

Decoding performance

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The purpose here is to derive mathematically the expected performance of a decoder on pre-stimulus data in a ordered sequence of stimuli such as implemented in Demarchi et al. 2019.

Expected performance under null hypothesis

We assume the null hypothesis is true: there is only sensory but no predictive/anticipatory activity.

Let p be the probability the decoder finds the correct input. We assume $p_A = p_B = p_C = p_D$, i.e. the decoder performs equally on all stimuli types. p is the *sensitivity* of the classifier.

Based on the transition matrix of the ordered sequence in Demarchi et al. 2019, we consider the following pairs of consecutive stimuli and their occurrence rates :

S_{-1}	S_0	occurrence rate
A	A	0.25
B	A	0
C	A	0

S_{-1}	S_0	occurrence rate
D	A	0.75

Equiprobable errors

We further assume that when the classifier fails, it might return any other class with equal probability, i.e.

$$\forall Y \neq X, p(Y|X) = \frac{1-p}{3}$$

Testing on S_{-1}

Then for any S_0 , the probability that a decoder tested on S_{-1} finds the identity of S_0 in the ordered sequence is:

$$p(Y = S_0) = \frac{1}{4}p(S_0|S_{-1} = S_0) + \frac{3}{4}p(S_0|S_{-1} \neq S_0) = \frac{1}{4}p + \frac{3}{4}\frac{1-p}{3} = \frac{1}{4}$$

In conclusion, under the assumption of equiprobable errors, the performance of the decoder tested on the previous trial is **at chance level**.

Testing on S_{-2}

We can extend the same reasoning to testing on S_{-2} :

$$p(Y = S_0) = \frac{1}{16}p(S_0|S_{-2} = S_0) + \frac{15}{16}p(S_0|S_{-2} \neq S_0) = \frac{1}{16}p + \frac{15}{16}\frac{1-p}{3} = \frac{5-4p}{16}$$

Considering that $p > \frac{1}{4}$, the latter quantity is necessarily smaller than $\frac{1}{4}$, i.e. the decoder performs **below chance level**.

Non-equiprobable misses

In practice, we might expect pairs of stimuli to be more often confounded if they are physically close:

$$p(B|A) > p(C|A) > p(D|A)$$

Fictitious confusion matrix

A complete confusion matrix can be obtained at any latency from training and testing on random sequences. Here, for convenience, we parametrize the 3 possible error rates as having a geometric relationship:

$$p(B|A) = kp(C|A) = k^2p(D|A)$$

where $k > 1$

We can extend this logic to all others X and $p(Y \neq X|X)$, with the additional constraint that:

$$\sum_{Y \neq X} p(Y|X) = 1 - p$$

An example of a plausible confounding matrix with $p = 0.35$ (based on fig.2a in Demarchi et al. 2019) and $k = 2$:

.35	.372	.186	.093
.26	.35	.26	.13
.13	.26	.35	.26
.093	.186	.372	.35

However, this confusion matrix is not plausible as we end up with some error rates being larger than the true positive rate (.372 vs. .35). Let us try with $k = 1.5$:

.35	.308	.205	.137
.244	.35	.244	.163
.163	.244	.35	.244
.137	.205	.308	.35

Using this confusion matrix and the transition matrix, we can now calculate the probability that a decoder tested on S_{-1} finds the identity of S_0 in the ordered sequence:

$$\begin{aligned}
p(Y = S_0) &= \sum_{x \in \{A, B, C, D\}} p(Y = x|x) \\
&= \frac{1}{4} \left(\frac{1}{4} p(A|S_{-1} = A) + \frac{3}{4} p(A|S_{-1} = D) \right) + \dots + \frac{1}{4} \left(\frac{1}{4} p(D|S_{-1} = D) + \frac{3}{4} p(D|S_{-1} = C) \right) \\
&= \frac{1}{4} \left(\frac{1}{4} .35 + \frac{3}{4} .137 \right) + \frac{1}{4} \left(\frac{1}{4} .35 + \frac{3}{4} .308 \right) + \frac{1}{4} \left(\frac{1}{4} .35 + \frac{3}{4} .244 \right) + \frac{1}{4} \left(\frac{1}{4} .35 + \frac{3}{4} .244 \right) \\
&= \frac{1}{4} .35 + \frac{3}{16} (.137 + .308 + .244 + .244) \\
&= .2624
\end{aligned}$$

Which is above chance level, and even above the accuracy reported by Demarchi et al. 2019 (fig. 3) !

