Solving the Mild-Slope and Helmholtz equations using the Virtual Element Method (VEM), dealing with high order Robin Boundary Condition

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Abstract

The numerical solution of the Mild-slope equation (MSE) is crucial in various fields, including coastal engineering, oceanography, and offshore structure design. In this article, we present a novel approach utilizing the Virtual Element Method (VEM) for the numerical solution of the MSE. The VEM offers significant advantages over traditional finite element methods, particularly in handling complex geometries and irregular meshes. We demonstrate the effectiveness of our approach through the validation of our model using the Helmholtz equation, a well-established benchmark problem in numerical analysis. By comparing our numerical results with analytical solutions and established numerical methods for the Helmholtz equation, we establish the accuracy and reliability of our VEM-based solver for the MSE. Our findings not only contribute to the advancement of numerical methods for coastal and ocean engineering but also showcase the potential of the Virtual Element Method in tackling challenging problems in computational fluid dynamics.

Keywords. Mild-slope equation, Helmholtz equation, Virtual Element Method, computational fluid dynamics, validation, Finite Element Methods, Numerical Analysis, Complex Geometries, Irregular Meshes, Robin Boundary Condition, Coastal Engineering.

Contents

1 Introduction **3**

2 Problem (section qui va pas ici) **3**

3 VEM **5**
3.1 Dofs 5
3.2 pb 5
3.3 Robin Boundary Condition 9

4 Numerical Validation **12**

5 Numerical Application **13**
5.1 Sensibilité de la pente, helmoltz vs Berkhoff 13
5.2 Application claquée - Résultats avec différents k 13
5.3 Sensibilité du coefficient de réflexion 13
5.4 Appli montrant intérêt Robin 14

6 Conclusion and Discussion **16**

7 Declarations **17**
7.1 Availability of data and material 17
7.2 Conflict of interest 17
7.3 Acknowledgements 17

A Roro pue **18**

1 Introduction

On introduit!

2 Problem (section qui va pas ici)

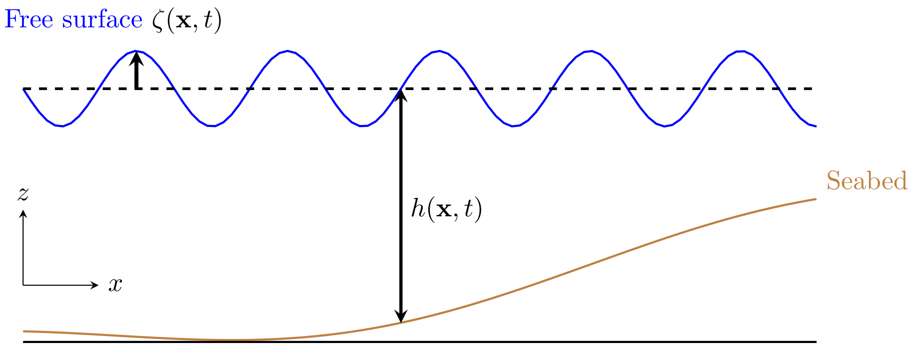


Figure 1: Sketch of a free-surface elevation ζ in the $(x,\, z)$-plane

We consider $\zeta(\mathbf{x}, t) = \Re\left\{\eta(x, y)\, e^{-i\omega t}\right\}$

$$u = u_i + u_r$$

with u_i the **incident wave** and u_r **the reflected wave**. We have,

$$u_i(\mathbf{x}, t) = a_i(\mathbf{x})e^{-i\sigma t} \quad \text{and} \quad u_r(\mathbf{x}, t) = a_r(\mathbf{x})e^{-i\sigma t}$$

with $\sigma = 2\pi/T_0$, the **angular frequency** and

$$a_i(\mathbf{x}) = a_{\max}e^{-i\mathbf{k}\mathbf{x}} \quad \text{with} \quad \mathbf{k} = k(\cos(\theta), \sin(\theta))$$

with θ the **incident wave angle**, $a_{\max}$ the **maximum wave amplitude**. a_r satisfies the following Berkhoff equation :

$$\begin{cases} \nabla(C_pC_g\nabla a) + k^2C_pC_g\, a = 0 & , \quad \text{in } \Omega\,, \\ a = -a_i & , \quad \text{on } \Gamma_D\,, \\ \dfrac{\partial a}{\partial n} + i\, k\, a = 0 & , \quad \text{on } \Gamma_{Inf}. \end{cases} \tag{1}$$

Assuming constant depth within the port and $C_g = C_p/2$ (as in shallow water) and noting that $C_p = \sigma/k = Cte$, equation (1) can be simplified to yield the following Helmholtz equation:

$$\begin{cases} \Delta a + k^2 a = 0 & , \quad \text{in } \Omega\,, \\ a = -a_i & , \quad \text{on } \Gamma_D\,, \\ \dfrac{\partial a}{\partial n} + i\, k\, a = 0 & , \quad \text{on } \Gamma_{Inf}. \end{cases} \tag{2}$$

