Preliminary Research Exam

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Contents

A Foundations

A.1 Measure Theory

Here, we state fundamental definitions and results of measure theory without proof, to be used primarily in Section A.2.

Notation We write $f : (\Omega, \mathcal{F}) \to (S, \mathcal{S})$ to mean $f : \Omega \to S$ is an $(\mathcal{F}, \mathcal{S})$-measurable function. We say f is $\mathcal{F}$-measurable if the codomain's σ-algebra $\mathcal{S}$ is understood from context. When no such $\mathcal{S}$ is specified, we understand it to be $\mathcal{R}$,the Borel σ-algebra.

Proposition A.1. *Given any measurable nonnegative function $f : \Omega \to [0, \infty]$, there is a sequence of nonnegative simple functions $(f_n)_n$ with $f_n \uparrow f$ pointwise.*

Theorem A.2 (Monotone convergence theorem). *Let $(\Omega, \mathcal{F}, \mu)$ be a common measure space, $(f_n)_{n\geq 0}$ be a sequence of measurable functions from Ω into $[0, \infty]$, which are increasing pointwise to limit function $f : \Omega \to \mathbb{R}$. Then, f is measurable, and*

$$\int f_n d\mu \uparrow \int f d\mu.$$

Theorem A.3 (Dominated convergence theorem). *Let $(\Omega, \mathcal{F}, \mu)$ be a common measure space, $(f_n)_{n\geq 0}$ be a sequence of measurable functions from Ω into $[-\infty, \infty]$, which converge pointwise to limit function $f : \Omega \to \mathbb{R}$. Suppose that there exists an integrable function h such that for each n, $|f_n| \leq h$ everywhere. Then,*

$$lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

Definition A.4 (Generated σ-algebra). For a measurable function $Z : \Omega \to \mathbb{R}$, we define $\sigma(Z)$ as the smallest σ-algebra (in the sense of inclusion) $\mathcal{F}$ on Ω such that Z is $\mathcal{F}$-measurable.

Definition A.5 (Absolute continuity). Let ν and μ be measures defined on the same measurable space. Then, *ν is absolutely continuous with respect to μ* (written $\nu \ll \mu$), if $\mu(A) = 0$ implies $\nu(A) = 0$ for all measurable sets A.

Definition A.6 (σ-finite measure). A measure μ on measurable space $(\Omega, \mathcal{F})$ is called σ-finite is there exists a sequence $A_1, A_2, ... \in \mathcal{F}$ such that

- $\mu(A_i) < \infty$ for each $i = 1, 2, ...$, and
- $\bigcup_{i=1}^{\infty} A_i = \Omega$.

In other words, the entire set can be written as a countable union of sets of finite measure.

Theorem A.7 (Radon-Nokodym theorem). *Let ν and μ be two σ-finite measures on $(\Omega, \mathcal{F})$. If $\nu \ll \mu$, then there is a $(\mathcal{F}, \mathcal{R})$-function $f : \Omega \to \mathbb{R}$ such that for all $A \in \mathcal{F}$,*

$$\int_A f d\mu = \nu(A).$$

THe function f is written $\frac{d\nu}{d\mu}$ and called the Radon-Nikodym derivative.

A.2 Conditional Probability

In this section, we define conditional probability in the language of measure theory. Let $(\Omega, \mathcal{F}_0, P)$ be a probability space, $\mathcal{F} \subseteq \mathcal{F}_0$ be a σ-algebra, $(\mathbb{R}, \mathcal{R})$ be the real line equipped with the Borel σ-algebra, and $X : \Omega \to \mathcal{R}$ be a $(\mathcal{F}_0, \mathcal{R})$-measurable random variable.

Definition A.8 (Conditional expectation)**.** Let $\mathbb{E}\left[|X|\right] < \infty$. We define the *conditional expectation of X given $\mathcal{F}$, or* $\mathbb{E}\left[X|\mathcal{F}\right]$, to be any random variable Y that satisfies

(i) Y is $(\mathcal{F}, \mathcal{R})$-measurable, and

(ii) for all $A \in \mathcal{F}$, $\int_A X \mathrm{d}P = \int_A Y \mathrm{d}P$.

Such a Y is called *a version of* $\mathbb{E}\left[X|\mathcal{F}\right]$.

The σ-algebra $\mathcal{F}$ represents a subset of events which is analogous to the conditioning random variable in the undergraduate probability notion of conditional expectation.

