Let's Tessellate: Fun and Wildly Impractical Applications of Graph Theory in Neo-Riemannian Theory

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1 An Introduction

Crash-Course in Classical Music Theory

We have 12 tones in Western music, and we assign a letter name to each of them. This cycle of 12 notes repeats on both ends into infinity, and each complete cycle is called an octave.

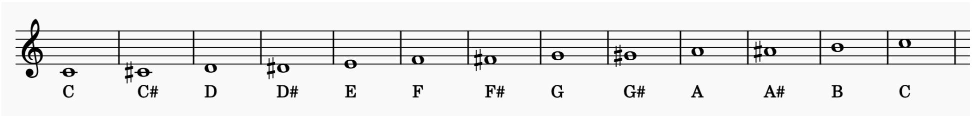

Figure 1: *A single octave, starting from C and ending on C in the next octave.*

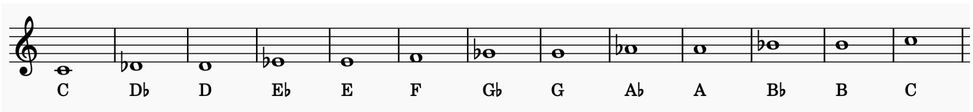

Figure 2: *Another way of writing the 12 tones. (For the sake of simplicity, we will assume that C♯ is the same as D♭, D♯ is the same as E♭, etc, which generally holds true unless you want to be fancy with tuning systems and whatnot.)*

We assign some of these 12 notes to sets called "scales". For the sake of simplicity, we will assume scales are sets of 8 notes. Classical music theory centers around a tonic, the first note of the scale. This is the most important note in the scale.

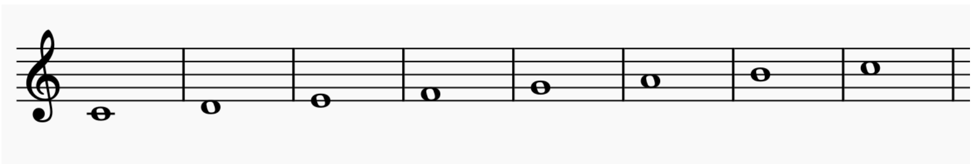

Figure 3: *A C major scale. C is the tonic of this scale.*

We can build "triads" on each one of these notes (we call these bottom notes the "roots") in the scale, created by stacking notes on top. We call the note directly above the root the "third" and the note above the third the "fifth", because of their distances to the roots in the scale. The triad built from the tonic is assigned a Roman numeral "I", and all other notes in the scale are also assigned Roman numerals relative to the tonic.

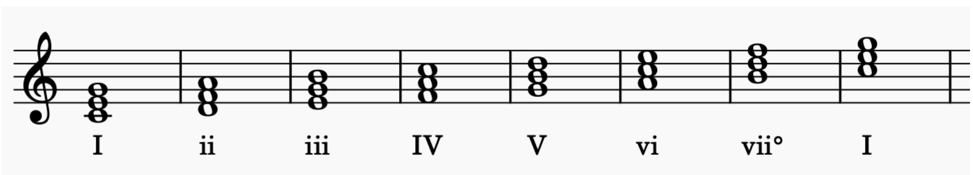

Figure 4: *Triads with Roman numerals in the C major scale.*

For my purposes, you need not know what these numerals mean, just that they have something to do with their harmonic function–that is, the purpose they serve. For instance, the V, called the dominant, sounds particularly consonant to us when leads into ("resolves to") the tonic. The point is, we *need* a tonic in order to assign harmonic functions to all other notes in the scale.

These are the designations and categorizations often found in Baroque music, and we still hear much of these rules in music we listen to today.

Leading to Neo-Riemannian Theory

German composer Richard Wagner premiered his opera *Trisan und Isolde* in 1865, rocking the music world as we know it. The reason? The now infamous Tristan Chord.