Nous considérons l'hypothèse suivante:
FAUX, à mieux redire:

$$\int_E C_p C_g = Cte \quad \text{and} \quad \int_E k^2 C_p C_g = Cte \tag{3}$$

3 VEM

3.1 Dofs

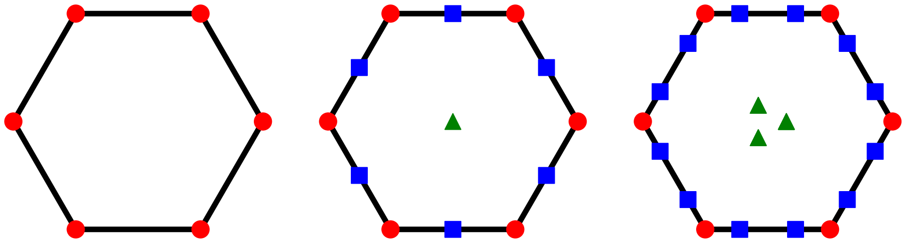


Figure 2: 2D element with ● : Summits dofs, ■ : Edges dofs, ▲ : Inner dofs.

3.2 pb

Nous considérons les problèmes suivant:

$$\begin{cases} \nabla(C_p C_g \nabla a) + k^2 C_p C_g \, a = 0 & , \quad \text{in } \Omega \, , \\ a = -a_i & , \quad \text{on } \Gamma_D \, , \\ \dfrac{\partial a}{\partial n} + i \, k \, a = 0 & , \quad \text{on } \Gamma_{Inf} \, . \end{cases} \tag{4}$$

On considère la formulation variationnelle:

$$\begin{cases} \text{find } u \in V = H_0^1(\Omega) \text{ such that} \\ \qquad a(u, v) \; = \; 0 \quad \forall v \in V, \end{cases} \tag{5}$$

where

$$a(u, v) = \int_\Omega \nabla(C_p C_g \nabla u \, v) + \int_\Omega k^2 C_p C_g \, u \, v \quad . \tag{6}$$

We will build a discrete problem in following form:

$$\begin{cases} \text{find } u_h \in V_h \subset V \text{ such that} \\ \qquad a(u_h, v_h) \; = \; 0 \quad \forall v \in V, \end{cases} \tag{7}$$

where

- $V_h \subset V$ is a finite dimensional space;
- $a_h(\,\cdot\,,\,\cdot\,) \,:\, V_h \,\times\, V_h \,\to \mathbb{R}$ is a discrete bilinear form approximating the continuous form $a(\,\cdot\,,\,\cdot\,)$;

and,

$$
\begin{aligned}
a_h(u_h, v_h) &= \sum_{E\in\Omega_h} \left[\int_E \nabla(C_p C_g \nabla u_h v_h) + \int_E k^2 C_p C_g u_h v_h\right], \\
&\underset{\substack{1/E\int_E C_p C_g = \mathcal{A}_E \\ 1/E\int_E k^2 C_p C_g = \mathcal{B}_E}}{\approx} \sum_{E\in\Omega_h} \left[\mathcal{A}_E \int_E (\Delta u_h\, v_h) + \mathcal{B}_E \int_E u_h v_h\right], \\
&\underset{\substack{green \\ \partial u/\partial n = -iku}}{=} \sum_{E\in\Omega_h} \left[-\mathcal{A}_E \int_E \nabla u_h \nabla v_h + \mathcal{B}_E \int_E u_h v_h - \mathbf{1}_{\Gamma_{\mathrm{Inf}}\subset E}\, i\mathcal{A}_E \int_{\Gamma_{\mathrm{Inf}}} k\, u_h v_h\right].
\end{aligned}
\tag{8}
$$

with $\mathbf{1}_{\Gamma_{\mathrm{Inf}}\subset E}$ the indicator function and $u = u + u_D$ and u_D is the lifting of $-a_i$.

Nous considérons les problèmes suivant:

$$\begin{cases} \Delta a + k^2\, a = 0 & , \quad \text{in } \Omega\,, \\ \qquad\qquad a = -a_i & , \quad \text{on } \Gamma_D\,, \\ \dfrac{\partial a}{\partial n} + i\,k\,a = 0 & , \quad \text{on } \Gamma_{Inf}. \end{cases} \tag{9}$$

On considère la formulation variationnelle:

$$\begin{cases} \text{find } u \in V = H_0^1(\Omega) \text{ such that} \\ \qquad\qquad a(u,v) \;=\; 0 \quad \forall v \in V, \end{cases} \tag{10}$$

where

$$\begin{aligned} a(u,v) &= \int_\Omega \Delta u\, v + k^2 \int_\Omega u\, v + k^2 \int_{\Gamma_{Inf}} \frac{\partial u}{\partial n}\, v \\ &= \int_\Omega \Delta u\, v + k^2 \int_\Omega u\, v - ik \int_{\Gamma_{Inf}} u\, v \end{aligned}. \tag{11}$$

