Computation Tree Logic

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- Different paths of the future;

In Computation Tree Logic (CTL) the model of time is a tree-like structure. This way, we cannot use Linear Temporal Logic (LTL) to express the existence of a certain path of time in which some event occurs.

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With p as a literal (atomic formula), AX, EX, AF, EF, AG e EG unary operators.

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- $\varphi\mathtt{U}\psi$: φ is true at least until ψ becomes true;

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- We assume that, similarly to the $\neg$ operator, the "new" unary operators ($\mathtt{AX}, \mathtt{EX}, \mathtt{AF}, \mathtt{EF}, \mathtt{AG}$, and $\mathtt{EG}$) have the first precedence. Next comes the $\wedge$ and $\vee$ operators. And at last the $\rightarrow$, $\mathtt{AU}$ and $\mathtt{EU}$

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A **transition system** $\mathcal{M}$ is a triple $\mathcal{M} = (S, \rightarrow, L)$ in which S is a set of states, $\rightarrow$ is a binary relation over S ($\rightarrow \subseteq S \times S$) and $L : S \rightarrow \mathcal{P}(Atoms)$ is a labelling function.

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Notation: we will use $\mathcal{M}, s \vDash \varphi$ to denote that the model $\mathcal{M}, s$ satisfies the formula φ

Take an arbitrary model $\mathcal{M}$. Let s, s_1, s_2, s_3 be states in S. Let $\varphi, \varphi_1, \varphi_2$ be well-formed formulas of CTL. And let p be an atom. The satisfaction of CTL formulas can be defined as follows:

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- $M, s, \vDash \mathtt{AG}\varphi$ iff for all paths $s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow ...$ in which $s = s_1$, for all s_i, $M, s_i \vDash \varphi$. Thus, AG says: "In all possible paths from now on in all next states..."

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- $M, s, \vDash \mathtt{AF}\varphi$ iff for all paths $s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow ...$ in which $s = s_1$, exists s_i, $M, s_i \vDash \varphi$. Thus, AF says: "In all possible paths from now on, in some next state..."

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- $M, s, \vDash \mathtt{A}[\varphi_1 \mathtt{U} \varphi_2]$ iff for all paths $s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow \ldots$ in which $s = s_1$, this path satisfies $\varphi_1 \mathtt{U} \varphi_2$, i.e., exists s_i in the path such that $\mathcal{M}, s_i \vDash \varphi_2$ and, for all $j < i$, $M, s_j \vDash \varphi_1$. Thus, AU says: "For all paths from now on, until some state..."

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- $M, s, \vDash \mathtt{E}[\varphi_1 \mathtt{U} \varphi_2]$ iff exists some path $s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow ...$

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Examples

- "It's possible to get to a state where something has started but it's not ready": $\mathtt{EF}(\mathit{started} \wedge \neg \mathit{ready})$
- "A certain process is enabled infinitely often on every computation path": $\mathtt{AG}(\mathtt{AF}\mathit{enabled})$

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- "An upwards travelling lift at the second floor does not change its direction when it has passengers wishing to go to the fifth floor": $\mathtt{AG}(\mathit{floor}2 \wedge \mathit{directionUp} \wedge \mathit{button}5 \rightarrow \mathtt{A}[\mathit{directionUp}\mathtt{U}\mathit{floor}5])$

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Two CTL formulas φ and ψ are said to be **semantically equivalent** if any state in any model which satisfies one of them also satisfies the other;

Equivalences

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Notation: we denote the equivalence of φ and ψ by $\varphi \equiv \psi$

Let φ be an arbitrary CTL formula.

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Example of equivalences

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- $\neg \mathtt{EF}\varphi \equiv \mathtt{AG}\neg\varphi$
- $\neg \mathtt{AX}\varphi \equiv \mathtt{EX}\neg\varphi$
- $\mathtt{AF}\varphi \equiv \mathtt{A}[\top \mathtt{U} \varphi]$
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Because of the equivalences shown and the ones in propositional logic, we can have some minimum sets of conectives for the CTL syntax. One of them is defined in Backus-Naur formalism below:

$$\phi ::= \top \mid p \mid \neg\phi \mid \phi \rightarrow \phi \mid \mathtt{AX}\phi \mid \mathtt{A}[\phi\mathtt{U}\phi] \mid \mathtt{E}[\phi\mathtt{U}\phi]$$

Even if CTL allow explicit quantification over paths, it cannot allow some expressions to be formed. For example, we cannot say, as in LTL: "All paths in which have p on them, also have q on them".
This expression can be translated in LTL as follows:

$$\mathtt{F}p \rightarrow \mathtt{F}q$$

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We can try to translate it as $\mathtt{AG}(p \rightarrow \mathtt{AF}q)$ which is closer, but not exactly the same. This one means "for all paths, in all states on the future, if they hold p then, all paths will eventualy hold q"

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For this, we can extend the CTL by dropping the constraint that every temporal operator (`X`, `U`, `F`, `G`) has to be associated with an unique path quantifier (`A`, `E`).
This allows us to generate some statements:

- "In all possible paths, q is true until r is true or p is true until r is true": $\mathtt{A}[q\mathtt{U}r \vee p\mathtt{U}r]$

Statements only possible with CTL*

- "In all possible paths, q is true until r is true or p is true until r is true": $\mathtt{A}[q\mathtt{U}r \vee p\mathtt{U}r]$
- "There is a path in which p eventually occurring will occur in all states": $\mathtt{E}[\mathtt{GF}p]$

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- "In all paths, p will occur in the next state or in the next of the next": $\mathtt{A}[\mathtt{X}p \vee \mathtt{XX}p]$

- "In all possible paths, q is true until r is true or p is true until r is true": $\mathtt{A}[q\mathtt{U}r \vee p\mathtt{U}r]$
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- "In all paths, p will occur in the next state or in the next of the next": $\mathtt{A}[\mathtt{X}p \vee \mathtt{XX}p]$

CTL* syntax

The syntax of CTL* can be defined with the BNF bellow:

$$\phi ::= \bot \mid \top \mid p \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \rightarrow \phi \mid \mathtt{A}[\alpha] \mid \mathtt{E}[\alpha] \mid$$

$$\alpha ::= \phi \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \alpha \rightarrow \alpha \mid \alpha \mathtt{U} \alpha \mid \mathtt{G}\alpha \mid \mathtt{F}\alpha \mid \mathtt{X}\alpha \mid$$

With the same meanings of each operator.

LTL $\subset$ CTL* and CTL $\subset$ CTL*

Although we don't define path operators to LTL we can assume that it consider in all paths. Therefore, we can say that a formula ϕ in LTL is a formula $\mathtt{A}[\phi]$ in CTL*;

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For CTL, it is trivial;