Graph Learning
SD212
2. PageRank

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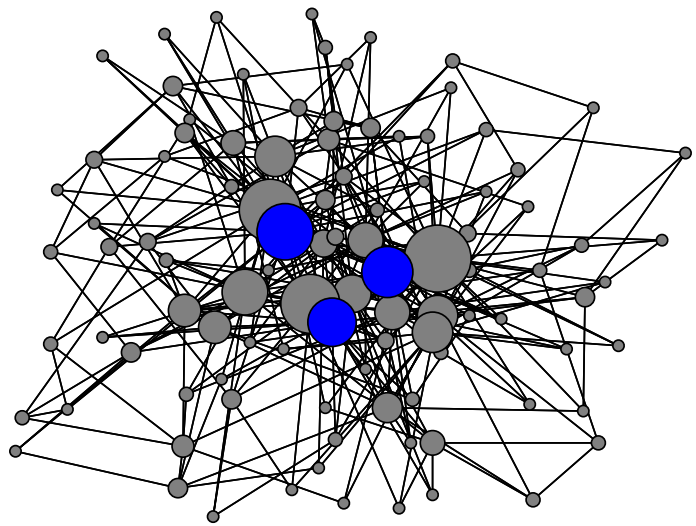
2023 – 2024

Motivation

How to identify the most "important" nodes in a graph?



We focus on **PageRank**[1], based on a **random walk** in the graph

Other methods exist (e.g., betweenness centrality)

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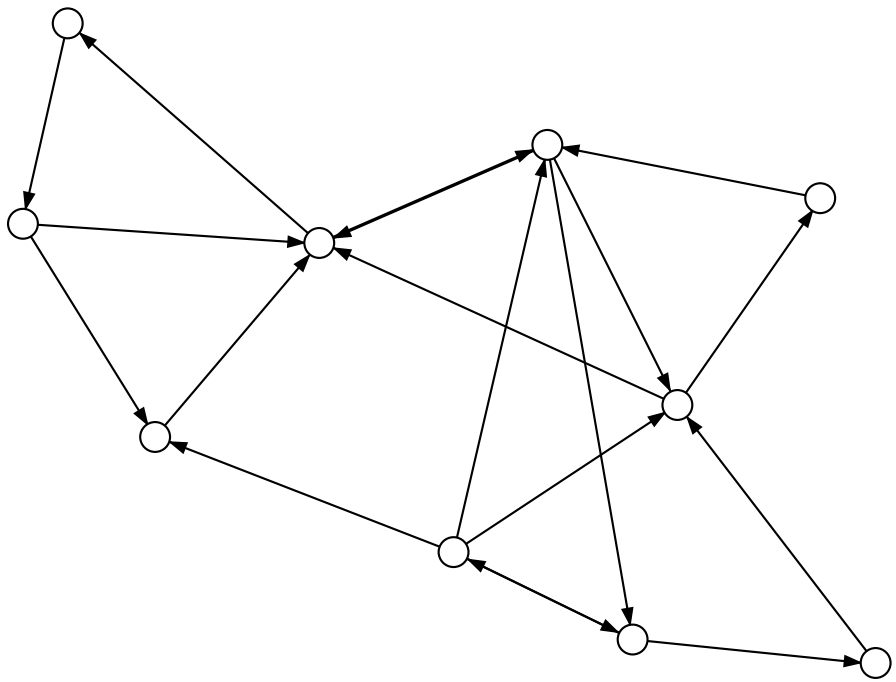
[1] Brin & Page (1998)
The anatomy of a large-scale hypertextual Web search engine

Outline

1. Random walk
2. PageRank
3. Personalized PageRank
4. Case of bipartite graphs
5. Applications

Setting

Consider a **directed** graph $G = (V, E)$
Let A be the **adjacency matrix**
Let $d^+ = A1$ and $d^- = A^T 1$ be the **out-degrees** and **in-degrees**



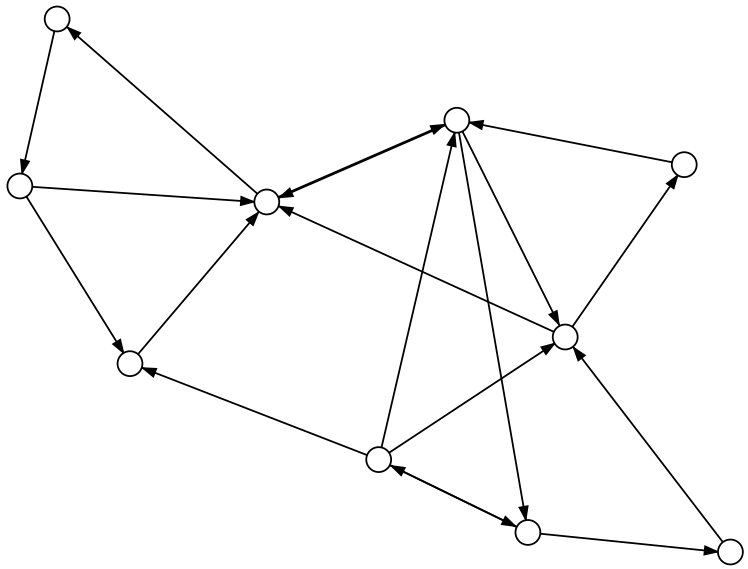
Random walk

Assume $d^+ > 0$ (no sink)

Definition

A **random walk** in the graph is a Markov chain $X_0, X_1, \ldots$ with transition matrix $P = D^{-1}A$ where $D = \mathrm{diag}(d^+)$

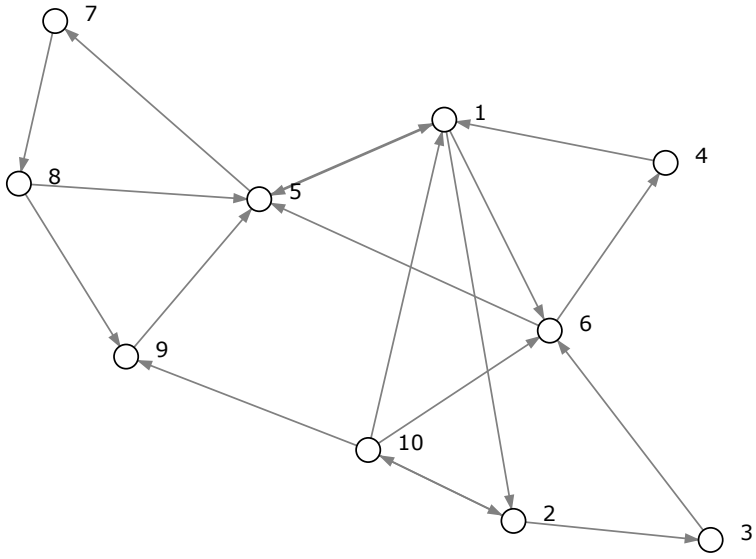
$$\forall i, j \in V, \quad P(X_{t+1} = j | X_t = i) = P_{ij}$$



Dynamics

Let $\pi(t)$ be the distribution of the random walk at time t:

$$\pi(t) = (P(X_t = 1), \ldots, P(X_t = n))$$



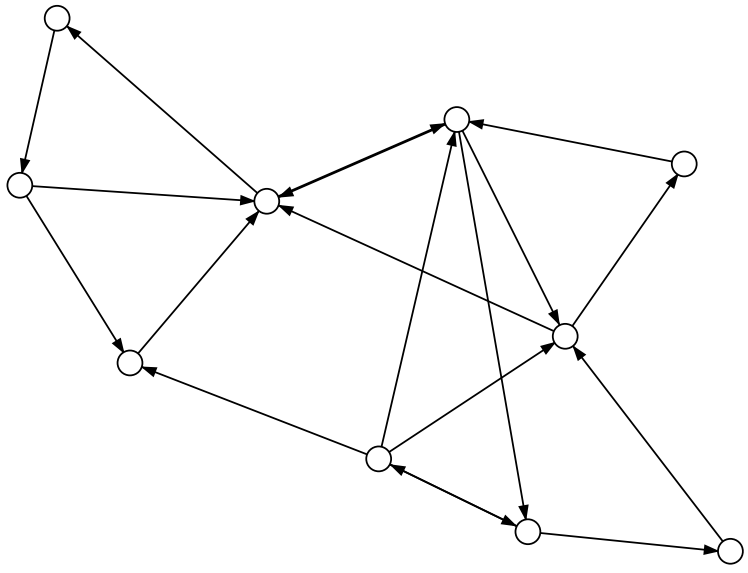
Proposition

The evolution of the random walk is given by

$$\boxed{\pi(t+1) = \pi(t)P}$$

Stationary distribution

Assume that the graph is **strongly connected**



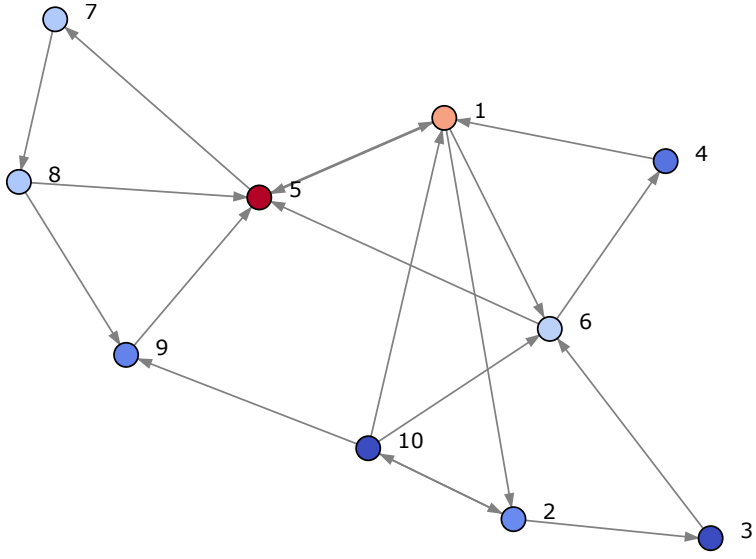
Theorem (Perron-Frobenius)