We will build a discrete problem in following form:

$$\begin{cases} \text{find } u_h \in V_h \subset V \text{ such that} \\ \qquad a(u_h, v_h) \;=\; 0 \quad \forall v \in V, \end{cases} \tag{12}$$

where

- $V_h \subset V$ is a finite dimensional space;
- $a_h(\,\cdot\,,\,\cdot\,) : V_h \times V_h \to \mathbb{R}$ is a discrete bilinear form approximating the continuous form $a(\,\cdot\,,\,\cdot\,)$;

and,

$$\begin{aligned} a_h(u_h, v_h) &= \sum_{E \in \Omega_h} \left[\int_E \Delta u_h v_h + \int_E k^2 u_h v_h \right], \\ \underset{\substack{green \\ \partial u/\partial n = -iku}}{=} &\sum_{E \in \Omega_h} \left[-\int_E \nabla u_h \nabla v_h + \int_E u_h v_h - \mathbf{1}_{\Gamma_{\mathrm{Inf}} \subset E}\, i \int_{\Gamma_{\mathrm{Inf}}} k\, u_h v_h \right]. \end{aligned} \tag{13}$$

with $\mathbf{1}_{\Gamma_{\mathrm{Inf}} \subset E}$ the indicator function and $u = u + u_D$ and u_D is the lifting of $-a_i$.

Let Ω_h be a simple polygonal mesh on Ω. This can be any decomposition of Ω in non overlapping polygons E with straight faces. The space V_h will be defined element-wise, by introducing

- local spaces $V_{h|E}$;
- the associated local degrees of freedom.

For all $E \in \Omega_h$:

$$V_{h|E} = \Big\{ v_h \in H_0^1(\Omega) \cap C^0(\partial E) \, : \, v_h|_{\partial E} \in \mathbb{P}_k(E), \Delta v_h \in \mathbb{P}_k(E) \\ \big(\Pi_k^\nabla v_h - v_h, p\big)_{0,E} = 0, \forall p \in \mathbb{P}_k(E)/\mathbb{P}_{k-2}(E) \Big\} \tag{14}$$

where $\mathbb{P}_k(E)/P_{k-2}(E)$ is the subspace of $\mathbb{P}_k(E)$ of the polynomials that are orthogonal in the sense of L^2 to $\mathbb{P}_{k-2}(E)$, or, alternatively, the polynomials of degree $k-1$ and k.

- the functions $V_{h|E}$ are continuous (and known) on ∂E;
- the functions $V_{h|E}$ are unknow on E!
-

1. Mesh Decomposition: We consider a polytopal decomposition $\{T_h\}_h$ of the domain Ω which is regular-that is, there exists $\rho \in (0,1)$, independent of h, such that every element $E \in T_h$ is star-shaped with respect to a ball of radius $\geq \rho h_E$, with h_E the diameter of E.

2. Local Projections: We denote by $\Pi_k^{\nabla,E} : H^1(E) \to \mathbb{P}_k(E)$ and $\Pi_k^{0,E} : L^2(E) \to \mathbb{P}_k(E)$ the usual lo elliptic projection and local L^2-projection respectively onto the space of polynomials of degree a most k.

3. Virtual Space: We define the local virtual space by

$$V_h^E = \Big\{ v_h \in H_0^1(\Omega) \cap C^0(\partial E) \, : \, v_h|_{\partial E} \in \mathbb{P}_k(E), \Delta v_h \in \mathbb{P}_k(E) \\ \big(\Pi_k^{\nabla} v_h - v_h, p\big)_{0,E} = 0, \forall p \in \mathbb{P}_k(E)/\mathbb{P}_{k-2}(E) \Big\}$$

where $\mathbb{P}_k(E)/P_{k-2}(E)$ is the subspace of $\mathbb{P}_k(E)$ of the polynomials that are orthogonal in the sense of L^2 to $\mathbb{P}_{k-2}(E)$, or, alternatively, the polynomials of degree $k-1$ and k.

3. DDL

4. Base:

$$m_{\alpha_1,\alpha_2} = \left(\frac{x - x_D}{h_D}\right)^{\alpha_1} \cdot \left(\frac{y - y_D}{h_D}\right)^{\alpha_2}$$

Soit $V = \{v \in H^1(\Omega), \ v_{|\Gamma_D} = 0\}$.

$$\int_\Omega \Delta u \, v + \int_\Omega k^2 \, u \, v = 0, \qquad \forall v \in V.$$

En appliquant la formule de Green, on a :

$$-\int_\Omega \nabla u \, \nabla v + \int_{\Gamma_{Inf}} \frac{\partial u}{\partial n} \, v + k^2 \int_\Omega u \, v = 0, \qquad \forall v \in V.$$

$$-\int_\Omega \nabla u \, \nabla v - ik \int_{\Gamma_{Inf}} u \, v + k^2 \int_\Omega u \, v = 0, \qquad \forall v \in V.$$

3.3 Robin Boundary Condition

Nous allons construire la matrice $B = \left(\int_{\partial\Omega} \alpha(\Phi_j(x,y), \Phi_i(x,y)\right)_{i,j}$ en parcourant les éléments frontières du bord. Ces éléments sont des segments joignant 2 points consécutifs du bord. La fonction de base Φ_i attachée au sommet i du bord, restreinte à l'élément de bord est une fonction $\mathbb{P}_k$ du bord.

