Contents: Chapter 4

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Chapter 4:
Counting

MATH TEAM

Counting

The regional Math Bowl is on October 30th.

We have a lot of work to do between now and then.

Open your books to page 74. We're going to begin the section on counting.

Counting? We all know how to count.

I could still use some practice.

The section we are working on today begins on page 74 and ends on page 84.

How many pages is that?

84 − 74 = 10 pages.

Are you sure?

Count them.

I think there are 11 pages from 74 to 84.

Huh?

Starting on page 74, we have...

Good.

When you're counting the number of pages, you can't just subtract the first page number from the last.

84
- 74
10

For example, if you read pages 1 through 10, that's a total of 10 pages...

...not 10-1=9 pages.

It's easy to count the number of pages you read if you start reading on page 1.

That's right. What could you do to make the number of pages from 74 to 84 easier to count?

74	75	76	77	78	79	80	81	82	83	84
-73	-73	-73	-73	-73	-73	-73	-73	-73	-73	-73
1	2	3	4	5	6	7	8	9	10	11

We could subtract 73 from every page number!

That gives us a list of numbers from 1 to 11.

So, there are 11 pages.

Great!

If you read pages 87 through 119, how many pages is that?

It will be easier to count the number of pages if we subtract 86 from every page number.

Why 86?

That way, the list of page numbers will start at 1!

How many pages are there from page 87 to page 119?

I get it.

A monster who reads pages 87 through 119...

...reads the same number of pages as a monster who starts on page 1 and ends on page 33!

33 pages!

$$\begin{array}{rrrcrrr} 87 & 88 & 89 & \cdots & 117 & 118 & 119 \\ -86 & -86 & -86 & \cdots & -86 & -86 & -86 \\ \hline 1 & 2 & 3 & \cdots & 31 & 32 & 33 \end{array}$$

THE THREE DOTS ABOVE MEAN THAT SOME NUMBERS ARE NOT WRITTEN IN THE MIDDLE, BUT THE PATTERN CONTINUES.

That's right. Let's try a similar problem.

Today is October 3rd. The Math Bowl is on October 30th. How many days do we have to prepare between now and the Math Bowl?

That depends. Do we count today?

What about the day of the contest? We always practice before every math meet.

Or, do we only count the full days, starting with tomorrow and ending the day before the competition?

What do you mean when you say ***between*** now and the Math Bowl?

Great questions! I wasn't clear about which days you should be counting.

If you are counting numbers in a list, sometimes you want to ***include*** the first and last numbers.

Other times, you want to ***exclude*** the first and last numbers.

Let's start by finding the number of days we have to practice from October 3rd to October 30th ***inclusive***.

That means we ***include*** the 3rd and the 30th.

INCLUDE MEANS TO KEEP IN. ***EXCLUDE*** MEANS TO LEAVE OUT.

To count the number of days from October 3rd to October 30th ***inclusive***, we can subtract 2 from each date.

That gives us a list from 1 to 28, so there are 28 days to practice.

3	4	5	...	28	29	30
-2	-2	-2	...	-2	-2	-2
1	2	3	...	26	27	28

Perfect.

Now, let's find the number of days we have to practice from October 3rd to October 30th ***exclusive***.

That means we ***exclude*** the 3rd and the 30th.

In that case, we start counting days on the 4th and stop on the 29th.

We subtract 3 from each date to get a list from 1 to 26.

So, if we don't include today, or the day of the math bowl, we only have 26 days to practice.

There's not much time! We better hit the books!

4	5	6	...	27	28	29
-3	-3	-3	...	-3	-3	-3
1	2	3	...	24	25	26

That's the spirit, Grogg! After you read pages 74 to 84, try to solve the section review problems.

All of them?

The problems are numbered 1 through 55.

Try all of the even-numbered problems. We'll save the odd-numbered problems for later.

How many even-numbered problems are there?

Try it.

We need to count the even-numbered problems from 2 through 54 ***inclusive***.

How can we turn this list of numbers into a list that is easy to count?

2 4 6 • • • 50 52 54

If we subtract 1 from every number, the list starts at 1...

...but the list still skips a lot of numbers, so that doesn't help.

2	4	6	• • •	50	52	54
-1	-1	-1	• • •	-1	-1	-1
1	3	5	• • •	49	51	53

What if we divide every number by 2?

That gives us a list of numbers from 1 to 27!

So, there are 27 even-numbered problems.

2	4	6	• • •	50	52	54
÷2	÷2	÷2	• • •	÷2	÷2	÷2
1	2	3	• • •	25	26	27

If we can turn a list of numbers into a list that starts at 1, and doesn't skip any numbers...

...then it's easy to count how many numbers are in the list.

It's like my dad always says. There are three types of monsters in this world...

...monsters who can count, and monsters who can't.

Ms. Q.

Choices

The annual fall festival is tomorrow. Everyone needs to fill out a meal order form for the cookout.

Does everyone have a form?

Ms. Q.?

Yes, Grogg?

COOKOUT ORDERS

This doesn't seem like a lot of choices.

There are only 2 choices of meat, 3 sides, 3 drinks, and 2 desserts to choose from.

Name _____

Fall Festival Cookout Order Form

Circle 1 item from each category:

Meat

Hot Dog

Hamburger

Add Ketchup

Add Mustard

Sides

Potato Salad

Green Beans

Corn

Drink

Milk

Juice

Lemonade

Dessert

Brownie

Pie

You can choose one of each.

How many different meals can you make with those choices?

Let's see...

I could have a hot dog, green beans, juice, and pie...

...or a hot dog, corn, milk, and a brownie...

...or a hamburger, potato sal--

Grogg!

You'll never count all of the possible meals that way!

Let's start with the meat and the sides.

We can make a diagram!

You can have a hot dog with any of the three sides...

...or a hamburger with any of the three sides.

That makes 6 ways to choose a meat and a side.

Hot Dog
- Potato Salad
- Green Beans
- Corn

Hamburger
- Potato Salad
- Green Beans
- Corn

Then, you can choose a drink!

