



NUMERICS2024

# Numerical solution of Mild-slope equation using Virtual Element Method

GÉOSCIENCES MONTPELLIER, IMAG, CNRS, UNIVERSITY OF MONTPELLIER

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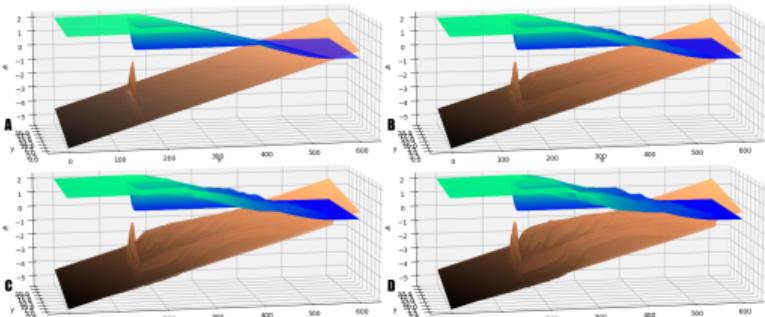
Biodiversité  
Agriculture  
Alimentation  
Environnement  
Terre  
Eau



# FOREWORD



- A parallel project to my PhD,
- Virtual element method of order  $k$  with Robin's Boundary condition,
- Application to a concrete problem.



My main PhD work



Photo of the port of Cherbourg (France)



# OVERVIEW



1. Model problem
2. Virtual Element Settings
3. Robin Boundary Condition
4. Numerical Results
5. Applications



# I) MODEL PROBLEM



We consider,

$$u = u_i + u_r$$

with  $u_i$  the **incident wave** and  $u_r$  the **reflected wave**. We have,

$$u_i(\mathbf{x}, t) = a_i(\mathbf{x})e^{-i\sigma t} \quad \text{and} \quad u_r(\mathbf{x}, t) = a_r(\mathbf{x})e^{-i\sigma t}$$

with  $\sigma = 2\pi/T_0$ , the **angular frequency** and

$$a_i(\mathbf{x}) = a_{\max}e^{-i\mathbf{k}\mathbf{x}} \quad \text{with} \quad \mathbf{k} = k(\cos(\theta), \sin(\theta))$$

with  $\theta$  the **incident wave angle**,  $a_{\max}$  the **maximum wave amplitude**.  $u_r$  is compute using ...

# I) MODEL PROBLEM



The Helmholtz equation:

$$\begin{cases} \Delta a + k^2 a = 0, & \text{in } \Omega, \\ +BC. \end{cases}$$

The Mild-Slope equation:

$$\begin{cases} \nabla(C_p C_g \nabla a) + k^2 C_p C_g a = 0, & \text{in } \Omega, \\ +BC. \end{cases}$$

with

$$C_p = \frac{\sigma}{k} \quad \text{and} \quad C_g = \frac{1}{2} C_p \left[ 1 + kh \frac{1 - \tanh^2(kh)}{\tanh(kh)} \right],$$

and the wave number  $k$ , solution of the dispersion relation:

$$\sigma^2 = g k \tanh(kh) \quad \text{with} \quad \sigma = \frac{2\pi}{T_0},$$

where  $T_0$  is the wave period and  $h$  the depth.

# I) MODEL PROBLEM



The Helmholtz equation:

$$\begin{cases} \Delta a + k^2 a = 0, & \text{in } \Omega, \\ +BC. \end{cases}$$

The Mild-Slope equation:

$$\begin{cases} \nabla(C_p C_g \nabla a) + k^2 C_p C_g a = 0, & \text{in } \Omega, \\ +BC. \end{cases}$$

- Works only with flat bottoms,
- Easy-to-calculate analytical solutions.

- Works with a non-constant seabed,
- Area of validity: maximum slope of 1/3,
- Difficult to obtain an analytical solution.