Les éléments du bord sont de longueur $\lambda(e)$. Pour un élément de référence de la forme $[\xi_0, \xi_0 + \lambda]$ entre deux points V_0 et V_1.

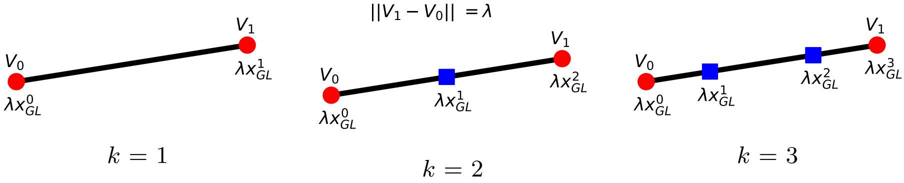


Figure 3: 1D element $[\xi_0, \xi_0 + \lambda]$ representation with ● : Summits dofs, ■ : Edges dofs.

La matrice élémentaire associée à ce terme de Robin est de la forme

$$elb = \left(\int_0^\lambda \alpha_*(\xi_0 + \xi)\varphi_j(\xi)\varphi_i(\xi)\, d\xi \right)_{1 \le i,j \le k+1}, \tag{15}$$

avec $\alpha_*(\xi_0 + \xi) = \alpha(V_0|_x + \xi\, t_x, V_0|_y + \xi\, t_y)$ et $\vec{t}$ est le vecteur unitaire orienté de V_0 à V_1. Pour $i, j \in [\![0, k]\!]$:

$$\varphi_i(\lambda\, x_{GL}^j) = \delta_j^i,$$

avec x_{GL}^j le $j - ieme$ point de quadrature de Gauss-Lobatto sur [0, 1]. On trouve les φ_i aisément avec les polynômes de Lagrange comme suivant:

$$\begin{aligned} \varphi_i(\xi) &= \sum_{j=0}^{k} \delta_j^i \left(\prod_{l=0, l\neq j}^{k} \frac{\xi - \lambda x_{\mathrm{GL}}^l}{\lambda x_{\mathrm{GL}}^j - \lambda x_{\mathrm{GL}}^l} \right) \\ &= \frac{1}{\lambda^k} \prod_{l=0, l\neq i}^{k} \frac{\xi - \lambda x_{\mathrm{GL}}^l}{x_{\mathrm{GL}}^i - x_{\mathrm{GL}}^l} \end{aligned}$$

Remark:

$$
\begin{aligned}
&\left(\int_0^\lambda \alpha_*(\xi_0+\xi)\varphi_j(\xi)\varphi_i(\xi)\,d\xi\right)_{0\le i,j\le k}\\
\underset{\alpha_*=\alpha=cte}{=}&\left(\frac{\alpha}{\lambda^{2k}}\int_0^\lambda\left[\prod_{l=0,l\neq i}^{k}\frac{\xi-\lambda x^l_{\mathrm{GL}}}{x^i_{\mathrm{GL}}-x^l_{\mathrm{GL}}}\prod_{l=0,l\neq i}^{k}\frac{\xi-\lambda x^l_{\mathrm{GL}}}{x^j_{\mathrm{GL}}-x^l_{\mathrm{GL}}}\right]d\xi\right)_{0\le i,j\le k}\\
\underset{\substack{\xi=\lambda\xi'\\ d\xi=\lambda d\xi'}}{=}&\left(\frac{\alpha}{\lambda^{2k}}\int_0^1\left[\prod_{l=0,l\neq i}^{k}\frac{\lambda\xi'-\lambda x^l_{\mathrm{GL}}}{x^i_{\mathrm{GL}}-x^l_{\mathrm{GL}}}\prod_{l=0,l\neq i}^{k}\frac{\lambda\xi'-\lambda x^l_{\mathrm{GL}}}{x^j_{\mathrm{GL}}-x^l_{\mathrm{GL}}}\right]\lambda d\xi'\right)_{0\le i,j\le k}\\
=&\left(\frac{\alpha}{\lambda^{2k}}\int_0^1\left[\prod_{l=0,l\neq i}^{k}\lambda\frac{\xi'-x^l_{\mathrm{GL}}}{x^i_{\mathrm{GL}}-x^l_{\mathrm{GL}}}\prod_{l=0,l\neq i}^{k}\lambda\frac{\xi'-x^l_{\mathrm{GL}}}{x^j_{\mathrm{GL}}-x^l_{\mathrm{GL}}}\right]\lambda d\xi'\right)_{0\le i,j\le k}\\
=&\left(\alpha\lambda\int_0^1\left[\prod_{l=0,l\neq i}^{k}\frac{\xi'-x^l_{\mathrm{GL}}}{x^i_{\mathrm{GL}}-x^l_{\mathrm{GL}}}\prod_{l=0,l\neq i}^{k}\frac{\xi'-x^l_{\mathrm{GL}}}{x^j_{\mathrm{GL}}-x^l_{\mathrm{GL}}}\right]d\xi\right)_{0\le i,j\le k}\\
=&\alpha\,\lambda\left(\int_0^1\tilde{\varphi}_j(\xi)\tilde{\varphi}_i(\xi)\,d\xi\right)_{0\le i,j\le k}
\end{aligned}
\tag{16}
$$