For each of the 6 ways that you can choose a meat and a side, you can choose milk, juice, or lemonade.

That makes 18 ways to choose a meat, a side, and a drink...

...starting with a hot dog, potato salad and milk--

Hot Dog
- Potato Salad
  - Milk
  - Juice
  - Lemonade
- Green Beans
  - Milk
  - Juice
  - Lemonade
- Corn
  - Milk
  - Juice
  - Lemonade

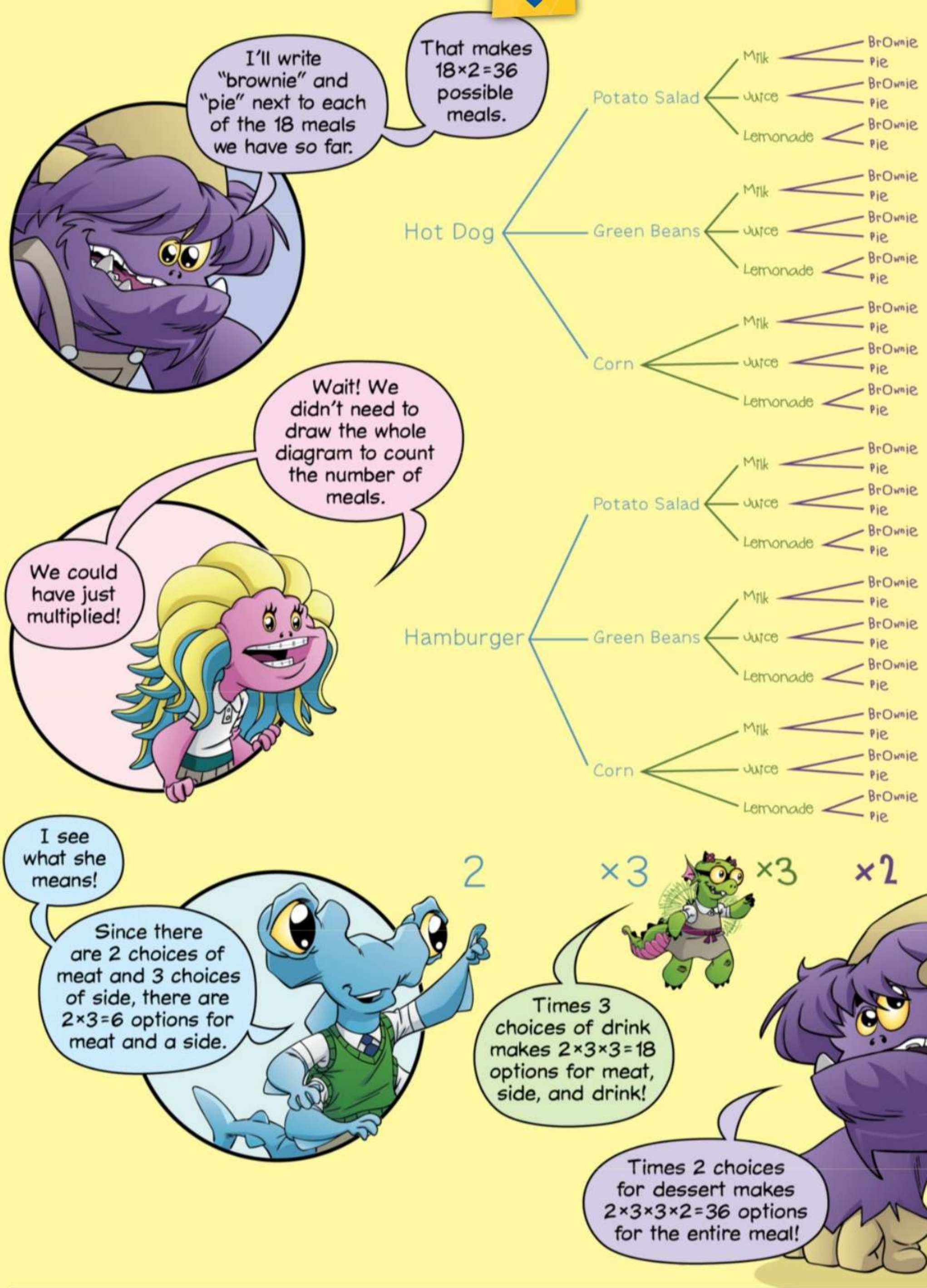
Hamburger
- Potato Salad
  - Milk
  - Juice
  - Lemonade
- Green Beans
  - Milk
  - Juice
  - Lemonade
- Corn
  - Milk
  - Juice
  - Lemonade

And ending with a hamburger, corn, and lemonade.

We've made 18 different meals...

...and we haven't even gotten to dessert!

For each of the 18 meals already on our diagram, there are 2 choices for dessert.



THIS IS CALLED A ***TREE DIAGRAM***. IT IS USED FOR COUNTING POSSIBILITIES, LIKE THE NUMBER OF MEALS YOU CAN MAKE WITH THE GIVEN CHOICES.

That's right!

With 2 choices of meat, 3 sides, 3 drinks, and 2 desserts, there are 2×3×3×2=36 possible meals.

But, I forgot to add a third choice of dessert! You can also choose to have a cookie.

How many meals are possible now?

COOKOUT ORDERS

Now, we have our choice of 2 meats, 3 sides, 3 drinks, and ***3*** desserts.

That makes 2×3×3×3=54 possible meals!

2×3×3×3=54

Practice: Pages 7-19

54 options!

What's the matter, Grogg?

Now I'll never be able to decide.

COOKOUT ORDERS

R&G
Venn Diagrams

I love the fall festival.

Here's the last hot dog.

That makes 25 hot dogs for 25 hungry little monsters.

Kiss the Cook

Time to add the mustard and ketchup.

20.

How many little monsters want mustard on their hot dog?

hot dogs: 25
with mustard: 20
with ketchup: 9
plain (no k or m): 1

Got it. 20 hot dogs with mustard.

