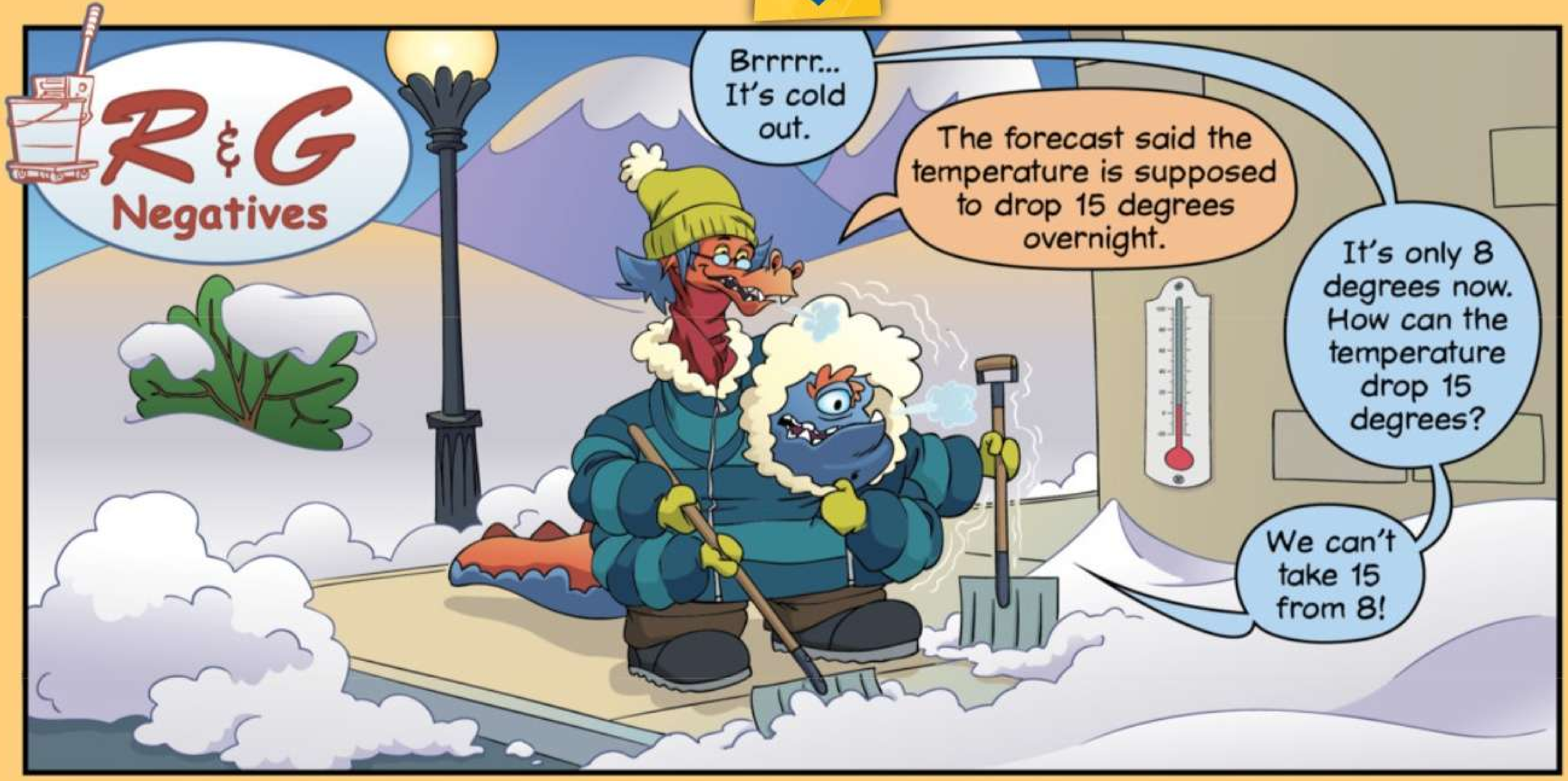
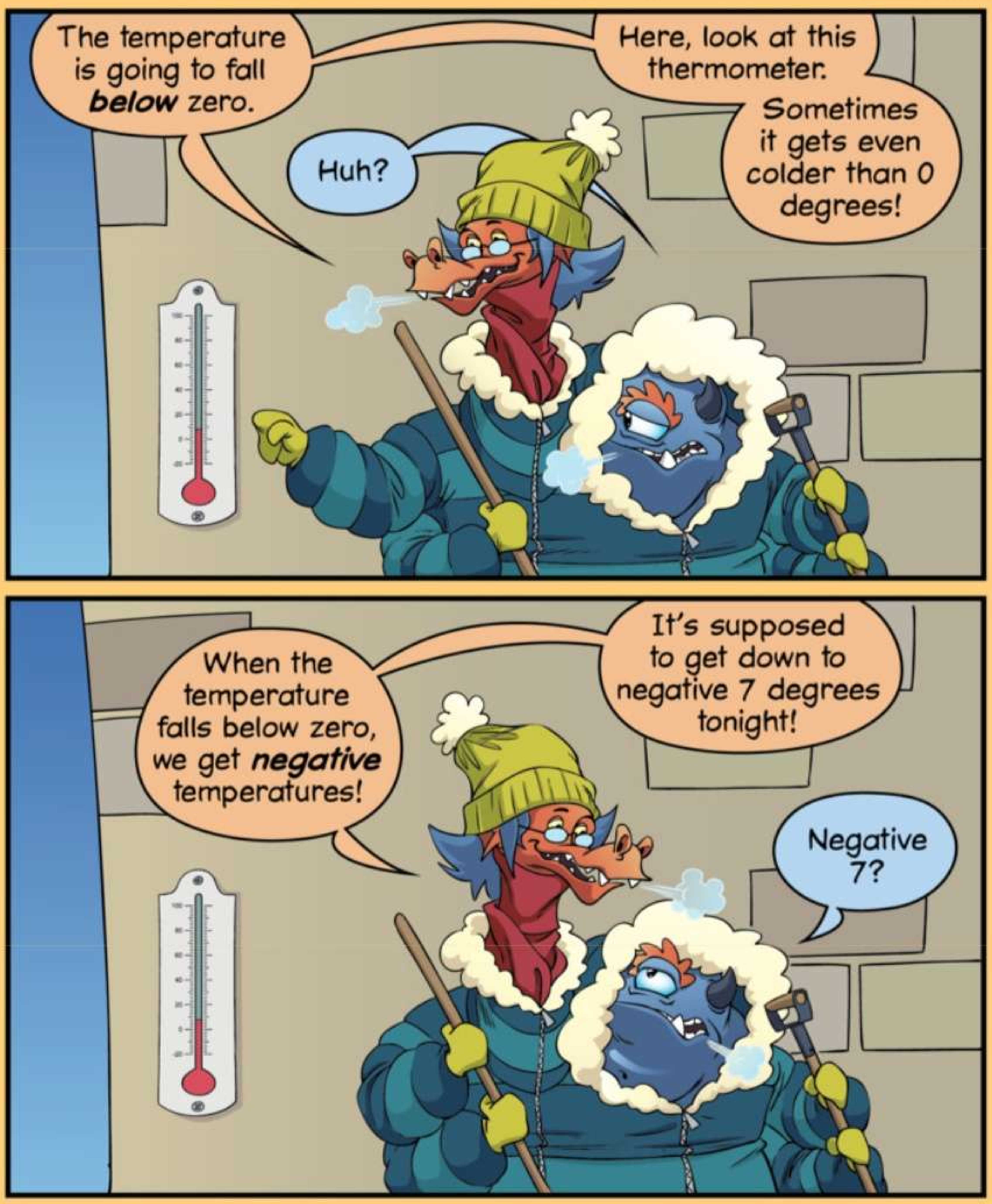
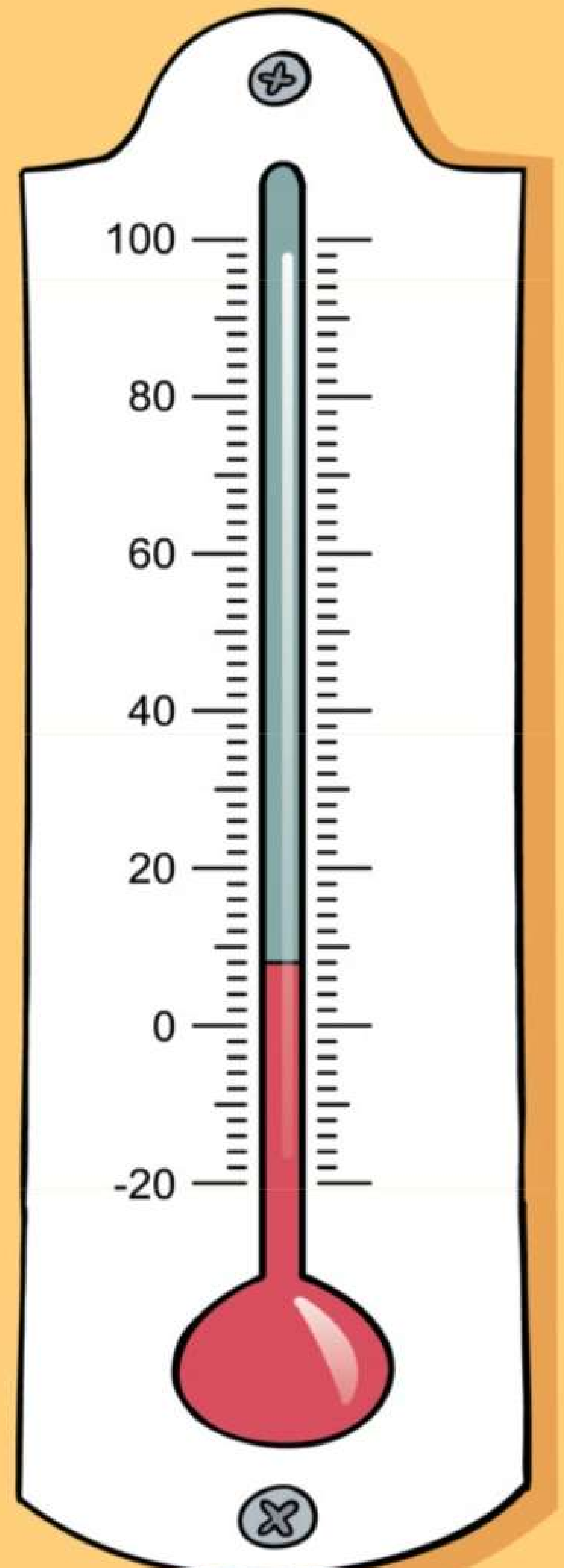
Contents: Chapter 9