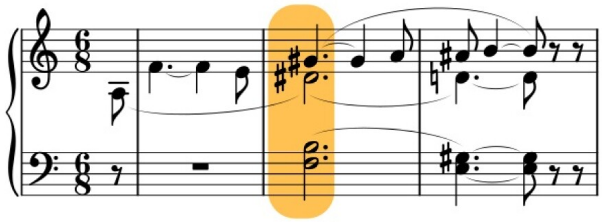

Figure 5: *The Tristan Chord. Listen here.*[1]

The reason for its notoriety is its blatant *atonality*–it didn't fit in with the scales or chords built around the tonic. For that reason, it sounded rather dissonant. So now that music has stopped adhering so strictly to traditional tonal rules or harmonic function, what are music theorists to do but to expand into a new field of music theory?

Building off the work of one Hugo Riemann, who had attempted to rationalize harmony, still in a tonal space, theorists were able to translate his ideas into chord-to-chord relationships outside of tonality.

[1]Image Credit: Vermont Public

2 Neo-Riemannian Theory

The Basics

Neo-Riemannian theory dabbles in analyzing chord progressions that don't follow traditional tonal rules. At its core, Neo-Riemannian theory is built on *transformations*–ways of relating chords to each other via shared notes and converting one to another. At its core, Neo-Riemannian theory has three such transformation:

- **P**arallel: lowers the third of the chord
- **R**elative: raises the fifth of the chord
- **L**eittonweschel: lowers the root of the chord

Note that these transformations are involutions–applying the same transformation twice brings you back to your original chord.

I will be referring to these transformations by their first letters from now on.

Consider the C major triad, which we saw earlier.

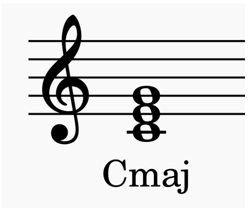

Figure 6: *The lowly C major triad.*

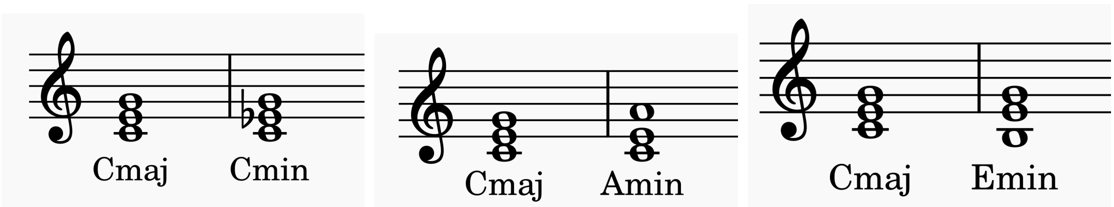

Figure 7: ***P**, **R**, and **L** being applied to the C major chord, respectively.*

From the E♭, we can see that we are clearly no longer within the confines of the C major scale. And yet–we've managed to relate all of these chords together.

More specifically, we have secondary transformations which allow us to make even more changes to our chords:

- **N**ebenverwandt: raises the third and the fifth
- **S**lide: raises the root and the fifth

- **H**exatonic pole: lowers the root and third and raises the fifth.[2]

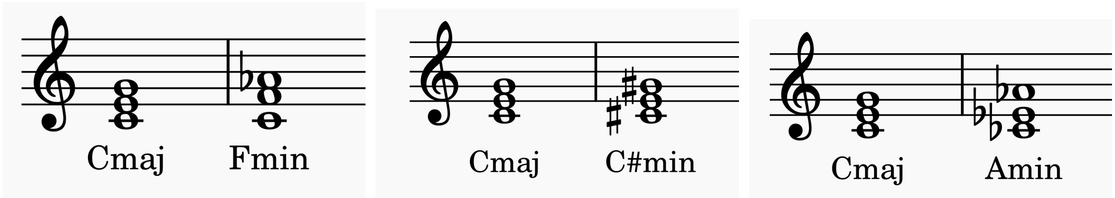

Figure 8: ***N**, **S**, and **H** being applied to the C major chord, respectively.*

Tonnetz

In 1738, Leonhard Euler proposed the concept of the *Tonnetz*, or a tone-net. It was a way of mapping the tonal space onto a lattice diagram, connecting notes by their triadic relationships to each other. We can think of the *Tonnetz* as a 6-regular planar graph, where each vertex corresponds to a pitch and each edge connects vertices that a major/minor third or fifth apart from it.

