $x(-1)$.

system is causal.

Causal and Non Causal Systems

Given that, $y(n) = \sum_{k=-\infty}^{n} x(k)$

When $n = 0$, $y(0) = \sum_{k=-\infty}^{0} x(k)$

$= \ldots x(-2) + x(-1) + x(0)$ Þ The response at $n = 0$, i.e., $y(0)$ depends on the present input $x(0)$ and past inputs $x(-1)$, $x(-2)$,.....

When $n = 1$, $y(1) = \sum_{k=-\infty}^{1} x(k)$

$= \ldots x(-2) + x(-1) + x(0) + x(1)$ Þ The response at $n = 1$, i.e., $y(1)$ depends on the present input $x(1)$ and past inputs $x(0)$, $x(-1)$, $x(-2)$,..

system is causal.

Causal and Non Causal Systems

$$y(n) = x(n) + 2\,x(n + 3)$$

When $n = 0$, $y(0) = x(0) + 2\,x(3)$ Þ The response at $n = 0$, i.e., $y(0)$ depends on the present input $x(0)$ and future input $x(3)$.

When $n = 1$, $y(1) = x(1) + 2\,x(4)$ Þ The response at $n = 1$, i.e., $y(1)$ depends on the present input $x(1)$ and future input $x(4)$.

system is noncausal.

Stable and Unstable

Definition : An arbitrary relaxed system is said to be ***BIBO stable*** (Bounded Input-Bounded Output stable) if and only if every bounded input produces a bounded output.

Condition for Stability of LTI System

The condition for stability of an LTI system is,

$$\sum_{n=-\infty}^{+\infty} |h(n)| < \infty$$

i.e., an LTI system is ***stable*** if the impulse response is absolutely summable.

Stable and Unstable

a) $h(n) = 0.2^n u(n)$

$$\therefore \quad \sum_{n=-\infty}^{+\infty} |h(n)| = \sum_{n=-\infty}^{+\infty} \left|0.2^n u(n)\right| = \sum_{n=0}^{\infty} 0.2^n$$

$$= \frac{1}{1-0.2} = 1.25$$

Since, $\sum_{n=-\infty}^{+\infty} |h(n)| < \infty$, system is stable.

Infinite geometric series sum formula

$$\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$$

if $0 < |C| < 1$

$$\boxed{\sum_{n=0}^{\infty} C^n = \infty \quad \text{if } C > 1}$$

b) $h(n) = 0.3^n\, u(n) + 2^n\, u(n)$

$$\therefore \quad \sum_{n=-\infty}^{+\infty} |h(n)| = \sum_{n=-\infty}^{+\infty} \left|0.3^n\, u(n) + 2^n\, u(n)\right|$$

$$= \sum_{n=0}^{\infty} 0.3^n + \sum_{n=0}^{\infty} 2^n\, u(n) = \frac{1}{1-0.3} + \infty = \infty$$

Since, $\displaystyle\sum_{n=-\infty}^{+\infty} |h(n)| = \infty$, system is unstable.

FIR and IIR Systems

- FIR system (Finite duration Impulse Response System) , the impulse response consists of finite number of samples.

- IIR system (Infinite duration Impulse Response System) , the impulse response consists of infinite number of samples.

Recursive and Nonrecursive Systems

A system whose output y(n) at time n depends on any number of past output values as well as present and past inputs is called a ***recursive system***. The past outputs are $y(n-1)$, $y(n-2)$, $y(n-3)$, etc.,.

$$y(n) = F\,[y(n-1), y(n-2), \ldots y(n-N), x(n), x(n-1), \ldots x(n-M)]$$

A system whose output does not depend on past output but depends only on the present and past input is called a ***nonrecursive system***.

$$y(n) = F\,[x(n), x(n-1), \ldots\ldots, x(n-M)]$$

Discrete or Linear Convolution

The ***Discrete*** or ***Linear convolution*** of two discrete time sequences $x_1(n)$ and $x_2(n)$ is defined as,

$$\boxed{x_3(n) = \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)} \quad \text{or} \quad \boxed{x_3(n) = \sum_{m=-\infty}^{+\infty} x_2(m)\, x_1(n-m)} \qquad .....(2.29)$$

where, $x_3(n)$ is the sequence obtained by convolving $x_1(n)$ and $x_2(n)$

m is a dummy variable

The convolution relation of equation (2.29) can be symbolically expressed as,

$$x_3(n) = x_1(n) * x_2(n) = x_2(n) * x_1(n) \qquad .....(2.30)$$

where, the symbol $*$ indicates convolution operation.

Discrete or Linear Convolution

- In linear convolution the x1 and x2 sequences are non periodic , hence convolution result x3 is also non periodic.
- Linear convolution is said to be aperiodic.

Procedure

1. Change of index : Change the index n in the sequences $x_1(n)$ and $x_2(n)$, to get the sequences $x_1(m)$ and $x_2(m)$.

2. Folding : Fold $x_2(m)$ about $m = 0$, to obtain $x_2(-m)$.

3. Shifting : Shift $x_2(-m)$ by q to the right if q is positive, shift $x_2(-m)$ by q to the left if q is negative to obtain $x_2(q - m)$.

4. Multiplication : Multiply $x_1(m)$ by $x_2(q - m)$ to get a product sequence. Let the product sequence be $v_q(m)$. Now, $v_q(m) = x_1(m) \times x_2(q - m)$.

5. Summation : Sum all the values of the product sequence $v_q(m)$ to obtain the value of $x_3(n)$ at $n = q$. [i.e., $x_3(q)$].

Properties

Commutative property : $x_1(n) * x_2(n) = x_2(n) * x_1(n)$

Associative property : $[x_1(n) * x_2(n)] * x_3(n) = x_1(n) * [x_2(n) * x_3(n)]$

Distributive property : $x_1(n) * [x_2(n) + x_3(n)] = [x_1(n) * x_2(n)] + [x_1(n) * x_3(n)]$

Method 1: Graphical Method

Let $x_1(n)$ and $x_2(n)$ be the input sequences and $x_3(n)$ be the output sequence.

1. Change the index "n" of input sequences to "m" to get $x_1(m)$ and $x_2(m)$.
2. Sketch the graphical representation of the input sequences $x_1(m)$ and $x_2(m)$.
3. Let us fold $x_2(m)$ to get $x_2(-m)$. Sketch the graphical representation of the folded sequence $x_2(-m)$.
4. Shift the folded sequence $x_2(-m)$ to the left graphically so that the product of $x_1(m)$ and shifted $x_2(-m)$ gives only one nonzero sample. Now multiply $x_1(m)$ and shifted $x_2(-m)$ to get a product sequence, and then sum up the samples of product sequence, which is the first sample of output sequence.
5. To get the next sample of output sequence, shift $x_2(-m)$ of previous step to one position right and multiply the shifted sequence with $x_1(m)$ to get a product sequence. Now the sum of the samples of product sequence gives the second sample of output sequence.

2. To get subsequent samples of output sequence, the step 5 is repeated until we get a nonzero product sequence.

$$\text{if,} \quad \text{Length of } x_1(n) = N_1$$
$$\text{Length of } x_2(n) = N_2$$
$$\text{then,} \quad \text{Length of } x_3(n) = N_1 + N_2 - 1$$

$$\text{if,} \quad x_1(n) \text{ start at } n = n_1$$
$$x_2(n) \text{ start at } n = n_2$$
$$\text{then,} \quad x_3(n) \text{ start at } n = n_1 + n_2$$
$$\text{and} \quad x_3(n) \text{ end at } n = (n_1 + n_2) + (N_1 + N_2 - 1) - 1$$
$$= (n_1 + n_2) + (N_1 + N_2 - 2)$$

Determine the response of the LTI system whose input x(n) and impulse response h(n) are given by,

$x(n) = \{\underline{1}, 2, 0.5, 1\}$ and $h(n) = \{\underline{1}, 2, 1, -1\}$

Method 1 : Graphical Method

The graphical representation of x(n) and h(n) after replacing n by m are shown below. The sequence h(m) is folded with respect to m = 0 to obtain h(–m).

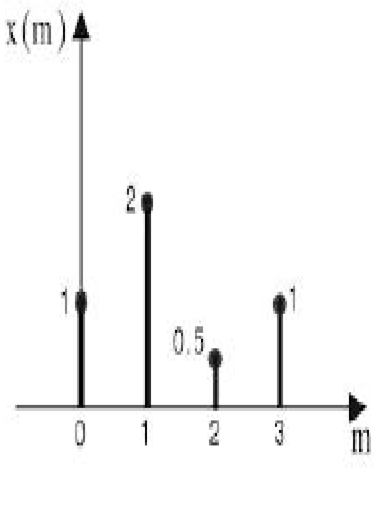


Fig 1 : Input sequence.

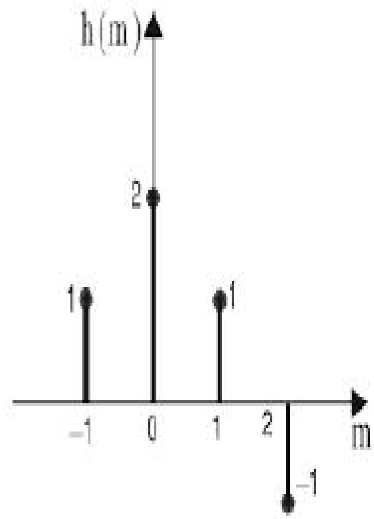


Fig 2 : Impulse response.

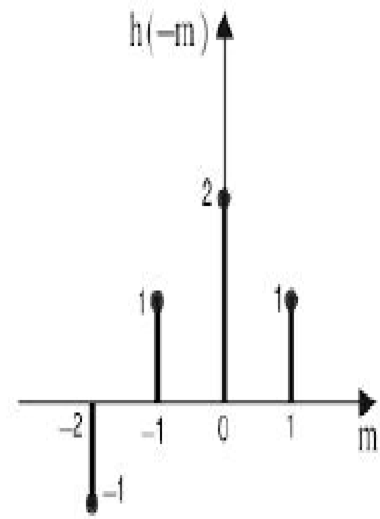


Fig 3 : Folded impulse response.

