Rami Wail Shoula
ID: 201600112

Nonono 512
Assignment 3

$\boxed{I}$ $G(T) = e^{AT} = L^{-1}(SI-A)^{-1}$ | $H(T) = \left(\int_0^T e^{A\lambda} d\lambda\right) B$

Taking the taylor series (powerseries) of the matrix exponential:-

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \dots + \frac{1}{k!}A^kt^k + \dots = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

$$\frac{d}{dt}e^{At} = A + \frac{2}{2}A^2t + \frac{3}{3}\cdot\frac{1}{2!}A^3t^2 + \dots + \frac{A^k t^{k-1}}{(k-1)!} + \dots$$

$$= A\left[I + At + \frac{A^2t^2}{2!} + \dots + \frac{A^{k-1}t^{k-1}}{(k-1)!} + \dots\right] = Ae^{At}$$

Also, $e^{A(t+s)} = e^{At}e^{As}$

$$\therefore e^{At}e^{As} = \left(\sum_{k=0}^{\infty}\frac{A^kt^k}{k!}\right)\left(\sum_{k=0}^{\infty}\frac{A^ks^k}{k!}\right) = \sum_{k=0}^{\infty} A^k\left[\sum_{i=0}^{k}\frac{t^i s^{k-i}}{i!\,(k-i)!}\right]$$

$$= \sum_{k=0}^{\infty} A^k \frac{(t+s)^k}{k!} = e^{A(t+s)}$$

when $s=-t$: $e^{At}e^{-At} = e^{A(t-t)} = I$

$\therefore$ inverse of e^{At} is e^{-At} and $e^{(A+B)t} = e^{At}e^{Bt}$ $AB=BA$

however, $e^{(A+B)t} \neq e^{At}e^{Bt}$ $AB \neq BA$

$\dot{x} = Ax + Bu \rightarrow \dot{x}(t) - Ax(t) = Bu(t)$

$\therefore e^{-At}[\dot{x}(t) - Ax(t)] = e^{-At}Bu(t)$ $\therefore \frac{d}{dt}[e^{-At}x(t)] = e^{-At}Bu(t)$

$\therefore e^{-At}x(t) = x(0) + \int_0^t e^{-At}Bu(t)\,d\tau$

$\therefore x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(t)\,d\tau$

$\therefore x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(t)\,d\tau$

| $\dot{x} = Ax + Bu$
| $y = Cx + Du$

$x((k+1)T) = G(T)x(kT) + H(T)u(kT)$

$u(t) = u(kT)$ for $kT \le t < kT+T$

$x((k+1)T) = e^{A(k+1)T}x(0) + e^{A(k+1)T}\int_0^{(k+1)T} e^{-At}Bu(\tau)\,d\tau$ ... (1)

$x(kT) = e^{AkT}x(0) + e^{AkT}\int_0^{kT} e^{-A\tau}Bu(\tau)\,d\tau$

multiplying both sides by e^{AT} and subtracting it from (1) gives:

$x((k+1)T) = e^{AT}x(kT) + e^{A(k+1)T}\int_{kT}^{(k+1)T} e^{-A\tau}Bu(\tau)\,d\tau$

$x((k+1)T) = e^{AT}x(kT) + e^{AT}\int_0^T e^{-At}Bu(t)\,dt$

$= e^{AT}x(kT) + \int_0^T e^{A\lambda}Bu(kT)\,d\lambda$ where $\lambda = T - t$

define $G(T) = e^{AT}$

$H(T) = \left(\int_0^T e^{A\lambda}\,d\lambda\right) B$

$\therefore x((k+1)T) = G(T)x(kT) + H(T)u(kT)$

$\therefore y(kT) = Cx(kT) + Du(kT)$

Here C, D are constant matricis and do not depend on T

if matrix A is nonsingular, then $H(T)$ can be simplified to

$$H(T) = \left(\int_0^T e^{A\lambda} d\lambda\right) B = A^{-1}(e^{AT} - I)B = (e^{AT} - I)A^{-1}B$$

and $G(T) = e^{AT} = \mathcal{L}^{-1}(sI - A)^{-1}$