How many little monsters want ketchup?

9.

Wait, *what?!*

How can we put mustard on 20 hot dogs and ketchup on 9 hot dogs if we only have 25 hot dogs?

Good question.

There's another problem.

One little monster doesn't want ketchup or mustard.

These numbers can't be right.

What should R&G do?

I know! Some of the little monsters probably want ketchup *and* mustard!

Hmmm... How will that work?

We've added mustard to 20 hot dogs, and we still have 5 plain hot dogs.

Let's set aside 1 of those for the monster who doesn't want ketchup or mustard.

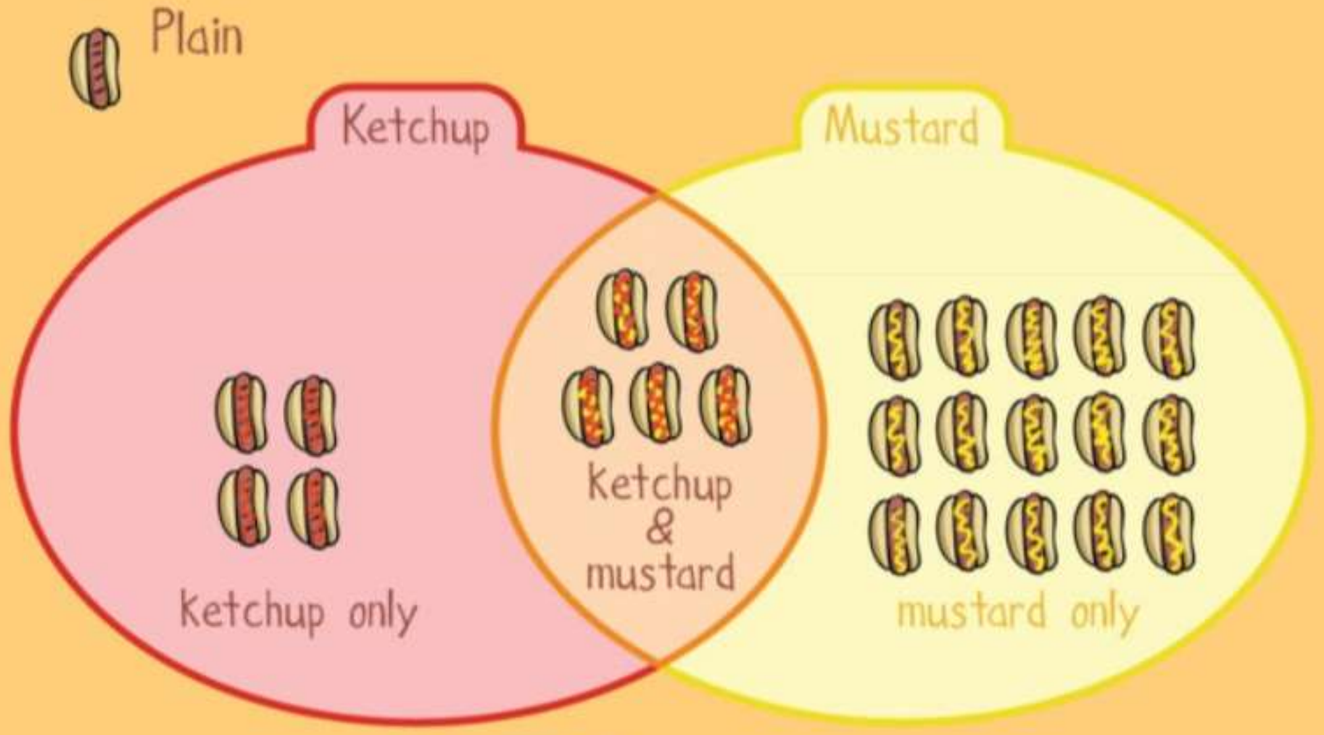
That leaves 4 hot dogs that don't have mustard.

If we put ketchup on all 4 of these hot dogs, we still need ketchup on 9−4=5 more hot dogs.

We can put ketchup on 5 of the hot dogs that already have mustard!

That makes sense!

We have 1 plain hot dog, 4 hot dogs with ketchup only, 15 with mustard only, and 5 with ketchup and mustard.



That gives us 4+5=9 hot dogs with ketchup and 5+15=20 hot dogs with mustard.

THIS DIAGRAM IS CALLED A ***VENN DIAGRAM***. EACH LABELED RING REPRESENTS A CATEGORY. THE ITEMS INSIDE EACH RING BELONG IN THAT CATEGORY. ITEMS IN THE REGION WHERE THE TWO RINGS OVERLAP ARE PART OF BOTH CATEGORIES.

Now for the hamburgers.

How many hamburgers do we need to grill?

I'm not sure.

Kiss the Cook

22 little monsters checked the box marked ketchup.

13 checked the box marked mustard.

And 3 didn't check either box.

But, 7 monsters checked both boxes.

hamburgers:
with ketchup: 22
with mustard: 13
plain(no k or m): 3
both(k & m): 7

Can't we just add these numbers up to figure out how many hamburgers we need to grill?

Nope.