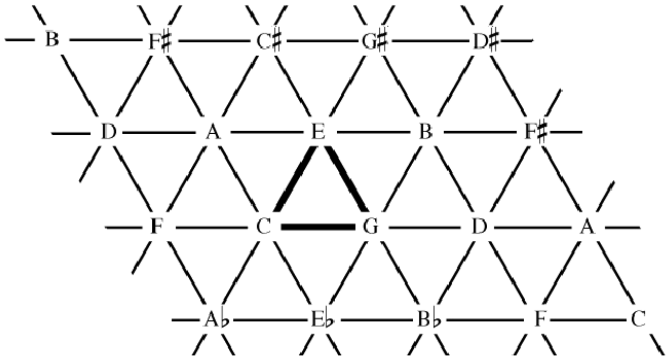

Figure 9: *A more modern rendering of the Tonnetz. The C major chord is bolded.*[3]

This lattice stretches out into infinity in all directions. In 2010, David Bulger projected it onto a torus–a genus-1 surface.

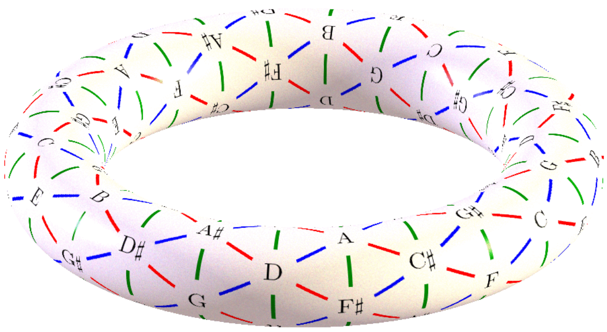

Figure 10: *Bulger's Tonnetz Torus.*[4]

[2]If you want to get more specific, **H** maps a major chord to the minor chord that lies the furthest leading-voice distance away from it in the same hexatonic system.

[3]Image Credit: Research Gate

Neo-Riemannian transformations can also be mapped onto a lattice, like so:

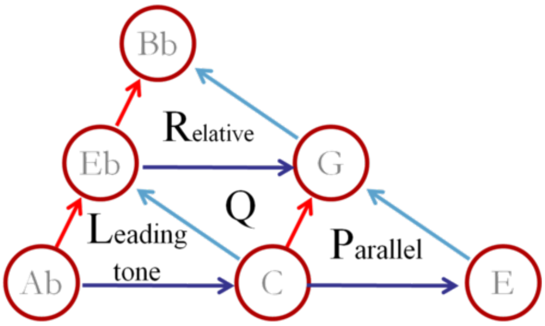

Figure 11: *The PLR transformations applied to the C minor chord, named "**Q**" for generalization.*[6]

If we construct the dual of the *Tonnetz*, we end up with another planar graph, where each vertex represents a triad and edges represent transformations. Indeed, if we repeat this process for the entire *Tonnetz*, we end up another planar graph, consisting of tiled hexagons.

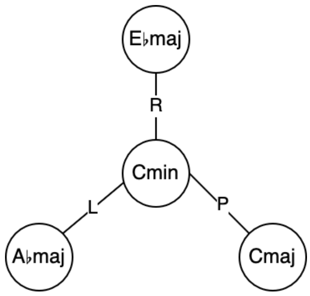

Figure 12: *The dual of the Tonnetz from Fig. 11.*

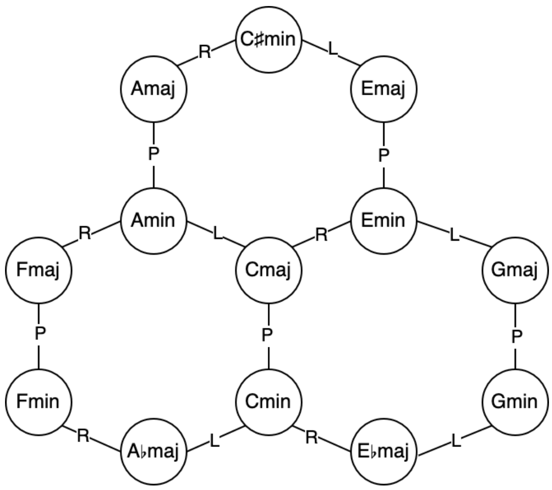

Figure 13: *More dual, this time from Fig. 9. Note that the Tonnetz in Fig. 9 and Fig. 11 are rotated by 180°.*