1. Model problem

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## II) VIRTUAL ELEMENT SETTINGS



- **Mesh Decomposition:** Decomposition  $\{T_h\}_h$  of the domain  $\Omega$  which is shape-regular.  $h_E$  the diameter of  $E$ ,  $(x_D, y_D)$  the centroid of  $E$ .
- **The standard scale monomial basis:**  $m_{\alpha_1, \alpha_2} = \left(\frac{x-x_D}{h_D}\right)^{\alpha_1} \cdot \left(\frac{y-y_D}{h_D}\right)^{\alpha_2}$  with  $\alpha_1 + \alpha_2 \leq k$ .
- **Local Projections:**
  - Local elliptic projector:  $\Pi_k^{\nabla, E} : H^1(E) \rightarrow \mathbb{P}_k(E)$
  - Local  $L^2$ -projector:  $\Pi_k^{0, E} : L^2(E) \rightarrow \mathbb{P}_k(E)$

## II) VIRTUAL ELEMENT SETTINGS



- **Virtual Space:**

$$V_h^E = \left\{ v_h \in H_0^1(\Omega) \cap C^0(\partial E) : v_h|_{\partial E} \in \mathbb{P}_k(E), \Delta v_h \in \mathbb{P}_k(E) \right.$$

$$\left. (\Pi_k^\nabla v_h - v_h, p)_{0,E} = 0, \forall p \in \mathbb{P}_k(E)/\mathbb{P}_{k-2}(E) \right\}$$

- **Local Degrees of Freedom:**

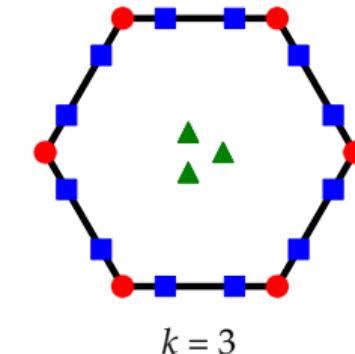
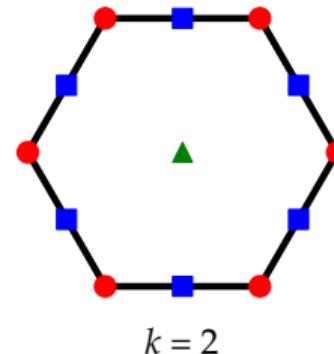
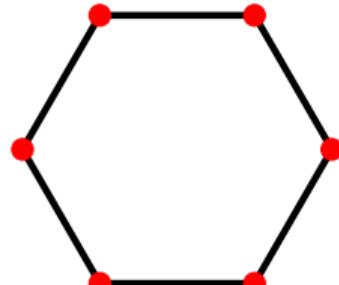


Figure 2: 2D element with ● : Summits dofs, ■ : Edges dofs, ▲ : Moments dofs.



1. Model problem

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### III) ROBIN BOUNDARY CONDITION



#### 1) VARIATIONAL FORMULATION

Helmholtz:

$$\begin{cases} \Delta a + k^2 a = 0 & , \quad \text{in } \Omega, \\ a = -a_i & , \quad \text{on } \Gamma_D, \\ \frac{\partial a}{\partial n} + i k a = 0 & , \quad \text{on } \Gamma_{Inf}. \end{cases}$$

Mild-Slope:

$$\begin{cases} \nabla(C_p C_g \nabla a) + k^2 C_p C_g a = 0, & , \quad \text{in } \Omega, \\ a = -a_i & , \quad \text{on } \Gamma_D, \\ \frac{\partial a}{\partial n} + i k a = 0 & , \quad \text{on } \Gamma_{Inf}. \end{cases}$$

$$\begin{cases} \text{find } u \in V = H_0^1(\Omega) \text{ such that} \\ a(u, v) = 0 \quad \forall v \in V, \end{cases}$$

$$\begin{aligned} a(u, v) &= \int_{\Omega} \Delta u v + k^2 \int_{\Omega} u v, \\ &\stackrel{green}{=} - \int_{\Omega} \nabla u \nabla v + k^2 \int_{\Omega} u v + \int_{\Gamma_{Inf}} \frac{\partial u}{\partial n} v, \\ &\stackrel{\partial u / \partial n = -iku}{=} - \int_{\Omega} \nabla u \nabla v + k^2 \int_{\Omega} u v - ik \int_{\Gamma_{Inf}} u v. \end{aligned}$$

### III) ROBIN BOUNDARY CONDITION

#### 1) VARIATIONAL FORMULATION - DISCRETE FORM

$$\begin{cases} \text{find } u_h \in V_h \subset V \text{ such that} \\ a_h(u_h, v_h) = 0 \quad \forall v \in V, \end{cases}$$

