CC0285 - Probabilidade II

Aula: 16/08/2023

Funções: Gama, Gama Generalizada e Beta.

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1. Função Gama.

0.1 Definição

A função Gama, introduzida pelo matemático alemão, Leonard Euler, é definida por:

$$\begin{aligned} T : I\!R^+ &\to I\!R^+ \\ \alpha &\to \Gamma(\alpha) \end{aligned}$$

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

Ela generaliza a função fatorial.

Propriedades da Função Gama:

(a) $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, para $\alpha > 0$.

Note que:

$$\Gamma(\alpha+1) = \int_0^\infty x^{\alpha} e^{-x} dx$$

(b) $\Gamma(n+1) = n!$ para $\alpha = n$.

(c) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

(d) $\Gamma(x)\Gamma(1-x) = \frac{\pi}{sen(\pi x)}$, para $0 < x < 1$.

(e) $\Gamma\left(\frac{2n+1}{2}\right) = \prod_{i=1}^{n}(2i-1) \quad \frac{\Gamma\left(\frac{1}{2}\right)}{2^n}$

(f) Para n grande podemos aproximar

$$\Gamma(n+1) = n! \approx \sqrt{2\pi}\; n^{n+1/2}\; e^{-n},$$

conhecida como fórmula de Stirling.

Às vezes queremos calcular a integral:
Sejam $a > 0$ e $b > 0$.

$$I = \int_0^\infty x^{a-1} e^{-bx} dx.$$

Mostre que esta integral é dada por:

$$I = \int_0^\infty x^{a-1} e^{-bx}\, dx = \frac{\Gamma(a)}{b^a}.$$

Prova: Vamos fazer em I a seguinte mudança de variável:

$$y = bx$$

Assim

$$dx = \frac{dy}{b} \quad , \quad e \quad x = \frac{y}{b} \quad e \quad x^{a-1} = \frac{y^{a-1}}{b^{a-1}}$$

Note que quando $x = 0$ temos $y = 0$. Quando $x \to \infty$ temos $y \to \infty$.
Logo,

$$I = \int_0^\infty \frac{y^{a-1}}{b^{a-1}}\, e^{-y}\, \frac{dy}{b},$$

$$I = \frac{1}{b^a} \int_0^\infty y^{a-1} e^{-y}\, dy = \frac{\Gamma(a)}{b^a}.$$

Exercícios Resolvidos

Calcular usando as propriedades da função gama:

1. $\dfrac{\Gamma(6)}{2\,\Gamma(3)} = \dfrac{5!}{2\;2!} = \dfrac{5.4.3.2}{2\;2} = 30.$

```
 gamma(6)/(2*gamma(3))
[1] 30
>
```

2. $\dfrac{\Gamma(5/2)}{\Gamma(1/2)} = \dfrac{\frac{3}{2}\,\Gamma(3/2)}{\Gamma(1/2)} = \dfrac{\frac{3}{2}\frac{1}{2}\,\Gamma(1/2)}{\Gamma(1/2)} = \dfrac{3}{4}.$

```
> fractions(gamma(5/2)/gamma(1/2))
[1] 3/4
>
```

3. $\dfrac{\Gamma(3)\,\Gamma(2,5)}{\Gamma(5,5)} = \dfrac{2!\frac{3}{2}\frac{1}{2}\,\Gamma(1/2)}{\frac{9}{2}\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\,\Gamma(1/2)} = \dfrac{16}{315}.$

```
> fractions(gamma(3)*gamma(5/2)/gamma(11/2))
[1] 16/315
```

4. $\dfrac{6\,\Gamma(8/3)}{5\Gamma(2/3)} = \dfrac{6\,\dfrac{5}{3}\dfrac{2}{3}\,\Gamma(2/3)}{5\,\Gamma(2/3)} = \dfrac{4}{3}$

```
> fractions(6*gamma(8/3))/(5*gamma(2/3))
[1] 4/3
>
```

Calcular as seguintes integrais.

