Solving the Mild-Slope and Helmholtz equations using the Virtual Element Method (VEM), dealing with high order Robin Boundary Condition

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July 18, 2024

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Abstract

The numerical solution of the Mild-slope equation (MSE) is crucial in various fields, including coastal engineering, oceanography, and offshore structure design. In this article, we present a novel approach utilizing the Virtual Element Method (VEM) for the numerical solution of the MSE. The VEM offers significant advantages over traditional finite element methods, particularly in handling complex geometries and irregular meshes. We first look at the implementation and validation of the model in the presence of Robin boundary conditions. We then apply the results to the calculation of eigenmodes for the port of Cherbourg.

Keywords. Mild-slope equation, Helmholtz equation, Virtual Element Method, computational fluid dynamics, validation, Finite Element Methods, Numerical Analysis, Complex Geometries, Irregular Meshes, Robin Boundary Condition, Coastal Engineering.

Contents

1 Introduction

Nowadays, coastal modeling has become a major challenge in the face of climate change. Coastal-related topics have become very numerous, including ocean modeling (large-scale), port modeling and numerous other topics such as morphodynamics. In this study, we are particularly interested in port modeling through the equation models developed by Helmholtz (1868) and Berkhoff (1972). These two equation models can be used in coastal modeling to calculate wave agitation inside a harbor. The Helmholtz (1868) equation is a very classical equation, which can be used in various fields such as electromagnetics or acoustics. In our study, it is used for flat sea bottoms, while the Berkhoff (1972) equation, also known as the Mild-Slope equation, is used for variable bottoms with a maximum slope of 1/3 (Booij 1983). In this study, we have chosen to solve these equations using the virtual element method (Beirão da Veiga et al. 2014) MATHIAS, tu peux ajouter qqls autres refs importantes stp. This method has the advantage of i) being a high-order finite element method, which enables wave phenomena to be accurately captured, where simple finite elements have difficulty capturing them RORO, tu peux ajouter des références pour ce que tu disais, que sur des méthodes FEM simple, même en raffinent, on arrivait pas à capturer tous les phénomènes ondulatoires., ii) handle polyhedral meshes as well as non-conforming meshes, enabling simple refinement in certain areas. Although a few studies have already dealt with the Helmholtz (1868) problem in virtual elements (Perugia et al. 2016; Mascotto et al. 2019), none of them has had any concrete application in the coastal sector. In this study, we will first express the modeling of the problem. Next, we will explain the virtual element strategy for approximating this problem. Finally, we'll look at a particular boundary condition, the Robin condition. After validating our model, we will apply it to the calculation of eigenvalues for the port of Cherbourg.

Vous pouvez étoffer un peu, c'est un draft.

2 Model Problem

In this section, we consider the wave problem described in figure 1.

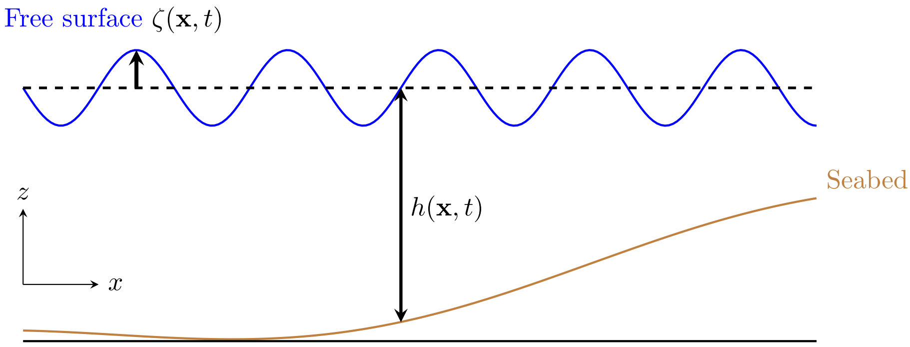

Figure 1: Sketch of a free surface elevation ζ in the $(x,\, z)$-plane.

with ζ the free surface defined by $\zeta(\mathbf{x}, t) = \Re\left\{\eta(x, y)\, e^{-i\omega t}\right\}$, η a complex-valued amplitude of ζ, $\omega = 2\pi/T_0$ the angular frequency, T_0 the wave period and h the depth.

The amplitude η can be split into its incident and reflective part,

$$\eta = \eta_I + \eta_R.$$

We thus have,

$$\eta_I(\mathbf{x}, t) = a_I(\mathbf{x})e^{-i\omega t} \quad \text{and} \quad \eta_R(\mathbf{x}, t) = a_R(\mathbf{x})e^{-i\omega t}$$

with the incident wave amplitude defined by,

$$a_I(\mathbf{x}) = a_{\max} e^{-i\mathbf{k}\mathbf{x}} \quad \text{with} \quad \mathbf{k} = k(\cos(\theta), \sin(\theta))^\intercal,$$

with θ the incident wave angle, $a_{\max}$ the maximum wave amplitude.
The amplitude of the reflected wave a_R is obtained by solving the Helmholtz (1868) equation, in the case of a flat bottom,

$$\begin{cases} \Delta a + k^2 a = 0 \quad , \quad \text{in } \Omega\,, \\ \quad +BC. \end{cases} \tag{1}$$

with k the wave number, solution of the dispersion relation at order 1 from linear theory (Airy 1845):

$$\omega^2 = g\, k\, \tanh(kh) \quad \text{with} \quad \omega = \frac{2\pi}{T_0}, \tag{2}$$

The amplitude of the reflected wave a_R can also be obtained by solving the Mild-Slope equation (Berkhoff 1972), in the case of a variable bottom,

$$
\begin{cases}
\nabla(C_p C_g \nabla a) + k^2 C_p C_g \, a = 0 \quad , \quad \text{in } \Omega \, , \\
\qquad\qquad\qquad\qquad +BC.
\end{cases}
\tag{3}
$$

(depending on whether the bottom is flat or not).

Remarks:

• In practice, k is obtained simply by using the Guo (2002) approximation.

• Assuming constant depth within the port and $C_g = C_p/2$ (as in shallow water) and noting that $C_p = \omega/k = Cte$, equation (3) can be simplified to yield the Helmholtz (1868) equation.