PH2101

Tutorial and Homework 1

January 15, 2016

1. **(Tutorial question)** Imagine a room containing 14 people, whose ages are as follows:
one person aged 14
one person aged 15
three people aged 16
two people aged 22
two people aged 24
five people aged 25

   (a) Compute the average of the squares of the ages $\langle j^2 \rangle$, and the square of the average of the ages $\langle j \rangle^2$.

   (b) Determine Δj for each j, and compute $\sigma = \sqrt{\langle (\Delta j)^2 \rangle}$.

   (c) Use your results in (a) and (b) to check that $\sigma = \sqrt{\langle j^2 \rangle - \langle j \rangle^2}$.

2. **(Tutorial question)** The needle on a broken car speedometer is free to swing, and bounces perfectly off the pins at either end, so that if you give it a flick, it is equally likely to come at rest at any angle between 0 and π.

   (a) What is the probability density, $\rho(\theta)$? Graph $\rho(\theta)$ as a function of θ, from $-\frac{\pi}{2}$ to $\frac{3\pi}{2}$.

   (b) Compute $\langle \theta \rangle$, $\langle \theta^2 \rangle$, and σ for this distribution.

   (c) Compute $\langle \sin(\theta) \rangle$, $\langle \cos(\theta) \rangle$, and $\langle \cos^2(\theta) \rangle$.

3. (**Homework Question**) Consider a Gaussian distribution,

$$\rho(x) = A\exp(-\lambda(x-a)^2) \tag{1}$$

where A, a and λ are constants.

(a) What is the value of A?

(b) Find $\langle x \rangle$, $\langle x^2 \rangle$, and σ

(c) Sketch the graph of $\rho(x)$

1. Let J be the age of the people.

J	14	15	16	22	24	25
N(J)	1	1	3	2	2	5
P(J=j)	$\frac{1}{14}$	$\frac{1}{14}$	$\frac{3}{14}$	$\frac{2}{14}$	$\frac{2}{14}$	$\frac{5}{14}$

Probability Distribution Fn (Table) (P.D.F.)

$$\sum_j N(j) = 1+1+3+2+2+5 = 14$$

(a) $\langle j^2 \rangle = \sum_{\forall j} j^2 P(J=j)$

$= 14^2\left(\frac{1}{14}\right) + 15^2\left(\frac{1}{14}\right) + 16^2\left(\frac{3}{14}\right) + 22^2\left(\frac{2}{14}\right) + 24^2\left(\frac{2}{14}\right) + 25^2\left(\frac{5}{14}\right)$

$= \frac{3217}{7}$

$\langle j \rangle^2 = \left[\sum_{\forall j} j P(J=j)\right]^2$

$= \left[14\left(\frac{1}{14}\right) + 15\left(\frac{1}{14}\right) + 16\left(\frac{3}{14}\right) + 22\left(\frac{2}{14}\right) + 24\left(\frac{2}{14}\right) + 25\left(\frac{5}{14}\right)\right]^2$

$= 21^2 = 441$

(b)

J	14	15	16	22	24	25
$\Delta j = j - \langle j \rangle$	-7	-6	-5	1	3	4
P(J=j)	$\frac{1}{14}$	$\frac{1}{14}$	$\frac{3}{14}$	$\frac{2}{14}$	$\frac{2}{14}$	$\frac{5}{14}$

1(b) $\sigma^2 = \langle (\Delta j)^2 \rangle = \sum_{\forall j} (\Delta j)^2 P(J=j)$

$= (-7)^2\left(\frac{1}{14}\right) + (-6)^2\left(\frac{1}{14}\right) + (-5)^2\left(\frac{3}{14}\right) + 1^2\left(\frac{2}{14}\right) + 3^2\left(\frac{2}{14}\right) + 4^2\left(\frac{5}{14}\right)$

$= \frac{130}{7}$

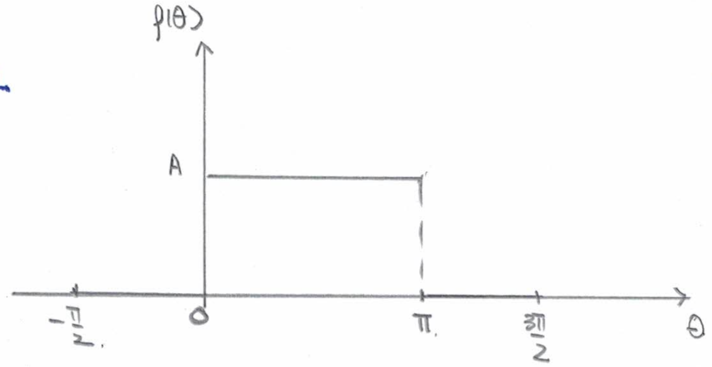
$\Rightarrow \quad \sigma = \sqrt{\frac{130}{7}}.$

(c) $\sigma^2 = \langle j^2 \rangle - \langle j \rangle^2$

$= \frac{3217}{7} - 441$

$= \frac{130}{7} \qquad \Rightarrow \qquad \sigma = \sqrt{\frac{130}{7}}$

2.