The unbalance can be made a little more dramatic with $k = 1.7$:

.35	.336	.198	.116
.251	.35	.251	.148
.148	.251	.35	.251
.116	.198	.336	.35

Now the probability that a decoder tested on S_{-1} finds the identity of S_0 in the ordered sequence is:

$$\begin{aligned}
p(Y = S_0) &= \sum_{x \in \{A, B, C, D\}} p(Y = x|x) \\
&= \frac{1}{4} \left(\frac{1}{4} \cdot .35 + \frac{3}{4} \cdot .116 \right) + \frac{1}{4} \left(\frac{1}{4} \cdot .35 + \frac{3}{4} \cdot .336 \right) + \frac{1}{4} \left(\frac{1}{4} \cdot .35 + \frac{3}{4} \cdot .251 \right) + \frac{1}{4} \left(\frac{1}{4} \cdot .35 + \frac{3}{4} \cdot .251 \right) \\
&= \frac{1}{4} \cdot .35 + \frac{3}{16} (.116 + .336 + .251 + .251) \\
&= .2664
\end{aligned}$$

which is even higher than before!

Observed confusion matrix

Training and testing on random sequences, we find the following confusion matrix:

.40	.26	.18	.23
.22	.29	.21	.18
.17	.25	.38	.25
.21	.20	.23	.35

Note that, surprisingly and unlike the fictitious matrices assumed above, stimuli A and D are confounded above chance level, as if they were neighbors.

Still, applying the same formula as above, we obtain:

$$\begin{aligned}
p(Y = S_0) &= \frac{1}{4} \left(\frac{1}{4} \cdot .40 + \frac{3}{4} \cdot .21 \right) + \frac{1}{4} \left(\frac{1}{4} \cdot .29 + \frac{3}{4} \cdot .26 \right) + \frac{1}{4} \left(\frac{1}{4} \cdot .38 + \frac{3}{4} \cdot .21 \right) + \frac{1}{4} \left(\frac{1}{4} \cdot .35 + \frac{3}{4} \cdot .25 \right) \\
&= .2631
\end{aligned}$$

which is very similar to results found with fictitious matrix.

Breaking the curse

In conclusion, we can find above chance decoding level in an ordered sequence even in absence of any predictive activity. This happened because the chosen ordering is such that the two stimuli that are most likely to follow one another are also the most confounded ones by the decoder, because they are the closest in sensory space. This condition may be broken in two ways:

- by using stimuli that are all equally distant in sensory space: this is not possible in auditory space but theoretically feasible in visual space
- by using transition probabilities that do not align with sensory distance. For example, the following transition matrix:

$$\begin{array}{cccc} .25 & 0 & .75 & 0 \\ 0 & .25 & 0 & .75 \\ .75 & 0 & .25 & 0 \\ 0 & .75 & 0 & .25 \end{array}$$

when combined with the putative confusion matrices above, yield a decoding performance of .2255 ($k = 1.5$) and .2173 ($k = 1.7$), i.e. *below* chance level. With the real confounding matrix, we find .2256, again very similar to results derived from fictitious matrices.

Expected performance in “fair” decoding

All Demarchi et al. and our calculations are based on the idea that predictive brain activity is reflected in the possibility to decode the current trial with pre-stimulus data. However, it is somewhat unfair to the brain which, having perceived S_{-1} , can only predict the most probable next stimulus \hat{S}_0 :

$$\hat{S}_0 = \operatorname{argmax}_{x \in \{A, B, C, D\}} (p(x|S_{-1}))$$

The identity of \hat{S}_0 is given by the transition matrix underlying the generation of the sequence of stimuli. In the highly ordered sequence (OR), $\hat{S}_0 = S_0$ in 75% of cases only.

A fair approach to detecting predictive brain activity would test the decoder at S_0 trying to decode \hat{S}_1 . Using the **observed** confusion matrix, we can estimate the performance under H_0 with fair decoding at 0.2325.