with $\tilde{\varphi}_i$ the polynomials for a unit element $[\xi_0,\xi_0+1]$

Et par exemple pour $k=1$, $\varphi_1(\xi)=\frac{\lambda-\xi}{\lambda}$, $\varphi_2(\xi)=\frac{\xi}{\lambda}$.

Pour le cas particulier où $\alpha=cte$, l'intégrale (15) revient à intégrer un polynôme de degrés $2k$. En évaluant celle-ci par $k+2$ points de GL (car exacte à $2n$ - 3), on obtient l'intégrale exacte.

Nous allons construire également le second membre SMB qui provient des termes de bord inhomogènes:

$$SMB_i=\int_{\partial\Omega}b(x,y)\Phi_i(x,y).$$

Ce terme est non trivialement nul si le sommet i appartient à $\partial\Omega=\Gamma_b\cup\Gamma_t\cup\Gamma_l\cup\Gamma_r$. Pour un élément de référence de la forme $[\xi_0,\xi_0+\delta]$, le vecteur élémentaire associé à ce terme inhomogène est de la forme

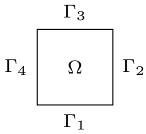
$$vecb=\left(\int_0^\lambda b_*(\xi_0+\xi)\varphi_i(\xi)\,d\xi\right)_{1\le i\le k+1},$$

avec $\varphi_1(\xi)=\frac{\delta-\xi}{\delta}$, $\varphi_2(\xi)=\frac{\xi}{\delta}$ et $\beta_*(\xi_0+\xi)$ coïncide avec $\beta_*(\xi_0+\xi)=\beta(V_0|_x+\xi\,t_x,V_0|_y+\xi\,t_y)$ et $\vec{t}$ est le vecteur unitaire orienté de V_0 à V_1. Également, cette intégrale pourra être intégrée par GL.

4 Numerical Validation

We consider,

$$\begin{cases} \Delta u + k^2\, u = f(x,y) & , \quad \text{in } \Omega\,, \\ \qquad\quad u = u_{\text{exact}} & , \quad \text{on } \Gamma_2 \cup \Gamma_3 \cup \Gamma_4\,, \\ \dfrac{\partial u}{\partial n} + i\,k\,u = g(x,y) & , \quad \text{on } \Gamma_1, \end{cases} \tag{17}$$



with,

$$\begin{aligned} u_{\text{exact}}(x,y) &= (x+y)\cdot(1+i) + \exp(x^2 + i\,y^2), \\ f(x,y) &= -\left((2x)^2 + (2\,i\,y)^2 + 2(1+i)\right)\cdot \exp(x^2 + i\,y^2) + k^2 \cdot u_{\text{exact}}(x,y), \\ g(x,y) &= (1+i) + (2\,i\,y)\cdot \exp(x^2 + i\,y^2) + i\,k \cdot u_{\text{exact}}(x,y). \end{aligned} \tag{18}$$

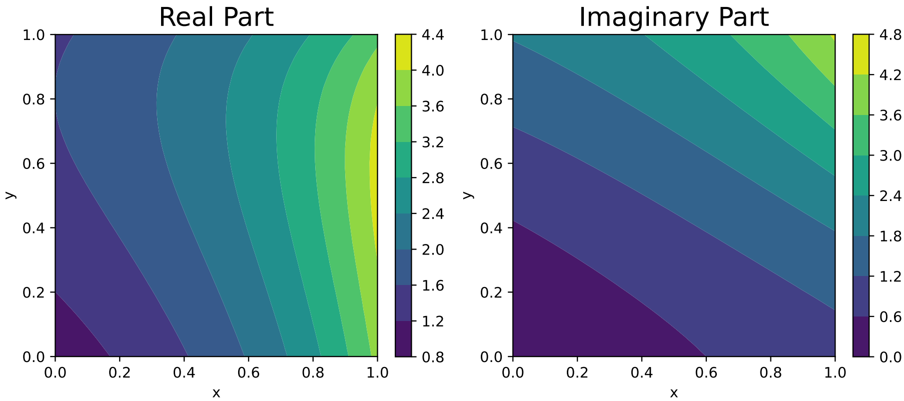


Figure 4: Real and Imaginary part of u_{exact}.

