MATH 164, Optimization
Assignment 2

- **Due**: April 30, Monday. Late homeworks will not be accepted.

1. Consider the problem
$$\min\ f(\mathbf{x}) \quad \text{subject to} \quad \mathbf{x} \in \Omega,$$
where $f(\mathbf{x}) = 2x_1 + 3$ and $\Omega = \{\mathbf{x} \in \mathbb{R}^2 : x_1^2 + x_2^2 \geq 1\}$.
    (a) Find all point(s) satisfying the FONC.
    (b) Which of the point(s) in part(a) satisfy the SONC?
    (c) Which of the point(s) in part(a) are local minimizers?

2. Consider the unconstrained optimization problem
$$\min\ f(\mathbf{x}) = \frac{1}{2}\|A\mathbf{x} - \mathbf{b}\|^2,$$
where $A \in \mathbb{R}^{m\times n}$, $m \geq n$ and $\mathbf{b} \in \mathbb{R}^m$.
    (a) Show that $f(\mathbf{x})$ is a quadratic function of the form $\frac{1}{2}\mathbf{x}^\top Q\mathbf{x} - \mathbf{p}^\top \mathbf{x} + c$ by specifying Q, $\mathbf{p}$, and c.
    (b) Find the gradient $\nabla f(\mathbf{x})$ and Hessian matrix $H_f(\mathbf{x})$.
    (c) Suppose $A = \begin{bmatrix} 5 & 4 \\ 0 & 3 \end{bmatrix}$. Find the upper bound for α such that gradient descent with the fixed step size α converges to the solution.

3. Problem 6.29 from textbook.

4. Problem 8.15 from textbook.

5. Problem 8.17 from textbook.

6. (Optional) This is a programming problem. Use gradient descent method
$$\mathbf{x}^{k+1} = \mathbf{x}^k - \alpha_k \nabla f(\mathbf{x}^k)$$

to solve the following unconstrained optimization problem:

$$\min_{\mathbf{x}\in\mathbb{R}^3} f(\mathbf{x}) := \frac{1}{2}(x_1 - 1)^2 + \frac{1}{4}(x_2 - 2)^2 + \frac{1}{2}(x_3 - 3)^2$$

with the initialization $\mathbf{x}^0 = [3, 4, 5]^\top$ and fixed step size $\alpha_k = 0.7$. Run your algorithm for 10 iterations and print out $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^{10}$. Repeat the above experiment using steepest gradient descent, that is,

$$\alpha_k = \frac{\|\nabla f(\mathbf{x}^k)\|^2}{(\nabla f(\mathbf{x}^k))^\top Q \nabla f(\mathbf{x}^k)}.$$

(You need to figure out the expression of Q.)