A Proof of Collatz Conjecture in Binary

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Collatz conjecture was first suggested in 1937, named after German mathematician Lothar Collatz. Given any positive integer n, perform the following operation and let $n = f(n)$, and continue to repeatedly apply the process, it would always end up at 1.

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \ldots\ldots (S1) \\ 3n+1, & \text{if } n \text{ is odd} \ldots\ldots (S2) \end{cases}$$

Nowadays, it still remains a conjecture [1]. However, a proof of the validity of the Collatz conjecture is found easily if we consider n in binary.

Let #n be the number of bits of n and #f be the number of bits of $f(n)$ in binary, respectively.

Case1. If n=1, then f(1)=100, f(100)=10, f(10)=1, the conjecture is valid.

Case2. If n>1 and is even, then #f = #n -1 by S1. If $f(n)$>1 and is even, then continue to apply S1 until n=1 or is greater than 1 and odd. When n=1, it is in Case1. When n is greater than 1 and odd, go to Case3.

Case3. Since n is greater than 1 and odd, its least significant bit (LSB) is 1. Let m=#n. After S2, #$f(n)$<= m+2 because $3n+1 < 4n$ when $n > 1$ and 4n makes #n=m+2, and $f(n)$<4n.

Case3.1. If the two LSBs are 01, the two LSBs of $f(n)$ become 00 after S2. In Case2, #n decreases by 2. If it is Case3 again, n decreases.

If the three LSBs are 011, the three LSBs of $f(n)$ become 010 after S2. In Case2, #n decreases by 1 and the two LSBs become 01, go to Case3.1 again.

If the four LSBs are 0111, the four LSBs of $f(n)$ become 0110 after S2. In Case2, #n decreases by 1 and the three LSBs become 011, go to Case3.1 again.

Following the same rule, when there is one bit 0 in n, n decreases in Case3.1.

Case3.2. If all bits of n are 1's, i.e., 11…11, #$f(n)$ increases by 2 after S2 and $f(n)$ becomes 101…110. In Case2, it becomes 101….11 and go to Case3.1.

As n decreases, it will go to Case1 and the conjecture is valid.

References

[1] Almost all orbits of the Collatz map attain almost bounded values, Terence Tao, Sep. 2019, ***https://arxiv.org/abs/1909.03562***