Graphical Method

$$y(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_n(m) \;;\; \text{where } h_n(m) = h(n-m)$$

When n = –1 ; $y(-1) = \sum_{m=-\infty}^{+\infty} x(m)\, h(-1-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_{-1}(m) = \sum_{m=-\infty}^{+\infty} v_{-1}(m)$

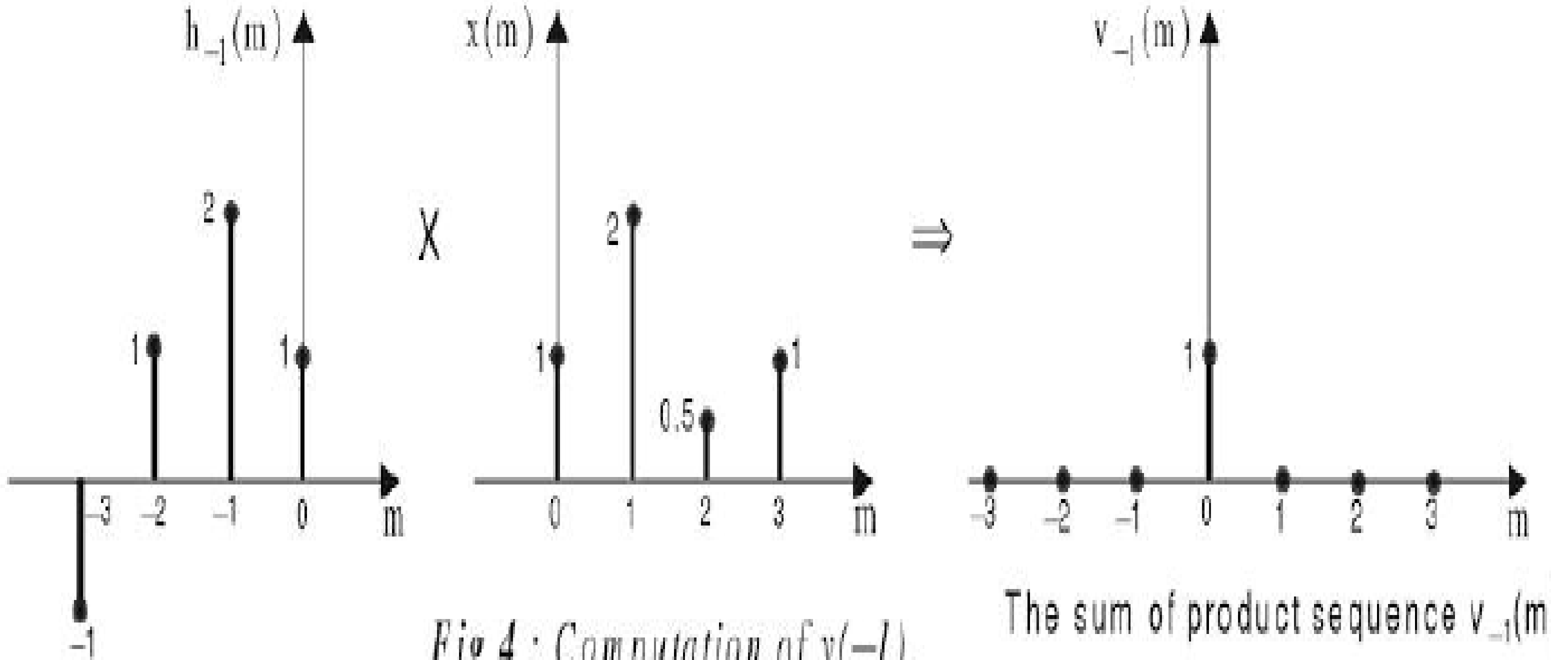


Fig 4 : Computation of y(–1).

The sum of product sequence $v_{-1}(m)$ gives $y(-1)$. $\therefore\ y(-1) = 1$

When $n = 0$; $y(0) = \sum_{m=-\infty}^{+\infty} x(m)\, h(0-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_0(m) = \sum_{m=-\infty}^{+\infty} v_0(m)$

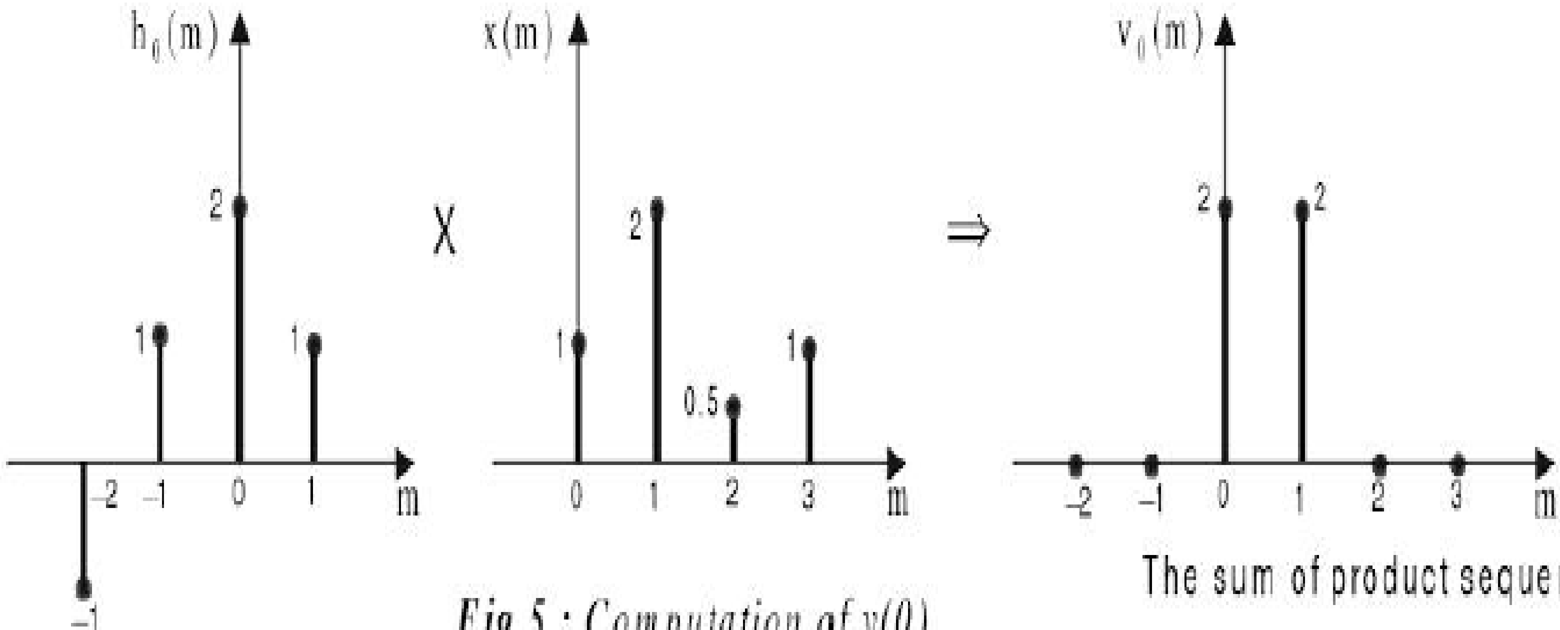
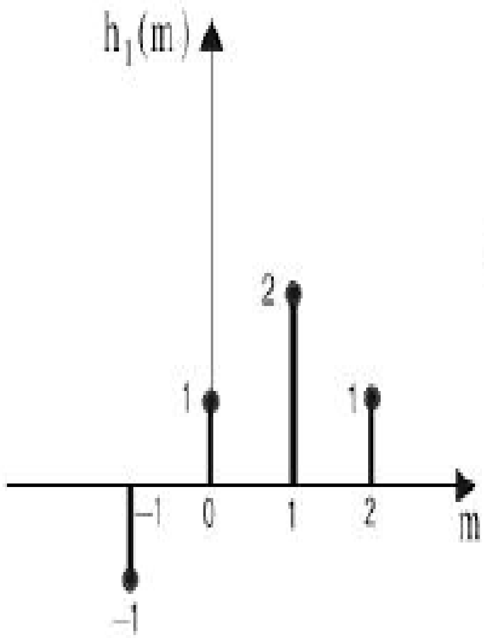


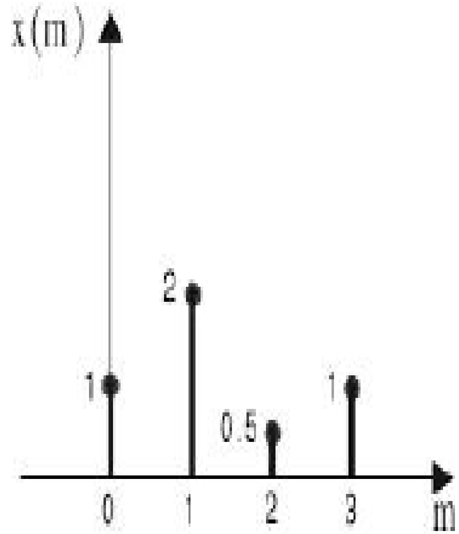
Fig 5 : Computation of y(0).

The sum of product sequence $v_0(m)$ gives $y(0)$. $\therefore\ y(0) = 2 + 2 = 4$

When n = 1 ; $y(1) = \sum_{m=-\infty}^{+\infty} x(m)\, h(1-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_1(m) = \sum_{m=-\infty}^{+\infty} v_1(m)$



X



$\Rightarrow$

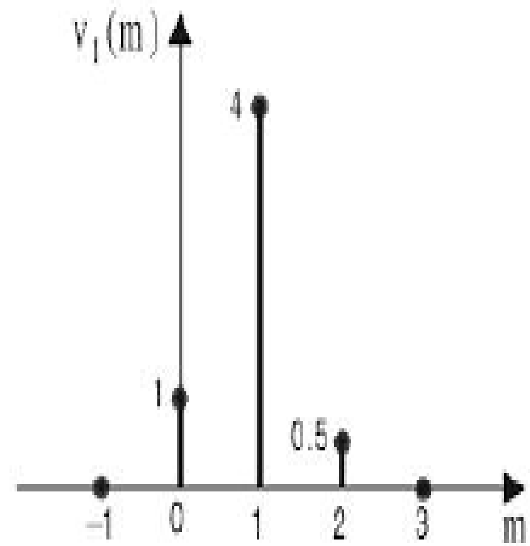


Fig 6 : Computation of y(1).

The sum of product sequence $v_1(m)$ gives $y(1)$. $\therefore$ $y(1) = 1 + 4 + 0.5 = 5.5$

When n = 2 ; $y(2) = \sum_{m=-\infty}^{+\infty} x(m)\, h(2-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_2(m) = \sum_{m=-\infty}^{+\infty} v_2(m)$

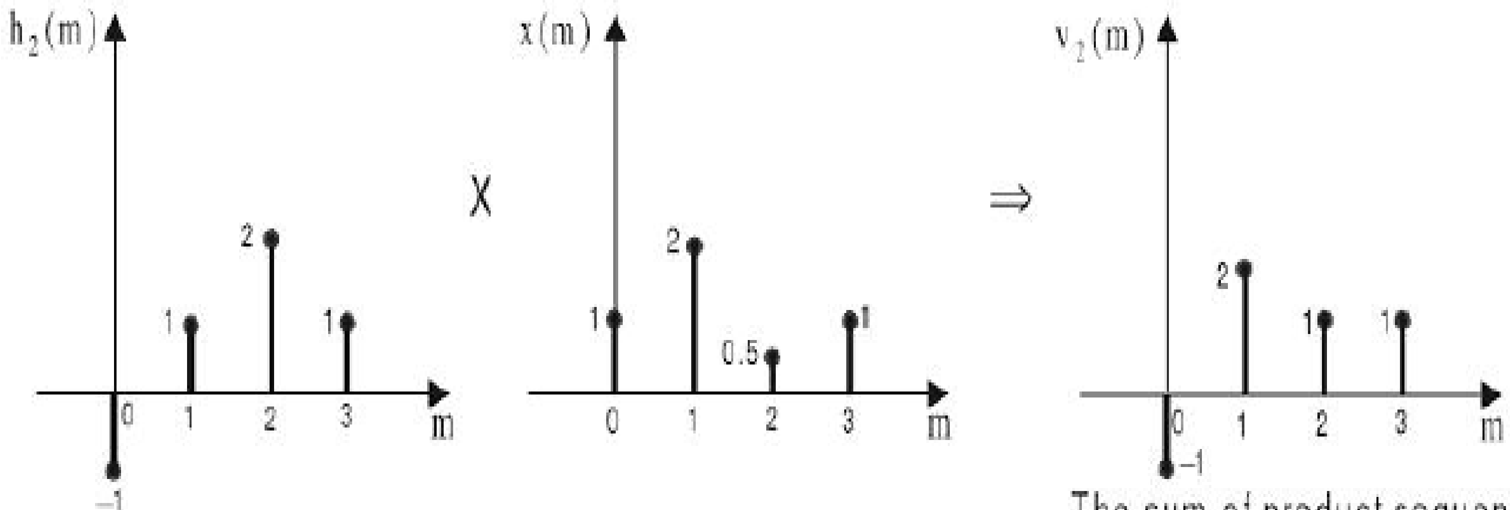


Fig 7 : Computation of y(2).