- $V_h \subset V$  is a finite dimensional space,
- $a_h(\cdot, \cdot) : V_h \times V_h \rightarrow \mathbb{R}$  is a discrete bilinear form approximating the continuous form  $a(\cdot, \cdot)$ .

$$a_h(u_h, v_h) = \sum_{E \in \Omega_h} \left[ \int_E \nabla(C_p C_g \nabla u_h v_h) + \int_E k^2 C_p C_g u_h v_h \right],$$

$\approx$   
 $\frac{1/E \int_E C_p C_g = \mathcal{A}_E}{1/E \int_E k^2 C_p C_g = \mathcal{B}_E}$   $\sum_{E \in \Omega_h} \left[ \mathcal{A}_E \int_E (\Delta u_h v_h) + \mathcal{B}_E \int_E u_h v_h \right],$

$$\stackrel{\text{green}}{=} \sum_{E \in \Omega_h} \left[ -\mathcal{A}_E \int_E \nabla u_h \nabla v_h + \mathcal{B}_E \int_E u_h v_h - \mathbf{1}_{\Gamma_{\text{Inf}} \subset E} i \mathcal{A}_E \int_{\Gamma_{\text{Inf}}} k u_h v_h \right].$$

### III) ROBIN BOUNDARY CONDITION

#### 1) VARIATIONAL FORMULATION - GENERAL CASE

$$\begin{cases} \Delta u + k^2 u = 0 & , \quad \text{in } \Omega , \\ \frac{\partial u}{\partial n} + k(x, y) u = g(x, y) & , \quad \text{on } \Gamma_{\text{Inf}} . \end{cases}$$

$$a_h(u_h, v_h) = - \int_{\Omega} \nabla u_h \nabla v_h + \int_{\Omega} k^2 u_h v_h - \underbrace{\int_{\Gamma_{\text{Inf}}} k u_h v_h}_B$$

$$b_h(v_h) = - \underbrace{\int_{\Gamma_{\text{Inf}}} g v_h}_G$$

By expressing in the classical shape functions basis:

$$B = \left( \int_{\Gamma_{\text{Inf}}} k(x, y) \Phi_i(x, y) \Phi_j(x, y) \right)_{i,j}$$

$$G = \left( \int_{\Gamma_{\text{Inf}}} g(x, y) \Phi_i(x, y) \right)_i$$

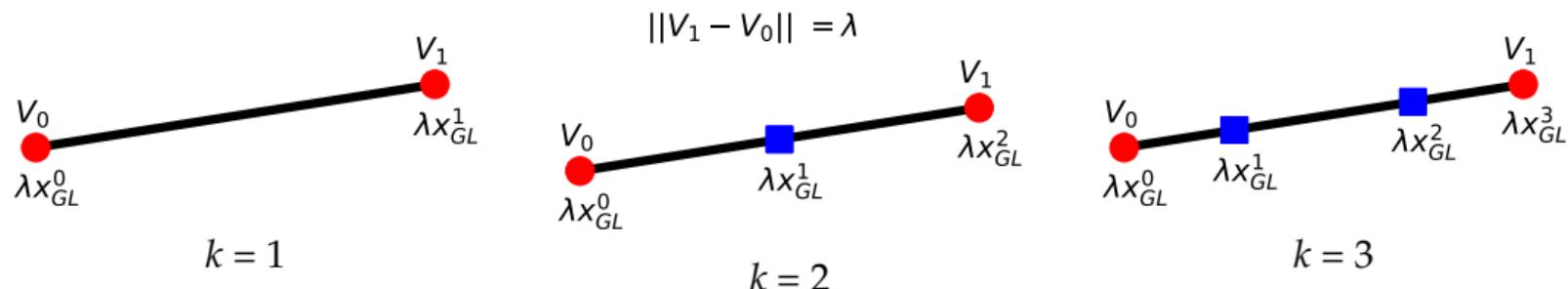
with  $\Phi_i$  the classical shape functions of order  $k$ .

### III) ROBIN BOUNDARY CONDITION



#### 2) ELEMENT PROPERTIES

$\Gamma_{Inf}$  can be expressed as a sum of 1D elements defined by  $[\xi_0, \xi_0 + \lambda]$ :



**Figure 3:** 1D element  $[\xi_0, \xi_0 + \lambda]$  representation with  
● : Summits dofs, ■ : Edges dofs.

with  $x_{GL}^j$  the  $j - th$  Gauss-Lobatto quadrature point on  $[0,1]$ .