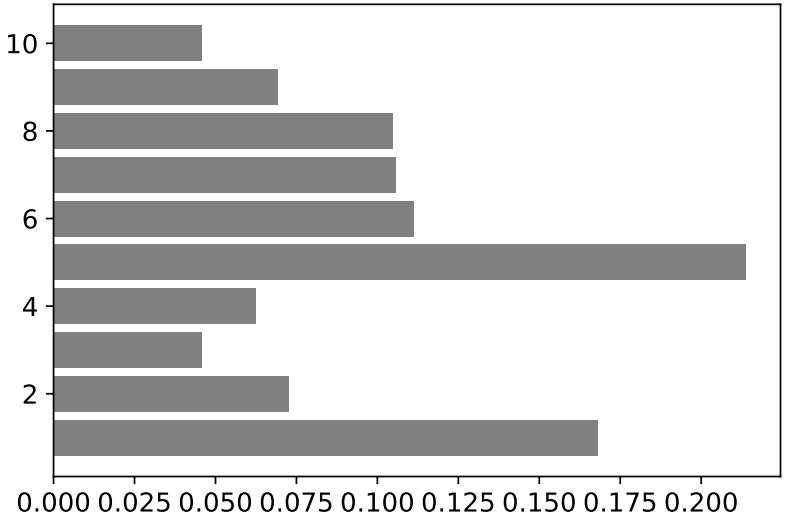
There is a unique solution π to the balance equations:

$$\boxed{\pi = \pi P}$$

This is the **frequency of visits** of each node in stationary regime

Example


7
1
4
8
5
6
9
10
2
3


10
8
6
4
2
0.000 0.025 0.050 0.075 0.100 0.125 0.150 0.175 0.200

Power iteration

Stationary distribution

Input:
P, transition matrix
K, number of iterations

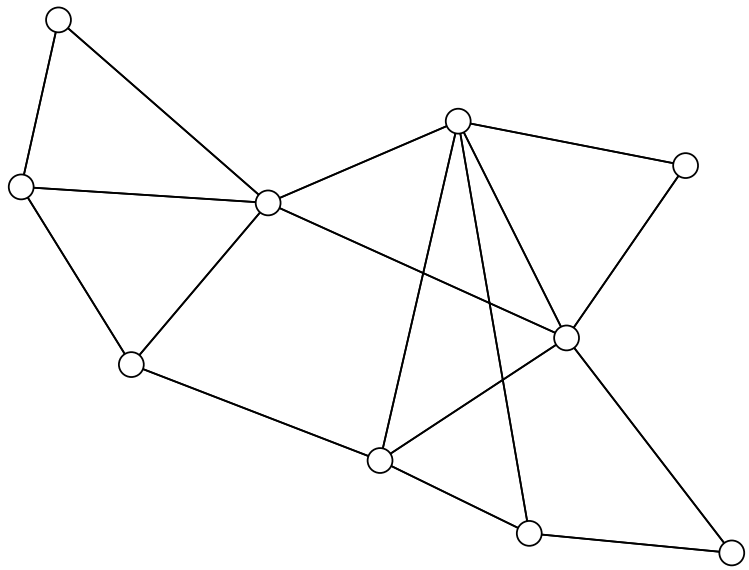
Do:
$\pi \leftarrow \frac{1}{n}(1, \ldots, 1)$
For $t = 1, \ldots, K$, $\pi \leftarrow \pi P$

