Preliminary Research Exam

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A.1 Conditional Probability

In this section, we define conditional probability in the language of measure theory. Let $(\Omega, \mathcal{F}_0, P)$ be a probability space, $\mathcal{F} \subseteq \mathcal{F}_0$ be a σ-algebra, $(\mathbb{R}, \mathcal{R})$ be the real line equipped with the Borel σ-algebra, and $X : \Omega \to \mathcal{S}$ be a $(\mathcal{F}_0, \mathcal{S})$-measurable random variable.

Definition A.1 (Conditional expectation)**.** Let $\mathbb{E}\left[|X|\right] < \infty$. We define the *conditional expectation of X given $\mathcal{F}$, or* $\mathbb{E}\left[X|\mathcal{F}\right]$, to be any random variable Y that satisfies

(i) Y is $(\mathcal{F}, \mathcal{R})$-measurable, and

(ii) for all $A \in \mathcal{F}$, $\int_A X \mathrm{d}P = \int_A Y \mathrm{d}P$.

Such a Y is called *a version of* $\mathbb{E}\left[X|\mathcal{F}\right]$.

Next, we establish some properties of conditional expectation.

Lemma A.2 (Integrability)**.** *If $Y = \mathbb{E}\left[X|\mathcal{F}\right]$ (a.e.), then Y is integrable.*

Proof. Let $A = \{\omega \in \Omega : Y(\omega) > 0\}$, which is a $\mathcal{F}$-measurable set, as $(0, \infty) \in \mathcal{R}$ and Y is a measurable function. Then, use property (ii) twice.

$$\int_A Y \mathrm{d}P = \int_A X \mathrm{d}P \leq \int_A |X| \, \mathrm{d}P.$$
$$\int_{A^c} -Y \mathrm{d}P = \int_{A^c} -X \mathrm{d}P \leq \int_{A^c} |X| \, \mathrm{d}P.$$

Then,

$$\mathbb{E}\left[|Y|\right] = \int_A Y \mathrm{d}P + \int_{A^c} -Y \mathrm{d}P \leq \int_A |X| \, \mathrm{d}P + \int_{A^c} |X| \, \mathrm{d}P = \mathbb{E}\left[|X|\right] < \infty.$$

□

Lemma A.3 (Uniqueness)**.** *If Y' also satisfies (i) and (ii),*

$$\int_A Y dP = \int_A Y' dP \text{ for all } A \in \mathcal{F}.$$

Proof. Take any $\epsilon > 0$ and let $A = \{\omega \in \Omega : Y(\omega) - Y'(\omega) \leq \epsilon\}$, which is $\mathcal{F}$ measurable because $Y - Y'$ is measurable and $[\epsilon, \infty) \in \mathcal{R}$. Then,

$$\begin{aligned}
0 &= \int_A X - X \mathrm{d}P \\
&= \int_A X \mathrm{d}P - \int_A X \mathrm{d}P \\
&= \int_A Y \mathrm{d}P - \int_A Y' \mathrm{d}P \\
&= \int_A Y - Y' \mathrm{d}P \\
&\geq \epsilon P(A),
\end{aligned}$$

indicating that $P(A) = P(Y - Y' \geq \epsilon) = 0$, or that $Y \geq Y'$ a.e.. Switching the roles of Y and Y' gives that $Y = Y'$ a.e.. $\square$

Before showing existence, we recall the following definitions and results.

Definition A.4 (Absolute continuity)**.** Let ν and μ be measures defined on the same measurable space. Then, ν *is absolutely continuous with respect to* μ (written $\nu \ll \mu$), if $\mu(A) = 0$ implies $\nu(A) = 0$ for all measurable sets A.

Definition A.5 (σ-finite measure)**.** A measure μ on measurable space $(\Omega, \mathcal{F})$ is called σ-finite is there exists a sequence $A_1, A_2, ... \in \mathcal{F}$ such that

- $\mu(A_i) < \infty$ for each $i = 1, 2, ...$, and
- $\bigcup_{i=1}^{\infty} A_i = \Omega$.

In other words, the entire set can be written as a countable union of sets of finite measure.

Theorem A.6 (Radon-Nokodym theorem)**.** *Let ν and μ be two σ-finite measures on $(\Omega, \mathcal{F})$. If $\nu \ll \mu$, then there is a $(\mathcal{F}, \mathcal{R})$-function $f : \Omega \to \mathbb{R}$ such that for all $A \in \mathcal{F}$,*

$$\int_A f d\mu = \nu(A).$$

THe function f is written $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ and called the Radon-Nikodym derivative.

Lemma A.7 (Existence)**.** *There exists a Y that satisfies the defining properties of conditional expectation.*

Proof. First, suppose that $X \geq 0$, let $\mu = P$, and define

$$\nu(A) := \int_A X \mathrm{d}P \text{ for any } A \in \mathcal{F}.$$

The monotone convergence theorem, along with a sequence of non-negative simple functions $X_n : \Omega \to [0, \infty)]$ that approach X pointwise, can be used to show that ν is a measure. Then, the definition clearly shows that $\mu(A) = 0 \implies \nu(A) = 0$, so $\nu \ll \mu$. Thus, by the Radon-Nikodym theorem,

$$\int_A X \mathrm{d}P = \nu(A) = \int_A \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \mathrm{d}P.$$

Letting $A = \Omega$, we have that $Y := \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ is $(\mathcal{F}, \mathcal{R})$-measurable and integrable, and because its non-negative, $\mathbb{E}\left[|Y|\right] < \infty$. Thus, both properties are satisfied, and Y is a version of $\mathbb{E}\left[X|\mathcal{F}\right]$.

For the general case, let X^+ and X^- be the nonnegative and nonpositive parts of X, and let $Y_1 = \mathbb{E}\left[X^+|\mathcal{F}\right]$ and $Y_2 = \mathbb{E}\left[X^-|\mathcal{F}\right]$. Now, $Y_1 - Y_2$ is integrable, and for all $A \in \mathcal{F}$, we have

$$\begin{aligned}
\int_A X \mathrm{d}P &= \int_A X^+ \mathrm{d}P - \int_A X^- \mathrm{d}P \\
&= \int_A Y_1 \mathrm{d}P - \int_A Y_2 \mathrm{d}P \\
&= \int_A (Y_1 - Y_2)\, \mathrm{d}P.
\end{aligned}$$

Thus, $Y_1 = Y_2$ is a version of $\mathbb{E}\left[X|\mathcal{F}\right]$. $\square$