### III) ROBIN BOUNDARY CONDITION

#### 3) DESCRIPTIVE EXPRESSION OF B AND G

On each  $[\xi_0, \xi_0 + \lambda]$  element:

$$B = \left( \int_0^\lambda k_*(\xi_0 + \xi) \varphi_j(\xi) \varphi_i(\xi) d\xi \right)_{0 \leq i,j \leq k},$$

$$G = \left( \int_0^\lambda g_*(\xi_0 + \xi) \varphi_i(\xi) d\xi \right)_{0 \leq i \leq k}.$$

- $\varphi_i, \varphi_i$  are polynomials of order  $k$ ,
- $k_*(\xi_0 + \xi) = k(V_0 + \xi \vec{t})$ ,
- $g_*(\xi_0 + \xi) = g(V_0 + \xi \vec{t})$ ,
- $\vec{t}$ : tangential unit vector ( $V_0$  to  $V_1$ ).



For  $i, j \in [|0, k|]$  :

$$\varphi_i(\lambda x_{GL}^j) = \delta_j^i.$$

Using Lagrange polynomials:

$$\begin{aligned} \varphi_i(\xi) &= \sum_{j=0}^k \delta_j^i \left( \prod_{l=0, l \neq j}^k \frac{\xi - \lambda x_{GL}^l}{\lambda x_{GL}^j - \lambda x_{GL}^l} \right) \\ &= \frac{1}{\lambda^k} \prod_{l=0, l \neq i}^k \frac{\xi - \lambda x_{GL}^l}{x_{GL}^i - x_{GL}^l} \end{aligned}$$

### III) ROBIN BOUNDARY CONDITION



#### 4) COMPUTATION OF B AND G

Case:  $k = \text{constant}$ ,  $g = \text{constant}$ :

On each  $[\xi_0, \xi_0 + \lambda]$  element:

$$B = \left( \int_0^\lambda k_*(\xi_0 + \xi) \varphi_j(\xi) \varphi_i(\xi) d\xi \right)_{0 \leq i,j \leq k},$$

$$G = \left( \int_0^\lambda g_*(\xi_0 + \xi) \varphi_i(\xi) d\xi \right)_{0 \leq i \leq k}.$$

- Approximated using Gauss-Lobatto quad of order  $2K + 1$  with  $K = 2k$ .

$$B = \lambda k \left( \int_0^1 \tilde{\varphi}_j(\xi) \tilde{\varphi}_i(\xi) d\xi \right)_{0 \leq i,j \leq k},$$

$$G = \lambda g \left( \int_0^1 \tilde{\varphi}_i(\xi) d\xi \right)_{0 \leq i \leq k}.$$

- $\tilde{\varphi}_i$ : polynomials for a unit element  $[\xi_0, \xi_0 + 1]$ ,
- Exact integration with  $4k - 3$  GL points,
- A single evaluation.



1. Model problem

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# IV) NUMERICAL RESULTS



## 1) ANALYTICAL SOLUTION

We consider,

$$\begin{cases} \Delta u + k^2 u = f(x, y) & , \quad \text{in } \Omega, \\ u = u_{\text{exact}} & , \quad \text{on } \Gamma_2 \cup \Gamma_3 \cup \Gamma_4, \\ \frac{\partial u}{\partial n} + ik u = g(x, y) & , \quad \text{on } \Gamma_1, \end{cases}$$

with:

$$\begin{cases} u_{\text{exact}}(x, y) = (x + y) \cdot (1 + i) + \exp(x^2 + iy^2), \\ f(x, y) = - \left( (2x)^2 + (2iy)^2 + 2(1+i) \right) \cdot \exp(x^2 + iy^2) + k^2 \cdot u_{\text{exact}}(x, y), \\ g(x, y) = (1+i) + (2iy) \cdot \exp(x^2 + iy^2) + ik \cdot u_{\text{exact}}(x, y). \end{cases}$$

