Computation Tree Logic

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- Some practical applications

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- How the model checking works?
- Some strengths and weaknesses of model checking

- Syntax and Semantics of LTL

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- ω-languages, Kripke structures, paths and traces
- Buchi automata and LTL model checking

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- Different paths of the future;

Intuition

In Computation Tree Logic (CTL) the model of time is a tree-like structure. This way, we cannot use Linear Temporal Logic (LTL) to express the existence of a certain path of time in which some event occurs.

CTL was defined by:

Figure 1: Mordechai Ben-Ari

Figure 2: Amir Pnueli

Figure 3: Zohar Manna

And, at the same time by:

Figure 4: Ernest Allen Emerson

Figure 5: Edmund Clarke

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$$AF\phi \mid EF\phi \mid AG\phi \mid EG\phi \mid A[\phi U \phi] \mid E[\phi U \phi]$$

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With p as a literal (atomic formula), AX, EX, AF, EF, AG e EG unary operators.

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- $\varphi U \psi$: φ is true at least until ψ becomes true;

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- Notice as well that the binary operators $A[\varphi U \psi]$ and $E[\varphi U \psi]$ can be represented as AU and EU, respectively;
- We assume that, similarly to the $\neg$ operator, the "new" unary operators (AX, EX, AF, EF, AG, and EG) have the first precedence. Next comes the $\wedge$ and $\vee$ operators. And at last the $\rightarrow$, AU and EU;

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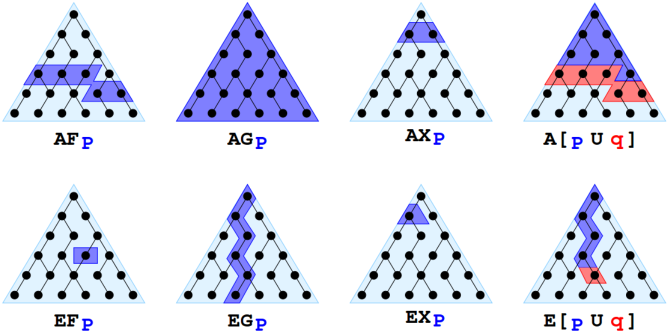
AF p
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AX p
A [p U q]
EF p
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EX p
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Definition

Let *Atoms* be a set of atomic formulas. A **transition system** or **model** $\mathcal{M}$ is a triple $\mathcal{M} = (S, \rightarrow, L)$ in which S is a set of states, $\rightarrow$ is a binary relation over S ($\rightarrow \subseteq S \times S$) such that for every state $s \in S$, exists a s' that $s \rightarrow s'$ and $L : S \rightarrow \mathcal{P}(Atoms)$ (or $L : S \rightarrow (Atoms \rightarrow \{0, 1\})$) is a labelling function.

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CTL formulas are satisfied by a transition system and a specific state.

Notation: we will use $\mathcal{M}, s \vDash \varphi$ to denote that the model $\mathcal{M}, s$ satisfies the formula φ

Definition

The **satisfaction** of a formula in CTL is recursive over the structure of the formula. It can be done as follows:

Take an arbitrary model $\mathcal{M}$. Let s, s_1, s_2, s_3 be states in S. Let $\varphi, \varphi_1, \varphi_2$ be well-formed formulas of CTL. And let p be an atom. The satisfaction of CTL formulas can be defined as follows:

- $\mathcal{M}, s \vDash \top$ and $\mathcal{M}, s \nvDash \bot$ for all $s \in S$

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- $\mathcal{M}, s \models \top$ and $\mathcal{M}, s \not\models \bot$ for all $s \in S$
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- $\mathcal{M}, s, \vDash AG\varphi$ iff for all paths $s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow \ldots$ in which $s = s_1$, for all s_i, $\mathcal{M}, s_i \vDash \varphi$. Thus, AG says: "In all possible paths from now on in all next states..."