5. $\displaystyle\int_0^\infty x^6 e^{-x} dx = \Gamma(7) = 6! = 720.$

Resolvendo no R tem-se:

```
>
> gamma(7)###6! que pode ser calculado assim:
[1] 720
> factorial(6)
[1] 720
>
```

6. $\displaystyle\int_0^\infty x^{5/2} e^{-x} dx = \Gamma(7/2) = \Gamma(5/2+1) = \frac{5}{2}\frac{3}{2}\frac{1}{2}\,\Gamma(1/2) = \frac{15\,\sqrt{\pi}}{8}.$

Resolvendo no R tem-se:

```
> gamma(7/2);15*sqrt(pi)/8
[1] 3.323351
[1] 3.323351
>
```

7. $\displaystyle\int_0^\infty x^6 e^{-2x} dx$

fazendo a mudança de variável $u = 2x$ então $du = 2dx$ e $x = \dfrac{u}{2}$,a integral ficará:

$$I = \int_0^\infty (\frac{u}{2})^6\; e^{-u}\; \frac{du}{2} = \frac{1}{2^7}\int_0^\infty u^6\; e^{-u} du = \frac{\Gamma(7)}{128} = \frac{720}{128} = \frac{45}{8} = 5,625.$$

Note que:

$$a - 1 = 6, \;\; a = 7 > 0 \quad e \quad b = 2 > 0.$$

e

$$I = \frac{\Gamma(7)}{2^7} = \frac{6!}{128} = 5,625.$$

Essa integral pedida pode ser calculada diretamente no R.

```
> f=function(x) x^6*exp(-2*x)
> I=integrate(f,0,Inf)$value;I
[1] 5.625
> gamma(7)/2^7
[1] 5.625
>
```

8. Quanto vale:

$$\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) \quad ?$$

Solução:

Fazendo $x = \frac{1}{3}$ temos $1 - x = \frac{2}{3}$

Sabemos que:

$$\Gamma\left(x\right)\Gamma\left(1-x\right) = \frac{\pi}{sen(\pi x)}.$$

Assim

$$\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{\pi}{sen(\frac{\pi}{3})} = \frac{2\,\pi}{\sqrt{3}}.$$

```
> gamma(1/3);gamma(2/3)
[1] 2.678939
[1] 1.354118
> gamma(1/3)*gamma(2/3); 2*pi/sqrt(3)
[1] 3.627599
[1] 3.627599
>
>
```

2. Função Gama Generalizada.

$$IGG(a,b,c) = \int_0^\infty x^{a-1} e^{-b\,x^c} dx = \frac{\Gamma(a/c)}{c\;\, b^{a/c}},\ a > 0, b > 0, c > 0.$$

Prova:

Fazendo a mudança de variável:

$$y = b\,x^c \quad x^c = \frac{y}{b} \quad x = \frac{y^{1/c}}{b^{1/c}}.$$

$$dx = \frac{1}{c}\,y^{1/c-1}\,\frac{1}{b^{1/c}}\,dy = \frac{1}{c\,b^{1/c}}\;\,y^{1/c-1}\,dy.$$

e

$$x^{a-1} = \frac{y^{(a-1)/c}}{b^{(a-1)/c}}.$$

Note que quando $x = 0$ temos $y = 0$. Quando $x \to \infty$ temos $y \to \infty$, pois $b > 0, c > 0$. Logo,

$$IGG(a, b, c) = \frac{1}{c\, b^{1/c}} \int_0^\infty \frac{y^{(a-1)/c}}{b^{(a-1)/c}}\, e^{-y}\, y^{1/c-1}\, dy.$$

$$IGG(a, b, c) = \frac{1}{c\, b^{a/c}} \int_0^\infty y^{a/c-1}\, e^{-y}\, dy.$$

$$IGG(a, b, c) = \frac{\Gamma(a/c)}{c\ \ b^{a/c}} \quad a > 0, b > 0, c > 0.$$

3. Função Beta. Sejam $a > 0, b > 0$ a função beta é definida por:

$$Beta(a,b) = \int_0^1 \ x^{a-1}\,(1-x)^{b-1}\ \ dx.$$

Mostre que:

a. $Beta(a,b) = Beta(b,a)$.