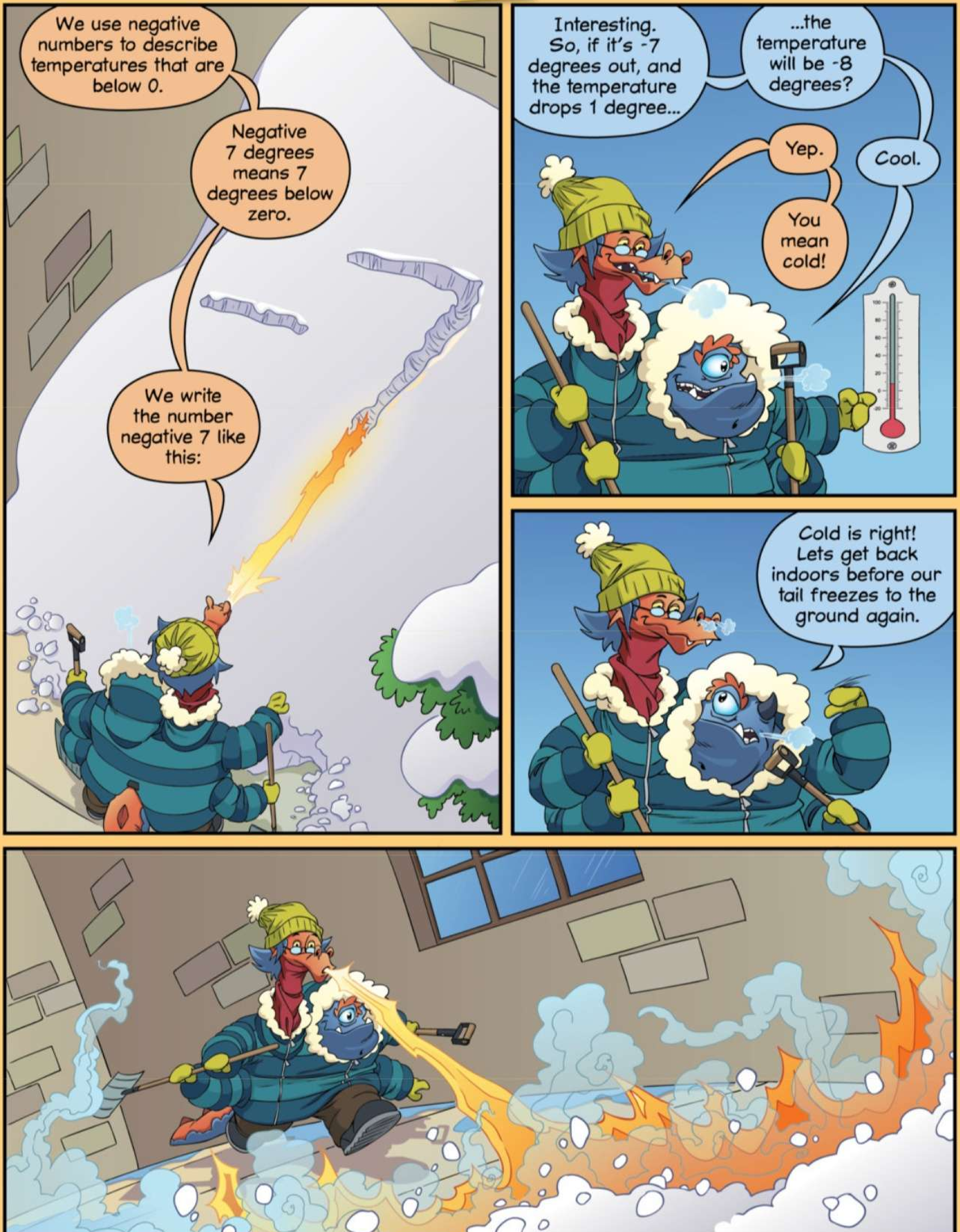
Click the Play List tab in the top-left to view a recommended reading/practice sequence.

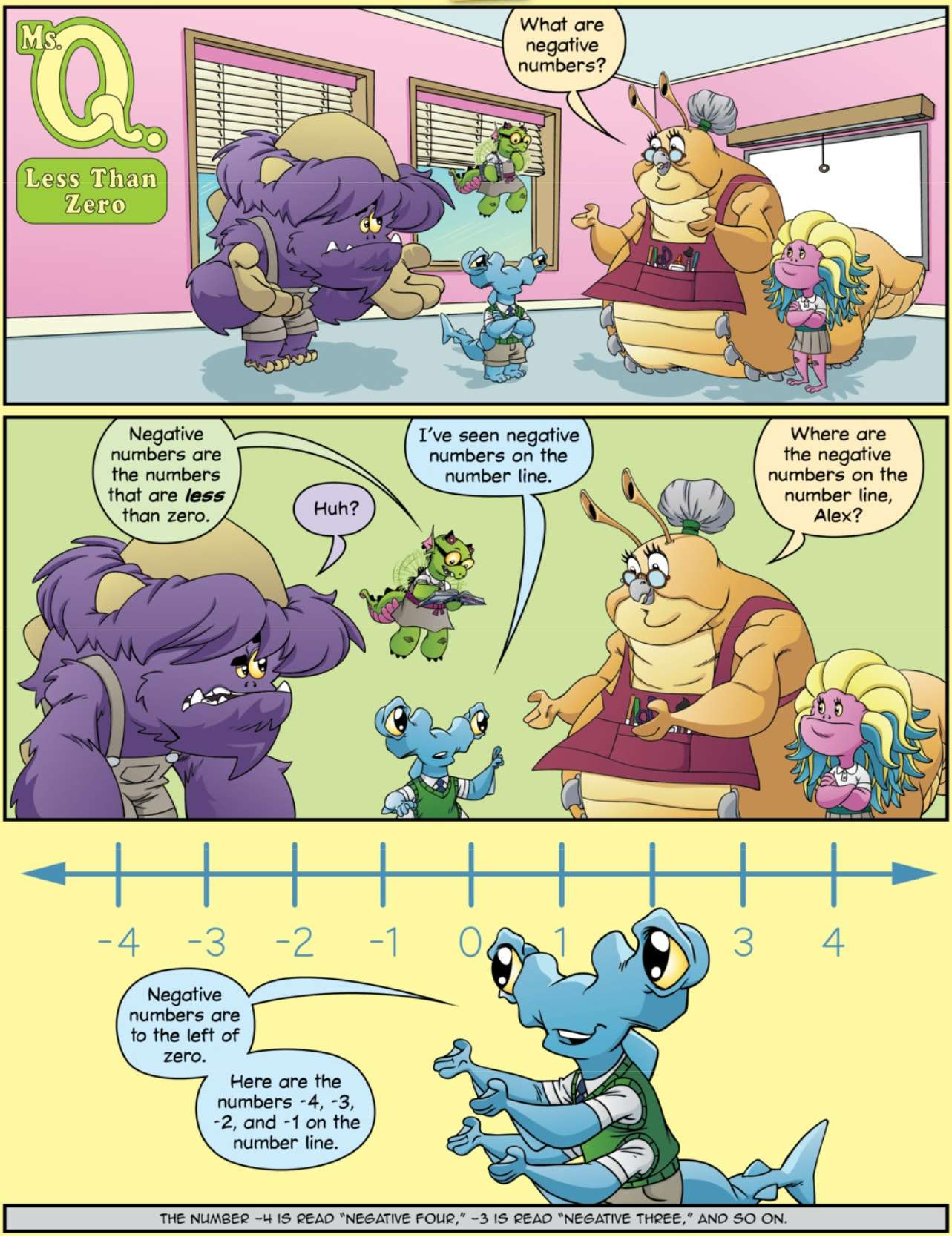
Chapter 9:
Integers

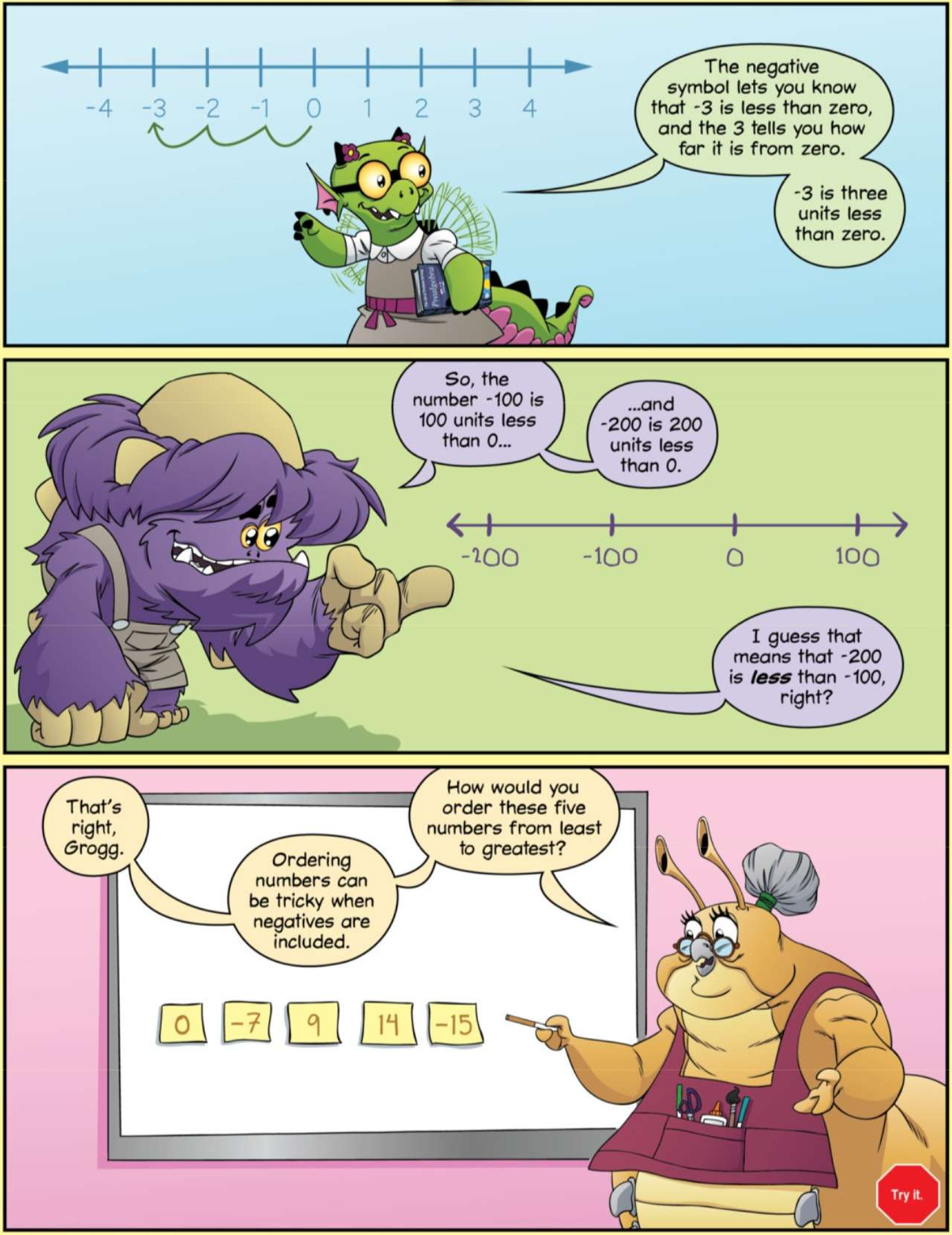

R&G
Negatives
Brrrrr... It's cold out.
The forecast said the temperature is supposed to drop 15 degrees overnight.
It's only 8 degrees now. How can the temperature drop 15 degrees?
We can't take 15 from 8!