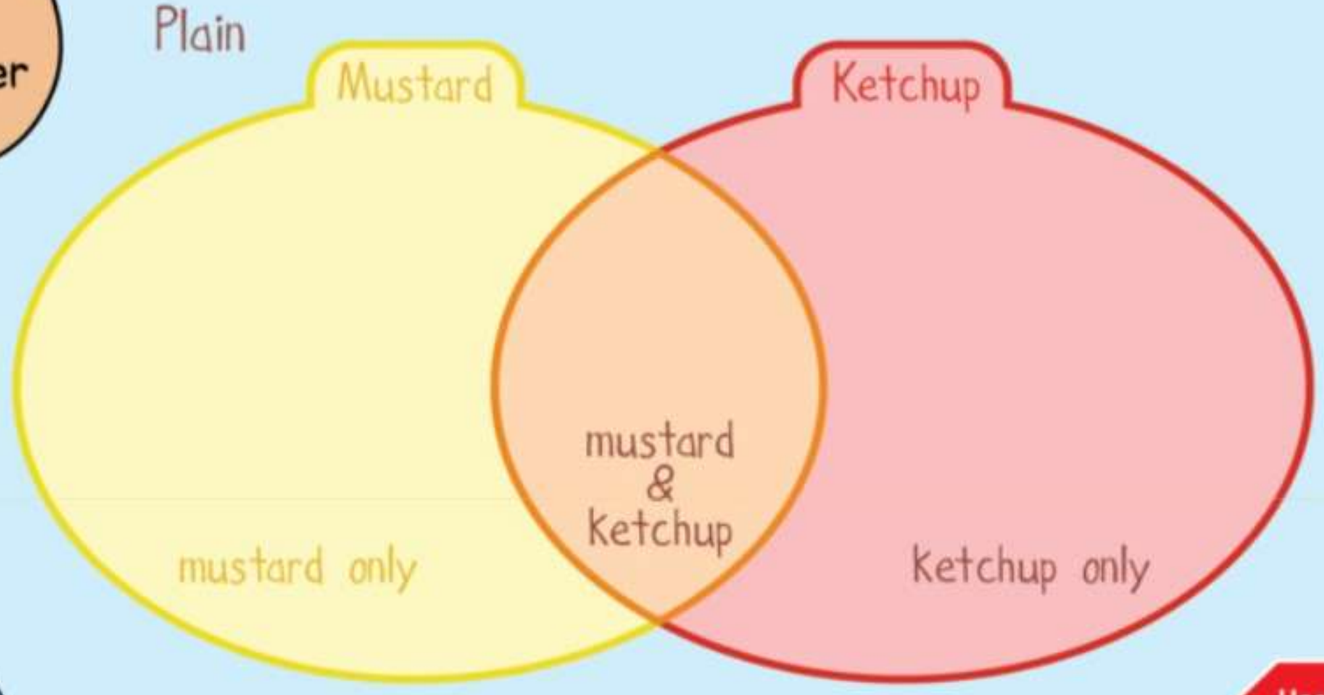
Some of the little monsters got counted more than once.

For example, Lizzie checked ketchup and mustard on her form. I counted her as a monster who picked ketchup...

...a monster who picked mustard...

...*and* a monster who picked both.

Maybe we should make another diagram.



Use this diagram to help you figure out how many hamburgers must be grilled.

We can start with the three plain burgers.

Got it. Plain burgers don't have ketchup or mustard, so they go out here. What's next?

Plain

Mustard

Ketchup

mustard & ketchup

mustard only

Ketchup only

We don't know how many burgers have just ketchup...

...or just mustard.

But, we do know how many have ketchup ***and*** mustard.

There are 7 monsters who want ketchup and mustard.

Plain

Mustard

Ketchup

mustard & ketchup

mustard only

Ketchup only

Burgers that need ketchup and mustard go in the middle of our diagram...

...where ketchup and mustard overlap.

Plain

Mustard

mustard only

mustard & ketchup

Ketchup

Ketchup only

We need 22 burgers with ketchup. Since 7 have mustard ***and*** ketchup...

...there must be 22−7=15 burgers with ketchup only.

We need a total of 13 burgers with mustard.

We already have 7 burgers with mustard and ketchup, so we need 13−7=6 burgers with mustard only.

Plain

Mustard

mustard only

mustard & ketchup

Ketchup

Ketchup only

That makes 3 plain burgers, 6 burgers with mustard only, 7 with mustard and ketchup, and 15 with ketchup only.

That's 3+6+7+15=31 burgers.

Plus this one I made for myself.

Want one?

No thanks, I'm a vegetarian.

WORKSHOP
FACTORIALS

Arrr! 'Tis me favorite math team.
Ahoy, little monsters!
Care to challenge me to a match of tug-o-war?
Sounds fun!
Let's do it!
I'm in.
I'll just watch.

Each of you should take a spot on the rope.
I'll give you a minute to arrange yourselves 'n' decide on a strategy.

How should we line up?
Grogg is the biggest and strongest, so he should stand in back, at the end of the rope.

Are you sure?
Maybe since I'm the biggest, I should stand in the front.
Where should I be?
There must be a dozen different ways that we can line up.
Let's figure it out!
How many different ways can you three little monsters be arrangin' yourselves on the rope?

How many ways can the three little monsters arrange themselves?

If I'm in front, then either Alex or Lizzie can stand behind me...

...and the other monster will stand in back.

G A L

G L A

If I stand in front, then there are 2 ways for Grogg and Lizzie to line up behind me.

A G L

A L G

And if I'm in front, then there are 2 ways for Grogg and Alex to line up.

L G A

L A G

I think we might have missed one.

Did the little monsters find all the possible ways?

What if I stand in back, with Lizzie ahead of me, then Alex in front?

We already counted that one when we found the ways we could line up with Alex in front.

We were careful not to miss any arrangements by staying organized.

There are two ways to line up if Grogg stands in front...

...two ways with me in front...

...and two ways if Lizzie stands in front.

Grogg - Alex - Lizzie
Grogg - Lizzie - Alex
Alex - Grogg - Lizzie
Alex - Lizzie - Grogg
Lizzie - Grogg - Alex
Lizzie - Alex - Grogg

So, the three of us can arrange ourselves 6 different ways!

Aye! There be 6 ways for 3 little monsters to arrange themselves in order.

Choose one o' the six ways, and we'll start the tug-o-war.

I'm all done with my snow cone. Can I be on the team now?

Aye. The more, the merrier!

Now, how many different ways can the ***four*** of you little monsters be arrangin' yourselves?

How many ways can the four little monsters arrange themselves?

Maybe making a tree diagram would help.

There are ***four*** of us, so any one of us can stand in the front.