Output:
π, (approximate) stationary distribution

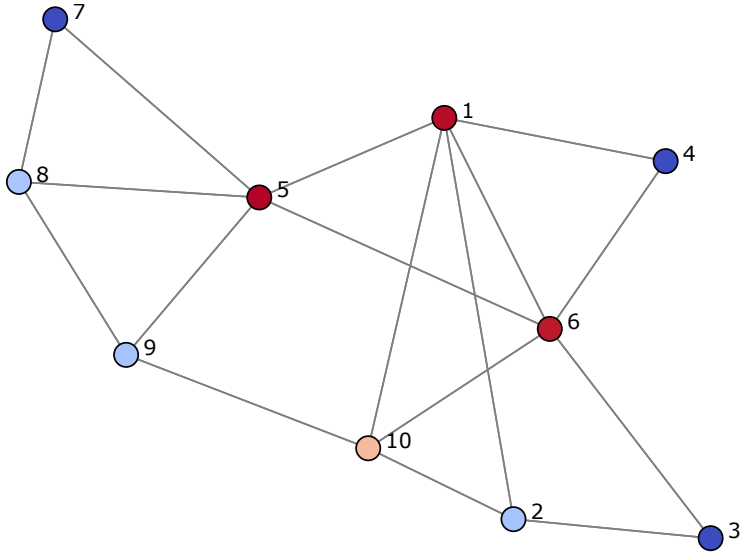
Time complexity: $O(Km)$ for m edges

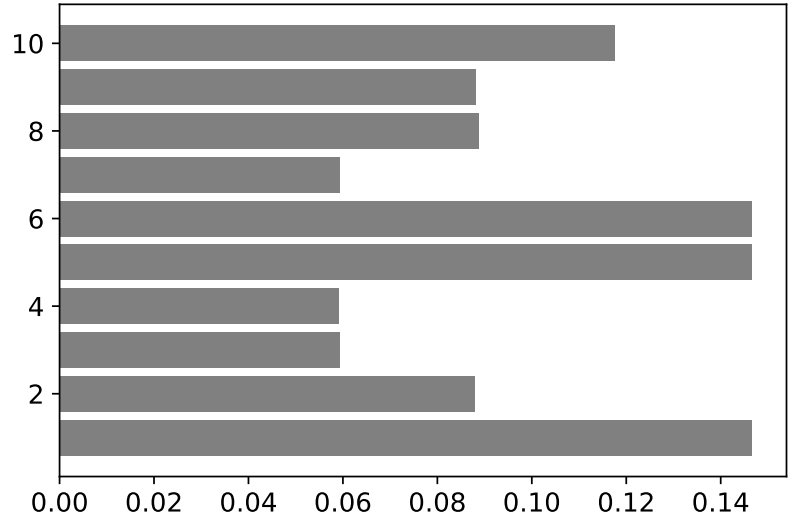
Undirected graphs

We have $d = d^+ = d^-$

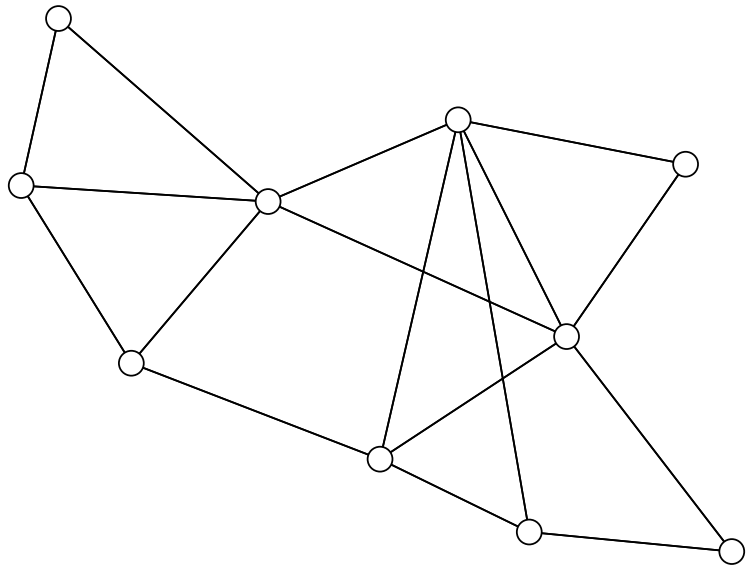


Example





Undirected graphs



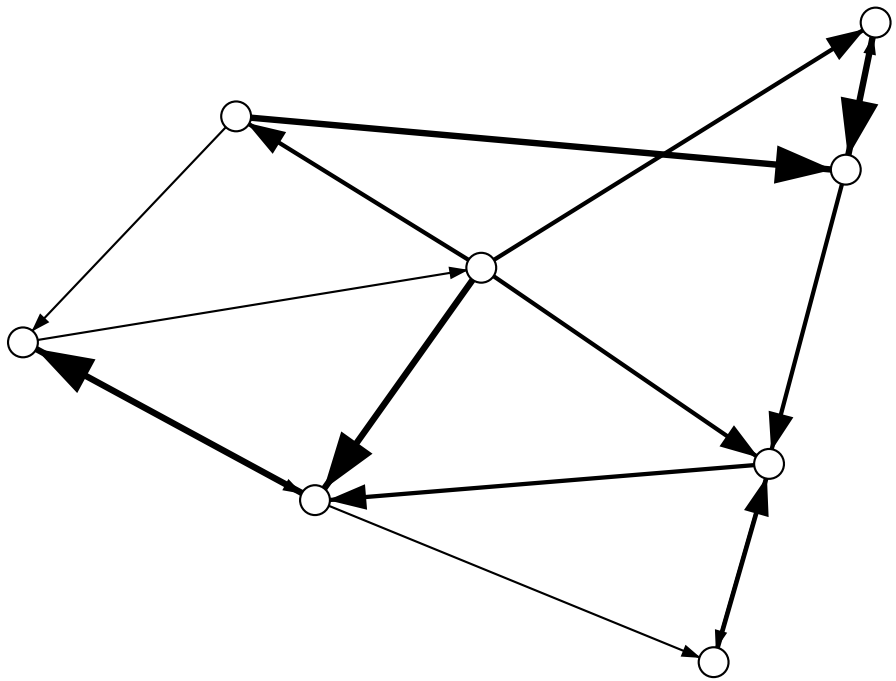
Proposition

If the graph is **undirected** and **connected**, the stationary distribution is proportional to the degrees:

$$\boxed{\pi \propto d}$$

Weighted graphs

Consider a directed, **weighted** graph $G = (V, E)$
Let A be the **weighted** adjacency matrix
Let $w^+ = A1$ and $w^- = A^T 1$ be the **out-weights** and **in-weights**



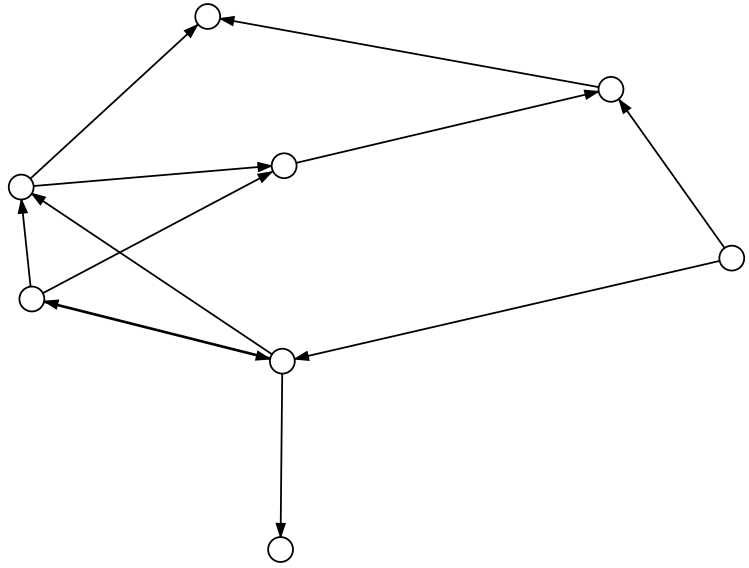
Same results with transition matrix $P = D^{-1}A$ where
$D = \mathrm{diag}(w^+)$

Outline

1. Random walk
2. **PageRank**
3. Personalized PageRank
4. Case of bipartite graphs
5. Applications

Accounting for sinks

Consider a directed graph $G = (V, E)$



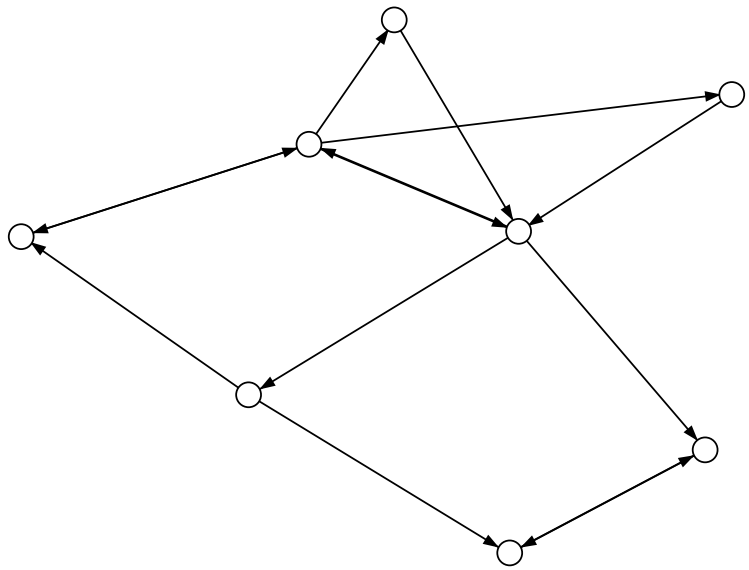
Definition

A random walk with **forced restarts** is a Markov chain with transition matrix:

$$P_{ij} = \begin{cases} \frac{A_{ij}}{d_i^+} & \text{if } d_i^+ > 0 \\ \frac{1}{n} & \text{otherwise} \end{cases}$$