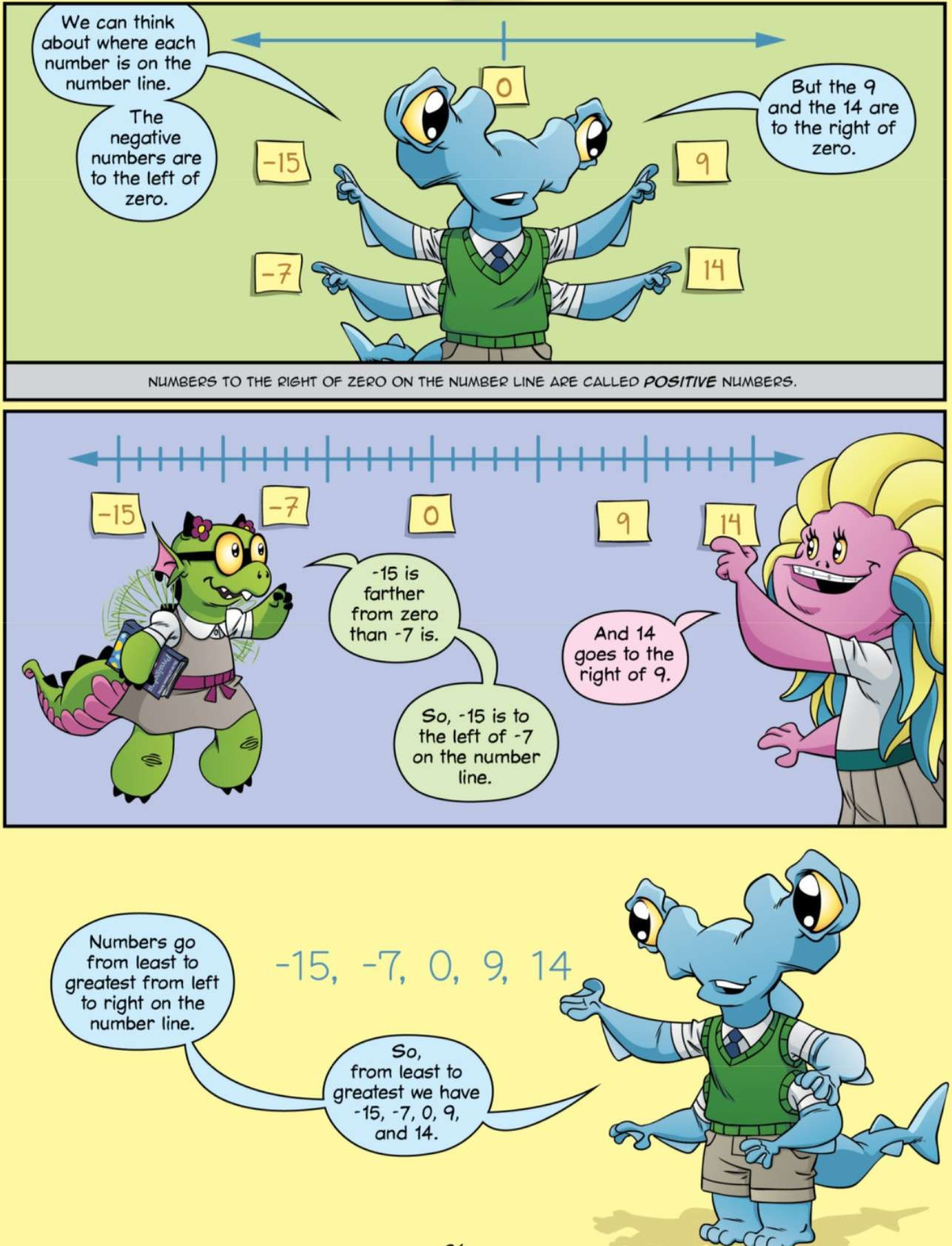

The temperature is going to fall ***below*** zero.
Huh?
Here, look at this thermometer.
Sometimes it gets even colder than 0 degrees!
When the temperature falls below zero, we get ***negative*** temperatures!
It's supposed to get down to negative 7 degrees tonight!
Negative 7?

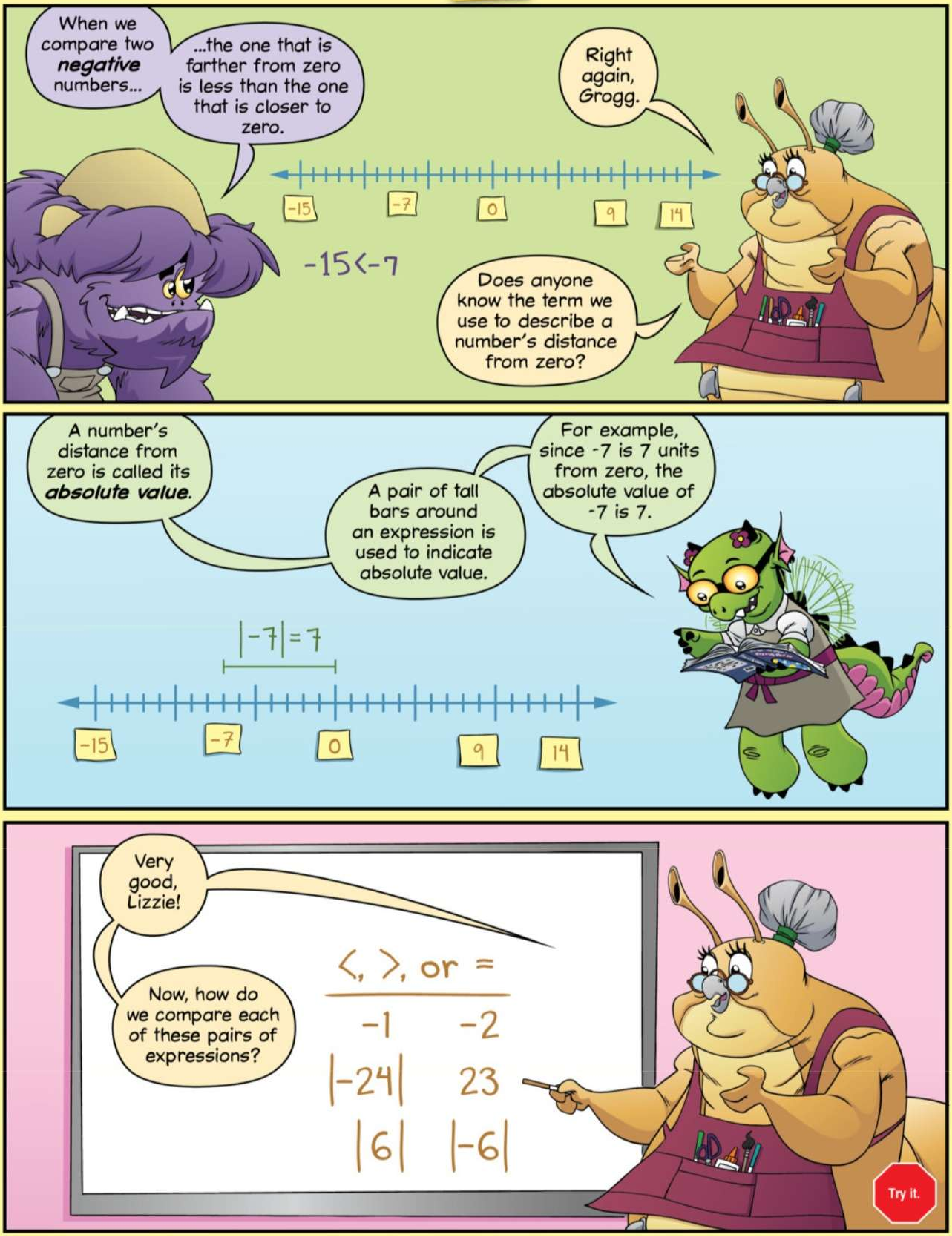

100
80
60
40
20
0
-20


We use negative numbers to describe temperatures that are below 0.
Negative 7 degrees means 7 degrees below zero.
We write the number negative 7 like this:
Interesting. So, if it's -7 degrees out, and the temperature drops 1 degree...
...the temperature will be -8 degrees?
Yep.
You mean cold!
Cool.
Cold is right! Lets get back indoors before our tail freezes to the ground again.


Ms. Q.
Less Than Zero
What are negative numbers?
Negative numbers are the numbers that are *less* than zero.
Huh?
I've seen negative numbers on the number line.
Where are the negative numbers on the number line, Alex?
-4 -3 -2 -1 0 1 3 4
Negative numbers are to the left of zero.
Here are the numbers -4, -3, -2, and -1 on the number line.
THE NUMBER –4 IS READ "NEGATIVE FOUR," –3 IS READ "NEGATIVE THREE," AND SO ON.