Grogg
- Alex
- Winnie
- Lizzie

Winnie
- GrOgg
- Alex
- Lizzie

Alex
- GrOgg
- Winnie
- Lizzie

Lizzie
- GrOgg
- Winnie
- Alex

Then, the ***three*** monsters who are left can choose who stands next.

After that, there are ***two*** monsters who can stand next.

Grogg
- Alex
  - Winnie — Lizzie
  - Lizzie — Winnie
- Winnie
  - Alex — Lizzie
  - Lizzie — Alex
- Lizzie
  - Alex — Winnie
  - Winnie — Alex

Winnie
- GrOgg
  - Alex — Lizzie
  - Lizzie — Alex
- Alex
  - Grogg — Lizzie
  - Lizzie — Grogg
- Lizzie
  - Grogg — Alex
  - Alex — Grogg

And the ***one*** monster who is left stands in back!

Alex
- GrOgg
  - Winnie — Lizzie
  - Lizzie — Winnie
- Winnie
  - Grogg — Lizzie
  - Lizzie — Grogg
- Lizzie
  - Grogg — Winnie
  - Winnie — Grogg

Lizzie
- GrOgg
  - Winnie — Alex
  - Alex — Winnie
- Winnie
  - Grogg — Alex
  - Alex — Grogg
- Alex
  - Grogg — Winnie
  - Winnie — Grogg

$4 \times 3 \times 2 \times 1 = 24$

So, there are 4 choices for the monster who stands in front, then 3 choices for the next monster, and 2 choices for the third monster.

That leaves just 1 choice for the monster who stands in back.

That makes $4 \times 3 \times 2 \times 1 = 24$ ways to arrange 4 monsters!

Excellent work, little monsters!

Four little monsters can line up in 4×3×2×1=24 ways on the rope.

There be a special shortcut for writin' 4×3×2×1.

We can put an exclamation point after a 4 to mean 4×3×2×1.

$4! = 4 \times 3 \times 2 \times 1$

Four!

Arrr. Havin' an exclamation point after the 4 doesn't mean you ***shout*** the number...

...'tis called ***4 factorial.***

Puttin' an exclamation point after a number tells you to multiply the number by each of the numbers below it...

...all the way down to 1.

For example, 7! be equal to 5,040.

And 8! be equal to 40,320.

9! be equal to 362,880.

And 10! be equal to 3,628,800.

$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5,040$

$8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40,320$

$9! = 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 362,880$

$10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 3,628,800$

Those numbers get big fast!

20! = 2,432,902,008,176,640,000

Aye, 20 factorial be 2,432,902,008,176,640,000.

You just did!

When countin' the number o' ways to arrange 4 little monsters in order, you found 4!=4×3×2×1=24 ways.

When would you ever use a factorial?

We need another little monster!

Blimey! That makes it five versus one.

How many different ways can the ***five*** of you little monsters arrange yourselves?

Plunk! Come play!

How many ways can the five little monsters arrange themselves?

There are 5 choices for the first monster, 4 choices for the second monster, 3 choices for the third, 2 choices for the fourth, and 1 choice for the last monster.

That makes 5×4×3×2×1=120 ways to arrange five monsters!

$5! = 5\times4\times3\times2\times1=120$

5×4×3×2×1 is 5 factorial!

THE EXCLAMATION POINT NOTATION CAN BE CONFUSING. FOR EXAMPLE, IF ALEX SAYS "2 PLUS 3 IS 5!" HE MEANS 5, NOT 5×4×3×2×1. WE WILL TRY TO MAKE IT CLEAR WHENEVER 5! MEANS 5 FACTORIAL.

Aye, 5 factorial gives us the number o' ways to arrange 5 little monsters in order.

With 6 little monsters, there are 6! ways to arrange ourselves.

Cammie! Come over to help!

$6! = 6\times(5\times4\times3\times2\times1)$

$= 6\times120$

$= 720$

That's 720 ways!

Very good, little buccaneers.

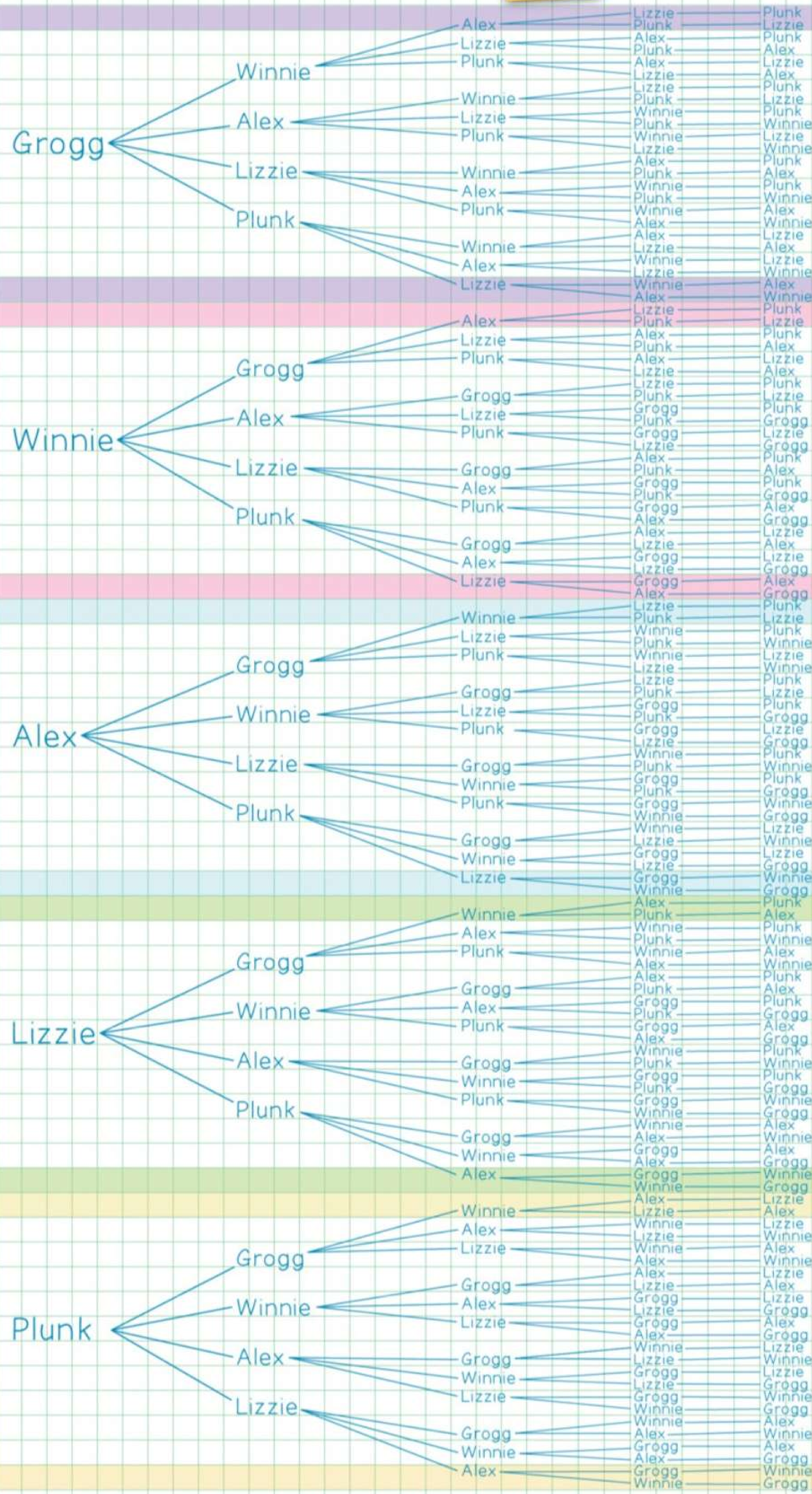
Heave Ho!