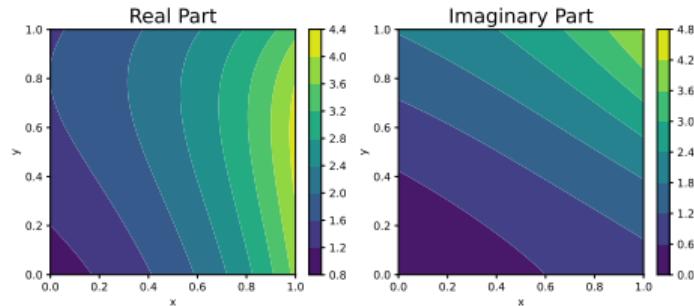
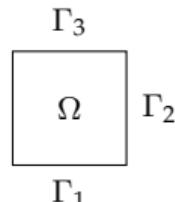


Figure 4: Real and Imaginary part of  $u_{\text{exact}}$ .



# IV) NUMERICAL RESULTS

## 2) CONVERGENCE OF ORDER $\mathcal{O}(h^{k+1})$

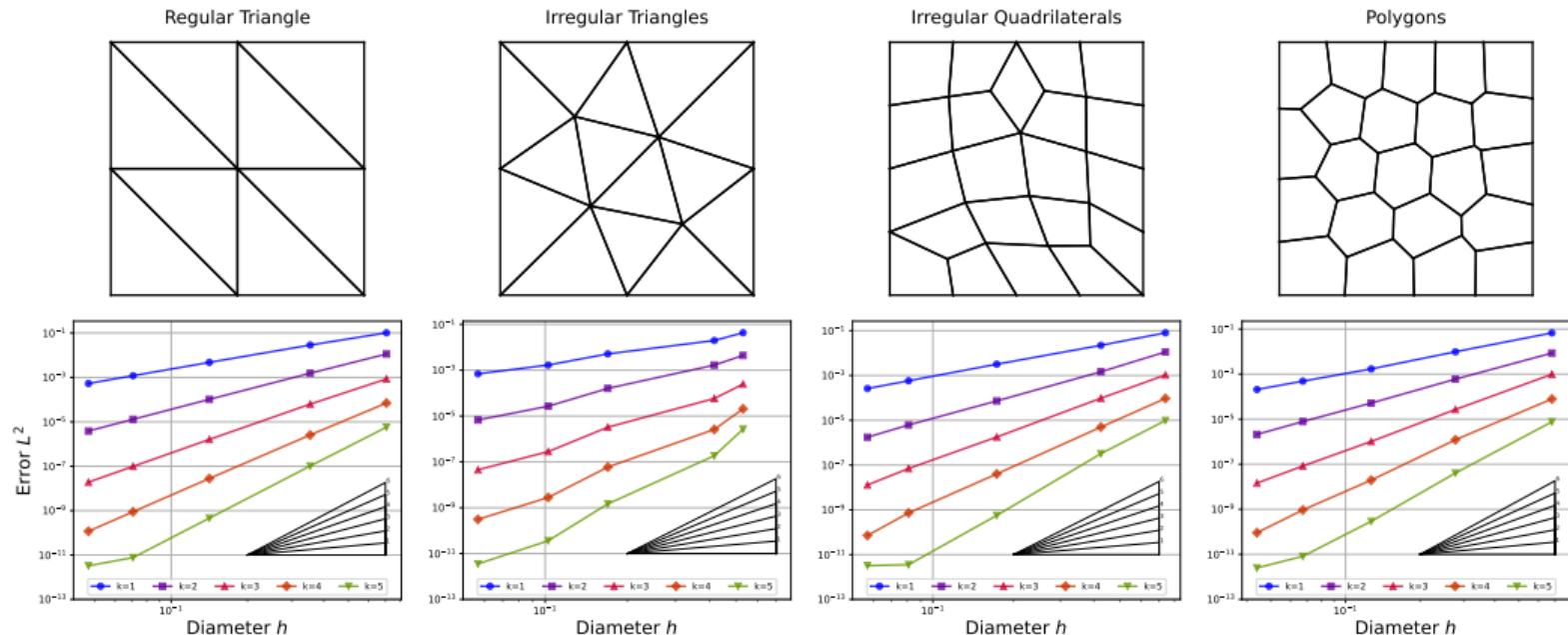


Figure 5: Convergence curves with different orders  $k$  and different types of elements