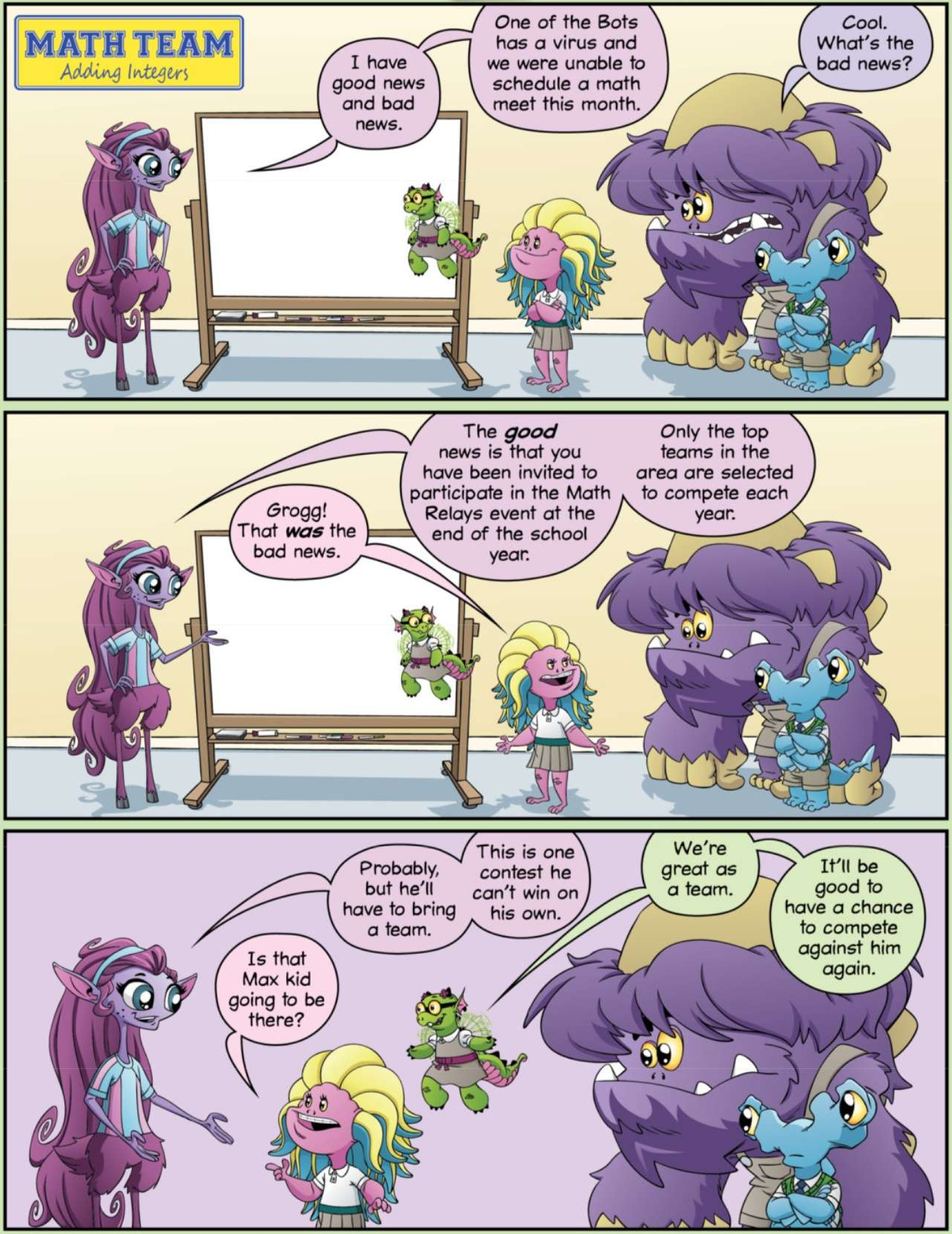

-4 -3 -2 -1 0 1 2 3 4
The negative symbol lets you know that -3 is less than zero, and the 3 tells you how far it is from zero.
-3 is three units less than zero.
So, the number -100 is 100 units less than 0...
...and -200 is 200 units less than 0.
-200 -100 0 100
I guess that means that -200 is ***less*** than -100, right?
That's right, Grogg.
Ordering numbers can be tricky when negatives are included.
How would you order these five numbers from least to greatest?
0 -7 9 14 -15
Try it.


We can think about where each number is on the number line.
The negative numbers are to the left of zero.
But the 9 and the 14 are to the right of zero.
0
-15
9
-7
14
NUMBERS TO THE RIGHT OF ZERO ON THE NUMBER LINE ARE CALLED *POSITIVE* NUMBERS.
-15
-7
0
9
14
-15 is farther from zero than -7 is.
So, -15 is to the left of -7 on the number line.
And 14 goes to the right of 9.
Numbers go from least to greatest from left to right on the number line.
-15, -7, 0, 9, 14
So, from least to greatest we have -15, -7, 0, 9, and 14.

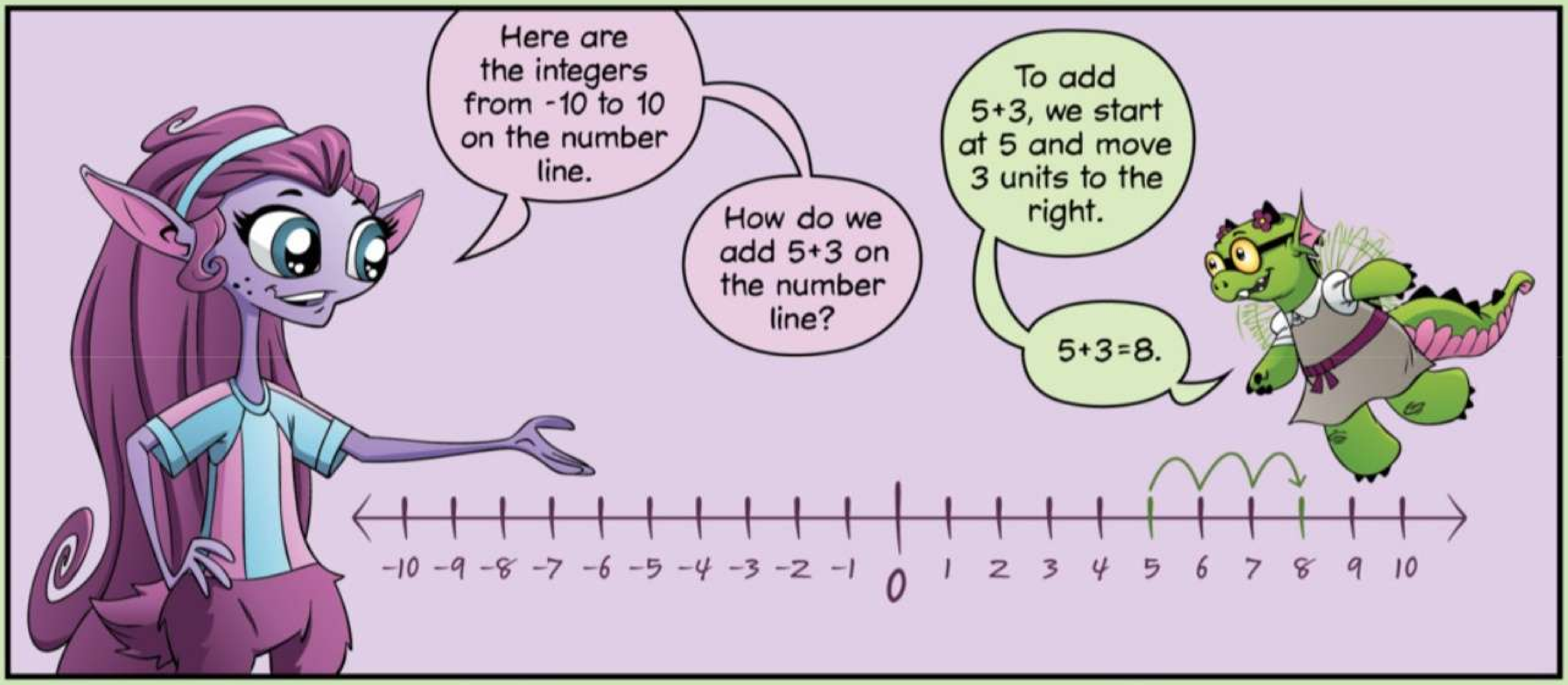


When we compare two ***negative*** numbers...
...the one that is farther from zero is less than the one that is closer to zero.
Right again, Grogg.
-15
-7
0
9
14
-15<-7
Does anyone know the term we use to describe a number's distance from zero?
A number's distance from zero is called its ***absolute value***.
A pair of tall bars around an expression is used to indicate absolute value.
For example, since -7 is 7 units from zero, the absolute value of -7 is 7.
|-7|=7
-15
-7
0
9
14
Very good, Lizzie!
Now, how do we compare each of these pairs of expressions?
<, >, or =
-1 -2
|-24| 23
|6| |-6|
Try it.


Since -1 is to the right of -2 on the number line, -1 is greater than -2.
-1 > -2
-2 -1 0 1
Since -24 is 24 units from zero, its absolute value is 24.
24 is greater than 23.
So, the absolute value of -24 is greater than 23.
|-24| = 24
|-24| > 23
6 and -6 are both 6 units from zero, but on opposite sides of zero.
So, the absolute values of 6 and -6 are equal.
6 = 6
|6| = |-6|
6 and -6 are called ***opposites***.
What a coincidence. Today is Opposite Day!
Don't you mean today is ***not*** Opposite Day?
TWO INTEGERS ARE OPPOSITES IF THEY ARE THE SAME DISTANCE FROM ZERO, BUT ON OPPOSITE SIDES OF ZERO.
What happened to Grogg?
Ms. Q. broke his brain again.