[4]Image Credit: Wikimedia Commons
[6]Image Credit: Wikimedia Commons

3 Applications of Graph Theory Topics

We now have a handful of tools we can take a closer look at: the transformations, the *Tonnetz* donut, and a handful of planar graphs.

Spanning Trees

Suppose some composition student John is trying to write a music composition C. John has an aversion to certain chord changes, perhaps because he dislikes abrupt tonal shifts or because Nebenverwandt killed his father. Furthermore, his composition teacher, say Professor Wadsworth[7], has challenged him to use a whole assortment of chords, *without any repeating chord progressions.*

We will tackle this problem with an excerpt of our dual *Tonnetz* graph. We can assign positive weights to all disliked chord changes–with larger edge weights correlated to stronger feelings of disdain–and weight 0 to every other chord progression. Name the starting vertex s. We can avoid cycles simply by running a minimum spanning tree algorithm, say Kruskal's. Here's an instance of this composition:

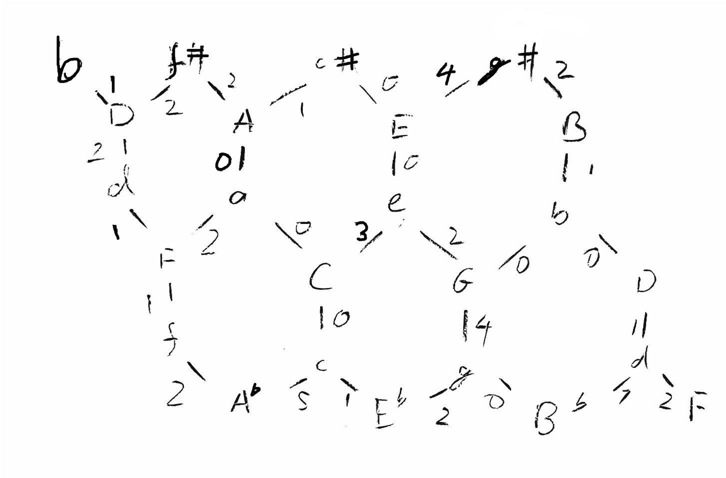

Figure 14: *An excerpt of the Tonnetz, with edge weights generated pseudo-randomly (my brain). Lowercase letters represent minor chords, and uppercase letters represent major chords.*

[7]No relation.

With the C major chord as s, we generate the following minimum spanning tree:

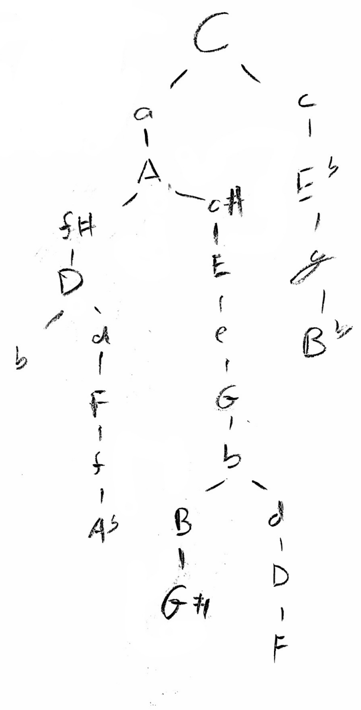

Figure 15: *The spanning tree generated from the above graph, with C as s.*

And then, with enough gumption, time, and determination, we can achieve a composition like this, using a portion of the spanning tree.

