MATH 164, Optimization
Assignment 4

- **Due**: May 28 (Monday). Late homeworks will not be accepted.

1. Consider the least squares problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}) := \|A\mathbf{x}-\mathbf{b}\|^2,$$

where $A \in \mathbb{R}^{m\times n}$ is **NOT** necessarily of full-rank. Then the FONC points can be non-unique. Show that any FONC point $\mathbf{x}^*$ is a global minimizer, i.e. $f(\mathbf{x}) \geq f(\mathbf{x}^*)$, $\forall\, \mathbf{x} \in \mathbb{R}^n$.

2. Problem 12.18 from textbook.

3. Problem 12.21 from textbook.

4. This problem derives the so-called *projected gradient descent* algorithm. Consider the following constrained problem:

$$\min_{\mathbf{x}\in\mathbb{R}^n} \ f(\mathbf{x}) \quad \text{subject to} \quad A\mathbf{x} = \mathbf{b},$$

where $A \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, $m < n$, and $\operatorname{rank}(A) = m$.

(a) Consider minimization of the following quadratic approximation to $f(\mathbf{x})$ around $\mathbf{x}^k$ without the constraint:

$$\mathbf{x}^{k+1} = \arg\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}^k) + (\nabla f(\mathbf{x}^k))^\top(\mathbf{x}-\mathbf{x}^k) + \frac{1}{2\alpha_k}\|\mathbf{x}-\mathbf{x}^k\|^2$$

for some $\alpha_k > 0$. Show that $\mathbf{x}^{k+1} = \mathbf{x}^k - \alpha_k \nabla f(\mathbf{x}^k)$.

(b) For any $\mathbf{y} \in \mathbb{R}^n$, show that the problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} \ \|\mathbf{x}-\mathbf{y}\|^2 \quad \text{subject to} \quad A\mathbf{x} = \mathbf{b}$$

has a unique solution given by $\mathbf{x}^* = \mathbf{\Pi}(\mathbf{y})$, where $\mathbf{\Pi}$ is a linear function on $\mathbb{R}^n$ defined as

$$\mathbf{\Pi} : \mathbf{x} \mapsto (I_n - A^\top(AA^\top)^{-1}A)\mathbf{x} + A^\top(AA^\top)^{-1}\mathbf{b}$$

with I_n being the identity matrix of order n.

(c) Consider minimization of the following quadratic approximation to $f(\mathbf{x})$ around $\mathbf{x}^k$ under the same constraint:

$$\mathbf{x}^{k+1} = \arg\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}^k)+(\nabla f(\mathbf{x}^k))^\top(\mathbf{x}-\mathbf{x}^k)+\frac{1}{2\alpha_k}\|\mathbf{x}-\mathbf{x}^k\|^2 \quad \text{subject to} \quad A\mathbf{x}=\mathbf{b}$$

for some $\alpha_k > 0$. Show that $\mathbf{x}^{k+1} = \mathbf{\Pi}(\mathbf{x}^k - \alpha_k \nabla f(\mathbf{x}^k))$. This gives the iteration for projected gradient descent.

5. Problem 12.25 from textbook.