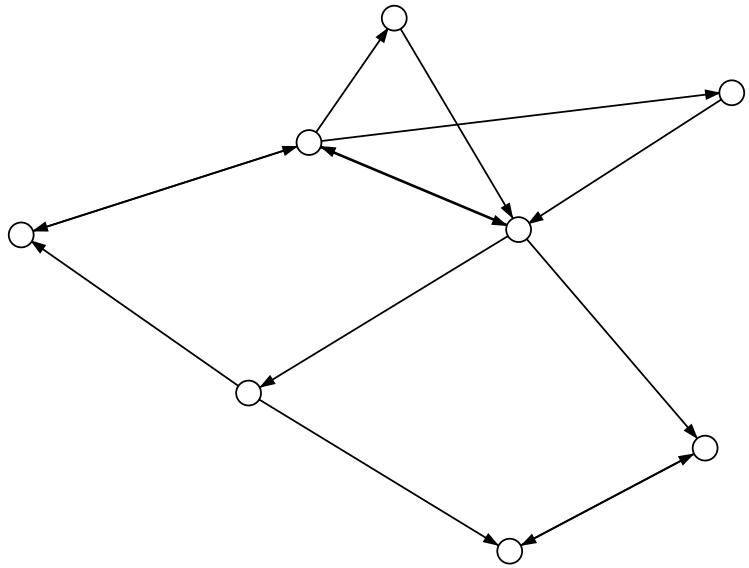
Accounting for traps

Even in the absence of sinks, the random walk can get **trapped**



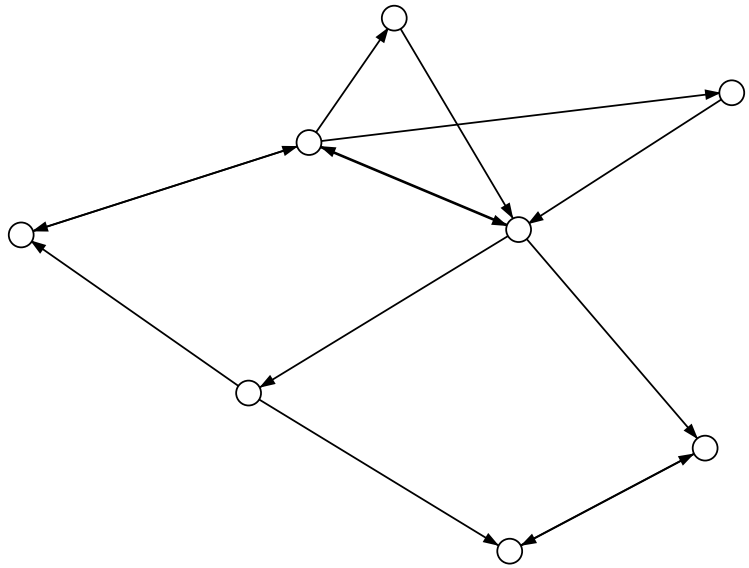
PageRank

Random restarts: walk with probability α, restart otherwise



PageRank

Random restarts: walk with probability α, restart otherwise



Definition

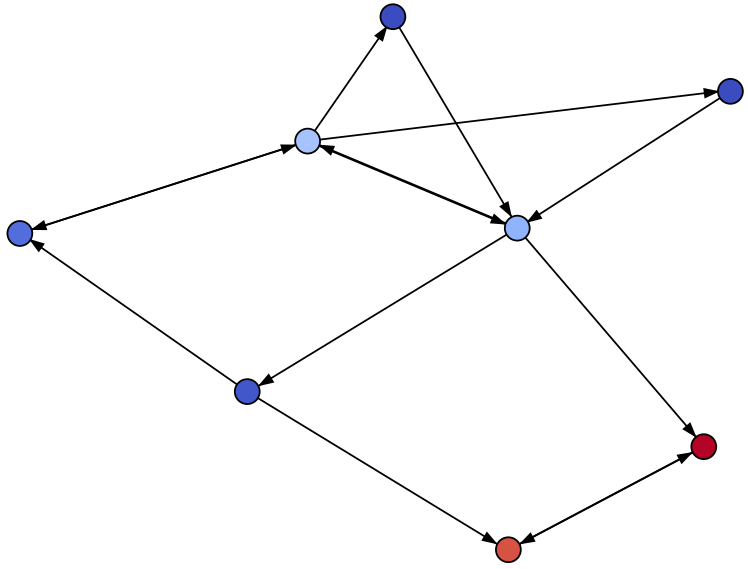
A random walk with **random restarts** is a Markov chain with transition matrix:

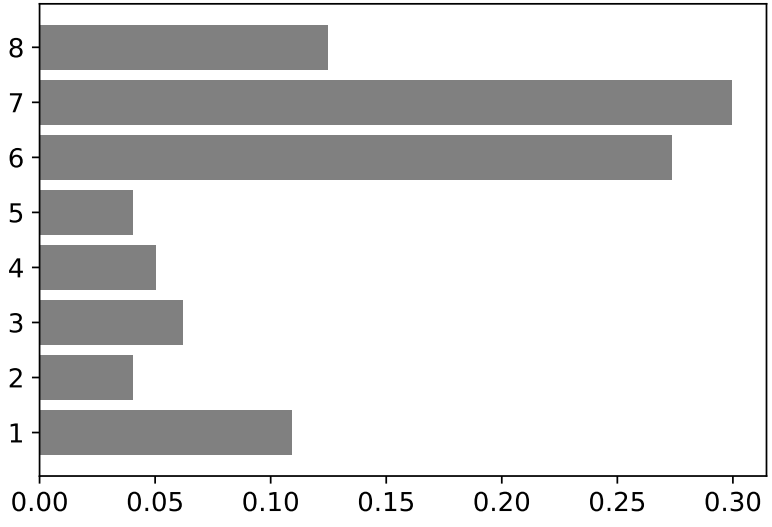
$$P^{(\alpha)} = \alpha P + (1-\alpha)\frac{11^T}{n}$$

The stationary distribution $\pi^{(\alpha)}$ is the **PageRank vector**

Example

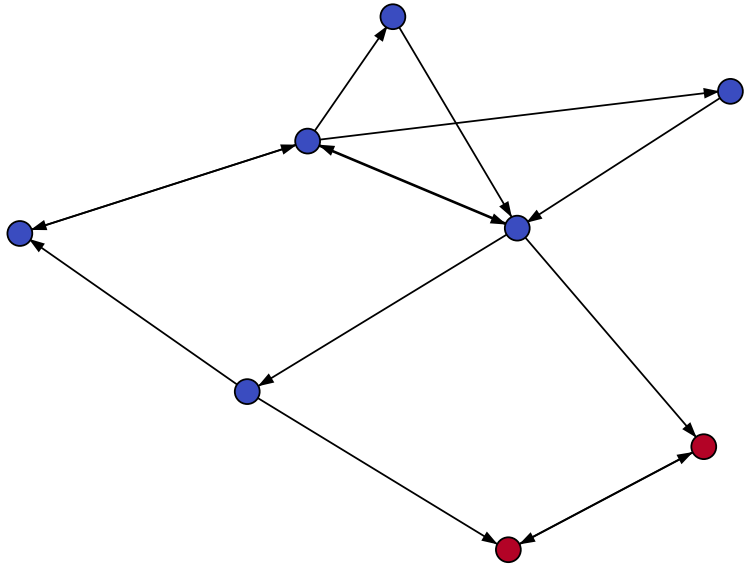
PageRank ($\alpha = 0.85$)

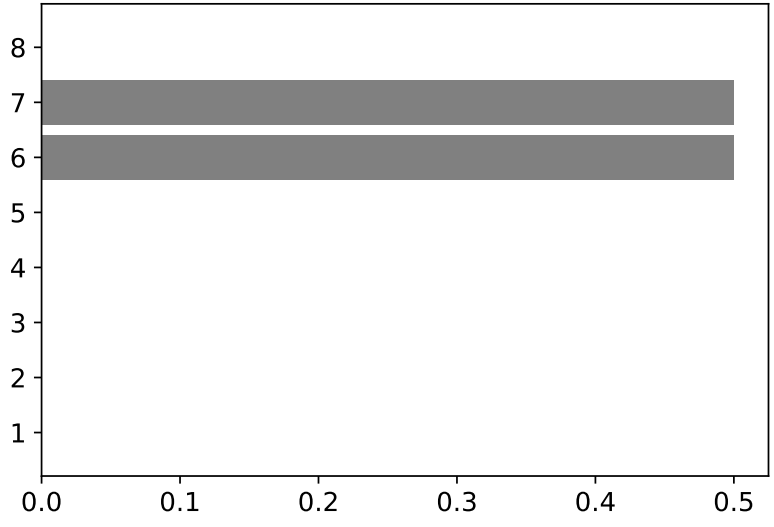




Example

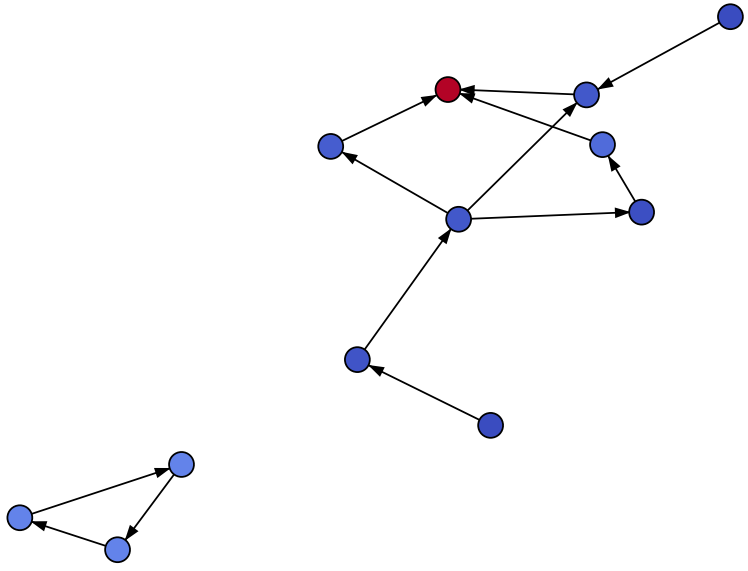
PageRank ($\alpha \to 1$)

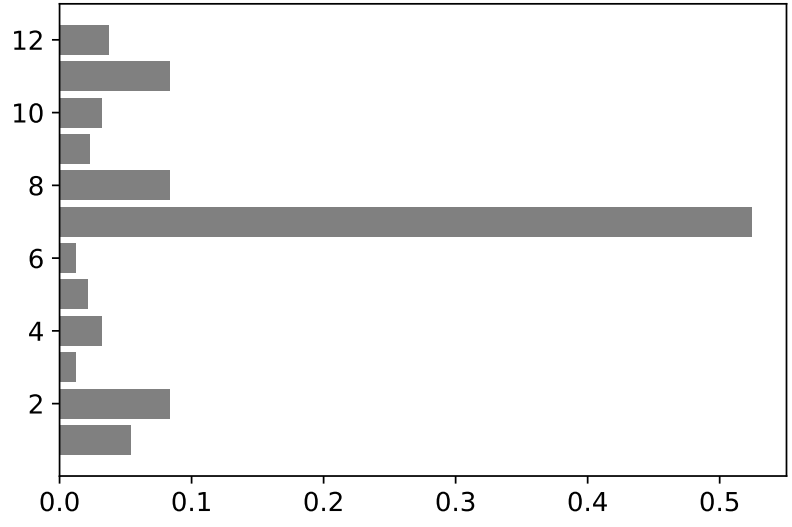




Disconnected graph

PageRank ($\alpha = 0.85$)





Power iteration

PageRank

Input:
P, transition matrix (with forced restarts)
α, damping factor
K, number of iterations