Prova: Na integral

$$Beta(a,b) = \int_0^1 \ x^{a-1}\,(1-x)^{b-1}\ \ dx$$

faça a mudança de variável $y = 1 - x$.
Assim

$$dy = -dx.$$

Para $x = 0$ temos $y = 1$ e para $x = 1$ temos $y = 0$
Logo,

$$Beta(a,b) = \int_1^0 \ (1-y)^{a-1}\,y^{b-1}\ \ (-1);dy.$$

$$Beta(a,b) = \int_0^1 \ y^{b-1}\,(1-y)^{a-1}\ \ \ dy = Beta(b,a).$$

b. $Beta(a,b) = 2\ \displaystyle\int_0^{\pi/2} [sen(\theta)]^{2a-1}[cos(\theta)]^{2b-1}d\theta.$

Prova: Na integral

$$Beta(a,b) = \int_0^1 \ x^{a-1}\,(1-x)^{b-1}\ \ dx$$

faça a mudança de variável $x = sen^2(\theta)$ e $1 - x = 1 - sen^2(\theta) = \cos^2(\theta)$.
Assim,

$$dx = 2\ sen(\theta)\ cos(\theta)\ d\theta.$$

Note que:

$$x^{a-1} = [sen^2(\theta)]^{a-1} = [sen(\theta)]^{2a-2}.$$

$$(1-x)^{b-1} = [\cos^2(\theta)]^{b-1} = [cos(\theta)]^{2b-2}.$$

Assim,

$$Beta(a,b) = \int_0^{\pi/2} \ [sen(\theta)]^{2a-2}\ [cos(\theta)]^{2b-2}\ 2\ sen(\theta)\ cos(\theta)\ d\theta.$$

$$Beta(a,b) = 2\ \int_0^{\pi/2} \ [sen(\theta)]^{2a-1}\ \ [cos(\theta)]^{2b-1}\ \ d\theta.$$

.

9. Provar que $Beta(a,b) = \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, $a > 0,\ b > 0$.

   Passo 1: Faça a transformação $z = x^2$ em

$$\Gamma(a) = \int_0^\infty z^{a-1} e^{-z} dz = 2\int_0^\infty x^{2a-1}\ e^{-x^2} dx$$

   Passo 2: Analogamente

$$\Gamma(b) = 2\int_0^\infty y^{2b-1} e^{-y^2} dy.$$

   Passo 3: Mostre que

$$\Gamma(a)\Gamma(b) = 4\int_0^\infty \int_0^\infty x^{2a-1} y^{2b-1}\ \ e^{-(x^2+y^2)} dxdy.$$

   Passo 4: Faça a transformação em coordenadas polares: $x = rcos(\theta)\ \ y = rsen(\theta)$.

   Note que:

$$x^2 + y^2 = r^2cos^2(\theta) + r^2sen^2(\theta) = r^2(cos^2(\theta) + sen^2(\theta)) = r^2.$$

$$e^{-(x^2+y^2)} = e^{-r}.$$

$$x^{2a-1} = r^{2a-1}\ [cos(\theta)^{2a-1}]$$

$$y^{2b-1} = r^{2b-1}\ [sen(\theta)]^{2b-1}$$

$$x^{2a-1} \times y^{2b-1}\ e^{-(x^2+y^2)} = r^{2a-1+2b-1}\ \ [cos(\theta)^{2a-1}] \times\ [sen(\theta)]^{2b-1}\ e^{-r}$$

   O módulo do jacobiano da transformação é dado por:

$$|J| = r.$$

$$\Gamma(a)\Gamma(b) = 4\int_0^\infty \int_0^{\pi/2} r^{2a-1+2b-1}\ \ e^{-r}\ [cos(\theta)]^{2a-1} \times\ [sen(\theta)]^{2b-1}\ \ \times rd\theta\ dr.$$

$$\Gamma(a)\Gamma(b) = 4\int_0^\infty r^{2a+2b-1}\ e^{-r}\ dr\ \ \int_0^{\pi/2} [cos(\theta)]^{2a-1} \times\ [sen(\theta)]^{2b-1}\ \ d\theta.$$

$$\Gamma(a)\Gamma(b) = 2\int_0^\infty r^{2(a+b)-1}\ e^{-r}\ dr\ \ \times 2\int_0^{\pi/2} [cos(\theta)]^{2a-1} \times\ [sen(\theta)]^{2b-1}\ \ d\theta.$$

$$\Gamma(a)\Gamma(b) = \Gamma(a+b) \times Beta(a,b),$$

$$Beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},\ a > 0,\ b > 0.$$

   Fato Queremos calcular a integral:

$$I = \int\int_A f(x,y)\, dxdy$$

em que $A = (0,\infty) \times (0,\infty)$.