Hamiltonian Cycles

Let's try something else that might turn out cool. Consider a similar *Tonnetz* dual to the earlier example, but this time completely unweighted and with some redundancies removed (the B minor and F major chords).

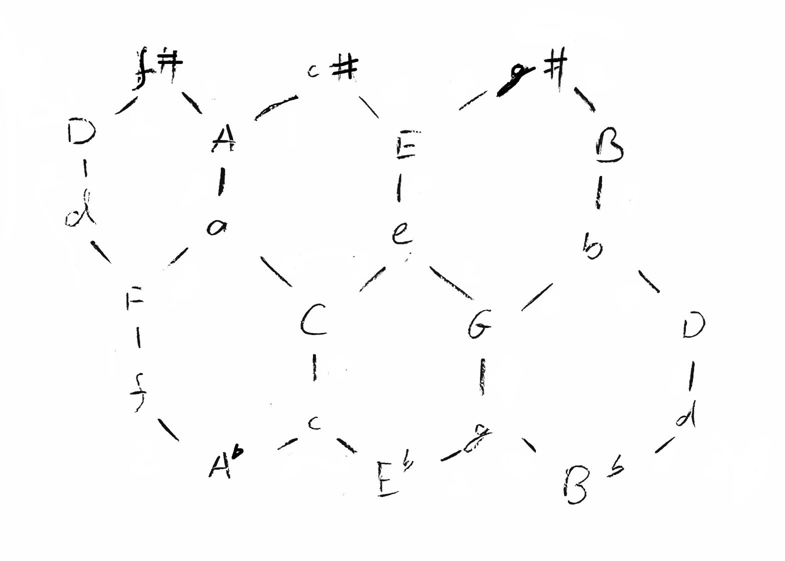

Figure 16: *The unweighted Tonnetz dual.*

As it turns out, it is possible to create a Hamiltonian path, as denoted by the red edges.

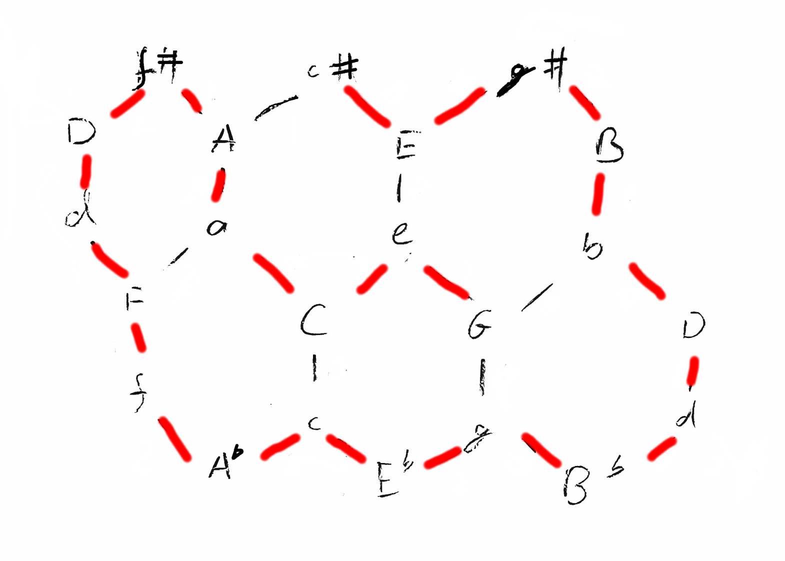

Figure 17: *The Hamiltonian path in the Tonnetz dual.*

Once again, we've managed to smoothly transition through every major and minor triad with no repeats. Neato!

References

David E. Cohen. (2019). Neo-riemannian theory (lecture snippet) [YouTube video]. https://www.youtube.com/watch?v=-L3-qtK4WkY

Lehman, F. (2015). Transformational analysis and the representation of genius in film music. *Music Theory Spectrum*, *37*(1), 1–22.

Nicholas Roumanis. (2021). Music theory: The tonnetz and neo-riemannian transformations [YouTube video]. https://www.youtube.com/watch?v=TY4awNdZ1y8