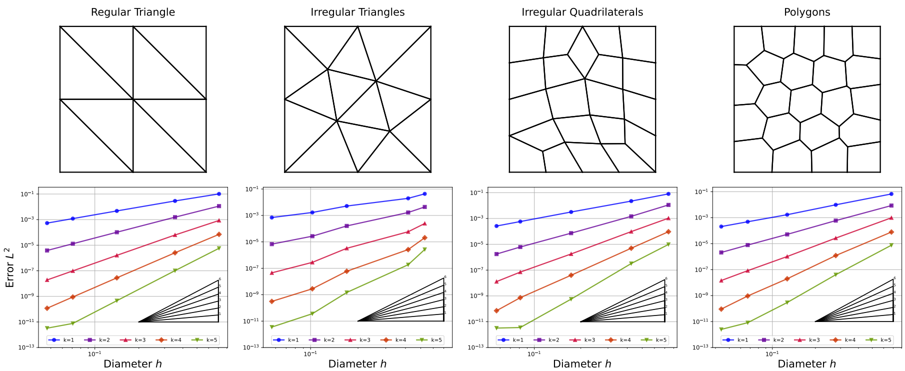
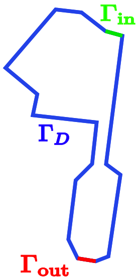


Figure 5: Convergence of order $\mathcal{O}(h^{k+1})$

Port location



Port boundary

The Helmholtz or Mild-Slope equation:

$$\begin{cases} \nabla(C_pC_g\nabla a) + k^2 C_pC_g\, a = 0, & \text{in } \boldsymbol{\Omega}\,, \\ a = 0, & \text{in } \boldsymbol{\Gamma_{in}}\,, \\ \dfrac{\partial a}{\partial n} + ik\, a = 0, & \text{in } \boldsymbol{\Gamma_{out}}\,, \\ a = \gamma\, a_i & \text{in } \boldsymbol{\Gamma_D}\,. \end{cases}$$

5 Numerical Application

On prend un truc bidon sur un port ou qqc du genre

5.1 Sensibilité de la pente, helmoltz vs Berkhoff

On introduit un déferlement selon (Munk 1949) Problem conditions:

- $a_{\max} = 2$ m,
- $T_0 = 8$ s,
- $\theta = 280$ °.

5.2 Application claquée - Résultats avec différents k

Problem conditions:

- $a_{\max} = 1$ m,
- $T_0 = 8$ s,
- $\theta = 250$ °.

5.3 Sensibilité du coefficient de réflexion

Problem conditions:

- $a_{\max} = 1$ m,
- $T_0 = 8$ s,
- $\theta = 280$ °.

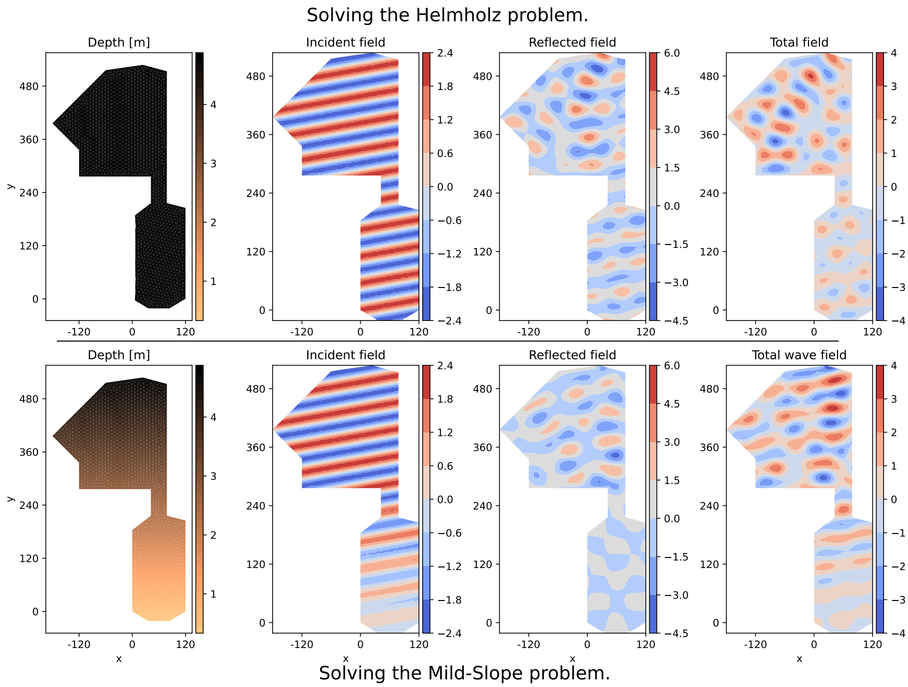


Figure 6: Caption

5.4 Appli montrant intérêt Robin

$$\begin{cases} \Delta u + k^2\, u = 0 & , \quad \text{in } \Omega\,, \\ u = -u_{\text{inc}} & , \quad \text{on } \Gamma_{\text{D}}\,, \\ \dfrac{\partial u}{\partial n} + i\,k\,u = 0 & , \quad \text{on } \Gamma_{\text{Inf}}. \end{cases}$$