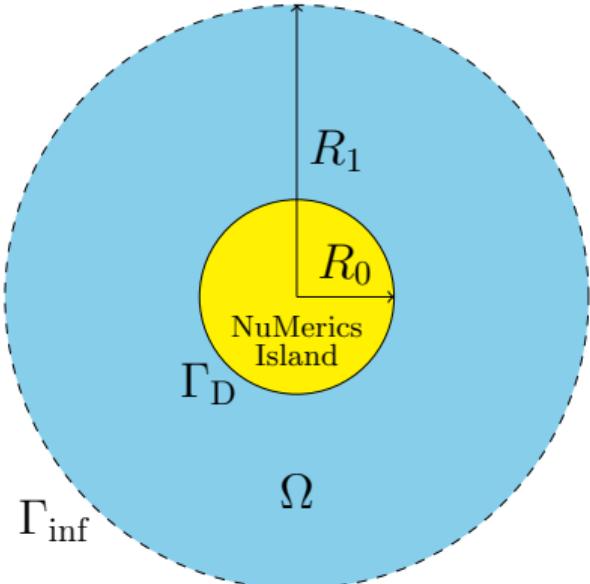
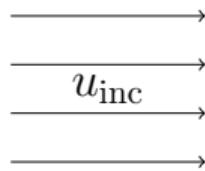
## IV) NUMERICAL RESULTS

### 3) INTEREST OF A ROBIN CONDITION

$$\begin{cases} \Delta u + k^2 u = 0 & , \text{ in } \Omega , \\ u = -u_{\text{inc}} & , \text{ on } \Gamma_D , \\ \frac{\partial u}{\partial n} + ik u = 0 & , \text{ on } \Gamma_{\text{Inf}} . \end{cases}$$

or

$$\begin{cases} \Delta u + k^2 u = 0 & , \text{ in } \Omega , \\ u = -u_{\text{inc}} & , \text{ on } \Gamma_D , \\ \frac{\partial u}{\partial n} = 0 & , \text{ on } \Gamma_{\text{Inf}} . \end{cases}$$



# IV) NUMERICAL RESULTS

## 3) INTEREST OF A ROBIN CONDITION



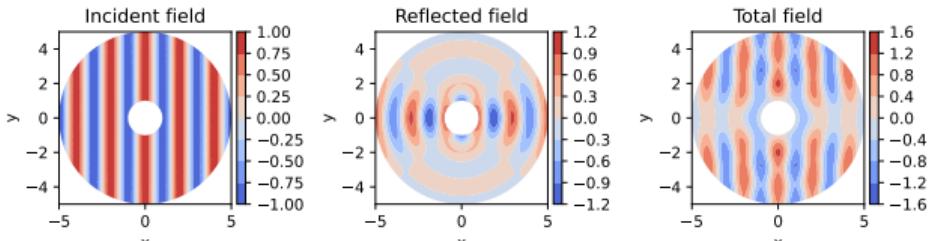
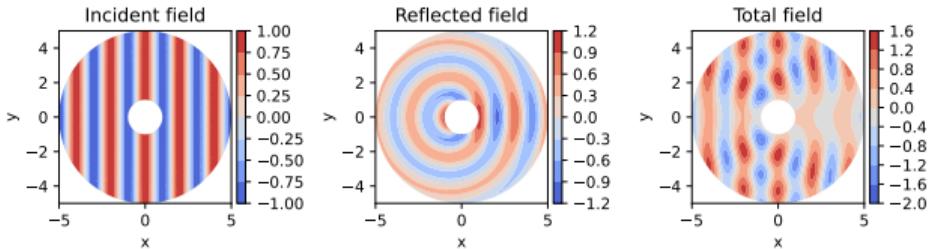
Problem conditions:

- $a_{\max} = 1 \text{ m}$ ,
- $T_0 = 20 \text{ s}$ ,
- $\theta = 0^\circ$ .

Points of interest:

- Disturbance of the reflected wave field.