Vamos fazer uma transformação de variáveis dada por:

$$T(x,y) = (u,v) = (h_1(x,y), h_2(x,y)).$$

$$u = h_1(x,y) \quad e\; v = h_2(x,y)$$

sujeita as seguintes condições:

i. $u = h_1(x,y)$ e $v = h_2(x,y)$ definam uma transformação biunívoca de Ai em B .

ii. As derivadas parciais de primeira ordem de $x = h_1^{-1}(x,y) = w_1(u,v)$ e $y = h_2^{-1}(x,y) = w_2(u,v)$, sejam funções contínuas em B.

iii. O jacobiano da transformação:

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix},$$

seja diferente de zero em B.

Assim:

$$\int_A\int f(x,y)\, dxdy = \int_B\int f(w_1(u,v), w_2(u,v))\, |J|\, du\, dv.$$

O jacobiano da transformação em coordenadas polares é dada por:

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} = cos(\theta) & \frac{\partial x}{\partial \theta} = -sen(\theta) \\ \frac{\partial y}{\partial \theta} = rsen(\theta) & \frac{\partial v}{\partial \theta} = rcos(\theta) \end{vmatrix} = rcos^2(\theta) + rsen^2(\theta) = r.$$

que é diferente de zero em $B = (0,\infty) \times (0, \pi/2)$.

10. Mostre que:

a. $I_a = \displaystyle\int_0^1 x^4(1-x)^3\, dx = \frac{1}{280}.$

Prova: olhando a integral I_a temos:

$$a - 1 = 4 \quad a = 5 \quad e \quad b - 1 = 3 \quad b = 4.$$

$$I_a = Beta(5,4) = \frac{\Gamma(4)\Gamma(5)}{\Gamma(9)} = \frac{3!\, 4!}{8!} = \frac{6}{8 \times 7 \times 6 \times 5} = \frac{1}{280}.$$

```
>
> require(MASS)
>
> a=5;b=4
> beta(5,4);beta(4,5)
```

```
[1] 0.003571429
[1] 0.003571429
> fractions(beta(5,4))
[1] 1/280
>
>
```

b. $I_b = \int_0^2 \frac{x^2}{\sqrt{2-x}}\, dx = \frac{64\sqrt{2}}{15}$.

Prova: Vamos manipular o integrando:

$$\frac{x^2}{\sqrt{2-x}} = x^2\ (2-x)^{-1/2}.$$

Precisamos de uma mudança de variável $y = h(x)$ com $h(0) = 0$ e $h(1) = 1$
Uma primeira tentativa é :

$$y = h(x) = \frac{x-0}{2-0} = \frac{x}{2}.$$

Note que:

$$h(0) = \frac{0}{2} = 0, \quad h(2) = \frac{2}{2} = 1.$$

Além disso

$$x = 2y\ ,\ dx = 2\ dy\ ,\ 2 - x = 2 - 2y = 2(1-y).$$

$$x^2\ (2-x)^{-1/2}\ dx = 4\ y^2\ 2^{-1/2}\ (1-y)^{-1/2}\ 2\ dy = \frac{8}{\sqrt{2}}\ y^2 \quad (1-y)^{-1/2},$$

$$x^2\ (2-x)^{-1/2}\ dx = 4\ \sqrt{2}\ y^2 \quad (1-y)^{-1/2}.$$

$$a - 1 = 2; \quad ,a = 3 \quad e \quad b - 1 = -\frac{1}{2}\ ,b = \frac{1}{2}.$$

Finalmente,

$$I_b = 4\ \sqrt{2} \quad \int_0^1 \quad y^2 \quad (1-y)^{-1/2}\ dy = 4\ \sqrt{2}\ Beta(3, 1/2).$$

$$Beta(3, 1/2) = \frac{\Gamma(3)\Gamma(1/2)}{\Gamma(7/2)}.$$

$$\Gamma(7/2) = \Gamma(1 + 5/2) = \frac{5}{2}\ \Gamma(5/2) = \frac{5}{2}\ \Gamma(1 + 3/2)$$

$$\Gamma(7/2) = \frac{5}{2}\ \frac{3}{2}\ \Gamma(3/2) = \frac{5}{2}\ \frac{3}{2}\ \Gamma(1 + 1/2)$$

$$\Gamma(3 + 1/2) = \frac{5}{2}\ \frac{3}{2}\ \frac{1}{2}\ \Gamma(1/2) = \frac{15\ \sqrt{\pi}}{8}.$$