or

$$\begin{cases} \Delta u + k^2\, u = 0 & , \quad \text{in } \Omega\,, \\ u = -u_{\text{inc}} & , \quad \text{on } \Gamma_{\text{D}}\,, \\ \dfrac{\partial u}{\partial n} = 0 & , \quad \text{on } \Gamma_{\text{Inf}}. \end{cases}$$

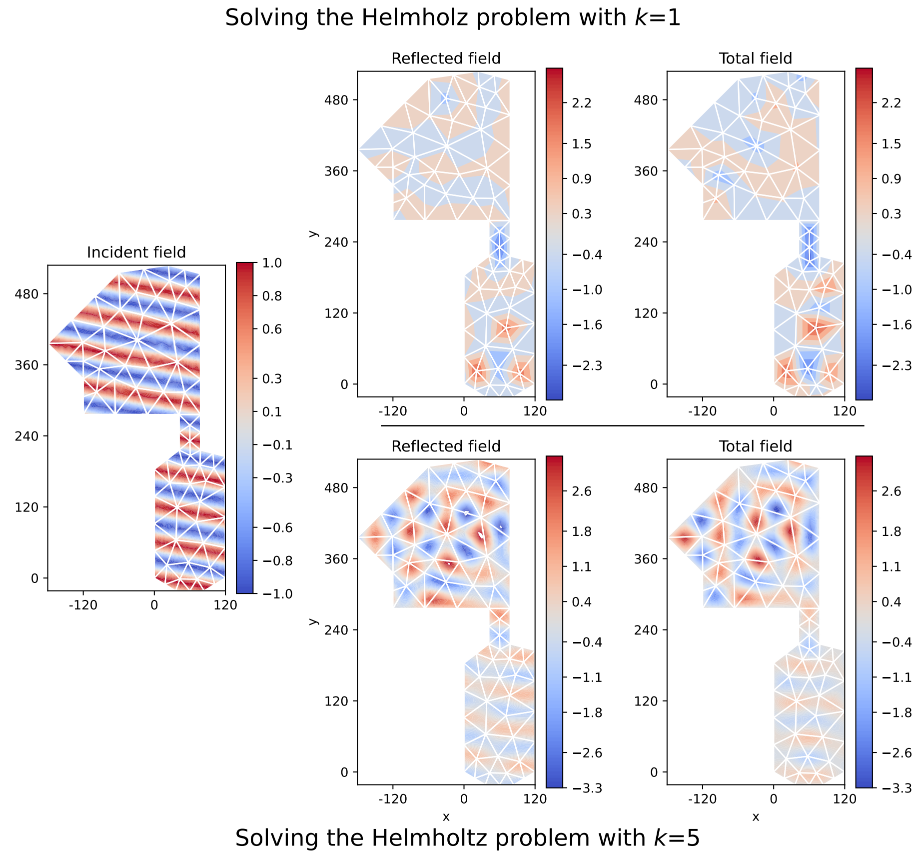
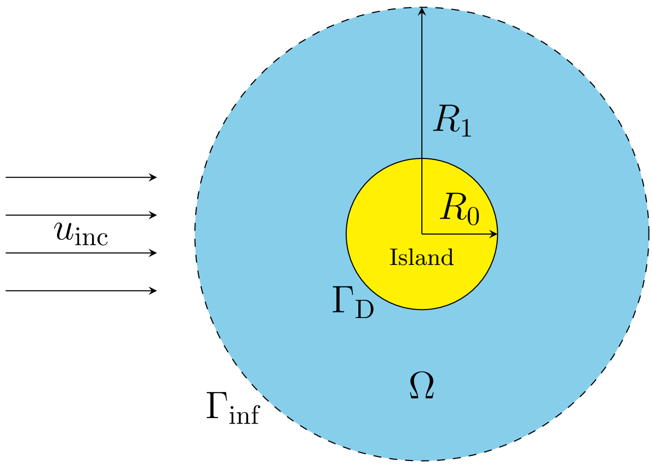