Definition A.9 (Variants of conditional expectation)**.** For conditional probability, we define the following function on $\mathcal{R}$:

$$A \mapsto \mathbb{P}\left[X \in A|\mathcal{F}\right] = \mathbb{E}\left[1_{X^{-1}(A)}|\mathcal{F}\right].$$

For an arbitrary measurable function $Z : \Omega \to \mathbb{R}$, we define

$$\mathbb{E}\left[X|Z\right] = \mathbb{E}\left[X|\sigma\left(Z\right)\right].$$

Next, we establish some properties of conditional expectation.

Lemma A.10 (Integrability)**.** *If $Y = \mathbb{E}\left[X|\mathcal{F}\right]$ (a.e.), then Y is integrable.*

Proof. Let $A = \{\omega \in \Omega : Y(\omega) > 0\}$, which is a $\mathcal{F}$-measurable set, as $(0, \infty) \in \mathcal{R}$ and Y is a measurable function. Then, use property (ii) twice.

$$\int_A Y \mathrm{d}P = \int_A X \mathrm{d}P \leq \int_A |X| \,\mathrm{d}P.$$
$$\int_{A^c} -Y \mathrm{d}P = \int_{A^c} -X \mathrm{d}P \leq \int_{A^c} |X| \,\mathrm{d}P.$$

Then,

$$\mathbb{E}\left[|Y|\right] = \int_A Y \mathrm{d}P + \int_{A^c} -Y \mathrm{d}P \leq \int_A |X| \,\mathrm{d}P + \int_{A^c} |X| \,\mathrm{d}P = \mathbb{E}\left[|X|\right] < \infty.$$

□

Lemma A.11 (Uniqueness)**.** *If Y' also satisfies (i) and (ii),*

$$\int_A Y dP = \int_A Y' dP \textit{ for all } A \in \mathcal{F}.$$

Proof. Take any $\epsilon > 0$ and let $A = \{\omega \in \Omega : Y(\omega) - Y'(\omega) \le \epsilon\}$, which is $\mathcal{F}$ measurable because $Y - Y'$ is measurable and $[\epsilon, \infty) \in \mathcal{R}$. Then,

$$\begin{aligned}
0 &= \int_A X - X \mathrm{d}P \\
&= \int_A X \mathrm{d}P - \int_A X \mathrm{d}P \\
&= \int_A Y \mathrm{d}P - \int_A Y' \mathrm{d}P \\
&= \int_A Y - Y' \mathrm{d}P \\
&\ge \epsilon P(A),
\end{aligned}$$

indicating that $P(A) = P(Y - Y' \ge \epsilon) = 0$, or that $Y \ge Y'$ a.e.. Switching the roles of Y and Y' gives that $Y = Y'$ a.e.. □

Lemma A.12 (Existence)**.** *There exists a Y that satisfies the defining properties of conditional expectation.*

Proof. First, suppose that $X \ge 0$, let $\mu = P$, and define

$$\nu(A) := \int_A X \mathrm{d}P \text{ for any } A \in \mathcal{F}.$$

The monotone convergence theorem, along with a sequence of non-negative simple functions $X_n : \Omega \to [0, \infty)]$ that approach X pointwise, can be used to show that ν is a measure. Then, the definition clearly shows that $\mu(A) = 0 \implies \nu(A) = 0$, so $\nu \ll \mu$. Thus, by the Radon-Nikodym theorem,

$$\int_A X \mathrm{d}P = \nu(A) = \int_A \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \mathrm{d}P.$$

Letting $A = \Omega$, we have that $Y := \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ is $(\mathcal{F}, \mathcal{R})$-measurable and integrable, and because its non-negative, $\mathbb{E}[|Y|] < \infty$. Thus, both properties are satisfied, and Y is a version of $\mathbb{E}[X|\mathcal{F}]$.

For the general case, let X^+ and X^- be the nonnegative and nonpositive parts of X, and let $Y_1 = \mathbb{E}[X^+|\mathcal{F}]$ and $Y_2 = \mathbb{E}[X^-|\mathcal{F}]$. Now, $Y_1 - Y_2$ is integrable, and for all $A \in \mathcal{F}$, we have

$$\begin{aligned}
\int_A X \mathrm{d}P &= \int_A X^+ \mathrm{d}P - \int_A X^- \mathrm{d}P \\
&= \int_A Y_1 \mathrm{d}P - \int_A Y_2 \mathrm{d}P \\
&= \int_A (Y_1 - Y_2) \mathrm{d}P.
\end{aligned}$$

Thus, $Y_1 - Y_2$ is a version of $\mathbb{E}[X|\mathcal{F}]$. □

Proposition A.13 (Properties of condtional expectation)**.** *The following hold:*