Assim,

$$Beta(3,1/2) = \frac{\Gamma(3)\Gamma(1/2)}{\Gamma(7/2)} = \frac{2\sqrt{\pi}}{\frac{15\sqrt{\pi}}{8}} = \frac{16}{15}.$$

$$I_b = 4\sqrt{2}\ \frac{16}{15} = \frac{64\sqrt{2}}{15}.$$

```
> f=function (x) x^2*(2-x)^(-1/2)
>
> I_b=integrate(f,0,2)$value;I_b
[1] 6.033978
>
>
>
> 64*sqrt(2)/15
[1] 6.033978
>
```

c. $I_c = \int_0^a y^4\sqrt{a^2-y^2}\ dy = \frac{\pi a^6}{16}.$

Solução:
Vamos manipular o integrando:

$$y^4\ \sqrt{a^2-y^2} = y^4\ (a^2-y^2)^{1/2}.$$

Precisamos de uma mudança de variável $u = h(y)$ com $h(0) = 0$ e $h(a) = 1$
Uma primeira tentativa é :

$$u = h(y) = \frac{y^2}{a^2}.$$

Note que:

$$h(0) = \frac{0}{a^2} = 0, \quad h(1) = \frac{a^2}{a^2} = 1.$$

Além disso

$$y^2 = a^2\ u \quad , \quad y = a\ u^{1/2}, \quad dy = \frac{a}{2}\ u^{-1/2}\ du$$

logo,

$$y^4 = a^4\ u^2.$$

$$(a^2-y^2) = a^2 - a^2\ u = a^2(1-u) \quad (a^2-y^2)^{1/2} = a\ (1-u)^{1/2}$$

$$y^4\sqrt{a^2-y^2}\ \ dy = a^4\ u^2\ a\ (1-u)^{1/2}\ a\ u^{-1/2}\ du = a^6\ u^{3/2}(1-u)^{1/2}\ du$$

$$I_c = a^6\ \int_0^1\ u^{3/2}(1-u)^{1/2}\ du = a^6\ \ Beta(5/2, 3/2)$$

Sabemos que

$$\Gamma(3/2) = \frac{1}{2}\,\Gamma(1/2) = \frac{1}{2}\,\sqrt{\pi} = \frac{\sqrt{\pi}}{2}.$$

$$\Gamma(5/2) = \frac{3}{2}\,\Gamma(3/2) = \frac{3}{2}\,\frac{\sqrt{\pi}}{2} = \frac{3\sqrt{\pi}}{4}.$$

$$\Gamma(5/2) \times \Gamma(3/2) = \frac{3\sqrt{\pi}}{4} \times \frac{\sqrt{\pi}}{2} = \frac{3\pi}{8}.$$

$$Beta(5/2, 3/2) = \frac{\frac{3\pi}{8}}{6} = \frac{\pi}{16}.$$

$$I_c = a^6\ \ \frac{\pi}{16} = \frac{\pi\ a^6}{16}.$$

11. Mostre que:

   a. $I_a = \int_0^{\pi/2} [sen(\theta)]^6 d\theta = \frac{5\pi}{32}.$

   Sabemos que:

$$\int_0^{\pi/2} [sen(\theta)]^{2a-1} [cos(\theta)]^{2b-1} d\theta = \frac{Beta(a,b)}{2}$$

Prova:
Fazendo o cotejo;

$$2a - 1 = 6, \quad a = \frac{7}{2} \quad e \quad 2b - 1 = 0 \quad b = \frac{1}{2}.$$

$$\Gamma(a+b) = \Gamma(4) = 3! = 6,$$

$$\Gamma(7/2) = \Gamma(1 + 5/2) = \frac{5}{2}\,\Gamma(5/2) = \frac{5}{2} \times \frac{3\sqrt{\pi}}{4} = \frac{15\sqrt{\pi}}{8}$$

$$\Gamma(a)\Gamma(b) = \Gamma(7/2)\Gamma(1/2) = \frac{15\sqrt{\pi}}{8} \times \sqrt{\pi} = \frac{15\pi}{8}$$