The sum of product sequence $v_2(m)$ gives $y(2)$. $\therefore\ y(2) = -1 + 2 + 1 + 1 = 3$

When n = 3 ; $y(3) = \sum_{m=-\infty}^{+\infty} x(m)\, h(3-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_3(m) = \sum_{m=-\infty}^{+\infty} v_3(m)$

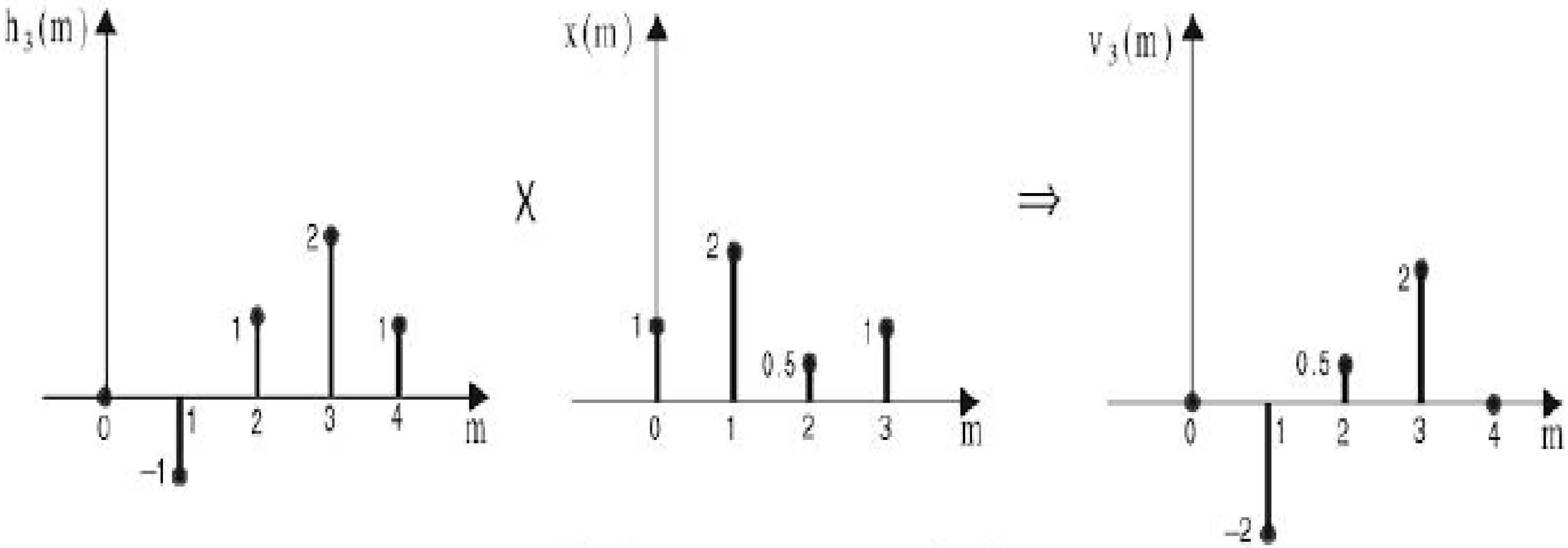


Fig 8 : Computation of y(3).

The sum of product sequence $v_3(m)$ gives y(3). $\therefore\ y(3) = -2 + 0.5 + 2 = 0.5$

When n = 4 ; $y(4) = \sum_{m=-\infty}^{+\infty} x(m)\, h(4-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_4(m) = \sum_{m=-\infty}^{+\infty} v_4(m)$

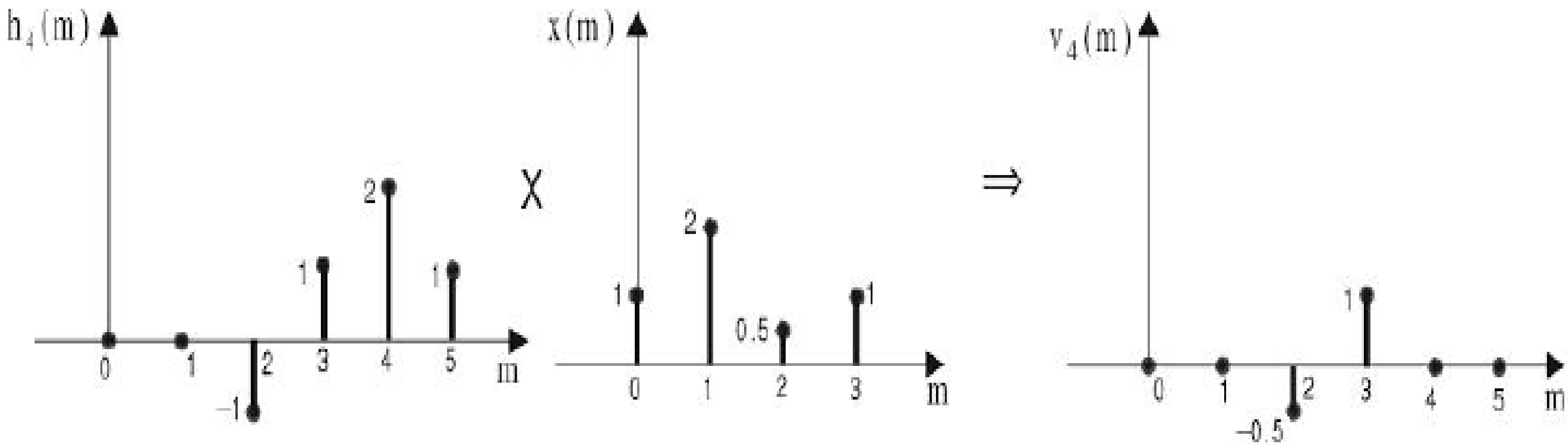


Fig 9 : Computation of y(4).

The sum of product sequence $v_4(m)$ gives $y(4)$. $\therefore\ y(4) = -0.5 + 1 = 0.5$

When n = 5 ; $y(5) = \sum_{m=-\infty}^{+\infty} x(m)\, h(5-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_5(m) = \sum_{m=-\infty}^{+\infty} v_5(m)$

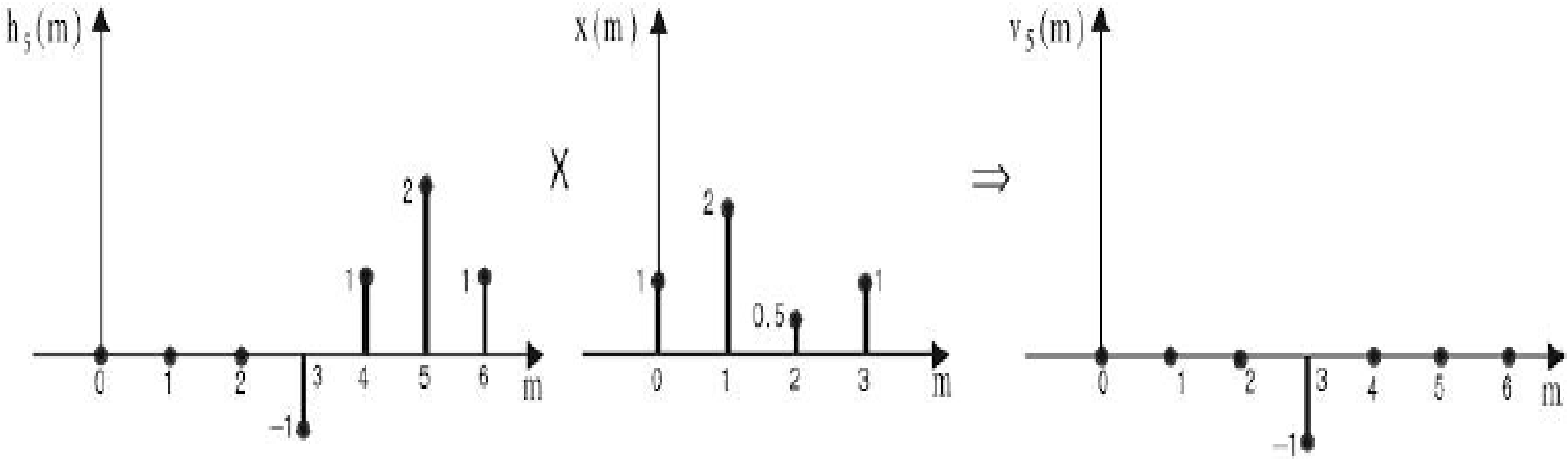


Fig 10 : Computation of y(5).

The sum of product sequence $v_5(m)$ gives $y(5)$. $\therefore\ y(5) = -1$

The output sequence, $y(n) = \left\{1,\ \underset{\uparrow}{4},\ 5.5,\ 3,\ 0.5,\ 0.5,\ -1\right\}$

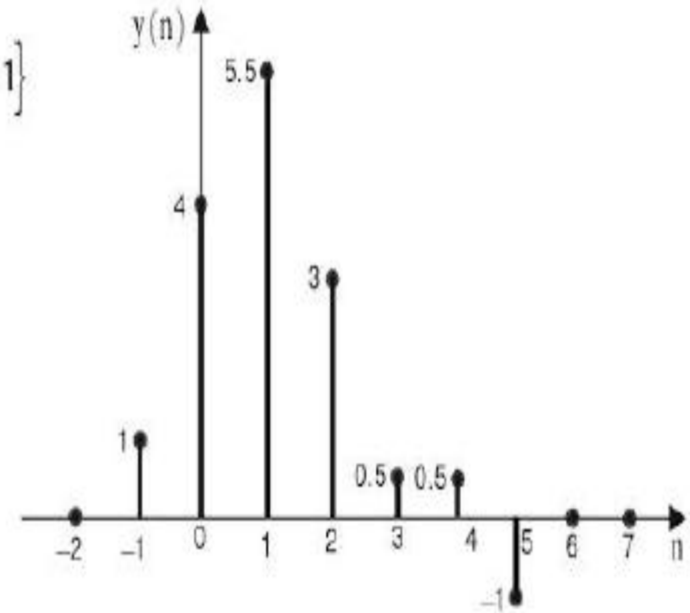


Fig 11 : Graphical representation of y(n).

Method 2 : Tabular Method

The given sequences and the shifted sequences can be represented in the tabular array as shown below.