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- $\mathcal{M}, s, \vDash AG\varphi$ iff exists some path $s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow \ldots$ in which $s = s_1$, for all s_i, $\mathcal{M}, s_i \vDash \varphi$ Thus, EG says: "Exists a path from now on in all next states..."

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- $\mathcal{M}, s, \vDash AF\varphi$ iff for all paths $s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow ...$ in which $s = s_1$, exists s_i, $\mathcal{M}, s_i \vDash \varphi$. Thus, AF says: "In all possible paths from now on, in some next state..."

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- $\mathcal{M}, s, \vDash EF\varphi$ iff exists some path $s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow \ldots$ in which $s = s_1$, that exists s_i, $\mathcal{M}, s_i \vDash \varphi$. Thus, EF says: "In some path from now on, in some next state..."

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- $\mathcal{M}, s, \models A[\varphi_1 U \varphi_2]$ iff for all paths $s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow \ldots$ in which $s = s_1$, this path satisfies $\varphi_1 U \varphi_2$, i.e., exists s_i in the path such that $\mathcal{M}, s_i \models \varphi_2$ and, for all $j < i$, $\mathcal{M}, s_j \models \varphi_1$. Thus, AU says: "For all paths from now on, until some state..."

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- $\mathcal{M}, s, \vDash E[\varphi_1 U \varphi_2]$ iff exists some path $s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow \ldots$ in which $s = s_1$, this path satisfies $\varphi_1 U \varphi_2$. Thus, EU says: "In some path from now on, until some state..."

Examples

- "It's possible to get to a state where something has started but it's not ready": $EF(\mathit{started} \wedge \neg \mathit{ready})$

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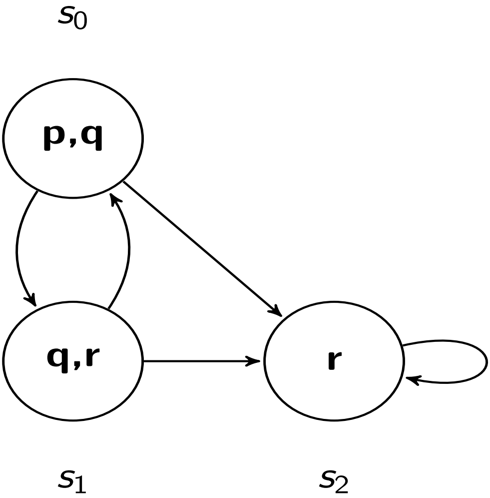
- "It's possible to get to a state where something has started but it's not ready": $EF(started \wedge \neg ready)$
- "A certain process is enabled infinitely often on every computation path": $AG(AFenabled)$
- "An upwards travelling lift at the second floor does not change its direction when it has passengers wishing to go to the fifth floor": $AG(floor2 \wedge directionup \wedge button5 \rightarrow A[directionup U floor5])$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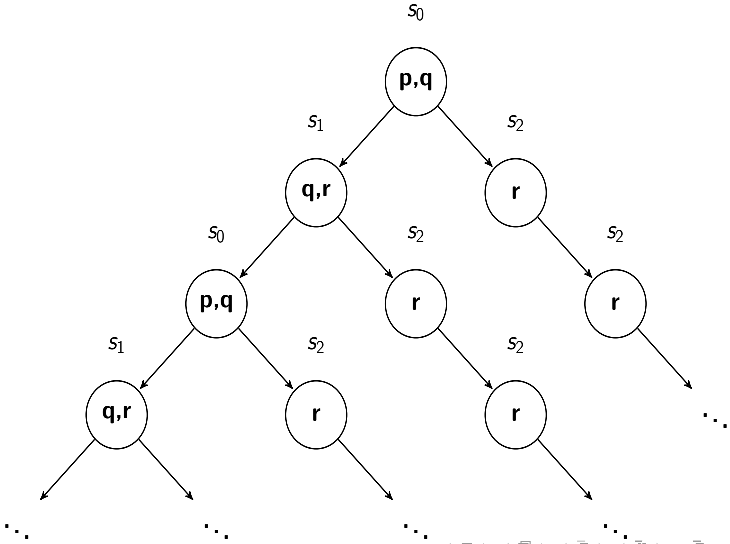
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Example of formulas that are satisfied by that model:

- $\mathcal{M}, s_0 \vDash p \wedge q$
- $\mathcal{M}, s_0 \vDash \neg r$
- $\mathcal{M}, s_0 \vDash EX(q \wedge r)$
- $\mathcal{M}, s_0 \vDash \neg AX(q \wedge r)$
- $\mathcal{M}, s_0 \vDash \neg EF(p \wedge q)$
- $\mathcal{M}, s_2 \vDash EGr$
- $\mathcal{M}, s_0 \vDash AFr$
- $\mathcal{M}, s_0 \vDash E[(p \wedge q)Ur]$
- $\mathcal{M}, s_0 \vDash A[pUr]$
- $\mathcal{M}, s_0 \vDash AG(p \vee q \vee r \rightarrow EFEGr)$

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Two CTL formulas φ and ψ are said to be **semantically equivalent** if any state in any model which satisfies one of them also satisfies the other.

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Notation: we denote the semantic equivalence of φ and ψ by $\varphi \equiv \psi$

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Let φ be an arbitrary CTL formula.

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Minimum set of CTL connectives

Because of the equivalences shown and the ones in propositional logic, we can have some minimum sets of conectives for the CTL syntax. One of them is defined in Backus-Naur formalism below:

$$\phi ::= \bot \mid p \mid \neg\phi \mid \phi \wedge \phi \mid EX\phi \mid AF\phi \mid E[\phi U \phi]$$

Even if CTL allow explicit quantification over paths, it cannot allow some expressions to be formed. For example, we cannot say, as in LTL: "All paths in which have p on them, also have q on them".
This expression can be translated in LTL as follows:

$$Fp \rightarrow Fq$$

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We can try to translate it as $AG(p \rightarrow AFq)$ which is closer, but not exactly the same. This one means "for all paths, in all states on the future, if they hold p then, all paths will eventualy hold q"

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This allows us to generate some statements:

- "In all possible paths, q is true until r is true or p is true until r is true": $A[qUr \vee pUr]$

- "In all possible paths, q is true until r is true or p is true until r is true": $A[qUr \vee pUr]$
- "There is a path in which p eventually occurring will occur in all states": $E[GFp]$

- "In all possible paths, q is true until r is true or p is true until r is true": $A[qUr \vee pUr]$
- "There is a path in which p eventually occurring will occur in all states": $E[GFp]$
- "In all paths, p will occur in the next state or in the next of the next": $A[Xp \vee XXp]$

The syntax of CTL* can be defined with the BNF bellow:

$$\phi ::= \bot \mid \top \mid p \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \rightarrow \phi \mid A[\alpha] \mid E[\alpha] \mid$$

$$\alpha ::= \phi \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \alpha \rightarrow \alpha \mid \alpha U \alpha \mid G\alpha \mid F\alpha \mid X\alpha \mid$$

With the same meanings of each operator.

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For CTL, it is trivial;

The CTL model-checking algorithm

We present an algorithm which, given a model and a CTL formula, outputs the set of states of the model that satisfy the formula.

The algorithm deals explicitly only with some of the CTL connectives; for the others, it tranforms them to their equivalent form in terms of the minimal set of connectives previously definied: $\{\bot, \neg, \wedge, AF, EU, EX\}$

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- First, rewrite ϕ in terms of $\perp, \neg, \wedge, AF, EU$ and EX.
- Next, label the states of $\mathcal{M}$ with the subformulas of ϕ that are satisfied there, starting with the smallest subformulas and working outwards towards ϕ.

Suppose ψ is a subformula of ϕ and states satisfying all the immediate subformulas of ψ have already been labelled. We determine by a case analysis which states to label with ψ. If ψ is

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- $\neg\psi_1$: label s with $\neg\psi_1$ if s is not already labelled with ψ_1.
- $AF\psi_1$:
  - If any state s is labelled with ψ_1, label it with $AF\psi_1$.
  - Repeat: label any state with $AF\psi_1$ if all successor states are labelled with $AF\psi_1$, until there is no change.