$$Beta(7/2, 1/2) = \frac{\frac{15\pi}{8}}{6} = \frac{5\pi}{16}.$$

$$I_a = \frac{Beta(7/2, 1/2)}{2} = \frac{5\pi}{32}.$$

```
> g=function (teta) sin(teta)^6
>
> I_a=integrate(g,0,pi/2)$value;I_a
[1] 0.4908739
>
> 5*pi/32
[1] 0.4908739
>
```

b. $I_b = \int_0^{\pi/2} [sen(\theta)]^4 [cos(\theta)]^5 d\theta = \frac{8}{315}.$

Prova:
Fazendo o cotejo;

$$2a-1=4, \quad a=\frac{5}{2} \quad e \quad 2b-1=5 \quad b=3.$$

Sabemos que:

$$\Gamma\left(\frac{2n+1}{2}\right) = \prod_{i=1}^{n}(2i-1) \quad \frac{\Gamma\left(\frac{1}{2}\right)}{2^n}.$$

$$\Gamma(a+b) = \Gamma(5/2+3) = \Gamma(11/2) = \Gamma(\frac{2\times 5+1}{2}).$$

$$\Gamma(11/2) = \Gamma(\frac{2\times 5+1}{2}) = \prod_{i=1}^{5}(2i-1) \quad \frac{\Gamma\left(\frac{1}{2}\right)}{2^5}$$

$$\Gamma(11/2) = 1\times 3\times 5\times 7 \ \times 9 \ \times \frac{\sqrt{\pi}}{32} = \frac{945\sqrt{\pi}}{32}.$$

$$\Gamma(5/2) = \frac{3\sqrt{\pi}}{4}.$$

$$\Gamma(a)\Gamma(b) = \Gamma(5/2)\Gamma(3) = \frac{3\sqrt{\pi}}{4}\times 2 = \frac{3\sqrt{\pi}}{2}.$$

$$Beta(5/2,3) = \frac{\frac{3\sqrt{\pi}}{2}}{\frac{945\sqrt{\pi}}{32}} = \frac{16}{135}.$$

$$I_b = \frac{Beta(5/2,3)}{2} = \frac{8}{315}.$$

```
> g=function (teta) sin(teta)^5*cos(teta)^4
> I_b=integrate(g,0,pi/2)$value;I_a
[1] 0.02539683
> require(MASS)
> fractions(I_b)
[1] 8/315
>
```

c. $\int_0^{\pi} [cos(\theta)]^4 d\theta = \frac{3\pi}{16}.$

Prova:
Fazendo o cotejo;

$$2a-1=0, \quad a=\frac{1}{2} \quad e \quad 2b-1=4 \quad b=\frac{5}{2}.$$

$$a + b = 3.$$

$$\Gamma(5/2) \times \Gamma(1/2) = \frac{3\pi}{4}.$$

$$beta(1/2, 5/2) = \frac{\Gamma(1/2)\,\Gamma(5/2)}{\Gamma(3)} = \frac{\frac{3\pi}{4}}{2} = \frac{3\pi}{8}.$$

$$I_c = \frac{\frac{3\pi}{8}}{2} = \frac{3\pi}{16}.$$

```
> g=function (teta) cos(teta)^4
>
> I_c=integrate(g,0,pi/2)$value;I_c
[1] 0.5890486
>
> 3*pi/16
[1] 0.5890486
>
```

d. $I_d = \int_0^\infty x^2 e^{-2x^3}\, dx.$

Prova:
Fazendo o cotejo;

$$a - 1 = 2, \quad a = 3 \quad , \quad b = 2 \;\; e \; c = 3.$$

$$I_d = IGG(3, 2, 3) = \frac{\Gamma(a/c)}{c \;\; b^{a/c}} = \frac{\Gamma(3/3)}{3 \;\; 2^{3/3}} = \frac{1}{6}.$$

```
>
> g=function(x) x^2* exp(-2*x^3)
>
> I_d=integrate(g,0,Inf)$value;I_d
[1] 0.1666667
> require(MASS)
> fractions(I_d)
[1] 1/6
>
```

Resolva os exercícios a seguir:

12. Calcule:

a. $\dfrac{\Gamma(8)}{2\,\Gamma(3)}$.

b. $\dfrac{\Gamma(9/2)}{\Gamma(1/2)}$.

c. $\dfrac{\Gamma(3)\,\Gamma(5/2)}{2\,\Gamma(11/2)}$

d. $\dfrac{6\,\Gamma\,(11/3)}{5\,\Gamma\,(2/3)}$.