Do:
$\pi \leftarrow \frac{1}{n}(1, \ldots, 1)$
For $t = 1, \ldots, K$, $\pi \leftarrow \alpha \pi P + (1 - \alpha)\frac{1}{n}(1, \ldots, 1)$

Output:
π, (approximate) PageRank vector

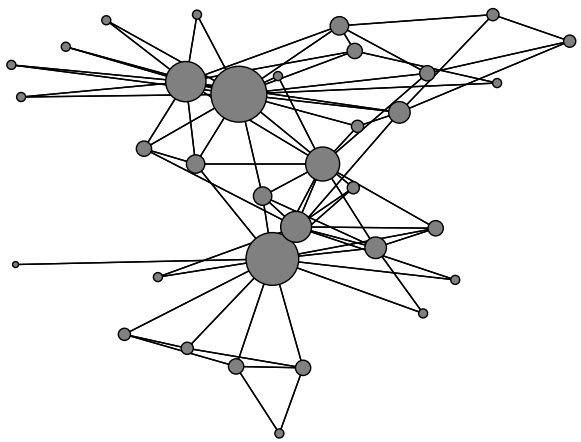
Time complexity: $O(Km)$ for m edges

Setting the damping factor α

The path length before restart (in the absence of sinks) is **geometric** with parameter $1 - \alpha$

$$\text{Mean path length} \rightarrow L = \frac{\alpha}{1 - \alpha}$$

For $\alpha = 0.85$, we get $L \approx 5.7$, a typical distance between two nodes in real graphs (cf. the **six degrees of separation**)



Expression of the PageRank vector

Proposition

$$\pi^{(\alpha)} = (1-\alpha)\sum_{t=0}^{+\infty} \alpha^t \pi(t) \quad \text{with } \pi(0) \text{ uniform}$$

Limiting cases

- **No restarts** $(\alpha \to 1)$

$$\pi^{(\alpha)} \to \lim_{t \to +\infty} \pi(t)$$

- **Frequent restarts** $(\alpha \to 0)$

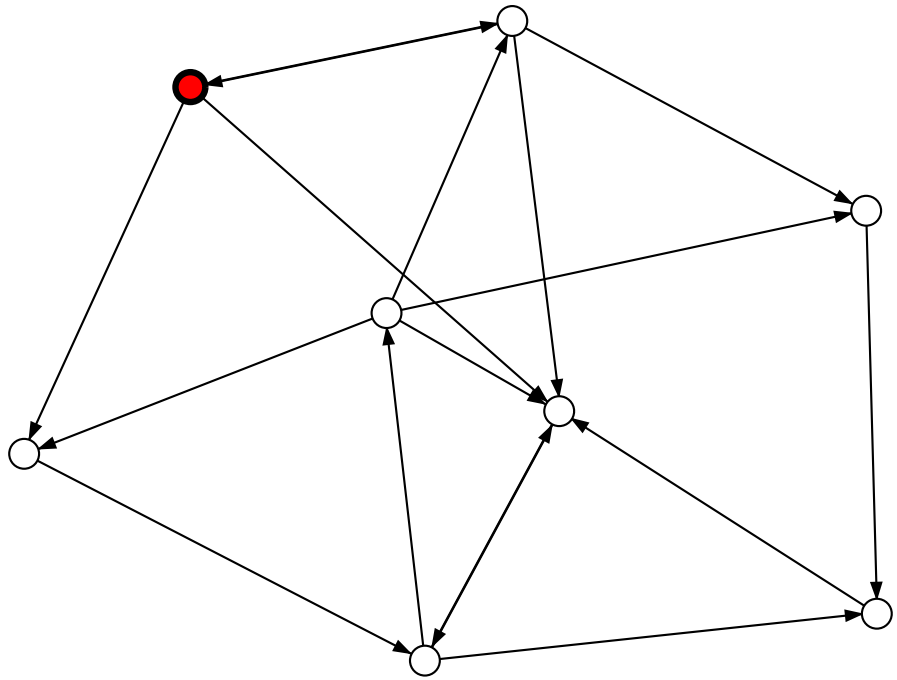
$$\pi^{(\alpha)} = (1-\alpha)\pi(0) + \alpha\pi(1) + o(\alpha)$$

Ranking equivalent to **neighbor sampling**

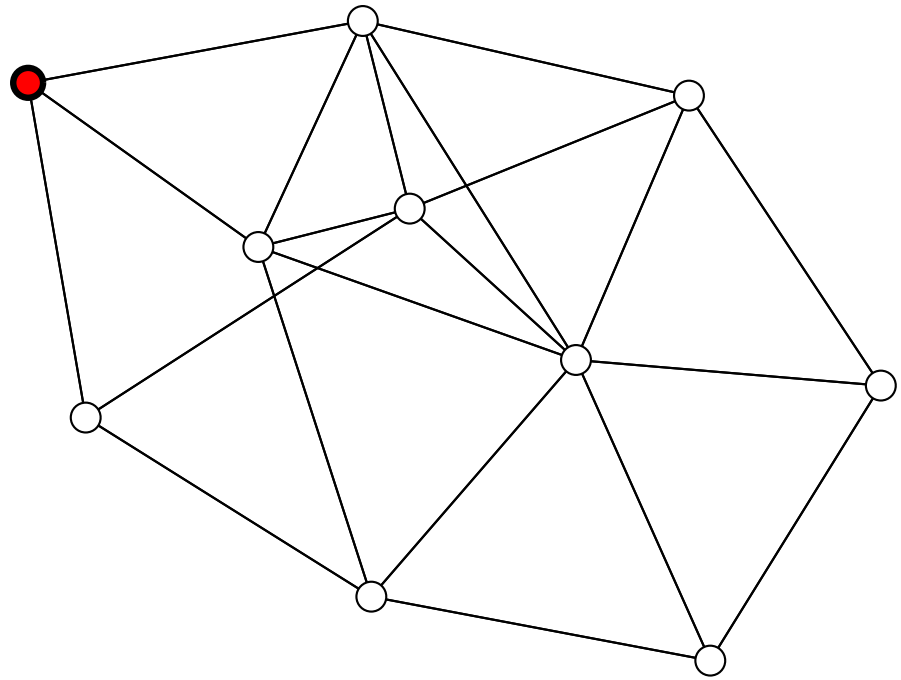
Outline

1. Random walk
2. PageRank
3. **Personalized PageRank**
4. Case of bipartite graphs
5. Applications

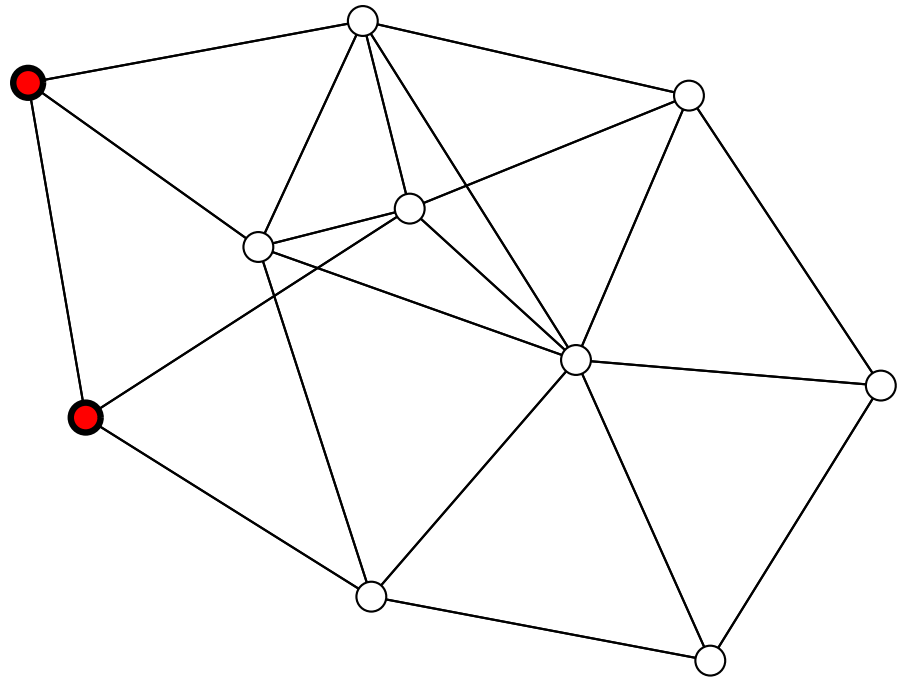
Personalization



Personalization



Personalization



Personalized PageRank

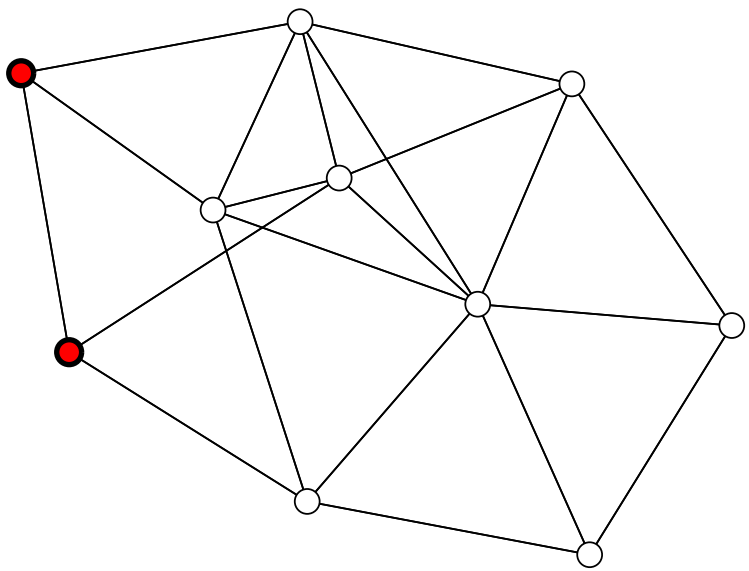
Let μ be some distribution on $S \subset V$ (e.g., uniform)