Figure 7: Caption



- $a_{\max} = 1$ m,

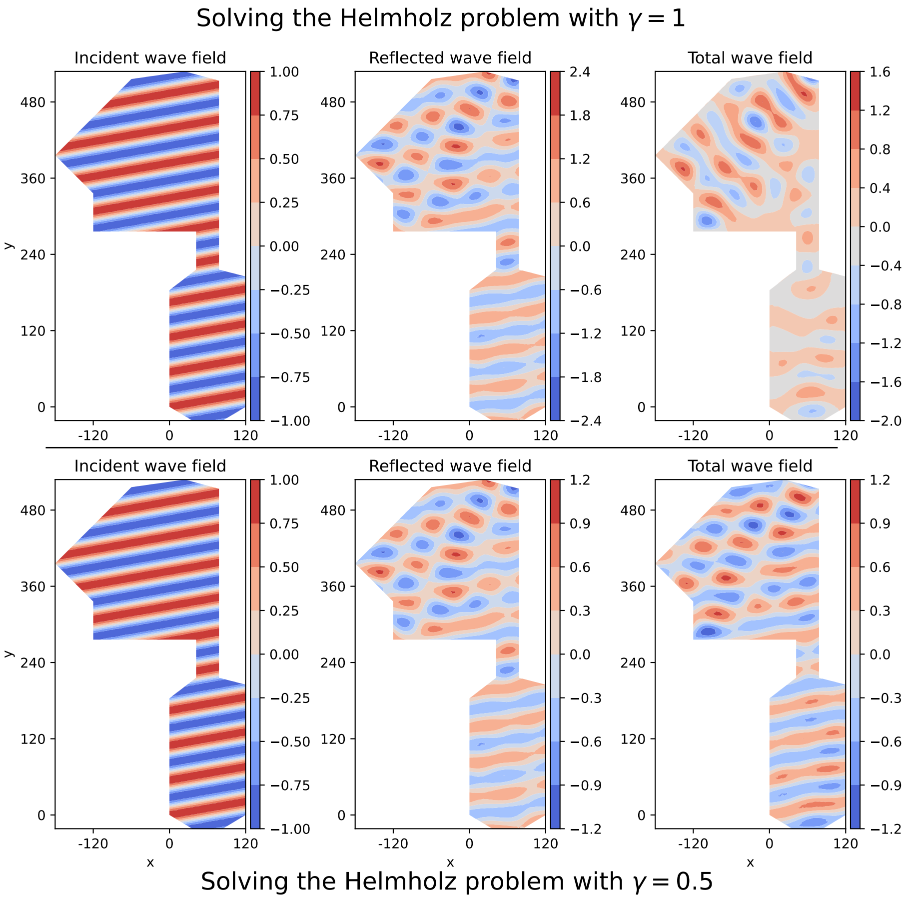


Figure 8: Caption

- $T_0 = 20$ s,
- $\theta = 0$ °.

6 Conclusion and Discussion

On est content

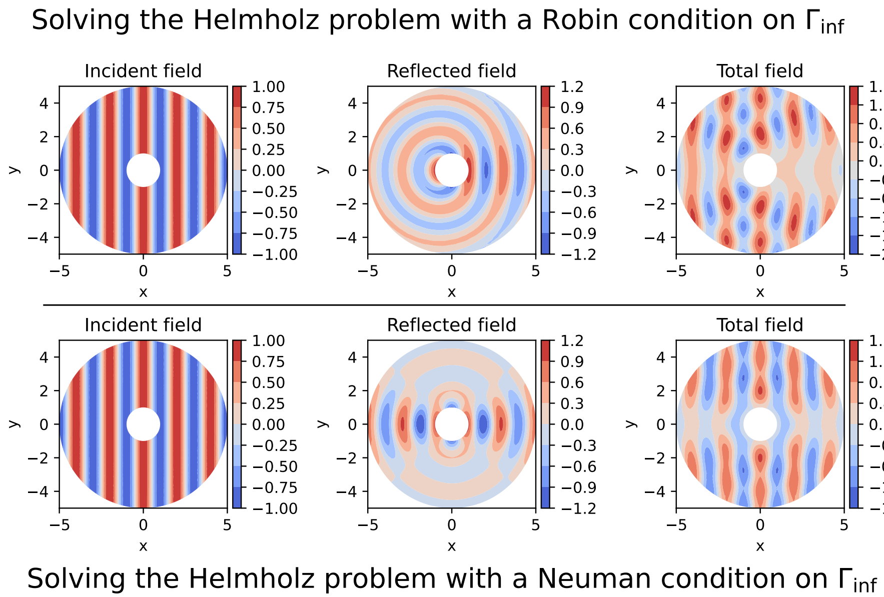


Figure 9: Caption

7 Declarations

7.1 Availability of data and material

All data, models, and code generated or used during the study are available on request.

7.2 Conflict of interest

The authors declare that they have no conflict of interest.

7.3 Acknowledgements

This work was conducted as part as M. Dupont's PhD studies which is funded by the CNRS with the MITI grant. We gratefully acknowledge funding from CNRS, OPTIBEACH projects and FEDER Europe.

Appendix

A Roro pue

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