Note : The unfilled boxes in the table are considered as zeros.

m	–3	–2	–1	0	1	2	3	4	5	6
x(m)				1	2	0.5	1			
h(m)			1	2	1	–1				
h(–m)		–1	1	2	1					
$h(-1-m) = h_{-1}(m)$	–1	1	2	1						
$h(0-m) = h_0(m)$		–1	1	2	1					
$h(1-m) = h_1(m)$			–1	1	2	1				
$h(2-m) = h_2(m)$				–1	1	2	1			
$h(3-m) = h_3(m)$					–1	1	2	1		
$h(4-m) = h_4(m)$						–1	1	2	1	
$h(5-m) = h_5(m)$							–1	1	2	1

$$y(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_n(m), \quad \text{where } h_n(m) = h(n-m)$$

When n = −1 ; $y(-1) = \sum_{m=-3}^{3} x(m)\, h_{-1}(m)$

$\boxed{\because \text{ The product is valid only for } m = -3 \text{ to } +3.}$

$$= x(-3)\, h_{-1}(-3) + x(-2) h_{-1}(-2) + x(-1) h_{-1}(-1) + x(0)\, h_{-1}(0) + x(1)\, h_{-1}(1) + x(2)\, h_{-1}(2) + x(3)\, h_{-1}(3)$$

$$= 0 + 0 + 0 + 1 + 0 + 0 + 0 = 1$$

The samples of y(n) for other values of n are calculated as shown for n = −1.

When n = 0 ; $y(0) = \sum_{m=-2}^{3} x(m)\,h_0(m) = 0 + 0 + 2 + 2 + 0 + 0 = 4$

When n = 1 ; $y(1) = \sum_{m=-1}^{3} x(m)\,h_1(m) = 0 + 1 + 4 + 0.5 + 0 = 5.5$

When n = 2 ; $y(2) = \sum_{m=0}^{3} x(m)\,h_2(m) = -1 + 2 + 1 + 1 = 3$

When n = 3 ; $y(3) = \sum_{m=0}^{4} x(m)\,h_3(m) = 0 - 2 + 0.5 + 2 + 0 = 0.5$

When n = 4 ; $y(4) = \sum_{m=0}^{5} x(m)\,h_4(m) = 0 + 0 - 0.5 + 1 + 0 + 0 = 0.5$

When n = 5 ; $y(5) = \sum_{m=0}^{6} x(m)\,h_5(m) = 0 + 0 + 0 - 1 + 0 + 0 + 0 = -1$

The output sequence, $y(n) = \{1,\ \underset{\uparrow}{4},\ 5.5,\ 3,\ 0.5,\ 0.5,\ -1\}$

Method 3 : Matrix Method

The input sequence x(n) is arranged as a column and the impulse response is arranged as a row as shown below. The elements of the two-dimensional array are obtained by multiplying the corresponding row element with the column element. The sum of the diagonal elements gives the samples of y(n).

$$
\begin{array}{c|cccc}
{}_{x(n)\downarrow}\diagdown^{h(n)\rightarrow} & 1 & 2 & 1 & -1 \\
\hline
1 & 1\times 1 & 1\times 2 & 1\times 1 & 1\times(-1) \\
2 & 2\times 1 & 2\times 2 & 2\times 1 & 2\times(-1) \\
0.5 & 0.5\times 1 & 0.5\times 2 & 0.5\times 1 & 0.5\times(-1) \\
1 & 1\times 1 & 1\times 2 & 1\times 1 & 1\times(-1)
\end{array}
\Rightarrow
\begin{array}{c|cccc}
{}_{x(n)\downarrow}\diagdown^{h(n)\rightarrow} & 1 & 2 & 1 & -1 \\
\hline
1 & 1 & 2 & 1 & -1 \\
2 & 2 & 4 & 2 & -2 \\
0.5 & 0.5 & 1 & 0.5 & -0.5 \\
1 & 1 & 2 & 1 & -1
\end{array}
$$

$y(-1) = 1$

$y(0) = 2 + 2 = 4$

$y(1) = 0.5 + 4 + 1 = 5.5$

$y(2) = 1 + 1 + 2 + (-1) = 3$

$y(3) = 2 + 0.5 + (-2) = 0.5$

$y(4) = 1 + (-0.5) = 0.5$

$y(5) = -1$

$\therefore y(n) = \{1, \underset{\uparrow}{4}, 5.5, 3, 0.5, 0.5, -1\}$

Circular Convolution

The ***circular convolution*** of two periodic discrete time sequences $x_1(n)$ and $x_2(n)$ with periodicity of N samples is defined as,

$$\boxed{x_3(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N} \quad \text{or} \quad \boxed{x_3(n) = \sum_{m=0}^{N-1} x_2(m)\, x_1((n-m))_N} \qquad .....(2.57)$$

where, $x_3(n)$ is the sequence obtained by circular convolution,

$x_1((n-m))_N$ represents circular shift of $x_1(n)$

$x_2((n-m))_N$ represents circular shift of $x_2(n)$

m is a dummy variable.

$$x_3(n) = x_1(n) \circledast x_2(n) = x_2(n) \circledast x_1(n)$$

Circular Convolution-Procedure

Let, $x_1(n)$ and $x_2(n)$ be periodic discrete time sequences with periodicity of N-samples. If $x_1(n)$ and $x_2(n)$ are non-periodic then convert the sequences to N-sample sequences and periodically extend the sequence $x_2(n)$ with periodicity of N-samples.

1. Change of index : Change the index n in the sequences $x_1(n)$ and $x_2(n)$, in order to get the sequences $x_1(m)$ and $x_2(m)$. Represent the samples of one period of the sequences on circles.

2. Folding : Fold $x_2(m)$ about $m = 0$, to obtain $x_2(-m)$.

3. Rotation : Rotate $x_2(-m)$ by q times in anti-clockwise if q is positive, rotate $x_2(-m)$ by q times in clockwise if q is negative to obtain $x_2((q-m))_N$.

4. Multiplication : Multiply $x_1(m)$ by $x_2((q-m))_N$ to get a product sequence. Let the product sequence be $v_q(m)$. Now, $v_q(m) = x_1(m) \times x_2((q-m))_N$.

5. Summation : Sum up the samples of one period of the product sequence $v_q(m)$ to obtain the value of $x_3(n)$ at $n = q$. [i.e., $x_3(q)$].

Perform circular convolution of the two sequences, $x_1(n) = \{2, 1, 2, -1\}$ and $x_2(n) = \{1, 2, 3, 4\}$

Method 1: Graphical Method of Computing Circular Convolution

The circular convolution of $x_1(n)$ and $x_2(n)$ is given by,

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N = \sum_{m=0}^{N-1} x_1(m)\, x_{2,n}(m)$$

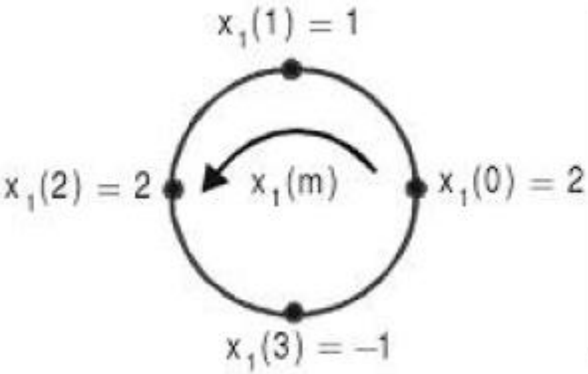


Fig 1.

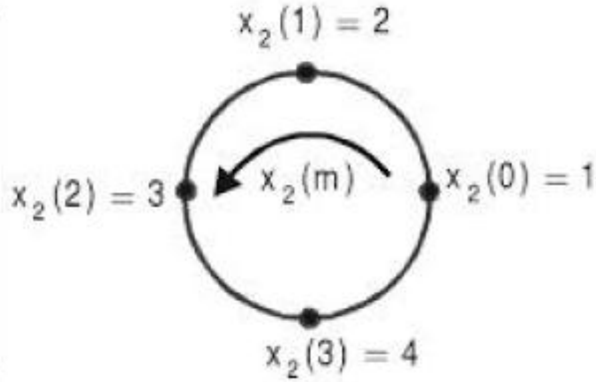


Fig 2.

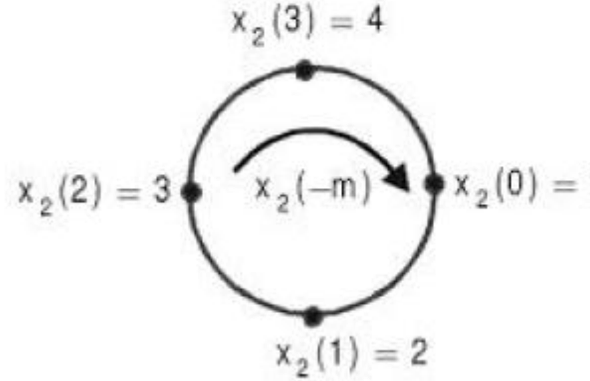


Fig 3.

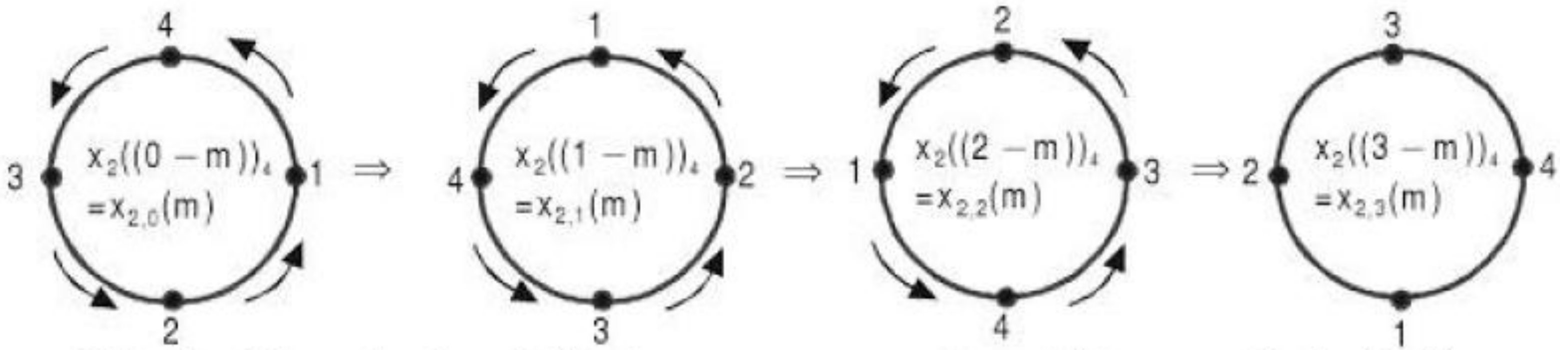


Fig 4: *Circularly shifted sequences $x_2(-m)$ for $n = 0, 1, 2, 3$.*

The given sequences are 4-point sequences . \ N = 4.