- Forced restarts:

$$P_{ij} = \begin{cases} \frac{A_{ij}}{d_i^+} & \text{if } d_i^+ > 0 \\ \mu_j & \text{otherwise} \end{cases}$$

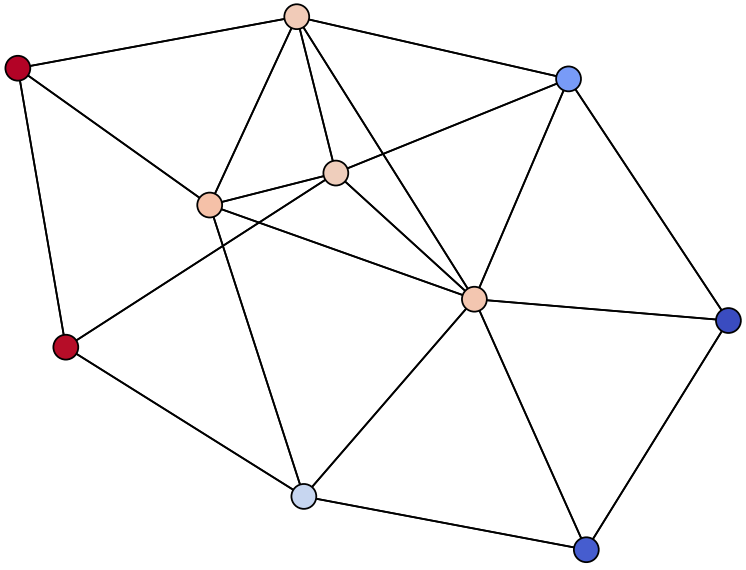
- Random restarts:

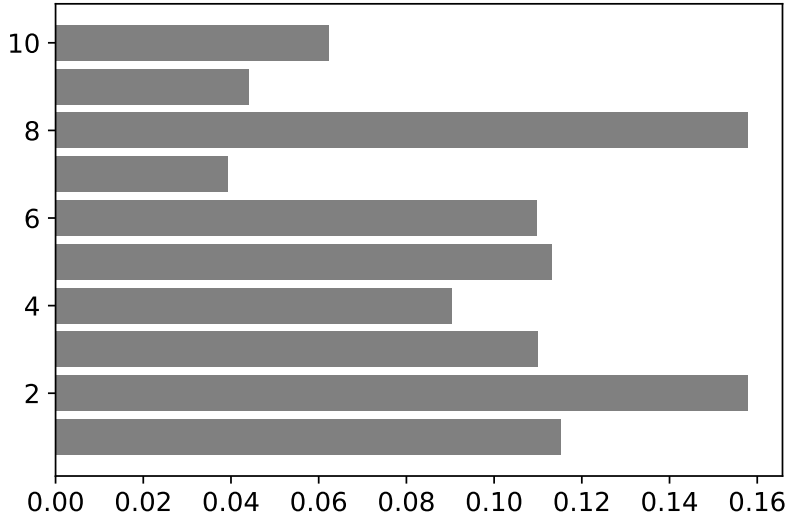
$$P^{(\alpha)} = \alpha P + (1 - \alpha) 1 \mu$$



Example

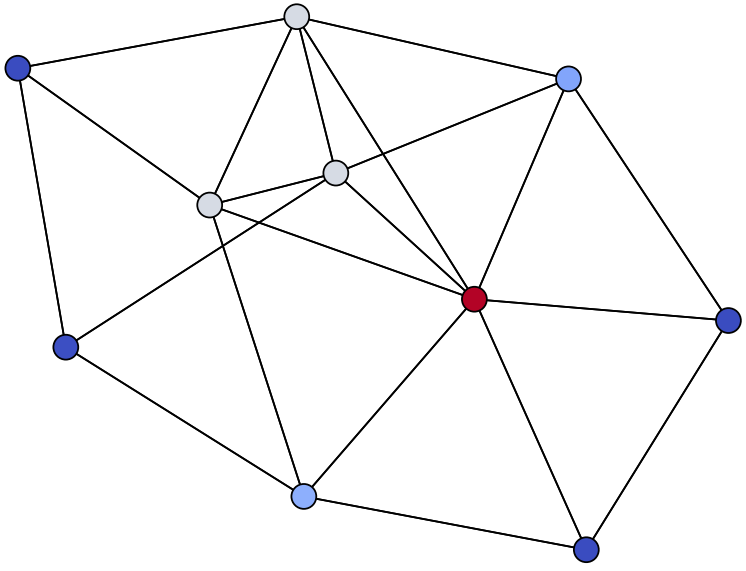
Personalized PageRank

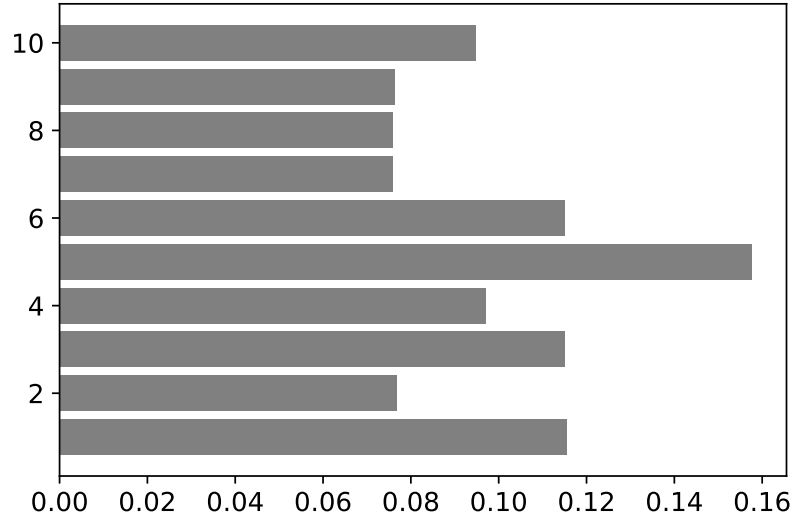




Example

PageRank (not personalized)





Power iteration

Personalized PageRank

Input:

P, transition matrix (with forced restarts)

μ, personalization row vector

α, damping factor

K, number of iterations

Do:

$\pi \leftarrow \mu$

For $t = 1, \ldots, K$, $\pi \leftarrow \alpha \pi P + (1 - \alpha)\mu$

Output:

π, (approximate) PageRank vector

Expression of the Personalized PageRank vector

Proposition

$$\pi^{(\alpha)} = (1-\alpha)\sum_{t=0}^{+\infty} \alpha^t \pi(t) \quad \text{with } \pi(0) = \mu$$