Solving the Helmholtz problem with a Robin condition on  $\Gamma_{\text{inf}}$



Solving the Helmholtz problem with a Neuman condition on  $\Gamma_{\text{inf}}$





1. Model problem

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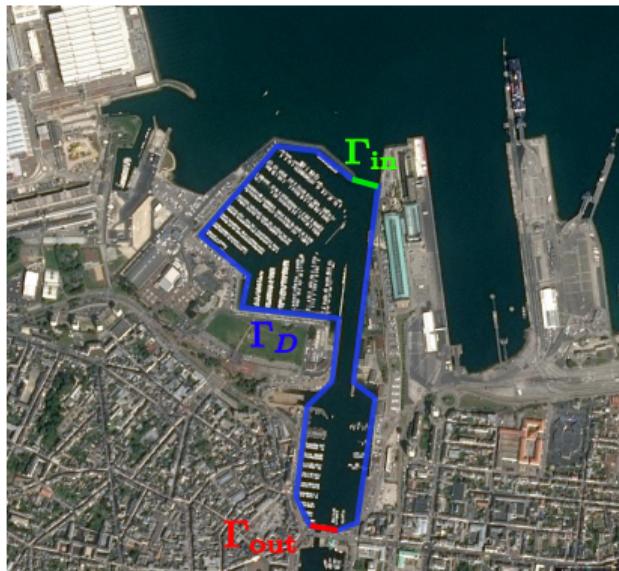
5. Applications



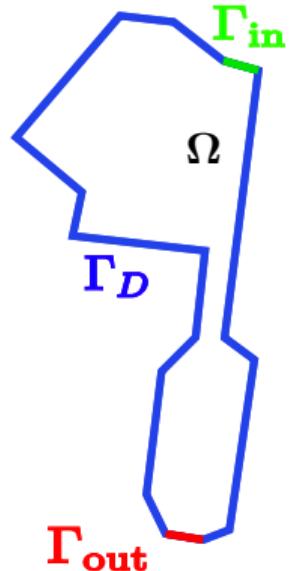


# V) APPLICATIONS

## 1) PROBLEM CONFIGURATION



Port location



Port boundary

The Helmholtz or Mild-Slope equation:

$$\left\{ \begin{array}{ll} \nabla(C_p C_g \nabla a) + k^2 C_p C_g a = 0, & \text{in } \Omega, \\ a = 0, & \text{in } \Gamma_{in}, \\ \frac{\partial a}{\partial n} + ika = 0, & \text{in } \Gamma_{out}, \\ a = \gamma a_i & \text{in } \Gamma_D. \end{array} \right.$$

# V) APPLICATIONS

## 2) SLOPE SENSITIVITY, HELMHOLTZ VS MILD-SLOPE

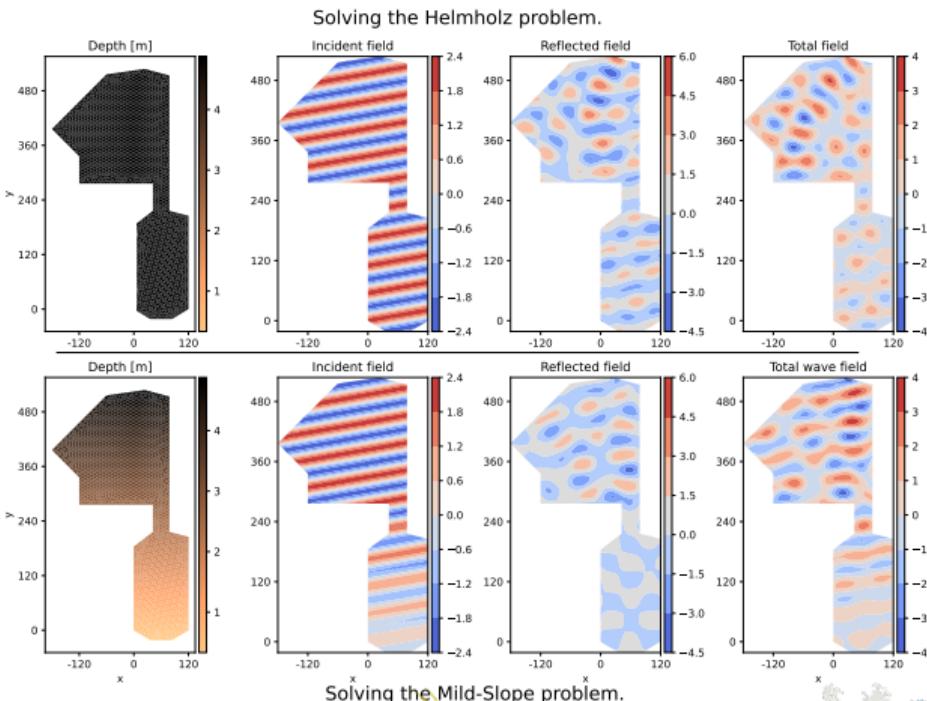


Problem conditions:

- $a_{\max} = 2 \text{ m}$ ,
- $T_0 = 8 \text{ s}$ ,
- $\theta = 280^\circ$ .