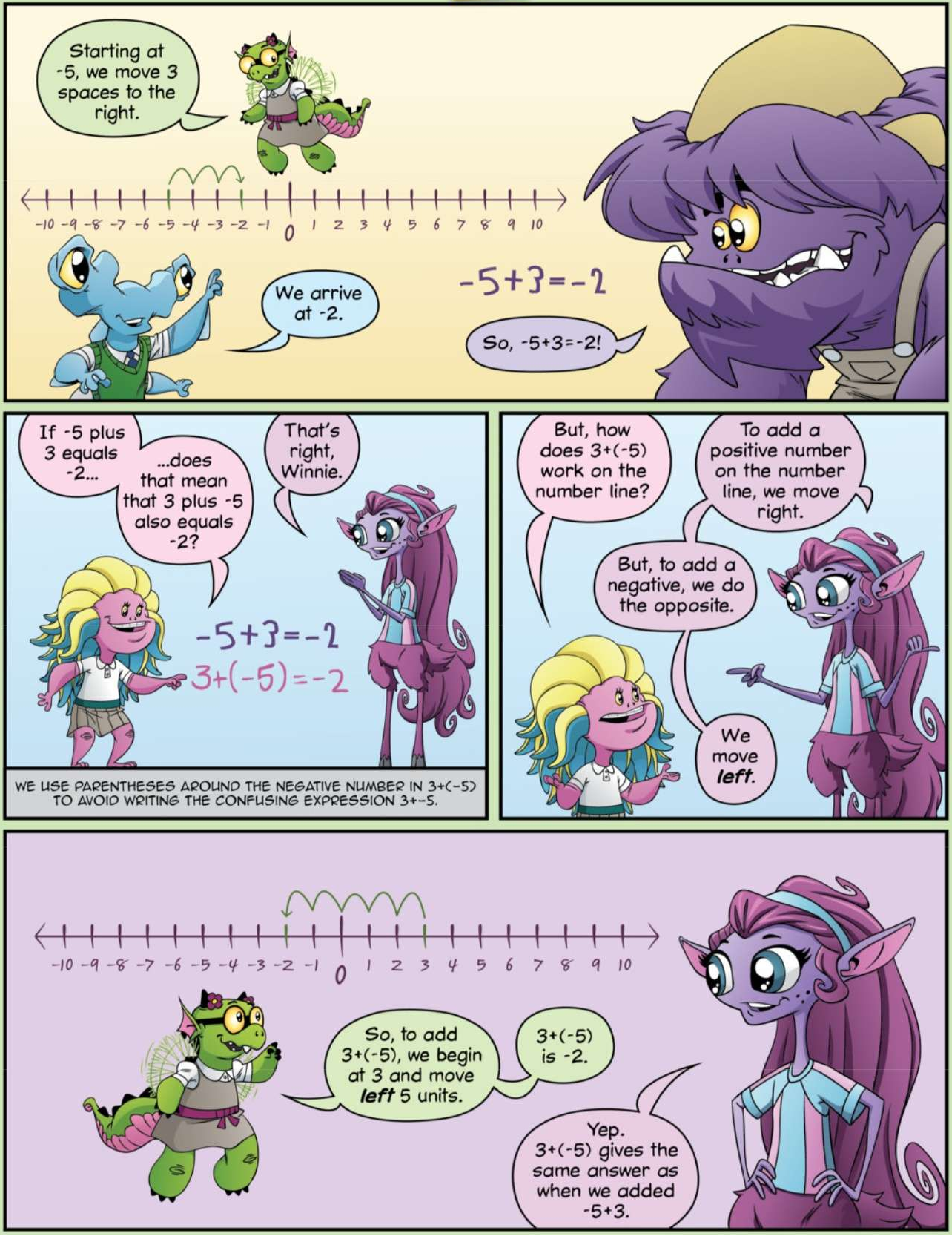

MATH TEAM
Adding Integers
I have good news and bad news.
One of the Bots has a virus and we were unable to schedule a math meet this month.
Cool. What's the bad news?
Grogg! That *was* the bad news.
The *good* news is that you have been invited to participate in the Math Relays event at the end of the school year.
Only the top teams in the area are selected to compete each year.
Is that Max kid going to be there?
Probably, but he'll have to bring a team.
This is one contest he can't win on his own.
We're great as a team.
It'll be good to have a chance to compete against him again.


Enough about competitions, let's learn how to add integers.
Integers?
A number that does not have a fractional part is called an ***integer***.

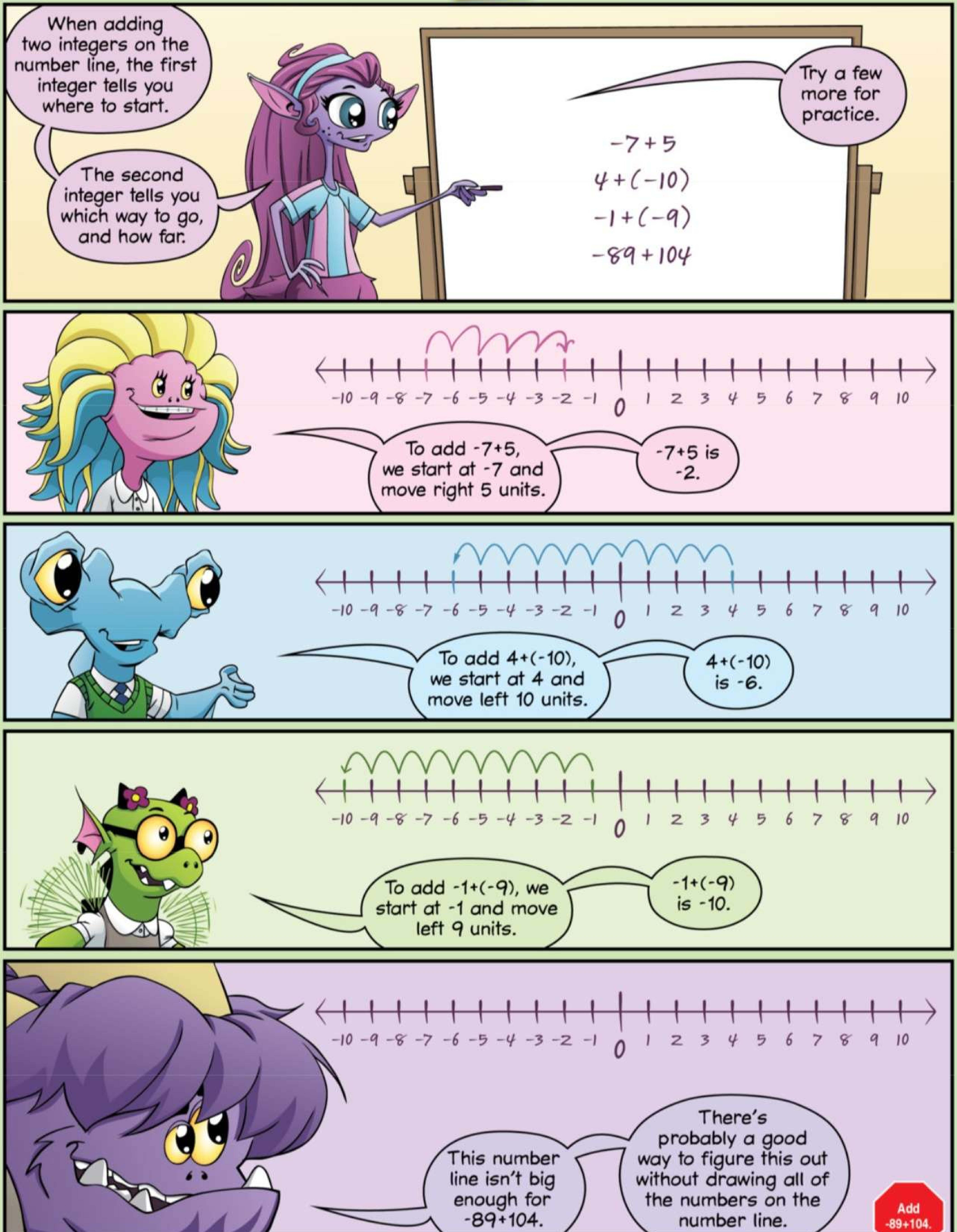

Here are the integers from -10 to 10 on the number line.
How do we add 5+3 on the number line?
To add 5+3, we start at 5 and move 3 units to the right.
5+3=8.
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10


To add 3 to any number on the number line, we start at the number and move 3 spaces to the right.
Exactly.
So, what do we get when we add -5+3?

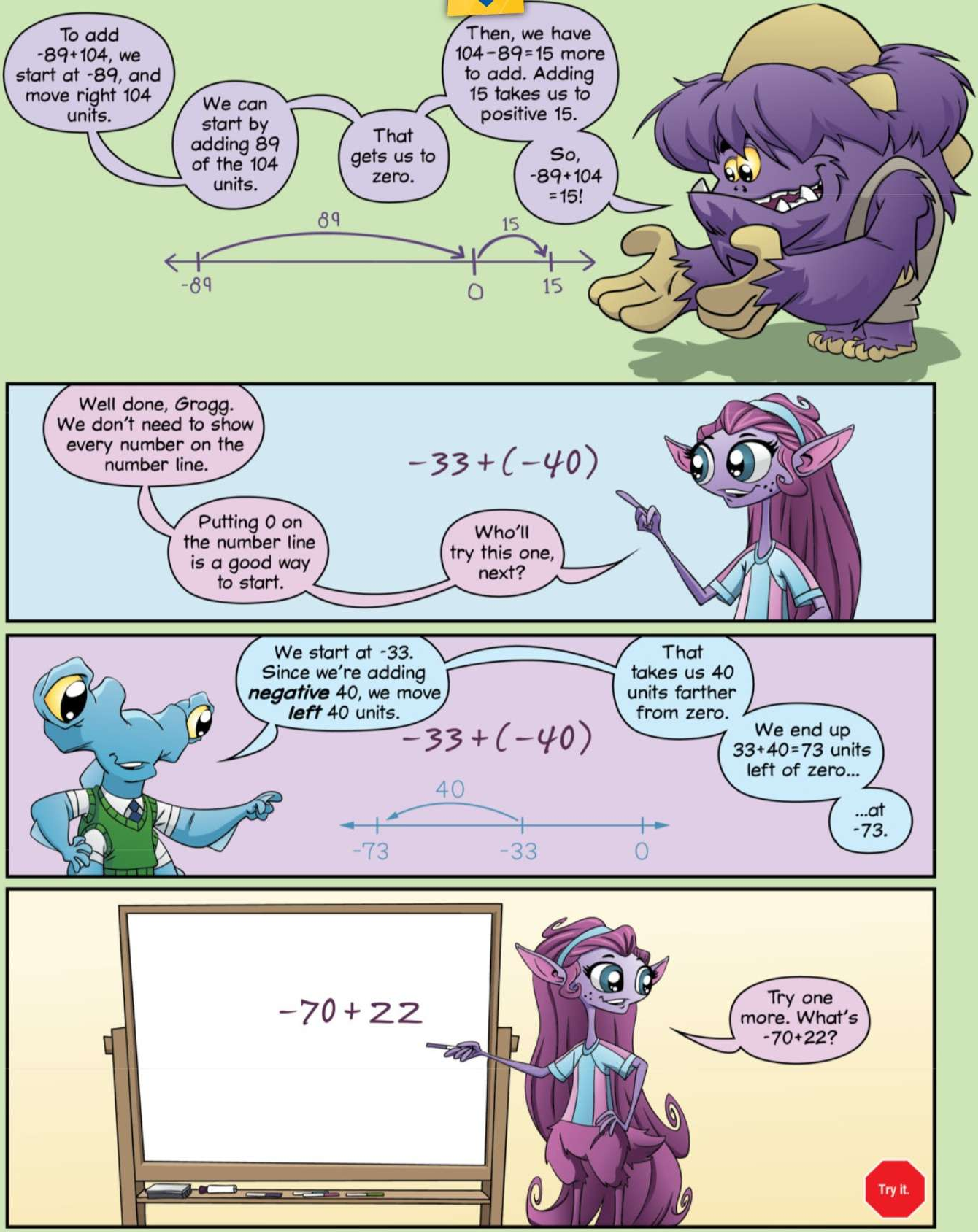
Try it.