Limiting cases

- **No restarts** $(\alpha \to 1)$

$$\pi^{(\alpha)} \to \lim_{t \to +\infty} \pi(t)$$

- **Frequent restarts** $(\alpha \to 0)$

$$\pi^{(\alpha)} = (1-\alpha)\mu + \alpha\pi(1) + o(\alpha)$$

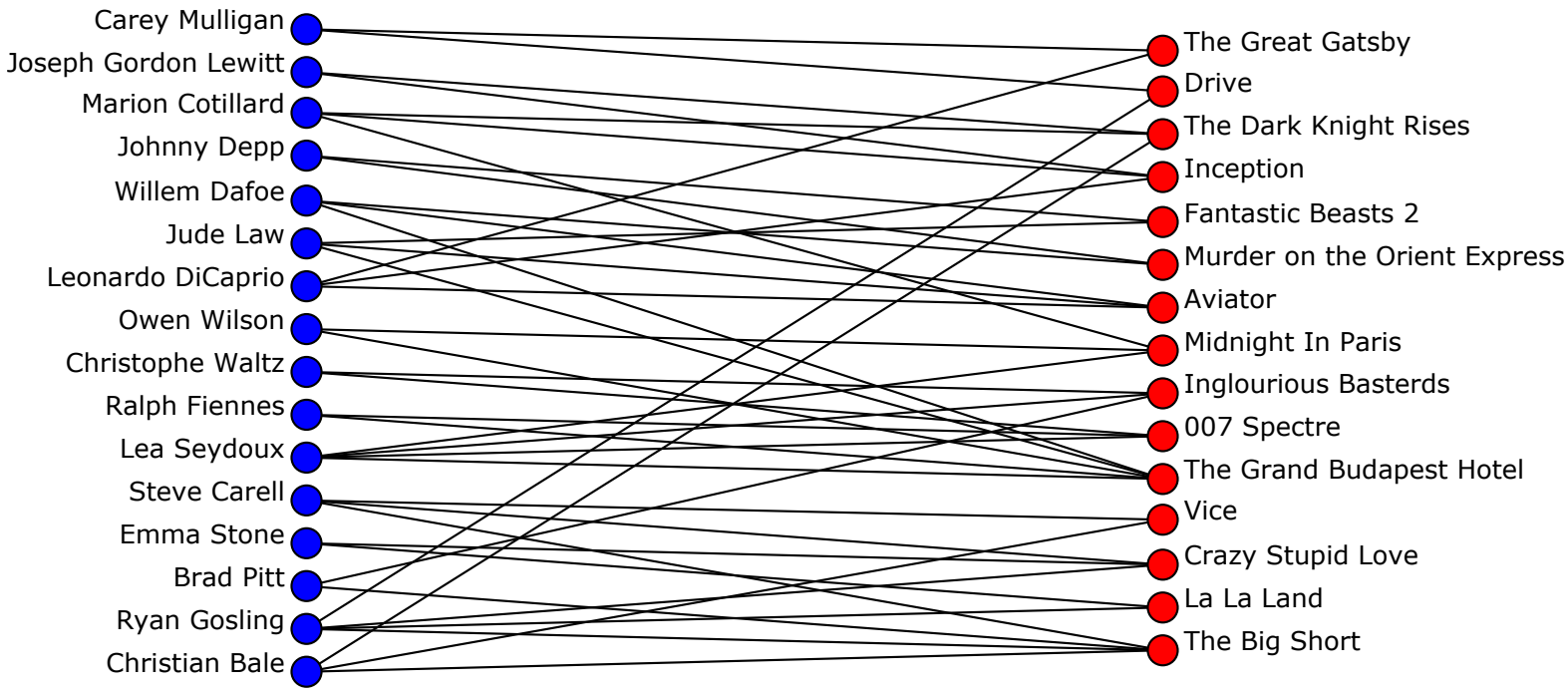
Ranking equivalent to **neighbor sampling**

Outline

1. Random walk
2. PageRank
3. Personalized PageRank
4. **Case of bipartite graphs**
5. Applications

Random walk in a bipartite graphs

The random walk (without restarts) has **period 2**



Actors starring in movies

Expression of the PageRank vector

Let $G = (V_1, V_2, E)$ be some connected **bipartite** graph
Assume **starts** and **restarts** in V_1
Let $\pi_1^{(\alpha)}$ and $\pi_2^{(\alpha)}$ be the PageRank vectors on V_1 and V_2

Proposition

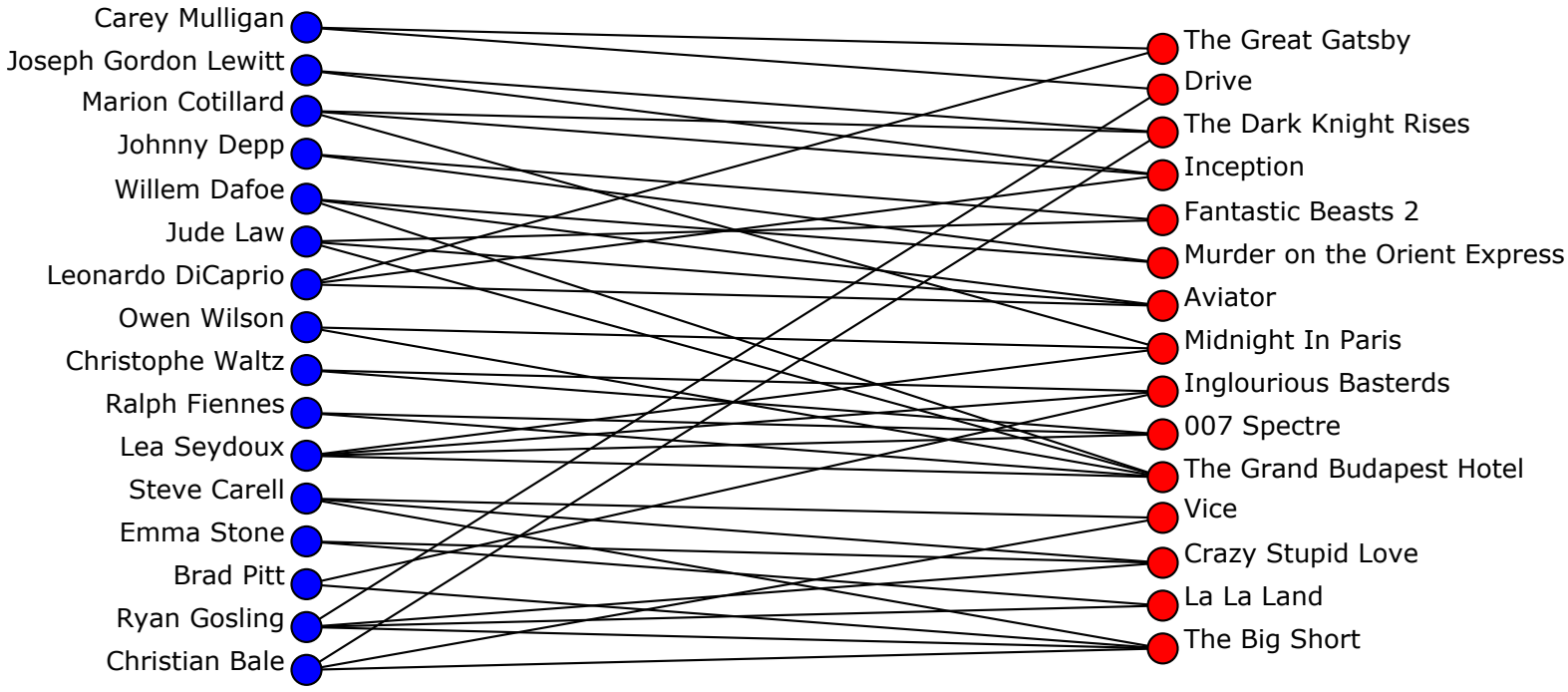
$$\pi_1^{(\alpha)} = (1-\alpha)\sum_{t\in 2\mathbb{N}} \alpha^t \pi_1(t), \quad \pi_2^{(\alpha)} = (1-\alpha)\sum_{t\in 2\mathbb{N}+1} \alpha^t \pi_2(t)$$

Note: The distribution over V_1 and V_2 is constant!

$$1^T \pi_1^{(\alpha)} = \frac{1}{\alpha+1}, \quad 1^T \pi_2^{(\alpha)} = \frac{\alpha}{\alpha+1}$$