(a) $A \cdot (\pi) = 1 \Rightarrow A = \frac{1}{\pi}$.

$$p(\theta) = \begin{cases} \frac{1}{\pi} & 0 \leq \theta \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

$\theta \sim \mathcal{U}(0, \pi)$ Uniform cts dist^n.

(b) $\langle\theta\rangle = E(\theta) = \int_{-\infty}^{\infty} \theta \cdot p(\theta)\, d\theta$

$$= \frac{1}{\pi}\int_0^{\pi} \theta\, d\theta = \frac{1}{2\pi}\left[\theta^2\right]_0^{\pi} = \frac{1}{2\pi}(\pi^2) = \frac{\pi}{2}.$$

OR $E(\theta) = \langle\theta\rangle = \frac{\pi + 0}{2} = \frac{\pi}{2}$.

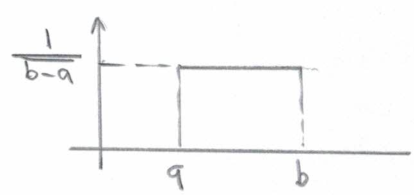
$$\langle\theta^2\rangle = E(\theta^2) = \int_{-\infty}^{\infty} \theta^2 p(\theta)\, d\theta = \frac{1}{\pi}\int_0^{\pi} \theta^2\, d\theta = \frac{1}{3\pi}\left[\theta^3\right]_0^{\pi} = \frac{\pi^2}{3}.$$

$$\sigma^2 = \text{Var}(\theta) = \langle\theta^2\rangle - \langle\theta\rangle^2 = \frac{\pi^2}{3} - \left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{12}.$$

$$\Rightarrow \sigma = \frac{\pi}{\sqrt{12}} = \frac{\pi}{2\sqrt{3}}.$$

Note: $X \sim \mathcal{U}(a, b)$ $E(X) = \frac{a+b}{2}$ $\text{Var}(X) = \frac{(b-a)^2}{12}$.

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$



Pf: $E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$

$$= \int_a^b \frac{x}{b-a}\, dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{b^2 - a^2}{2} \cdot \frac{1}{b-a} = \frac{a+b}{2}.$$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_a^b \frac{x^2}{b-a}\, dx$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b = \frac{b^3 - a^3}{3(b-a)}$$

$$= \frac{(b-a)(a^2 + ab + b^2)}{3(b-a)} = \frac{a^2 + ab + b^2}{3}$$

Hence $\quad Var(X) = E(X^2) - (E(X))^2$

$$= \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{4(a^2+ab+b^2) - 3(a^2+2ab+b^2)}{12}$$

$$= \frac{a^2-2ab+b^2}{12} = \frac{(b-a)^2}{12}$$

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2 (c) $\quad \langle \sin(\theta) \rangle = \int_{-\infty}^{\infty} \sin(\theta).\, \rho(\theta)\, d\theta = \int_0^{\pi} \frac{\sin\theta}{\pi}\, d\theta = \frac{-1}{\pi}\left[\cos\theta\right]_0^{\pi} = \frac{2}{\pi}$

$\langle \cos(\theta) \rangle = \int_{-\infty}^{\infty} \cos(\theta).\, \rho(\theta)\, d\theta = \int_0^{\pi} \frac{\cos(\theta)}{\pi}\, d\theta = \frac{1}{\pi}\left[\sin\theta\right]_0^{\pi} = 0.$

$\langle \cos^2(\theta) \rangle = \int_{-\infty}^{\infty} \cos^2(\theta)\, \rho(\theta)\, d\theta = \int_0^{\pi} \frac{\cos^2(\theta)}{\pi}\, d\theta = \frac{1}{\pi}\int_0^{\pi} \frac{\cos(2\theta)+1}{2}\, d\theta$

$\cos(2\theta) = 2\cos^2(\theta) - 1 \qquad = \frac{1}{2\pi}\left[\frac{1}{2}\sin(2\theta) + \theta\right]_0^{\pi}$

$$= \frac{1}{2\pi}\left\{\pi - 0\right\} = \frac{1}{2}.$$