13. Mostre que:

a. $\displaystyle\int_0^\infty x^3 e^{-x} dx = 6.$

b. $\displaystyle\int_0^\infty x^6 e^{-2x} dx = \frac{45}{8}.$

c. $\displaystyle\int_0^\infty y^{1/2} e^{-y^3} dy = \frac{\sqrt{\pi}}{3}.$

d. $\displaystyle\int_0^\infty 3^{-4z^2} dz = \frac{\sqrt{\pi}}{4\sqrt{ln 3}}.$

e. $\displaystyle\int_0^\infty x^2 e^{-2x^2} dx = \frac{\sqrt{2\pi}}{16}.$

f. $\displaystyle\int_0^1 (lnx)^4 dx = 24.$

g. $\displaystyle\int_0^1 (xlnx)^3 dx = -\frac{3}{128}.$

h. $\displaystyle\int_0^\infty \frac{e^{-st}}{\sqrt{t}} dt = \sqrt{\frac{\pi}{s}}, \quad s > 0.$

14. Provar que $\displaystyle\Gamma\,(n) = \int_0^1 \left[ln\left(\frac{1}{x}\right)\right]^{n-1} dx, \; n > 0.$

15. Calcular: $\displaystyle\int_0^\infty x^m e^{-ax^n} dx, m > 0, n > 0, a > 0.$

16. Provar que $\displaystyle\int_0^1 x^m (lnx)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}, m > -1,$ n inteiro positivo.

17. Dado que

$$\int_0^\infty \frac{x^{p-1}}{1+x}\, dx = \frac{\pi}{sen(p\,\pi)}, 0 < p < 1,$$

mostre que

$$\Gamma(p)\,\Gamma(1-p) = \frac{\pi}{sen(p\,\pi)}.$$

Dica: Faça a mudança de variável $y = \dfrac{x}{1+x}$.

18. Mostrar que: $\displaystyle\int_0^2 x\sqrt[3]{8-x^3}\, dx = \frac{16\pi}{9\sqrt{3}}.$

19. Mostrar que: $\displaystyle\int_0^{\pi/2} \sqrt{tg(\theta)}\, d\theta = \frac{\pi}{\sqrt{2}}.$

20. Calcular:

a. $Beta(3, 5)$.

b. $Beta(3/2, 2)$.

c. $Beta(1/3, 2/3)$.

21. Mostrar que:

    a. $\displaystyle\int_0^1 x^2(1-x)^3\,dx = \frac{1}{60}.$

    b. $\displaystyle\int_0^1 \sqrt{\frac{1-x}{x}}\,dx = \frac{\pi}{2}.$

    c. $\displaystyle\int_0^2 (4-x^2)^{3/2}\,dx = 3\pi.$

22. Provar que:

    a. $\displaystyle\int_0^\infty \frac{x}{1+x^6}\,dx = \frac{\pi}{3\sqrt{3}}.$

    b. $\displaystyle\int_0^\infty \frac{y^2}{1+y^4}\,dy = \frac{\pi}{2\sqrt{2}}.$

23. Provar que :

$$\int_{-\infty}^{\infty} \frac{e^{2x}}{ae^{3x}+b}\,dx = \frac{2\pi}{3\sqrt{3}\,a^{2/3}\,b^{1/3}}\ a>0,\ b>0.$$

24. Provar que :

$$\int_{-\infty}^{\infty} \frac{e^{2x}}{(e^{3x}+1)^2}\,dx = \frac{2\pi}{9\sqrt{3}}.$$

    Dica: Derivar em relação a b na questão 16 e.........!!!!!!!!!!!!!!!!!

25. Mostre que $\displaystyle\int_0^\infty \frac{x^a}{(1+x)^b}\,dx = \ Beta(a+1, b-a-1)$ para $a>-1$ e $b>a+1$.

26. Para z negativo $\Gamma(z)$ é definida por:

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}.$$

    Calcule $\Gamma(-5/2)$.