$\rightarrow$ Ranking makes sense only **within** sets V_1 and V_2

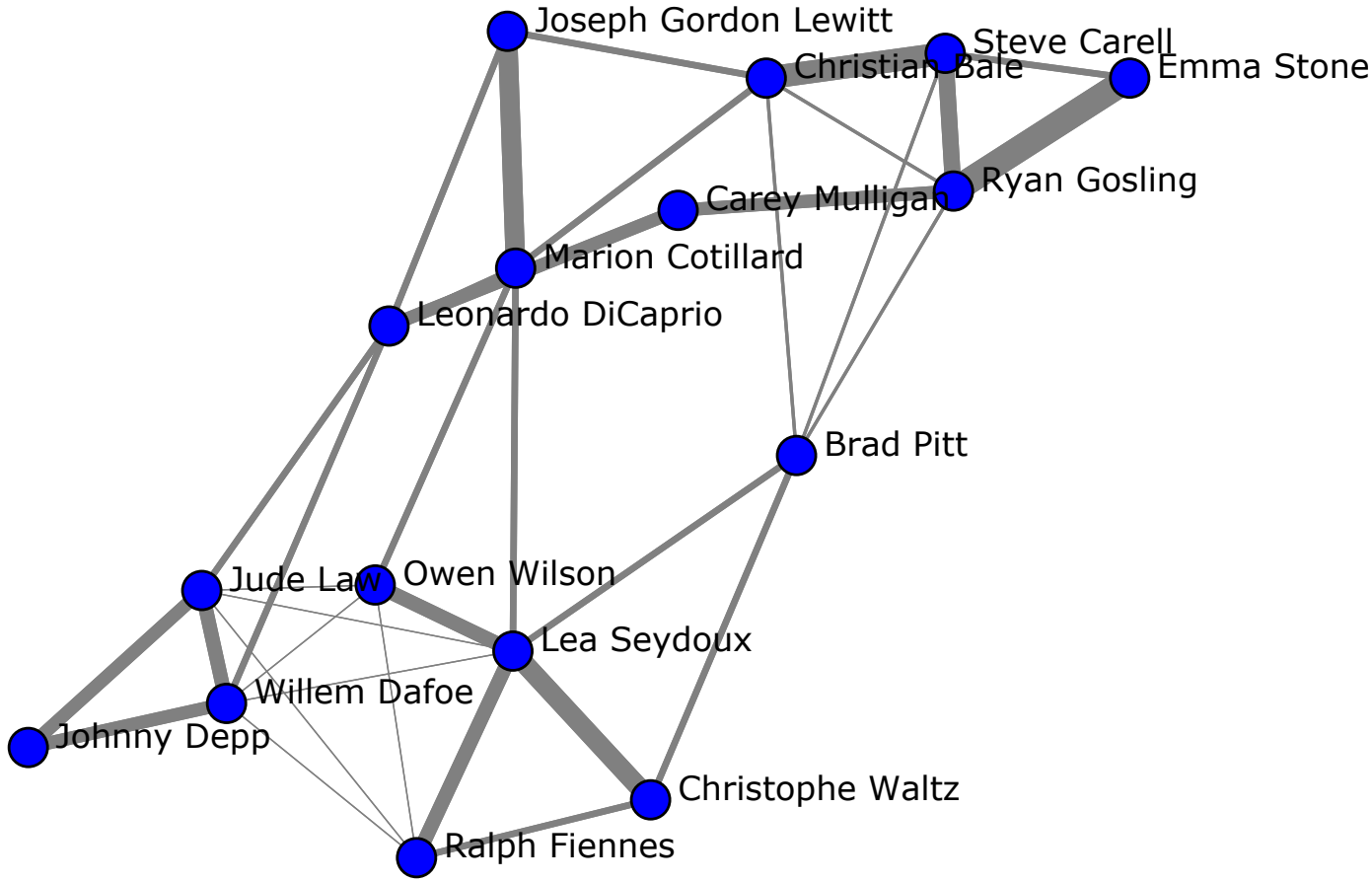
Co-neighbor graph



Actors starring in movies

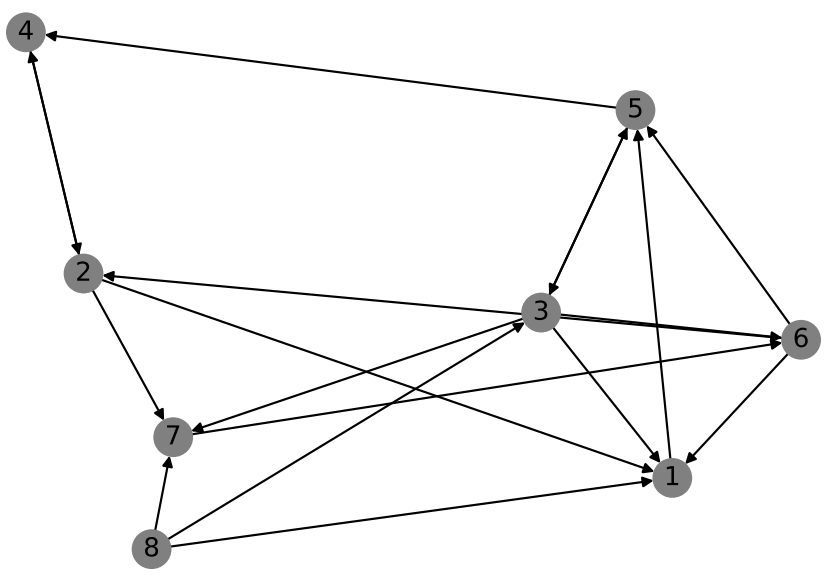
Co-neighbor graph

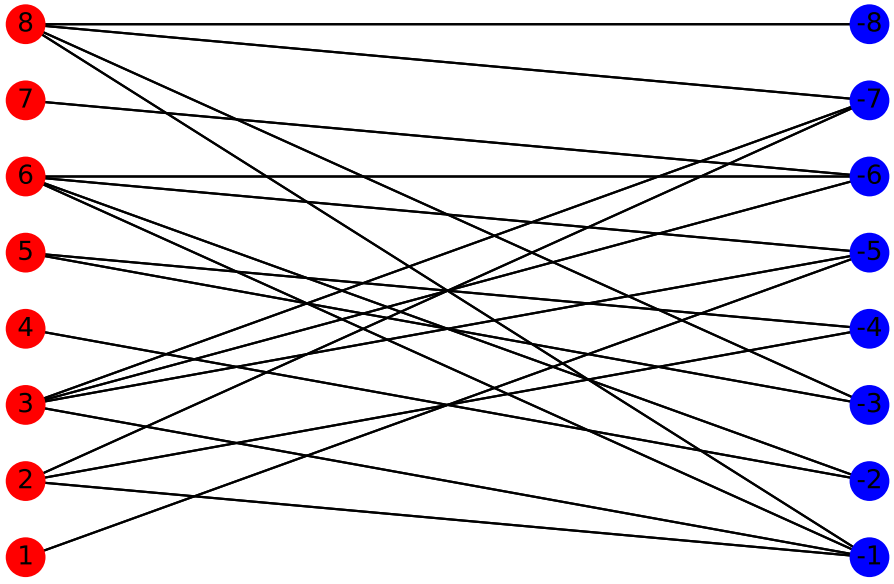
Graph of actors linked by movies



PageRank of actors in the **bipartite graph** with damping factor α = PageRank in the **co-neighbor graph** with damping factor α^2

Directed graphs as bipartite graphs





Question: How to interpret PageRank in the bipartite graph?

Answer: PageRank associated with the **forward-backward** random walk in the directed graph!

Outline

1. Random walk
2. PageRank
3. Personalized PageRank
4. Case of bipartite graphs
5. **Applications**

Recommendation

Consider the **MovieLens** dataset
Rating of movies by users

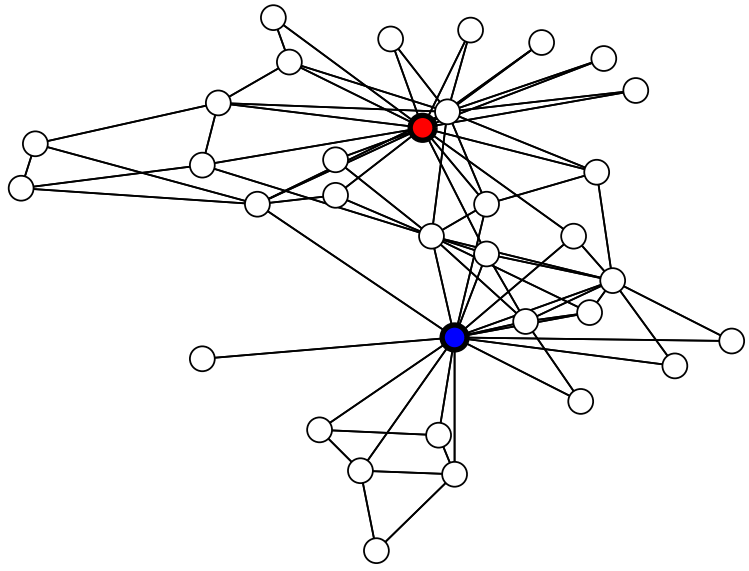
Which movies to recommend to a target user u?

Algorithm

- Construct the **bipartite graph** between users and movies with **positive** reviews (possibly weighted)
- Compute the **Personalized PageRank** vector π starting from the target user u
- **Rank** movies with respect to π

Classification

Consider the **Karate Club** graph
You know the class of 2 nodes



How to predict the label of the other nodes?

Algorithm

- Compute the **Personalized PageRank** vectors $\pi^{(k)}$ starting from each class $k = 1, 2$
- **Predict** $\arg\max_{k=1,2} \pi^{(k)}$ for each node

Clustering

Consider the **Wikivitals** dataset
Links between articles of Wikipedia

How to form k clusters?

Algorithm

- Select k seeds (e.g., at random)
- **Label** these seeds $1, \ldots, k$
- **Classify** the other nodes by **Personalized PageRank** (one-against-all)

Summary

PageRank

A key tool for graph analysis

- Useful to quantify the **importance** of nodes, possibly relatively to other nodes $\rightarrow$ **Personalized PageRank**
- **Fast** computation through matrix-vector multiplications
- **Applications**: search, ranking, classification, clustering, ...