Starting at -5, we move 3 spaces to the right.
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
We arrive at -2.
-5+3=-2
So, -5+3=-2!
If -5 plus 3 equals -2...
...does that mean that 3 plus -5 also equals -2?
That's right, Winnie.
-5+3=-2
3+(-5)=-2
WE USE PARENTHESES AROUND THE NEGATIVE NUMBER IN 3+(-5) TO AVOID WRITING THE CONFUSING EXPRESSION 3+-5.
But, how does 3+(-5) work on the number line?
To add a positive number on the number line, we move right.
But, to add a negative, we do the opposite.
We move *left*.
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
So, to add 3+(-5), we begin at 3 and move *left* 5 units.
3+(-5) is -2.
Yep. 3+(-5) gives the same answer as when we added -5+3.



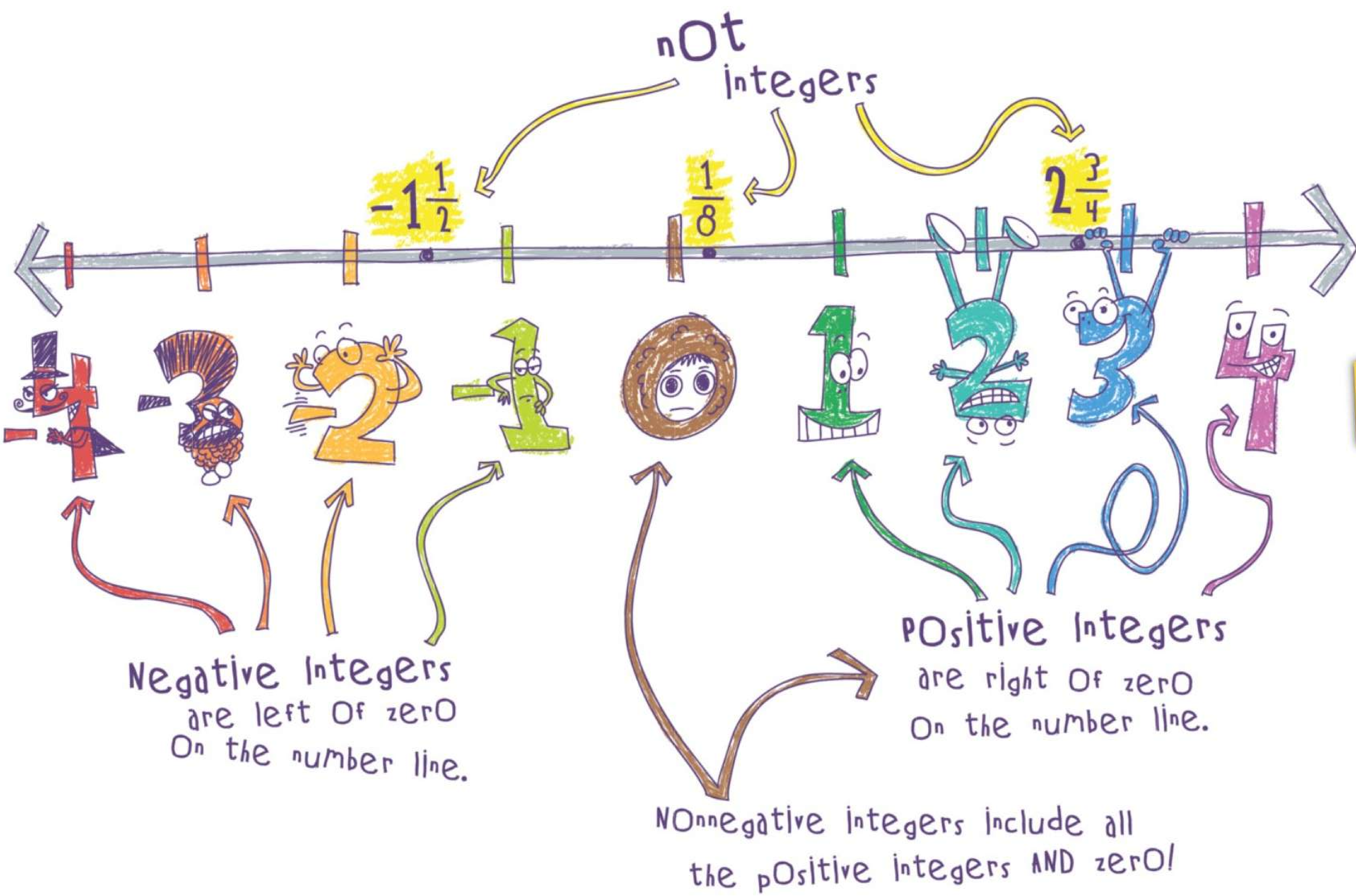
When adding two integers on the number line, the first integer tells you where to start.
The second integer tells you which way to go, and how far.
Try a few more for practice.
-7+5
4+(-10)
-1+(-9)
-89+104
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
To add -7+5, we start at -7 and move right 5 units.
-7+5 is -2.
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
To add 4+(-10), we start at 4 and move left 10 units.
4+(-10) is -6.
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
To add -1+(-9), we start at -1 and move left 9 units.
-1+(-9) is -10.
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
This number line isn't big enough for -89+104.
There's probably a good way to figure this out without drawing all of the numbers on the number line.
Add -89+104.



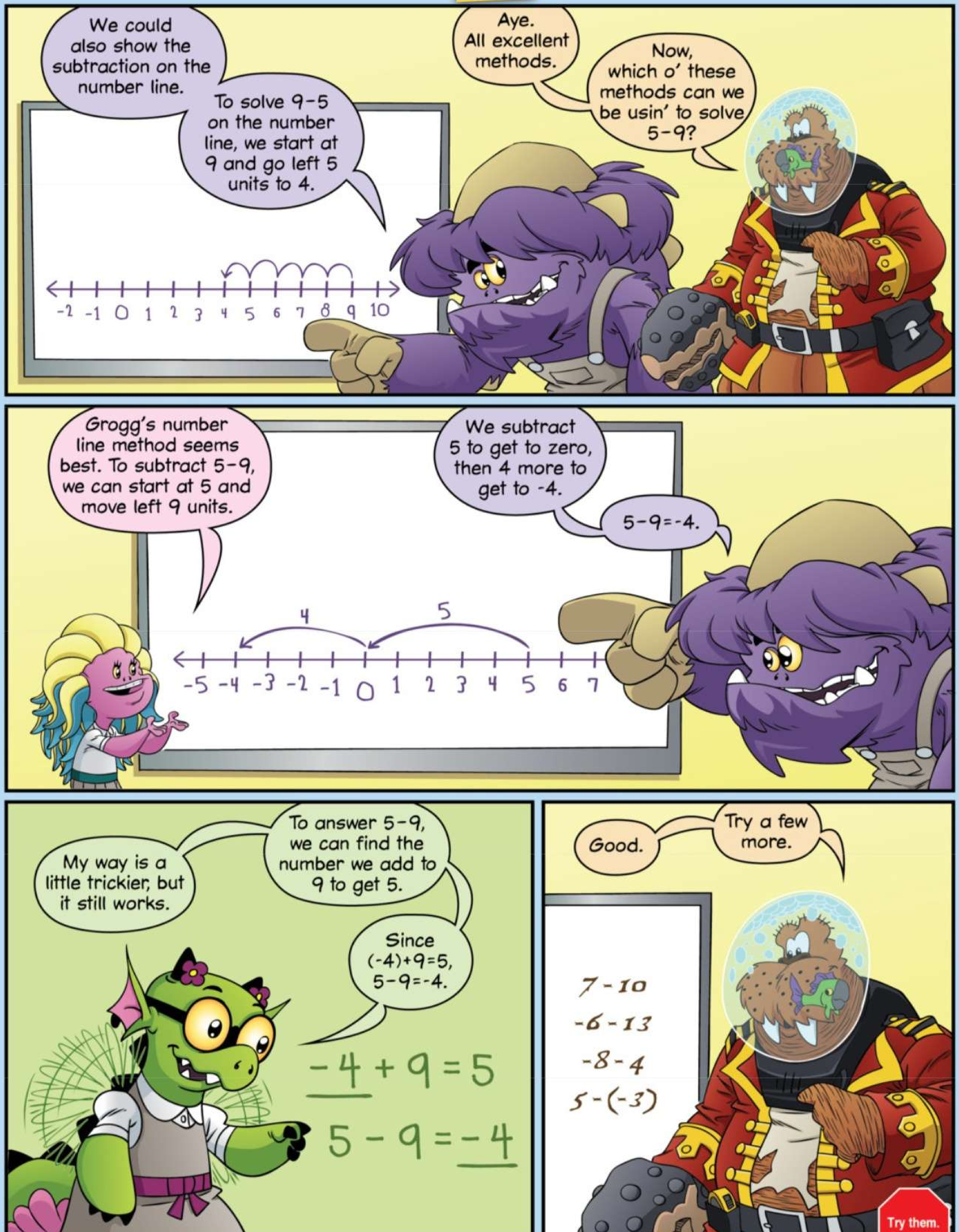
To add -89+104, we start at -89, and move right 104 units.
We can start by adding 89 of the 104 units.
That gets us to zero.
Then, we have 104−89=15 more to add. Adding 15 takes us to positive 15.
So, -89+104 =15!
89
15
-89
0
15
Well done, Grogg. We don't need to show every number on the number line.
Putting 0 on the number line is a good way to start.
Who'll try this one, next?
−33+(−40)
We start at -33. Since we're adding **negative** 40, we move **left** 40 units.
That takes us 40 units farther from zero.
We end up 33+40=73 units left of zero...
...at -73.
−33+(−40)
40
-73
-33
0
−70+22
Try one more. What's -70+22?
Try it.