SPLOSH!

TITLE All of the 5!=120 ways to arrange 5 monsters in order:

NAME Alex

DATE 10/12



1. GWALP
2. GWAPL
3. GWLAP
4. GWLPA
5. GWPAL
6. GWPLA
7. GAWLP
8. GAWPL
9. GALWP
10. GALPW
11. GAPWL
12. GAPLW
13. GLWAP
14. GLWPA
15. GLAWP
16. GLAPW
17. GLPWA
18. GLPAW
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65. ALPGW
66. ALPWG
67. APGWL
68. APGLW
69. APWGL
70. APWLG
71. APLGW
72. APLWG
73. LGWAP
74. LGWPA
75. LGAWP
76. LGAPW
77. LGPWA
78. LGPAW
79. LWGAP
80. LWGPA
81. LWAGP
82. LWAPG
83. LWPGA
84. LWPAG
85. LAGWP
86. LAGPW
87. LAWGP
88. LAWPG
89. LAPGW
90. LAPWG
91. LPGWA
92. LPGAW
93. LPWGA
94. LPWAG
95. LPAGW
96. LPAWG
97. PGWAL
98. PGWLA
99. PGAWL
100. PGALW
101. PGLWA
102. PGLAW
103. PWGAL
104. PWGLA
105. PWAGL
106. PWALG
107. PWLGA
108. PWLAG
109. PAGWL
110. PAGLW
111. PAWGL
112. PAWLG
113. PALGW
114. PALWG
115. PLGWA
116. PLGAW
117. PLWGA
118. PLWAG
119. PLAGW
120. PLAWG

THE LAB

COUNTING PAIRS

Hi, little monsters.

Looks like you challenged Captain Kraken in tug-o-war.

Yep.

Maybe a game of badminton will lift your spirits.

Are you going to play against all *four* of us?

I only have three racquets...

...so I can only play against two little monsters at a time.

Who wants to play against Professor Grok first?

There must be at least ten different two-monster teams we can make.

Let's count them!

Splendid idea!

How many different 2-monster teams can you make from the four of you little monsters?

How many different 2-monster teams are possible?

I could pair up with Alex, Winnie, or Lizzie.

GrOgg & Alex

That makes three possible teams that I am on.

GrOgg & Winnie

GrOgg & Lizzie

I could team up with Grogg, Winnie, or Lizzie.

Alex & Grogg

That makes three possible teams that I'm on.

Alex & Winnie

Alex & Lizzie

Winnie & Grogg

I could pair up with Grogg, Alex, or Lizzie.

Winnie & Alex

Winnie & Lizzie

And I could pair up with Grogg, Alex, or Winnie.

Lizzie & Grogg

Each of the four of us can pair up with any of the other three. That makes a total of 4×3=12 possible teams.

Lizzie & Alex

Wait. Something is wrong.

Lizzie & Winnie

What's wrong?

I think we overcounted.

If we count Grogg and Alex as a team...

....we shouldn't count the team of Alex and Grogg.

It's the same team!

~~Grogg & Alex~~	~~Grogg & Winnie~~	~~Grogg & Lizzie~~
~~Alex & Grogg~~	Alex & Winnie	Alex & Lizzie
~~Winnie & Grogg~~	~~Winnie & Alex~~	Winnie & Lizzie
~~Lizzie & Grogg~~	~~Lizzie & Alex~~	~~Lizzie & Winnie~~

Every team got counted twice!

So, instead of 4×3=12 possible teams, there are only 4×3÷2=6 possible teams.

I see another way to count the total number of possible teams!

G & A

G & W

G & L

There are 3 monsters I can pair up with to make a team.

Since we already counted the team of me and Alex...

...there are only two monsters left to pair with Alex.

G & A A & W

G & W A & L

G & L

I can team up with Winnie or Lizzie.

Then, since I've already been paired with Grogg and Alex, there is only one monster left for me to pair up with.

G & A
G & W
G & L

A & W
A & L

W & L

Me!

And since I have already been paired with each of you, there are no more possible teams.

That makes a total of 3+2+1+0=6 possible teams.

3 + 2 + 1 + 0 = 6

That's right! So, who would like to play first?