Each sample of $x_3(n)$ is given by sum of the samples of product sequence defined by the equation,

$$x_3(n) = \sum_{m=0}^{3} x_1(m)\ x_{2,n}(m) = \sum_{m=0}^{3} v_n(m) \quad ; \quad \text{where } v_n(m) = x_1(m)\ x_{2,n}(m) \qquad .....(1)$$

Using the above equation (1), graphical method of computing each sample of $x_3(n)$ are shown in fig 5 to fig 8.

$$\text{When } n = 0\ ;\ x_3(0) = \sum_{m=0}^{3} x_1(m)\ x_2((0-m))_4 = \sum_{m=0}^{3} x_1(m)\ x_{2,0}(m) = \sum_{m=0}^{3} v_0(m)$$

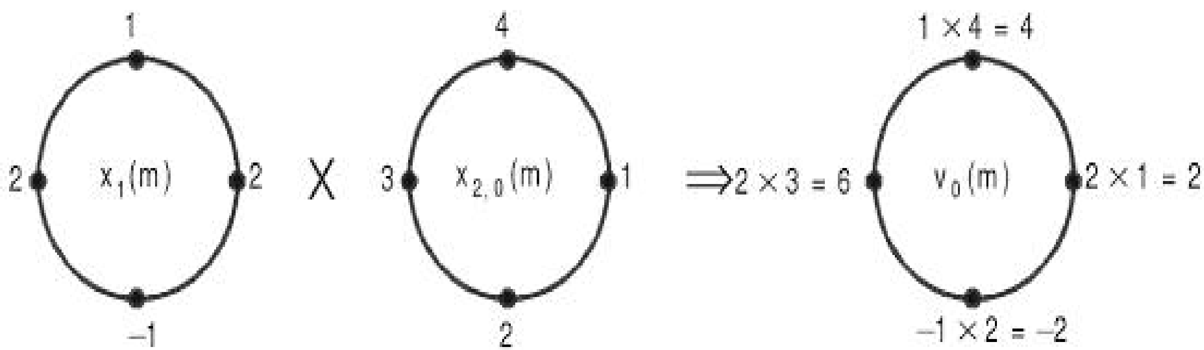


Fig 5: Computation of $x_3(0)$.

The sum of samples of $v_0(m)$ gives $x_3(0)$

$\therefore\ x_3(0) = 2 + 4 + 6 - 2 = 10$

$$\text{When } n = 1\ ;\ x_3(1) = \sum_{m=0}^{3} x_1(m)\, x_2((1-m))_4 = \sum_{m=0}^{3} x_1(m)\, x_{2,1}(m) = \sum_{m=0}^{3} v_1(m)$$

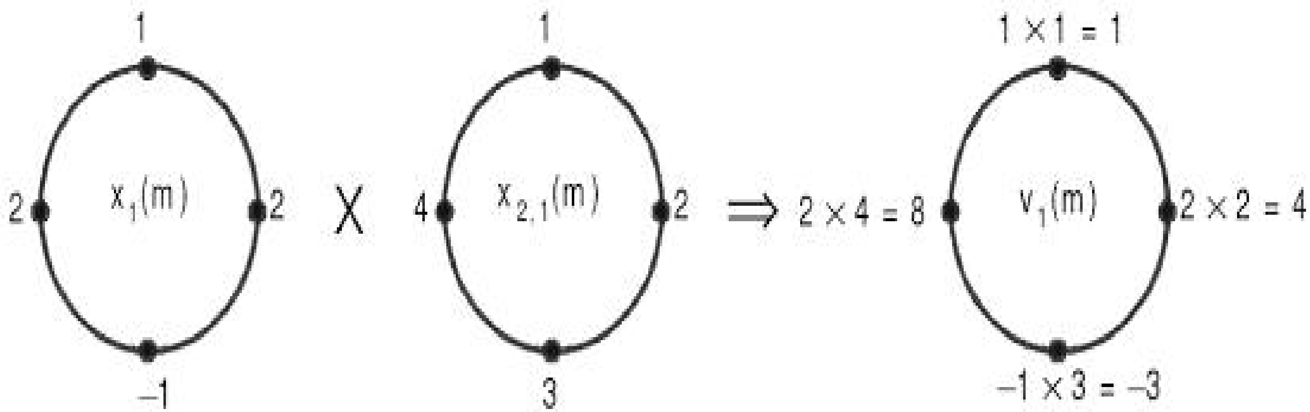
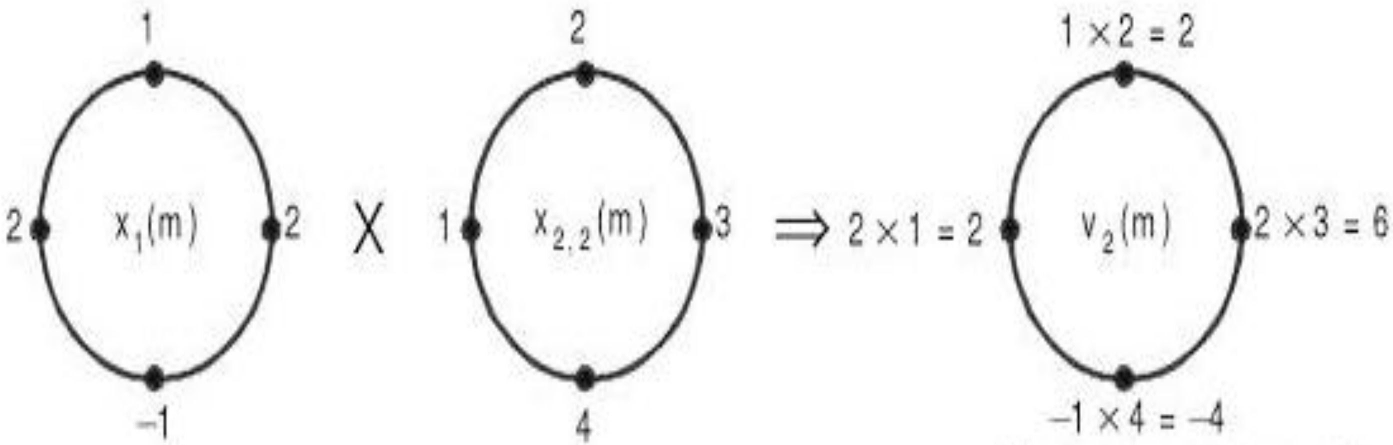


Fig 6: Computation of $x_3(1)$.

The sum of samples of $v_1(m)$ gives $x_3(1)$

$\therefore\ x_3(1) = 4 + 1 + 8 - 3 = 10$

When n = 2 ; $x_3(2) = \sum_{m=0}^{3} x_1(m)\, x_2((2-m))_4 = \sum_{m=0}^{3} x_1(m)\, x_{2,2}(m) = \sum_{m=0}^{3} v_2(m)$



The sum of samples of $v_2(m)$ gives $x_3(2)$

Fig 7: Computation of $x_3(2)$.

$\therefore\ x_3(2) = 6 + 2 + 2 - 4 = 6$

When n = 3 ; $x_3(3) = \sum_{m=0}^{3} x_1(m)\, x_2((3-m))_4 = \sum_{m=0}^{3} x_1(m)\, x_{2,3}(m) = \sum_{m=0}^{3} v_3(m)$

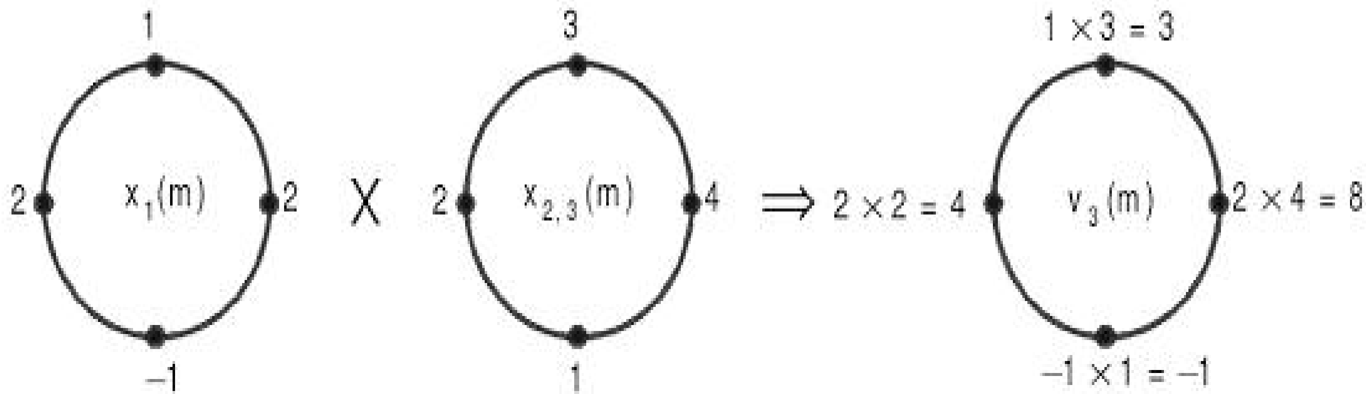


Fig 8: Computation of $x_3(3)$.

The sum of samples of $v_3(m)$ gives $x_3(3)$

$\therefore \quad x_3(3) = 8 + 3 + 4 - 1 = 14$

\ $x_3(n) = \{10, 10, 6, 14\}$

Method 2 : Circular Convolution Using Tabular Array

Note : The boldfaced numbers are samples obtained by periodic extension.

m	–3	–2	–1	0	1	2	3
$x_1(m)$				2	1	2	–1
$x_2(m)$				1	2	3	4
$x_2((-m))_4 = x_{2,0}(m)$	4	3	2	1	**4**	**3**	**2**
$x_2((1-m))_4 = x_{2,1}(m)$		4	3	2	1	**4**	**3**
$x_2((2-m))_4 = x_{2,2}(m)$			4	3	2	1	**4**
$x_2((3-m))_4 = x_{2,3}(m)$				4	3	2	1

When n = 0 ; $x_3(0) = \sum_{m=0}^{3} x_1(m)\, x_{2,0}(m)$

$$= x_1(0)\, x_{2,0}(0) + x_1(1)\, x_{2,0}(1) + x_1(2)\, x_{2,0}(2) + x_1(3)\, x_{2,0}(3)$$

$$= 2 \times 1 + 1 \times 4 + 2 \times 3 + (-1) \times 2 = 2 + 4 + 6 - 2 = 10$$

The samples of $x_3(n)$ for other values of n are calculated as shown for n = 0.