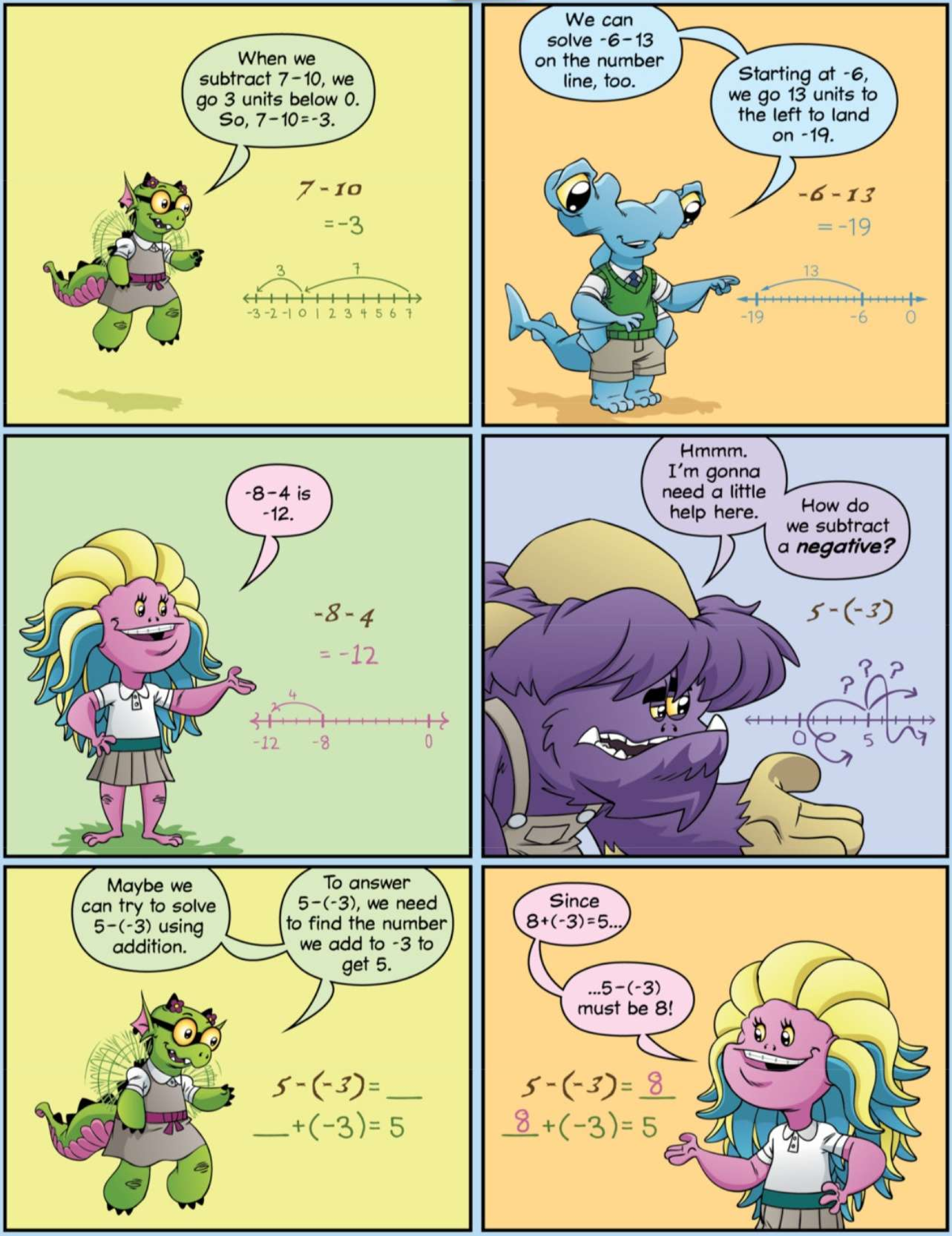
We start at -70 and move right 22 units.
That takes us 22 units closer to zero.
We end up 70−22=48 units left of zero...
...at -48.
−70 + 22
22
-70
-48
0
Adding integers can be pretty tough.
Are there rules we can memorize?
You could come up with some on your own...
...but you probably shouldn't spend time trying to memorize rules.
A little practice using the number line will help make adding integers natural.
Don't memorize rules? Try telling that to my mom!
She made me memorize a whole list of them.
Your shopping cart stays outside.
Don't make funny faces at grandma.
Stay out of my makeup drawer.
If you want to dig a hole, do it outside.
Keep all four shopping cart wheels on the ground.
"Ewoks made me do it" is not an excuse.
All activity preceded by "Hey mom, watch this!" is prohibited.
Replace the roll!
Put it back where you got it.
Dragons did not burn your homework.
Chocolate mousse doesn't go in your hair.

Integer: A number without a fractional part

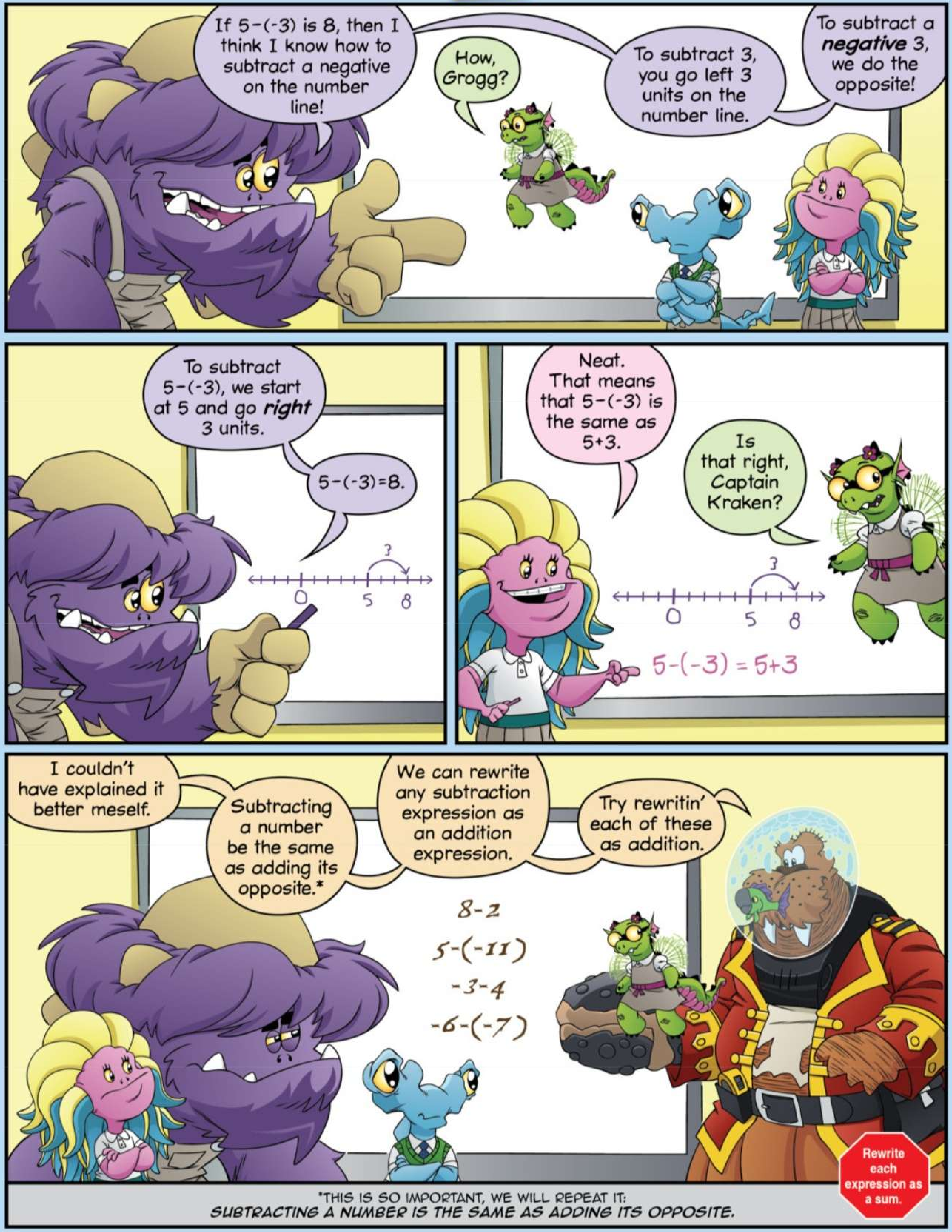


SUBTRACTING INTEGERS
Captain Kraken?
Aye, lad.
How do we subtract 5−9?
Arrr... Let's try some examples 'n' see if we can figure it out.
Start with one ye' already know. What's 9−5?
That's easy. If I start with 9 and I take away 5, I have 4 left over. So, 9−5=4.
Be there other ways to think about subtraction?
We can use addition to solve a subtraction problem.
To answer 9−5, we can find the number we add to 5 to get 9.
Since 4+5=9, 9−5=4.
4 + 5 = 9
9 − 5 = 4


We could also show the subtraction on the number line.
To solve 9−5 on the number line, we start at 9 and go left 5 units to 4.
-2 -1 0 1 2 3 4 5 6 7 8 9 10
Aye. All excellent methods.
Now, which o' these methods can we be usin' to solve 5−9?
Grogg's number line method seems best. To subtract 5−9, we can start at 5 and move left 9 units.
We subtract 5 to get to zero, then 4 more to get to -4.
5−9=-4.
4
5
-5 -4 -3 -2 -1 0 1 2 3 4 5 6 7
My way is a little trickier, but it still works.
To answer 5−9, we can find the number we add to 9 to get 5.
Since (-4)+9=5, 5−9=-4.
-4+9=5
5-9=-4
Good.
Try a few more.
7-10
-6-13
-8-4
5-(-3)
Try them.