I will.

Me too.

Here comes the first serve!

If you beat me, I'll buy you each an ice cream.

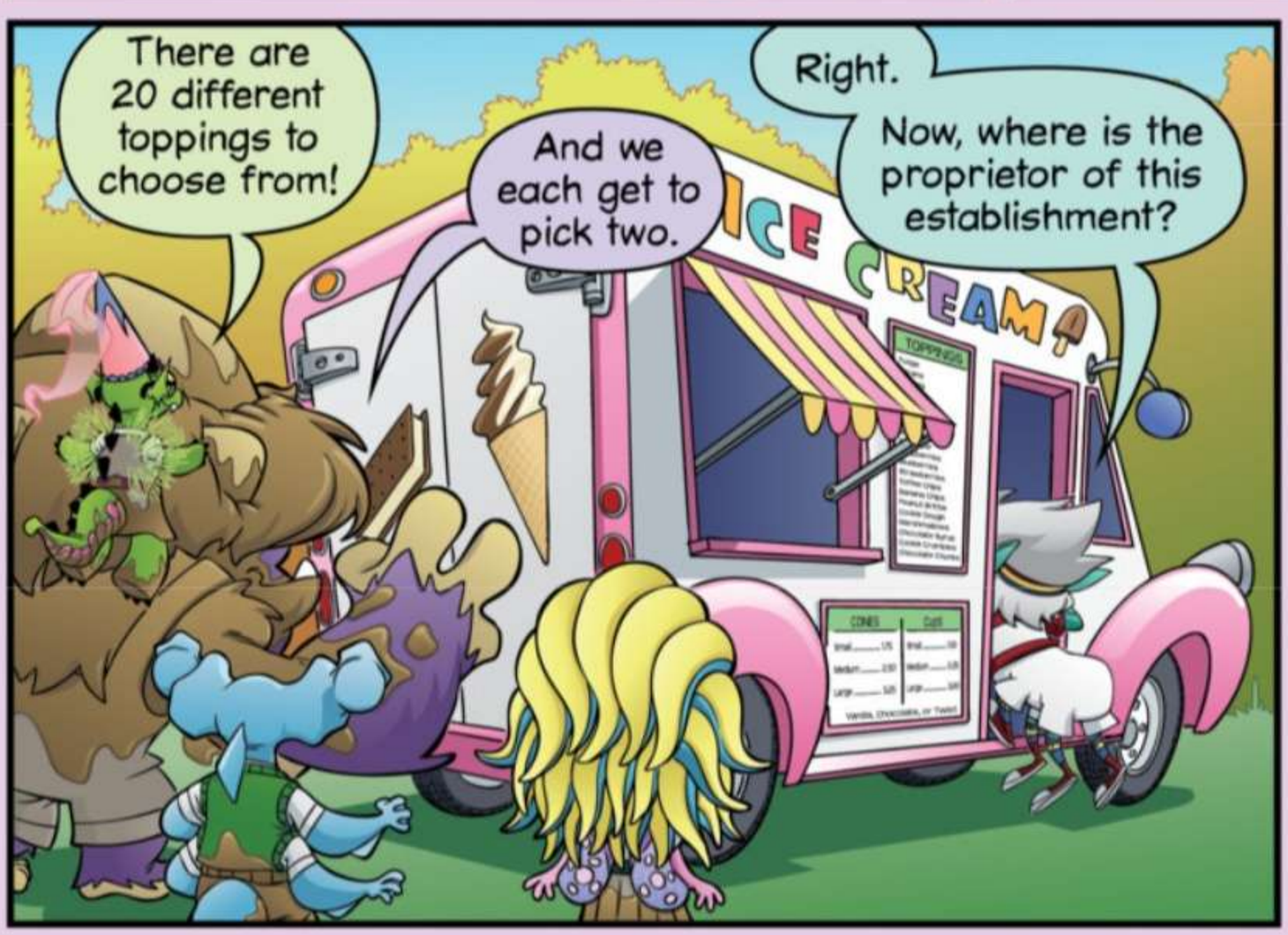

7
9
Grok
Nex & Wendo


15
14
Grok
Nex & Wendo


19
21
Grok
Nex & Wendo


Excellent form, little monsters.
Let's go get some ice cream.


You can each order a cup of vanilla ice cream with two toppings mixed in.
ICE CREAM
TOPPINGS
CONES
CUPS


There are 20 different toppings to choose from!
And we each get to pick two.
Right.
Now, where is the proprietor of this establishment?
ICE CREAM


KA-FLOMP!
SKWEEECH!
BWORMP

Bwah Ha Ha! Professor Grok is gone. I've abducted your educator. It's time for something much more diabolically difficult!

Tallying two-monster teams formed from a flock of four is lamentably elementary.

But identifying the number of tasty two-topping treats given a total of twenty toppings is a terrifically troubling task.

Professor Grok will remain captured in my cooler until you have counted every candied confection concoction.

The correct number is the code to the ice cream cooler.

Solve swiftly, or your shivering schoolteacher will surely become...

...Professor Popsicle!

There are twenty toppings!

How are we ever going to count every possible combination of two toppings?!

Maybe we can count the pairs of toppings the same way we counted the pairs of monsters that could make a badminton team.

TOPPINGS

Fudge
Pecans
Granola
Caramel
Cherries
Coconut
Almonds
Sprinkles
Pineapple
Raspberries
Blueberries
Strawberries
Toffee Chips
Banana Chips
Peanut Brittle
Cookie Dough
Marshmallows
Chocolate Syrup
Cookie Crumbles
Chocolate Chunks

Cups

Small 1.50
Medium 2.25
Large ..... 3.

Vanilla, Chocolate, or Twist

How many pairs of two different toppings are possible?

Starting with fudge...

...we can combine fudge with any of the other 19 toppings.

That gives us 19 different options.