1. **Linearity:** *For any $a \in \mathbb{R}$,*

$$\mathbb{E}[aX + Y|\mathcal{F}] = a\mathbb{E}[X|\mathcal{F}] + \mathbb{E}[Y|\mathcal{F}].$$

2. **Monotonicity:** *If $X \leq Y$ a.e., then*

$$\mathbb{E}\left[X|\mathcal{F}\right] \leq \mathbb{E}\left[Y|\mathcal{F}\right] \text{ a.e.}.$$

3. **Preservation of limits:** *If $X_n \geq 0$ and $X_n \uparrow X$, then*

$$\mathbb{E}\left[X_n|\mathcal{F}\right] \uparrow \mathbb{E}\left[X|\mathcal{F}\right].$$

Proof. 1. The RHS is $\mathcal{F}$-measurable, as it is a linear combination of $\mathcal{F}$-measurable functions. Then, for any $A \in \mathcal{F}$,

$$\begin{aligned}
\int_A a\mathbb{E}\left[X|\mathcal{F}\right] + \mathbb{E}\left[Y|\mathcal{F}\right] \mathrm{d}P &= a \int_A \mathbb{E}\left[X|\mathcal{F}\right] \mathrm{d}P + \int_A \mathbb{E}\left[Y|\mathcal{F}\right] \mathrm{d}P \\
&= a \int_A X \mathrm{d}P + \int_A Y \mathrm{d}P \\
&= \int_A aX + Y \mathrm{d}P.
\end{aligned}$$

2. Take any ϵ, and let $A = \{\omega : \mathbb{E}\left[X|\mathcal{F}\right](w) - \mathbb{E}\left[y|\mathcal{F}\right](w) \geq \epsilon\}$. Then,

$$\int_A \mathbb{E}\left[X|\mathcal{F}\right] \mathrm{d}P = \int_A X \mathrm{d}P \leq \int_A Y \mathrm{d}P = \int_A \mathbb{E}\left[Y|\mathcal{F}\right] \mathrm{d}P.$$

Then,

$$\begin{aligned}
0 &\geq \int_A \mathbb{E}\left[X|\mathcal{F}\right] \mathrm{d}P - \int_A \mathbb{E}\left[Y|\mathcal{F}\right] \mathrm{d}P \\
&\geq \epsilon P(A),
\end{aligned}$$

so the set set A has zero measure for all ϵ. This means that $\mathbb{E}\left[X|\mathcal{F}\right] \leq \mathbb{E}\left[Y|\mathcal{F}\right]$ a.e..

3. First, let $Y_n = X - X_n$. Due to linearity and monotonicity, it sufficies to show that $Z_n := \mathbb{E}\left[Y_n|\mathcal{F}\right] \downarrow 0$. Because $Y_n(\omega)$ in monotonically non-increasing in n, there exists a set Ω_0 of measure one such that for each $\omega \in \Omega_0$, we have that $Z_n(\omega)$ is bounded (within $[0, X(\omega)]$) and non-increasing, indicating convergence. String together these limits on Ω_0 to be the function Z_∞, defined arbitrarily on Ω_0^c.

Then, for any $A \in \mathcal{F}$,

$$\lim_{n\to\infty} \int_A Z_n \mathrm{d}P = \lim_{n\to\infty} \int_A Y_n \mathrm{d}P \overset{\text{DCT}}{=} \int_A \lim_{n\to\infty} Y_n \mathrm{d}P = 0.$$

The dominated convergence theorem applies because $0 \leq Y_n(\omega) \leq X(\omega)$, with X integrable by definition of conditional expectation.

□

Definition A.14 (Regular conditional distribution)**.** Let $(\Omega, \mathcal{F}.P)$ be a probability space, $\mathcal{G} \subseteq \mathcal{F}$ be a σ-algebra, $X : \Omega \to S$ be a $(\mathcal{F}, \mathcal{S})$-measurable map. The function $\mu : \Omega \times \mathcal{S} \to [0, 1]$ is said to be a *regular conditional distribution (r.c.d.)* for X given $\mathcal{G}$ if

(i) For each A, the function $\omega \mapsto \mu(\omega, A)$ is a version of $\mathbb{P}\left[X \in A|\mathcal{G}\right]$.

(ii) For a.e. ω, the function $A \mapsto \mu(\omega, A)$ is a probability measure on $(S, \mathcal{S})$.

Theorem A.15. *If $(S, \mathcal{S}) = (\mathbb{R}, \mathcal{R})$, that is that X is measurable with respect to $\mathcal{F}$ and the Borel σ-algebra, then there exists an r.c.d..*

Proof. lol $\square$