When n = 1; $x_3(1) = \sum_{m=0}^{3} x_1(m)\, x_{2,1}(m) = 4 + 1 + 8 - 3 = 10$

When n = 2; $x_3(2) = \sum_{m=0}^{3} x_1(m)\, x_{2,2}(m) = 6 + 2 + 2 - 4 = 6$

When n = 3; $x_3(3) = \sum_{m=0}^{3} x_1(m)\, x_{2,3}(m) = 8 + 3 + 4 - 1 = 14$

$\therefore \quad x_3(n) = \{\underset{\uparrow}{10},\ 10,\ 6,\ 14\}$

Method 3 : Circular Convolution Using Matrices

$$\begin{bmatrix} x_2(0) & x_2(3) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & x_2(3) \\ x_2(3) & x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \end{bmatrix} = \begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ x_3(3) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1\times 2+4\times 1+3\times 2+2\times -1 \\ 2\times 2+1\times 1+4\times 2+3\times -1 \\ 3\times 2+2\times 1+1\times 2+4\times -1 \\ 4\times 2+3\times 1+2\times 2+1\times -1 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 6 \\ 14 \end{bmatrix}$$

\ $x_3(n) = \{10, 10, 6, 14\}$

Perform the circular convolution of the two sequences $x_1(n)$ and $x_2(n)$, where,

$$x_1(n) = \{0.2,\ 0.4,\ 0.6,\ 0.8,\ 1.0,\ 1.2,\ 1.4,\ 1.6\}$$

$$x_2(n) = \{0.1,\ 0.3,\ 0.5,\ 0.7,\ 0.9,\ 1.1,\ 1.3,\ 1.5\}$$

m	–7	–6	–5	–4	–3	–2	–1	0	1	2	3	4	5	6	7
$x_1(m)$								0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
$x_2(m)$								0.1	0.3	0.5	0.7	0.9	1.1	1.3	1.5
$x_2((-m))_8 = x_{2,0}(m)$	1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	1.5	**1.3**	**1.1**	**0.9**	**0.7**	**0.5**	**0.3**
$x_2((1-m))_8 = x_{2,1}(m)$		1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	1.5	**1.3**	**1.1**	**0.9**	**0.7**	**0.5**
$x_2((2-m))_8 = x_{2,2}(m)$			1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	1.5	**1.3**	**1.1**	**0.9**	**0.7**
$x_2((3-m))_8 = x_{2,3}(m)$				1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	1.5	**1.3**	**1.1**	**0.9**
$x_2((4-m))_8 = x_{2,4}(m)$					1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	1.5	**1.3**	**1.1**
$x_2((5-m))_8 = x_{2,5}(m)$						1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	1.5	**1.3**
$x_2((6-m))_8 = x_{2,6}(m)$							1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	1.5
$x_2((7-m))_8 = x_{2,7}(m)$								1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1

$$\text{When } n = 0\,;\; x_3(n) = \sum_{m=0} x_1(m)\, x_2((0-m))_8 = \sum_{m=0} x_1(m)\, x_{2,0}(m)$$

$$= x_1(0)\, x_{2,0}(0) + x_1(1)\, x_{2,0}(1) + x_1(2)\, x_{2,0}(2) + x_1(3)\, x_{2,0}(3)$$

$$+ x_1(4)\, x_{2,0}(4) + x_1(5)\, x_{2,0}(5) + x_1(6)\, x_{2,0}(6) + x_1(7)\, x_{2,0}(7)$$

$$= 0.02 + 0.6 + 0.78 + 0.88 + 0.9 + 0.84 + 0.7 + 0.48 = 5.20$$

The samples of $x_3(n)$ for other values of n are calculated as shown for $n = 0$.

$$\text{When } n = 1;\; x_3(1) = \sum_{m=0}^{7} x_1(m)\, x_2((1-m))_8 = \sum_{m=0}^{7} x_1(m)\, x_{2,1}(m) = 6.00$$

$$\text{When } n = 2;\; x_3(2) = \sum_{m=0}^{7} x_1(m)\, x_2((2-m))_8 = \sum_{m=0}^{7} x_1(m)\, x_{2,2}(m) = 6.48$$

$$\text{When } n = 3;\; x_3(3) = \sum_{m=0}^{7} x_1(m)\, x_2((3-m))_8 = \sum_{m=0}^{7} x_1(m)\, x_{2,3}(m) = 6.64$$

$$\text{When } n = 4;\; x_3(4) = \sum_{m=0}^{7} x_1(m)\, x_2((4-m))_8 = \sum_{m=0}^{7} x_1(m)\, x_{2,4}(m) = 6.48$$

$$\text{When } n = 5;\; x_3(5) = \sum_{m=0}^{7} x_1(m)\, x_2((5-m))_8 = \sum_{m=0}^{7} x_1(m)\, x_{2,5}(m) = 6.00$$

$$\text{When } n = 6;\; x_3(6) = \sum_{m=0}^{7} x_1(m)\, x_2((6-m))_8 = \sum_{m=0}^{7} x_1(m)\, x_{2,6}(m) = 5.20$$

$$\text{When } n = 7;\; x_3(7) = \sum_{m=0}^{7} x_1(m)\, x_2((7-m))_8 = \sum_{m=0}^{7} x_1(m)\, x_{2,7}(m) = 4.08$$

$$\therefore x_3(n) = \{\underset{\uparrow}{5.20},\; 6.00,\; 6.48,\; 6.64,\; 6.48,\; 6.00,\; 5.20,\; 4.08\}$$

The input x(n) and impulse response h(n) of a LTI system are given by,

$$x(n) = \{\underset{\uparrow}{-1}, 1, 2, -2\} \ ; \ h(n) = \{0.5, \underset{\uparrow}{1}, -1, 2, 0.75\}$$

Determine the response of the system **a)** using linear convolution and **b)** using circular convolution.

m	-4	-3	-2	-1	0	1	2	3	4	5	6	7
$x(m)$					-1	1	2	-2				
$h(m)$				0.5	1	-1	2	0.75				
$h(-m)$		0.75	2	-1	1	0.5						
$h(-1-m) = h_{-1}(m)$	0.75	2	-1	1	0.5							
$h(0-m) = h_0(m)$		0.75	2	-1	1	0.5						
$h(1-m) = h_1(m)$			0.75	2	-1	1	0.5					
$h(2-m) = h_2(m)$				0.75	2	-1	1	0.5				
$h(3-m) = h_3(m)$					0.75	2	-1	1	0.5			
$h(4-m) = h_4(m)$						0.75	2	-1	1	0.5		
$h(5-m) = h_5(m)$							0.75	2	-1	1	0.5	
$h(6-m) = h_6(m)$								0.75	2	-1	1	0.5

i.e., $y(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_n(m)$

When n = –1 ; $y(-1) = \sum_{m=-4}^{3} x(m)\, h_{-1}(m)$

$$= x(-4)\, h_{-1}(-4) + x(-3)\, h_{-1}(-3) + x(-2)\, h_{-1}(-2) + x(-1)\, h_{-1}(-1) + x(0)\, h_{-1}(0) + x(1)\, h_{-1}(1) + x(2)\, h_{-1}(2) + x(3)\, h_{-1}(3)$$

$$= 0 + 0 + 0 + 0 + (-0.5) + 0 + 0 + 0 = -0.5$$

The samples of y(n) for other values of n are calculated as shown for n = –1.

When n = 0 ; $y(0) = \sum_{m=-3}^{3} x(m)\, h_0(m) = 0 + 0 + 0 + (-1) + 0.5 + 0 + 0 = -0.5$

When n = 1 ; $y(1) = \sum_{m=-2}^{3} x(m)\, h_1(m) = 0 + 0 + 1 + 1 + 1 + 0 = 3$

When n = 2 ; $y(2) = \sum_{m=-1}^{3} x(m)\, h_2(m) = 0 + (-2) + (-1) + 2 + (-1) = -2$

When $\mathbf{n} = 3$; $y(3) = \sum_{m=0}^{4} x(m)\, h_3(m) = -0.75 + 2 + (-2) + (-2) + 0 = -2.75$

When $\mathbf{n} = 4$; $y(4) = \sum_{m=0}^{5} x(m)\, h_4(m) = 0 + 0.75 + 4 + 2 + 0 + 0 = 6.75$

When $\mathbf{n} = 5$; $y(5) = \sum_{m=0}^{6} x(m)\, h_5(m) = 0 + 0 + 1.5 + (-4) + 0 + 0 + 0 = -2.5$

When $\mathbf{n} = 6$; $y(6) = \sum_{m=0}^{7} x(m)\, h_6(m) = 0 + 0 + 0 + (-1.5) + 0 + 0 + 0 + 0 = -1.5$

The response of LTI system y(n) is,

$$y(n) = \{-0.5, \underset{\uparrow}{-0.5}, 3, -2, -2.75, 6.75, -2.5, -1.5\}$$

b) Response of LTI System Using Circular Convolution

m	–7	–6	–5	–4	–3	–2	–1	0	1	2	3	4	5	6	7
$x(m)$							0	–1	1	2	–2	0	0	0	
$h(m)$							0.5	1	–1	2	0.75	0	0	0	
$h(-m)$		0	0	0	0.75	2	–1	1	0.5						
$h((-1-m))_8 = h_{-1}(m)$	0	0	0	0.75	2	–1	1	0.5	0	0	0	0.75	2	–1	1
$h((0-m))_8 = h_0(m)$		0	0	0	0.75	2	–1	1	0.5	0	0	0	0.75	2	–1
$h((1-m))_8 = h_1(m)$			0	0	0	0.75	2	–1	1	0.5	0	0	0	0.75	2
$h((2-m))_8 = h_2(m)$				0	0	0	0.75	2	–1	1	0.5	0	0	0	0.75
$h((3-m))_8 = h_3(m)$					0	0	0	0.75	2	–1	1	0.5	0	0	0
$h((4-m))_8 = h_4(m)$						0	0	0	0.75	2	–1	1	0.5	0	0
$h((5-m))_8 = h_5(m)$							0	0	0	0.75	2	–1	1	0.5	0
$h((6-m))_8 = h_6(m)$	0	0	0.75	2	–1	1	0.5	0	0	0	0.75	2	–1	1	0.5

When $n = -1$; $y(-1) = \sum_{m=-1}^{6} x(m)\, h_{-1}(n) = x(-1)\, h_{-1}(-1) + x(0)\, h_{-1}(0) + x(1)\, h_{-1}(1) + x(2)\, h_{-1}(2)$

$$+ x(3)\, h_{-1}(3) + x(4)\, h_{-1}(4) + x(5)\, h_{-1}(5) + x(6)\, h_{-1}(6)$$

$$= 0 + (-0.5) + 0 + 0 + 0 + 0 + 0 + 0 = -0.5$$

The samples of $y(n)$ for other values of n are calculated as shown for $n = -1$.

When $n = 0$; $y(0) = \sum_{m=-1}^{6} x(m)\, h_0 m = 0 + (-1) + 0.5 + 0 + 0 + 0 + 0 + 0 = -0.5$

When $n = 1$; $y(1) = \sum_{m=-1}^{6} x(m)\, h_1 m = 0 + 1 + 1 + 1 + 0 + 0 + 0 + 0 = 3$

When $n = 2$; $y(2) = \sum_{m=-1}^{6} x(m)\, h_2 m = 0 + (-2) + (-1) + 2 + (-1) + 0 + 0 + 0 = -2$

When $n = 3$; $y(3) = \sum_{m=-1}^{6} x(m)\, h_3 m = 0 + (-0.75) + 2 + (-2) + (-2) + 0 + 0 + 0 = -2.75$

When $n = 4$; $y(4) = \sum_{m=-1}^{6} x(m)\, h_4 m = 0 + 0 + 0.75 + 4 + 2 + 0 + 0 + 0 = 6.75$

When $n = 5$; $y(5) = \sum_{m=-1}^{6} x(m)\, h_5 m = 0 + 0 + 0 + 1.5 + (-4) + 0 + 0 + 0 = -2.5$

When $n = 6$; $y(6) = \sum_{m=-1}^{6} x(m)\, h_6 m = 0 + 0 + 0 + 0 + (-1.5) + 0 + 0 + 0 = -1.5$

The response of LTI system $y(n)$ is,

$$y(n) = \{-0.5, \underset{\uparrow}{-0.5}, 3, -2, -2.75, 6.75, -2.5, -1.5\}$$

Correlation

- Used when two signals are to be compared.
- Measure of degree to which two signals are similar.
- **Cross Correlation** – correlation of two separate signals
- **Auto Correlation** – correlation of signal with itself.

