MATH 164 Optimization
Assignment 1

- Due: April 16. Late homeworks will not be accepted.

1. Let $\mathbf{a}, \mathbf{x} \in \mathbb{R}^n$, consider the functions $f(\mathbf{x}) = \mathbf{a}^\top \mathbf{x}$ and $g(\mathbf{x}) = (\mathbf{a}^\top \mathbf{x})^2$.
   (a) Find $\nabla f(\mathbf{x})$ and the Hessian $H_f(\mathbf{x})$.
   (b) Show that $g(\mathbf{x})$ is a quadratic form.
   (c) Find $\nabla g(\mathbf{x})$ and $H_g(\mathbf{x})$ using part (b).

2. Let $f(\mathbf{x}) = \mathbf{x}^\top Q \mathbf{x}$ be quadratic form, where $Q \in \mathbb{R}^{n \times n}$ is **NOT** symmetric. Show that $\nabla f(\mathbf{x}) = (Q + Q^\top)\mathbf{x}$ and $H_f(\mathbf{x}) = Q + Q^\top$. You may use the result for symmetric matrix without proof. (Hint: $\mathbf{x}^\top Q \mathbf{x} = \mathbf{x}^\top Q^\top \mathbf{x}$)

3. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$:
$$f(\mathbf{x}) = \mathbf{x}^\top \begin{bmatrix} 1 & 0 \\ 6 & 5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 4 \\ 8 \end{bmatrix}^\top \mathbf{x} + \pi.$$
   and the point $\mathbf{x}^* = [1, 1]^\top$.
   (a) Find the $\nabla f(\mathbf{x}^*)$ and $H_f(\mathbf{x}^*)$.
   (b) Find the **unit-normed** vector (direction) $\mathbf{d}$ that minimizes the directional derivative $\frac{\partial f}{\partial \mathbf{d}}(\mathbf{x}^*)$.
   (c) Find a point that satisfies the FONC for f. Does this point satisfy the SONC?

4. Consider the problem
$$\min_{x_1, x_2} \; -x_2 + \pi \quad \text{subject to} \quad x_1 \geq x_2^2, \; |x_2| \geq 1.$$
   (a) Find all point(s) satisfying the FONC.
   (b) Which of the point(s) in part(a) satisfy the SONC?
   (c) Which of the point(s) in part(a) are local minimizers?

5. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable. Let $\mathbf{d} \in \mathbb{R}^n$ with $\|\mathbf{d}\| = 1$, show that
$$\langle \nabla f(\mathbf{x}), \mathbf{d} \rangle \geq -\|\nabla f(\mathbf{x})\|.$$
   For what $\mathbf{d}$ the equality holds?