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- $E[\psi_1 U \psi_2]$:
  - If any state s is labelled with ψ_2, label it with $E[\psi_1 U \psi_2]$.
  - Repeat: label any state with $E[\psi_1 U \psi_2]$ if it is labelled with ϕ_1 and at least one of its successors is labelled with $E[\psi_1 U \psi_2]$, until there is no change.

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  - Repeat: label any state with $E[\psi_1 U \psi_2]$ if it is labelled with ϕ_1 and at least one of its successors is labelled with $E[\psi_1 U \psi_2]$, until there is no change.
- $EX\psi_1$: label any state with $EX\psi_1$ if one of its successors is labelled with ψ_1.

- Having performed the labelling for all the subformulas of ϕ (including ϕ itself), we output the states which are labelled ϕ.

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- The complexity of this algorithm is $O(fV(V+E))$, where f is the number of connectives in the formula, V is the number of states and E is the number of transitions.

- Here, we present a simple, pretty pseudocode for the labelling algorithm.

- Here, we present a simple, pretty pseudocode for the labelling algorithm.
- The program *SAT* expects a tree-structured CTL formula constructed by means of the BNF showed earlier.

function SAT(ϕ) **begin**

case

ϕ is $\top$: **return** S

ϕ is $\bot$: **return** $\emptyset$

ϕ is atomic: **return** $\{s \in S | \phi \in L(s)\}$

ϕ is $\neg\phi_1$: **return** $S - SAT(\phi_1)$

ϕ is $\phi_1 \wedge \phi_2$: **return** $SAT(\phi_1) \cap SAT(\phi_2)$

ϕ is $\phi_1 \vee \phi_2$: **return** $SAT(\phi_1) \cup SAT(\phi_2)$

ϕ is $\phi_1 \rightarrow \phi_2$: **return** $SAT(\neg\phi_1 \vee \phi_2)$

ϕ is $AX\phi_1$: **return** $SAT(\neg EX\neg\phi_1)$

ϕ is $EX\phi_1$: **return** $SAT_{EX}(\phi_1)$

ϕ is $A[\phi_1 U \phi_2]$: **return** $SAT(\neg(E[\neg\phi_2 U(\neg\phi_1 \wedge \phi_2)] \vee EG\neg\phi_2))$

ϕ is $E[\phi_1 U \phi_2]$: **return** $SAT_{EU}(\phi_1, \phi_2)$

ϕ is $EF\phi_1$: **return** $SAT(E(\top U\phi_1))$
ϕ is $EG\phi_1$: **return** $SAT(\neg AF\neg\phi_1)$
ϕ is $AF\phi_1$: **return** $SAT_{AF}(\phi_1)$
ϕ is $AG\phi_1$: **return** $SAT(\neg EF\neg\phi_1)$
end case
end function

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- 'pre' denotes travelling backwards along the transition relation.
- $pre_{\exists}$ returns set of states of S which *can* make a transition into S.
- $pre_{\forall}$ returns the set of states of S which make transitions *only* into Y.

The pseudocode for the special procedures of SAT are the following.

function $SAT_{EX}(\phi)$
local var X, Y
begin
    $X := SAT(\phi)$
    $Y := pre_{\exists}(X)$
    return Y
end

function $SAT_{AF}(\phi)$
local var X, Y
begin
    $X := S$
    $Y := SAT(X)$
    repeat until $X = Y$
    begin
        $X := Y$
        $Y := Y \cup pre_{\forall}(Y)$
    end
    return Y
end

function $SAT_{EU}(\phi, \psi)$
local var W, X, Y
begin
    $W := SAT(\phi)$
    $X := S$
    $Y := SAT(\psi)$
    repeat until $X = Y$
    begin
        $X := Y$
        $Y := Y \cup (W \cap pre_{\exists}(Y))$
    end
    return Y
end

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