**Mathematics Review for Probability and Statistics**
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## 1 Introduction

This note is contains review materials for the mathematics background necessary for undergraduate probability and statistics courses, directed toward pure/applied mathematics, statistics, and engineering majors.

## 2 Set Notation

Please review set notation, including sets represented by enumeration, by intervals, unions and intersections, and set-builder notation. Additionally, review the following common sets:

(a) The real numbers: $\mathbb{R} = (-\infty, \infty)$

(b) The integers: $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$

(c) The natural numbers (nonnegative integers): $\mathbb{N} = \{0, 1, 2, ...\}$

Pay specific attention to set-builder notation. For example:

(a) $\{x \in \mathbb{R} : 0 < x \leq 1\} = (0, 1]$

(b) $\{x : x \subset \{1, 2\}\} = \{\emptyset, \{1\}, \{2\}\}$

(c) $\{x \in \mathbb{Z} : x \geq 0\} = \mathbb{N}$

## 3 Functions

- A function $f$ takes elements of set $A$ and maps/corresponds them to elements of set $B$. We write this as $f : A \to B$.
- If $x \in A$ corresponds to $y \in B$, we write $y = f(x)$. For example, take a function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$; it maps $x \in \mathbb{R}$ to $y = x^2 \in \mathbb{R}$.
- $A$ is called the *domain* of $f$ and $B$ is called the *codomain* of $f$.
- The *image* or *range* of $f$ is the set of all points that can be achieved by plugging in values into $f$, or all possible outputs of $f$. That is, $\text{image}(f) = \{f(x) : x \in \text{domain}(f)\}$.
- $\text{image}(f) \subseteq \text{codomain}(f)$. The reason that they are not necessarily equal is that all points in the codomain need not be achieved by plugging in values into $f$. For $f(x) = x^2$ as defined above, the codomain is $\mathbb{R}$, but only the nonnegative numbers in $\mathbb{R}$ are possible outputs of $f$.
- Functions can be of multiple variables, such as $z = f(x, y) = 2x + y$. Here, we usually call $z$ the *output* or *value* of the function, while $x$ and $y$ are called *inputs* or *arguments* to the function.

- A function is *one-to-one* if different points in the domain map to different points in the codomain. That is, $x \neq y \implies f(x) \neq f(y)$, or that $f$ preserves distinctness. Another way to write this is that $f(x) = f(y) \implies x = y$.

- Finally, $f$ is *onto* if $\text{image}(f) = \text{codomain}(f)$, or that for every $y \in \text{codomain}(f)$, there exists an $x \in \text{domain}(f)$ such that $y = f(x)$.

Understand the following examples:

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not onto. If we define the function instead as $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ where $\mathbb{R}_{\geq 0} = [0, \infty)$, then $f$ is onto. More generally, a function can always be made onto by restricting the codomain to just the range/image.

(b) $f : \mathbb{R} \to \mathbb{R}$ is one-to-one if it passes the horizontal line test, i.e. no horizontal line crosses the plot of the function more than once.

(c) For any constant $R$, take $f : [-R, R] \to \mathbb{R}$ defined by $f(x) = \sqrt{R^2 - x^2}$. image$(f) = [0, R]$. Why is domain$(f)$ restricted to $[-R, R]$?

(d) If you have taken linear algebra: take $f : \mathbb{R}^n \to \mathbb{R}^m$ defined by $f(x) = Ax$ for some $m \times n$ matrix $A$. Why is image$(f) = \text{span}\{\text{columns of } A\}$?

## 4 Sums and Products

A *list* is an ordered collection of elements, as opposed to a set, which is an unordered, collection of distinct elements. We write this as $\{a_i\}_{i=1}^n = (a_1, a_2, ..., a_n)$. The subscript $i$ for $i = 1, 2, ..., n$ indexes the $i$-th element of this list. If this was a list of real numbers, you can also think of $a$ as a function $a : \{1, 2, ..., n\} \to \mathbb{R}$, with $a_i = a(i)$. (Can this function be onto $\mathbb{R}$?) Assuming that we are dealing with a list of real numbers, the sum of the list is written:

$$S = \sum_{i=1}^{n} a_i = a_1 + a_2 + ... + a_n$$

$i$ is called the index variable, and any variable can be used for this purpose; $i$, $j$, $k$, and $l$ are common ones (similar to a "dummy" variable in integration). If we multiply each element by a constant $k$, the sum also increases by factor $k$. Using this information, we can pull constants (values that do not have the index variable as a subscript) out of the sum. If a list $\{c_i\}$ can be decomposed into a sum of two lists $\{a_i\}$ and $\{b_i\}$, we can sum them individually.

$$\sum_{i=1}^{n} ka_i = ka_1 + ka_2 + ... + ka_n = k(a_1 + a_2 + ... + a_n) = k \sum_{i=1}^{n} a_i$$
$$\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

The sum of an empty list of numbers has size 0.

Similar to sums, we may want to multiply all elements of a list into one product. We write this as:

$$P = \prod_{i=1} a_i = a_1 \cdot a_2 \cdot ... \cdot a_n$$

When pulling out a constant $k$, we must raise it to the $n$-th power, because it scales the total product once *per element*. Similarly, if a list $\{c_i\}$ is an element-wise product of lists $\{a_i\}$ and $\{b_i\}$, then we can take the products individually and multiply them.

$$\prod_{i=1}^{n} ka_i = (ka_1) \cdot (ka_2) \cdot ... \cdot (ka_n) = k^n \cdot (a_1 \cdot a_2 \cdot ... \cdot a_n) = k^n \prod_{i=1}^{n} a_i$$

$$\prod_{i=1}^{n} c_i = \prod_{i=1}^{n} (a_i b_i) = \left(\prod_{i=1}^{n} a_i\right) \cdot \left(\prod_{i=1}^{n} b_i\right)$$

The product of an empty list has value 1.

## 5 Additional Review

A few other concepts that you should know, with test exercises:

(a) Integration techniques, such as integration by parts.

- Show $\int_0^1 \sqrt{1-x^2}dx = \frac{\pi}{4}$.
- Show $\int \sec^3(x)dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\log|\sec(x) + \tan(x)| + C$

(b) The algebra and calculus of $e^x$ and $\log(x)$ (base $e$).

- Show $\log\left(\prod_{i=1}^{n} a_i\right) = \sum_{i=1}^{n} \log(a_i)$.
- Show that for $f(x) = \prod_{i=1}^{n} e^{a_i x}$, $f^{(k)}(x) = \left(\sum_{i=1}^{n} a_i\right)^k \cdot e^{\left(\sum_{i=1}^{n} a_i\right)x}$, where $f^{(k)}(x)$ is the $k$-th derivative of $f(x)$ with respect to $x$.

(c) Geometric series and their sums.

- Show that $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{2k+1} = \left(\frac{1}{2}\right) \cdot \left(\frac{1}{1-\frac{1}{4}}\right)$.
- Using the formula $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ for $|r| < 1$, show that $\sum_{k=m}^{\infty} r^k = \frac{r^m}{1-r}$, and show that $\sum_{k=0}^{m} r^k = \frac{1-r^{m+1}}{1-r}$.