Points of interest:

- Eigenmode position.



# V) APPLICATIONS

## 3) REFLECTION COEFFICIENT SENSITIVITY $\gamma$

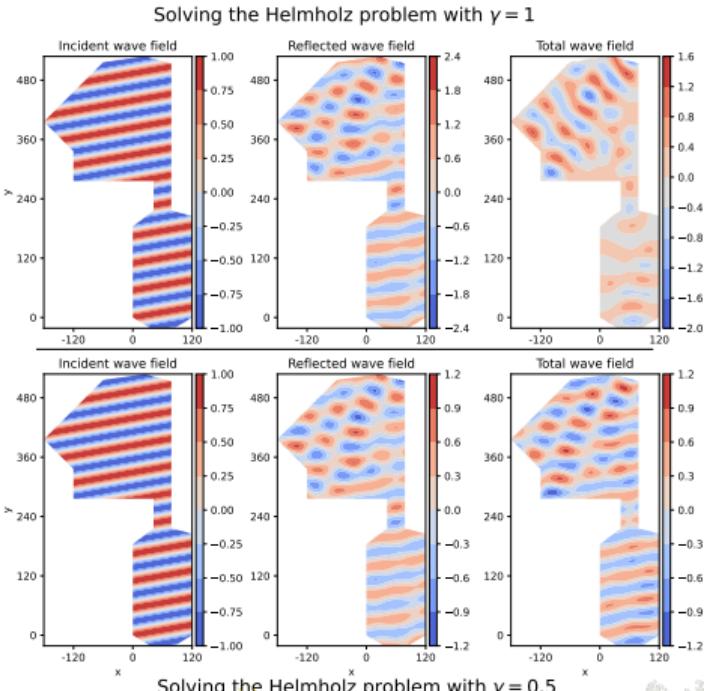


Problem conditions:

- $a_{\max} = 1 \text{ m}$ ,
- $T_0 = 8 \text{ s}$ ,
- $\theta = 280^\circ$ .

Points of interest:

- Eigenmode position,
- Amplitude of reflected wave field.



# V) APPLICATIONS

## 4) RESULTS AT DIFFERENT ORDERS $k$



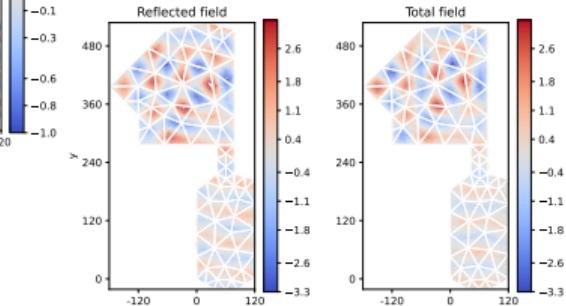
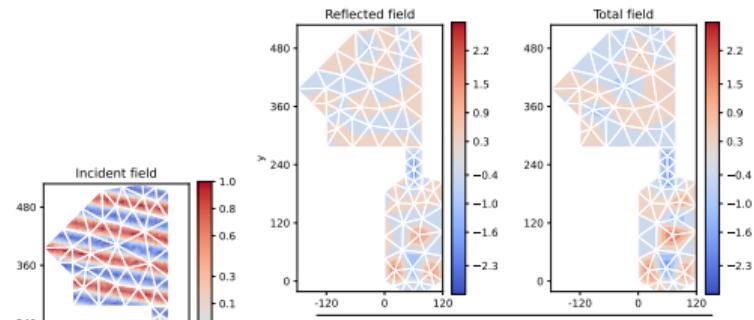
Problem conditions:

- $a_{\max} = 1 \text{ m}$ ,
- $T_0 = 8 \text{ s}$ ,
- $\theta = 250^\circ$ .

Points of interest:

- Eigenmode position.

Solving the Helmholtz problem with  $k=1$



Solving the Helmholtz problem with  $k=5$



# Grazie per l'attenzione!



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