Cross Correlation

- Cross-correlation of x(n) and y(n) is a sequence, $r_{xy}(l)$

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\,y(n-l) \qquad l = 0, \pm 1, \pm 2, \ldots$$

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n-l)\,x(n) \qquad l = 0, \pm 1, \pm 2, \ldots$$

- => $r_{xy}(l) = r_{yx}(-l)$

Cross Correlation

- Number of Samples = N1+N2-1
- Start of Sequence mi = n1-(n2+N2-1)
- End of Sequence mf = mi+(N1+N2-2)

Auto Correlation

- Correlation of a signal with itself

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)x(n-l) = r_{xx}(-l) \qquad l = 0, \pm 1, \pm 2, \ldots$$

Auto Correlation

- Number of Samples = 2N-1
- Start of Sequence

  initial value of m:mi = -(N-1)

- End of Sequence

  final value of m: mf = mi+(2N-2)

Perform crosscorrelation of the sequences, $x(n) = \{1, 1, 2, 2\}$ and $y(n) = \{1, 0.5, 1\}$.

Solution

Let $r_{xy}(m)$ be the crosscorrelation sequence obtained by crosscorrelation of x(n) and y(n).

The crosscorrelation sequence $r_{xy}(m)$ is given by,

$$r_{xy} = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-m)$$

The x(n) starts at n = 0 and has 4 samples.

$$\therefore\ n_1 = 0,\ N_1 = 4$$

The y(n) starts at n = 0 and has 3 samples.

$$\therefore\ n_2 = 0,\ N_2 = 3$$

Now, $r_{xy}(m)$ will have $N_1 + N_2 - 1 = 4 + 3 - 1 = 6$ samples.

The initial value of $m = m_i = n_1 - (n_2 + N_2 - 1)$
$= 0 - (0 + 3 - 1) = -2$

The final value of $m = m_f = m_i + (N_1 + N_2 - 2)$
$= -2 + (4 + 3 - 2) = 3$

Method 1 : Graphical Method

The graphical representation of x(n) and y(n) are shown below.

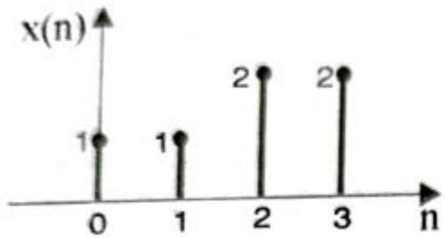


Fig 1.

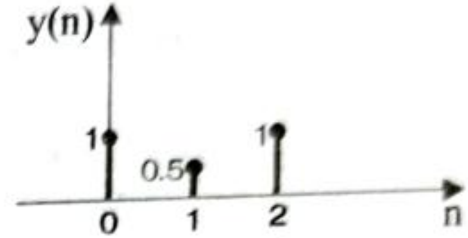
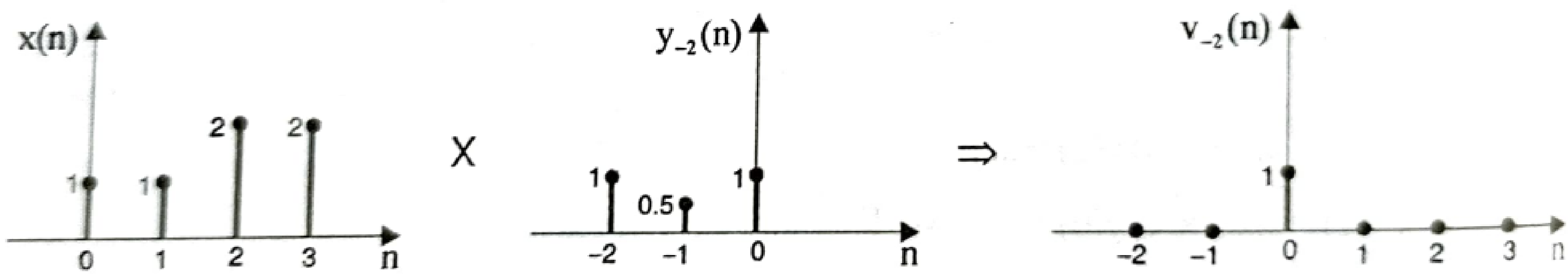


Fig 2.

The 6 samples of $r_{xy}(m)$ are computed using the equation,

$$r_{xy}(m) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-m) = \sum_{n=-\infty}^{+\infty} x(n)\, y_m(n)\ ;\ \text{where } y_m(n) = y(n-m)$$

When m = −2 ; $r_{xy}(-2) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-(-2)) = \sum_{n=-\infty}^{+\infty} x(n)\, y_{-2}(n) = \sum_{n=-\infty}^{+\infty} v_{-2}(n)$



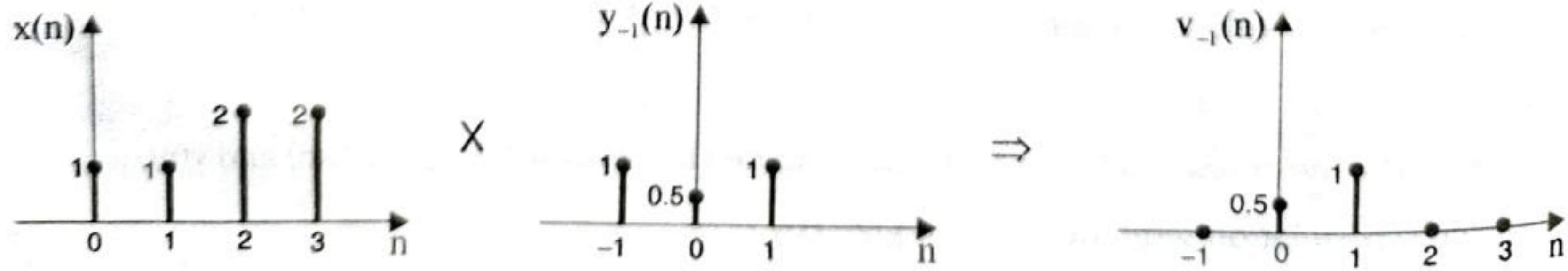
The sum of product sequence

$v_{-2}(n)$ gives $r_{xy}(-2)$

Fig 3: Computation of $r_{xy}(-2)$.

$\therefore\ r_{xy}(-2) = 0 + 0 + 1 + 0 + 0 + 0 = 1$

When $m = -1$; $r_{xy}(-1) = \sum\limits_{n=-\infty}^{+\infty} x(n)\, y(n-(-1)) = \sum\limits_{n=-\infty}^{+\infty} x(n)\, y_{-1}(n) = \sum\limits_{n=-\infty}^{+\infty} v_{-1}(n)$

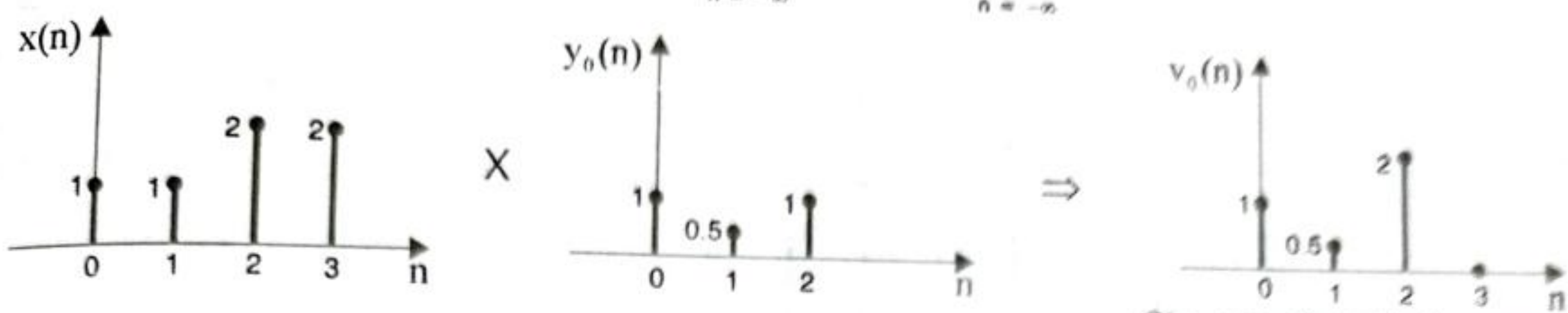


The sum of product sequence $v_{-1}(n)$ gives $r_{xy}(-1)$

$\therefore\ r_{xy}(-1) = 0 + 0.5 + 1 + 0 + 0 = 1.5$

Fig 4 : Computation of $r_{xy}(-1)$.

When $m = 0$; $r_{xy}(0) = \sum\limits_{n=-\infty}^{+\infty} x(n)\, y(n) = \sum\limits_{n=-\infty}^{+\infty} x(n)\, y_0(n) = \sum\limits_{n=-\infty}^{+\infty} v_0(n)$

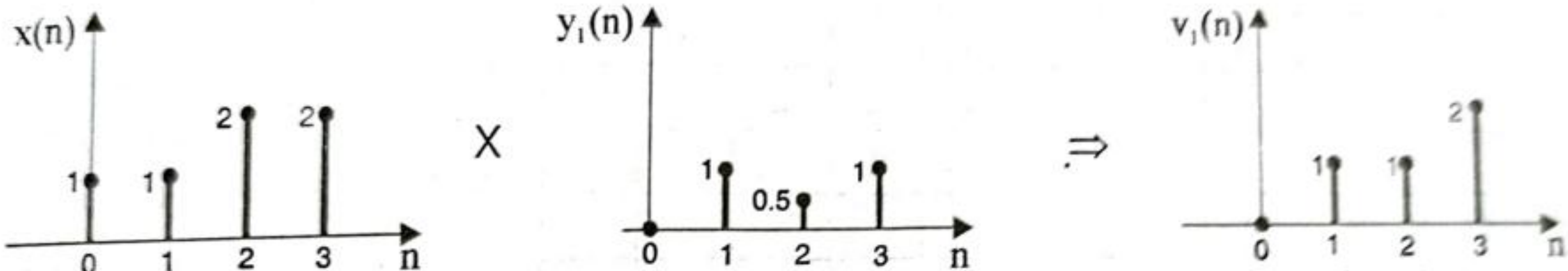


The sum of product sequence $v_0(n)$ gives $r_{xy}(0)$

$\therefore\ r_{xy}(0) = 1 + 0.5 + 2 + 0 = 3.5$

Fig 5 : Computation of $r_{xy}(0)$.