TOPPINGS

Fudge
Pecans
Granola
Caramel
Cherries
Coconut
Almonds
Sprinkles
Pineapple
Raspberries
Blueberries
Strawberries
Toffee Chips
Banana Chips
Peanut Brittle
Cookie Dough
Marshmallows
Chocolate Syrup
Cookie Crumbles
Chocolate Chunks

Then, since pecans have already been paired with fudge...

...there are only 18 toppings left to pair with pecans.

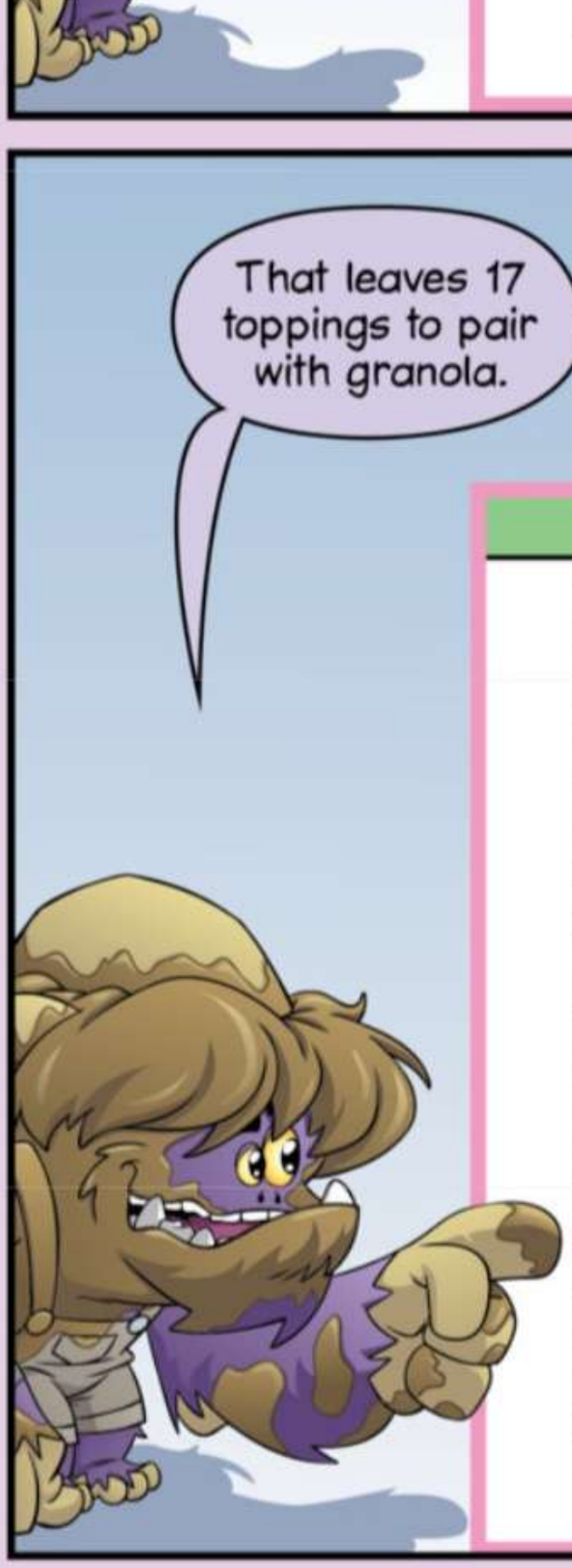
TOPPINGS

Fudge
Pecans
Granola
Caramel
Cherries
Coconut
Almonds
Sprinkles
Pineapple
Raspberries
Blueberries
Strawberries
Toffee Chips
Banana Chips
Peanut Brittle
Cookie Dough
Marshmallows
Chocolate Syrup
Cookie Crumbles
Chocolate Chunks

That leaves 17 toppings to pair with granola.

TOPPINGS

Fudge
Pecans
Granola
Caramel
Cherries
Coconut
Almonds
Sprinkles
Pineapple
Raspberries
Blueberries
Strawberries
Toffee Chips
Banana Chips
Peanut Brittle
Cookie Dough
Marshmallows
Chocolate Syrup
Cookie Crumbles
Chocolate Chunks



If we pair each of the toppings with every topping below it, that will give us every possible pair of toppings.

We get 19+18+17+16+15+14+13+12+11+10+9+8+7+6+5+4+3+2+1+0 pairs of toppings.

TOPPINGS

Fudge 19
Pecans +18
Granola +17
Caramel +16
Cherries +15
Coconut +14
Almonds +13
Sprinkles +12
Pineapple +11
Raspberries +10
Blueberries +9
Strawberries +8
Toffee Chips +7
Banana Chips +6
Peanut Brittle +5
Cookie Dough +4
Marshmallows +3
Chocolate Syrup +2
Cookie Crumbles +1
Chocolate Chunks +0

Let's start adding!

Wait! I know another way.

If we pair each of the 20 toppings with ***every*** one of the other 19 toppings...

...that gives us 20×19=380 possible pairs of toppings, right?

I see. When you pick two different toppings, there are 20 choices for the first topping.

And after you pick the first topping, there are 19 choices for the second topping.

TOPPINGS

Fudge
Pecans
Granola
Caramel
Cherries
Coconut
Almonds
Sprinkles
Pineapple
Raspberries
Blueberries
Strawberries
Toffee Chips
Banana Chips
Peanut Brittle
Cookie Dough
Marshmallows
Chocolate Syrup
Cookie Crumbles
Chocolate Chunks

Right, but when we pair each topping with every other topping, we get every pair of toppings twice.

CONES

Small ...
2.50
Medium
25
Large ...

For example, pairing coconut with sprinkles...

...is the same as pairing sprinkles with coconut.

Exactly! Multiplying 20×19 counts each pair of toppings twice.

What could you do to make sure each pair is only counted once?

So, to get the actual number of pairs...

...we divide 20×19 by 2!

20×19=380, and 380÷2=190.

That's the same answer I got by adding all of the numbers from 1 to 19!

Quick, open the lock!

KLIK

Great work! Now, who's ready for ice cream?

Vanilla