When m = 1 ; $r_{xy}(1) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-1)) = \sum_{n=-\infty}^{+\infty} x(n)\, y_1(n) = \sum_{n=-\infty}^{+\infty} v_1(n)$

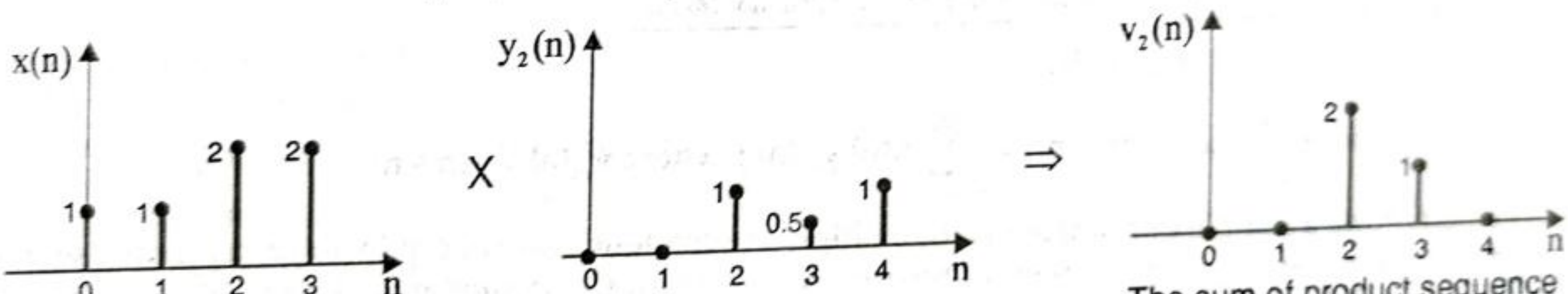


The sum of product sequence $v_1(n)$ gives $r_{xy}(1)$

$\therefore\ r_{xy}(1) = 0 + 1 + 1 + 2 = 4$

Fig 6 : Computation of $r_{xy}(1)$.

When m = 2 ; $r_{xy}(2) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-2) = \sum_{n=-\infty}^{+\infty} x(n)\, y_2(n) = \sum_{n=-\infty}^{+\infty} v_2(n)$

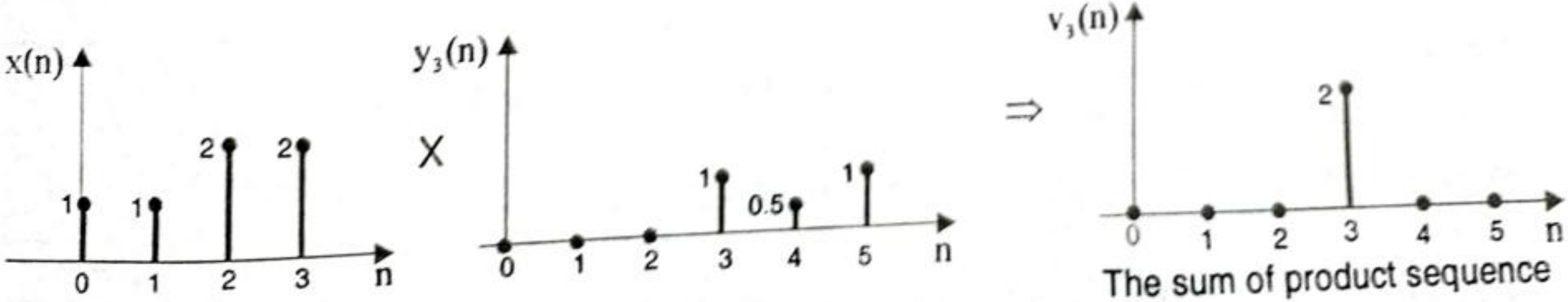


The sum of product sequence $v_2(n)$ gives $r_{xy}(2)$

$\therefore\ r_{xy}(2) = 0 + 0 + 2 + 1 + 0 = 3$

Fig 7 : Computation of $r_{xy}(2)$.

When m = 3 ; $r_{xy}(3) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-3) = \sum_{n=-\infty}^{+\infty} x(n)\, y_3(n) = \sum_{n=-\infty}^{+\infty} v_3(n)$



The sum of product sequence $v_3(n)$ gives $r_{xy}(3)$

Fig 8 : Computation of $r_{xy}(3)$.

$\therefore\ r_{xy}(3) = 0 + 0 + 0 + 2 + 0 + 0 = 2$

The crosscorrelation sequence, $r_{xy}(m) = \{1, 1.5, \underset{\uparrow}{3.5}, 4, 3, 2\}$

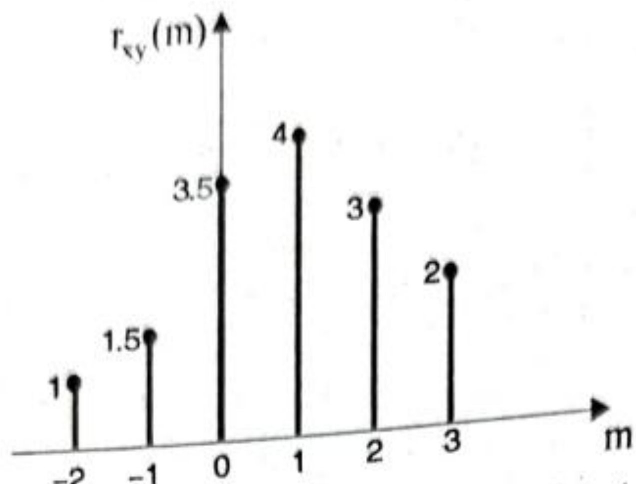


Fig 9 : Graphical representation of $r_{xy}(m)$.

Method 2: Tabular Method

n	−2	−1	0	1	2	3	4	5
$x(n)$			1	1	2	2		
$y(n)$			1	0.5	1			
$y(n-(-2)) = y_{-2}(n)$	1	0.5	1					
$y(n-(-1)) = y_{-1}(n)$		1	0.5	1				
$y(n) = y_0(n)$			1	0.5	1			
$y(n-1) = y_1(n)$				1	0.5	1		
$y(n-2) = y_2(n)$					1	0.5	1	
$y(n-3) = y_3(n)$						1	0.5	1

When $m = -2$; $r_{xy}(-2) = \sum_{n=-2}^{3} x(n)\, y_{-2}(n) = 0 + 0 + 1 + 0 + 0 + 0 = 1$

When $m = -1$; $r_{xy}(-1) = \sum_{n=-1}^{3} x(n)\, y_{-1}(n) = 0 + 0.5 + 1 + 0 + 0 \quad = 1.5$

When $m = 0$; $r_{xy}(0) = \sum_{n=0}^{3} x(n)\, y_0(n) = 1 + 0.5 + 2 + 0 \quad = 3.5$

When $m = 1$; $r_{xy}(1) = \sum_{n=0}^{3} x(n)\, y_1(n) = 0 + 1 + 1 + 2 \quad = 4$

When $m = 2$; $r_{xy}(2) = \sum_{n=0}^{4} x(n)\, y_2(n) = 0 + 0 + 2 + 1 + 0 \quad = 3$

When $m = 3$; $r_{xy}(3) = \sum_{n=0}^{5} x(n)\, y_3(n) = 0 + 0 + 0 + 2 + 0 + 0 = 2$

$\therefore$ Crosscorrelation sequence, $r_{xy}(m) = \{1,\ 1.5,\ \underset{\uparrow}{3.5},\ 4,\ 3,\ 2\}$

Method 3: Matrix Method

Given that, $x(n) = \{\underset{\uparrow}{1}, 1, 2, 2\}$; $y(n) = \{\underset{\uparrow}{1}, 0.5, 1\}$; $\therefore\ y(-n) = \{1, 0.5, \underset{\uparrow}{1}\}$

The sequence x(n) is arranged as a column and the folded sequence y(–n) is arranged as a row as shown below. The elements of the two-dimensional array are obtained by multiplying the corresponding row element with column element. The sum of the diagonal elements gives the samples of the crosscorrelation sequence, $r_{xy}(m)$.

x(n) ↓ \ y(–n) →	1	0.5	1
1	1 × 1	1 × 0.5	1 × 1
1	1 × 1	1 × 0.5	1 × 1
2	2 × 1	2 × 0.5	2 × 1
2	2 × 1	2 × 0.5	2 × 1

⇒

x(n) ↓ \ y(–n) →	1	0.5	1
1	1	0.5	1
1	1	0.5	1
2	2	1	2
2	2	1	2

$r_{xy}(-2) = 1$; $r_{xy}(-1) = 1 + 0.5 = 1.5$; $r_{xy}(0) = 2 + 0.5 + 1 = 3.5$

$r_{xy}(1) = 2 + 1 + 1 = 4$; $r_{xy}(2) = 1 + 2 = 3$; $r_{xy}(3) = 2$

$\therefore\ r_{xy}(m) = \{1, 1.5, \underset{\uparrow}{3.5}, 4, 3, 2\}$

Determine the autocorrelation sequence for $x(n) = \{1, 2, 3, 4\}$.

Solution

Let, $r_{xx}(m)$ be the autocorrelation sequence.

The autocorrelation sequence $r_{xx}(m)$ is given by,

$$r_{xx}(m) = \sum_{n=-\infty}^{+\infty} x(n)\, x(n-m)$$

The $x(n)$ starts at $n = 0$ and has 4 samples.

$$\therefore n_x = 0 \quad \text{and} \quad N = 4$$

Now, $r_{xx}(m)$ will have, $2N - 1 = 2 \times 4 - 1 = 7$ samples.

The initial value of $m = m_i = -(N-1) = -(4-1) = -3$

The final value of $m = m_f = m_i + (2N - 2) = -3 + (2 \times 4 - 2) = 3$

n	−3	−2	−1	0	1	2	3	4	5	6
x(n)				1	2	3	4			
$x(n-(-3)) = x_{-3}(n)$	1	2	3	4						
$x(n-(-2)) = x_{-2}(n)$		1	2	3	4					
$x(n-(-1)) = x_{-1}(n)$			1	2	3	4				
$x(n) = x_0(n)$				1	2	3	4			
$x(n-1) = x_1(n)$					1	2	3	4		
$x(n-2) = x_2(n)$						1	2	3	4	
$x(n-3) = x_3(n)$							1	2	3	4

When $m = -3$; $r_{xx}(-3) = \sum_{n=-3}^{3} x(n)\, x_{-3}(n) = 0 + 0 + 0 + 4 + 0 + 0 + 0 = 4$

When $m = -2$; $r_{xx}(-2) = \sum_{n=-2}^{3} x(n)\, x_{-2}(n) = 0 + 0 + 3 + 8 + 0 + 0 \quad = 11$

When $m = -1$; $r_{xx}(-1) = \sum_{n=-1}^{3} x(n)\, x_{-1}(n) = 0 + 2 + 6 + 12 + 0 \quad = 20$

When $m = 0$; $r_{xx}(0) = \sum_{n=0}^{3} x(n)\, x_{0}(n) = 1 + 4 + 9 + 16 \quad = 30$

When $m = 1$; $r_{xx}(1) = \sum_{n=0}^{4} x(n)\, x_{1}(n) = 0 + 2 + 6 + 12 + 0 \quad = 20$

When $m = 2$; $r_{xx}(2) = \sum_{n=0}^{5} x(n)\, x_{2}(n) = 0 + 0 + 3 + 8 + 0 + 0 \quad = 11$

When $m = 3$; $r_{xx}(3) = \sum_{n=0}^{6} x(n)\, x_{3}(n) = 0 + 0 + 0 + 4 + 0 + 0 + 0 = 4$

$\therefore$ Autocorrelation sequence, $r_{xx}(m) = \{4,\ 11,\ 20,\ \underset{\uparrow}{30},\ 20,\ 11,\ 4\}$

THANK YOU