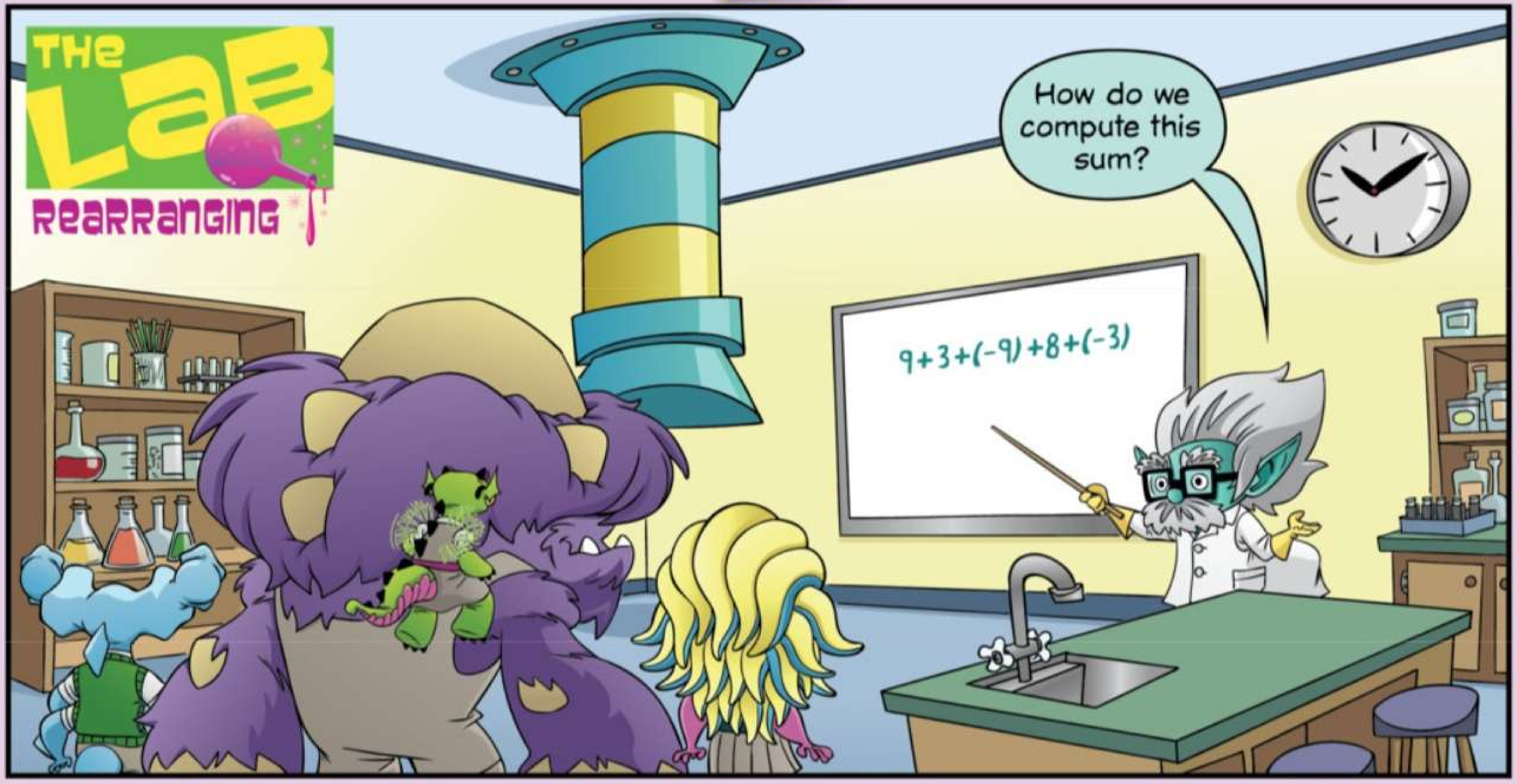
When we subtract 7−10, we go 3 units below 0. So, 7−10=-3.
7 - 10
= -3
3
7
-3 -2 -1 0 1 2 3 4 5 6 7
We can solve -6−13 on the number line, too.
Starting at -6, we go 13 units to the left to land on -19.
-6 - 13
= -19
13
-19
-6
0
-8−4 is -12.
-8 - 4
= -12
4
-12
-8
0
Hmmm. I'm gonna need a little help here.
How do we subtract a ***negative?***
5 - (-3)
? ? ? ?
0
5
Maybe we can try to solve 5−(-3) using addition.
To answer 5−(-3), we need to find the number we add to -3 to get 5.
5 - (-3) = __
__ + (-3) = 5
Since 8+(-3)=5...
...5−(-3) must be 8!
5 - (-3) = 8
8 + (-3) = 5

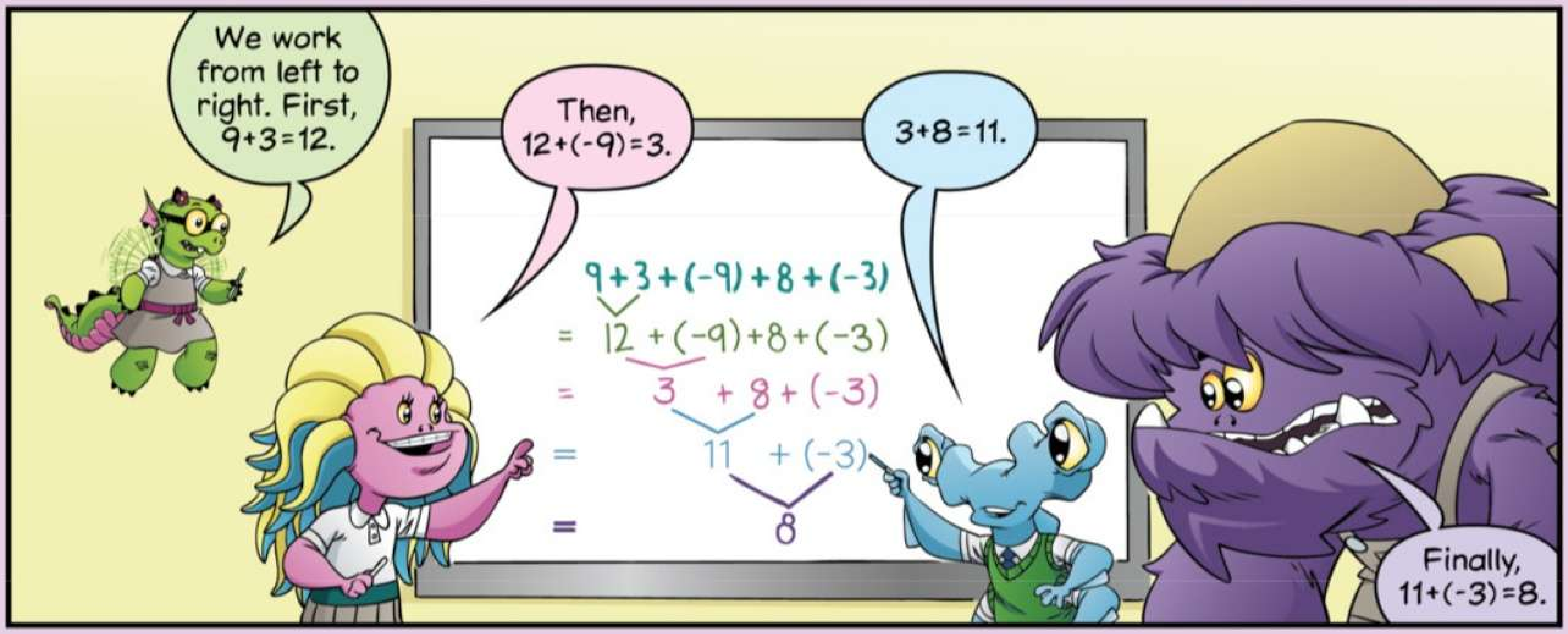


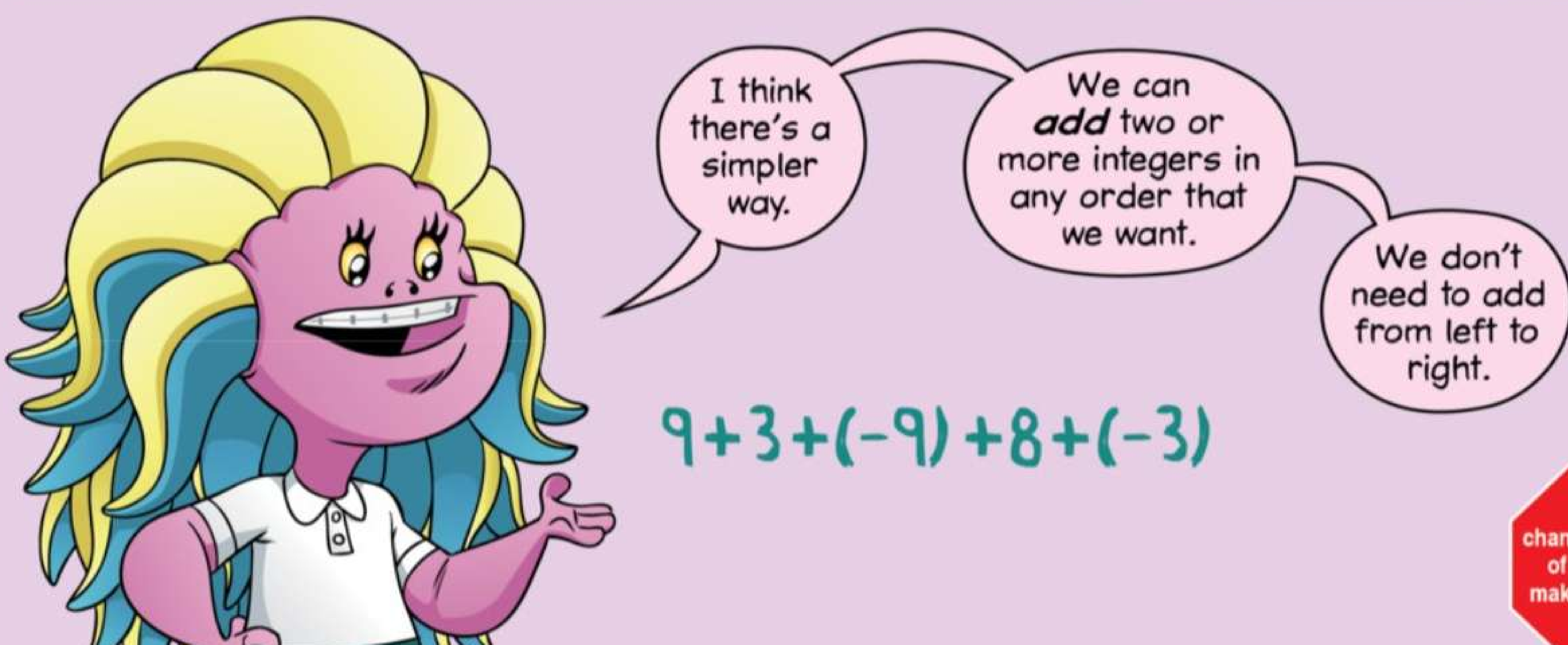
If 5−(-3) is 8, then I think I know how to subtract a negative on the number line!
How, Grogg?
To subtract 3, you go left 3 units on the number line.
To subtract a ***negative*** 3, we do the opposite!
To subtract 5−(-3), we start at 5 and go ***right*** 3 units.
5−(-3)=8.
3
0
5
8
Neat. That means that 5−(-3) is the same as 5+3.
Is that right, Captain Kraken?
3
0
5
8
5-(-3) = 5+3
I couldn't have explained it better meself.
Subtracting a number be the same as adding its opposite.*
We can rewrite any subtraction expression as an addition expression.
Try rewritin' each of these as addition.
8-2
5-(-11)
-3-4
-6-(-7)
Rewrite each expression as a sum.
*THIS IS SO IMPORTANT, WE WILL REPEAT IT:
SUBTRACTING A NUMBER IS THE SAME AS ADDING ITS OPPOSITE.

To subtract positive 2, we can add negative 2.
8-2
=8+(-2)
To subtract negative 11, we can add positive 11.
5-(-11)
=5+11
To subtract positive 4, we can add negative 4.
-3-4
=-3+(-4)
And to subtract negative 7, we can add positive 7.
-6-(-7)
=-6+7
Beautiful work.
Now, compute the value of each expression usin' what you've learned about addition.
8-2
=8+(-2)
=6
5-(-11)
=5+11
=16
-3-4
=-3+(-4)
=-7
-6-(-7)
=-6+7
=1
5+11 makes a lot more sense than 5-(-11).
5-(-11)
=5+11
=16
But to compute 8-2, it's better to just subtract.
8-2
~~=8+(-2)~~
=6

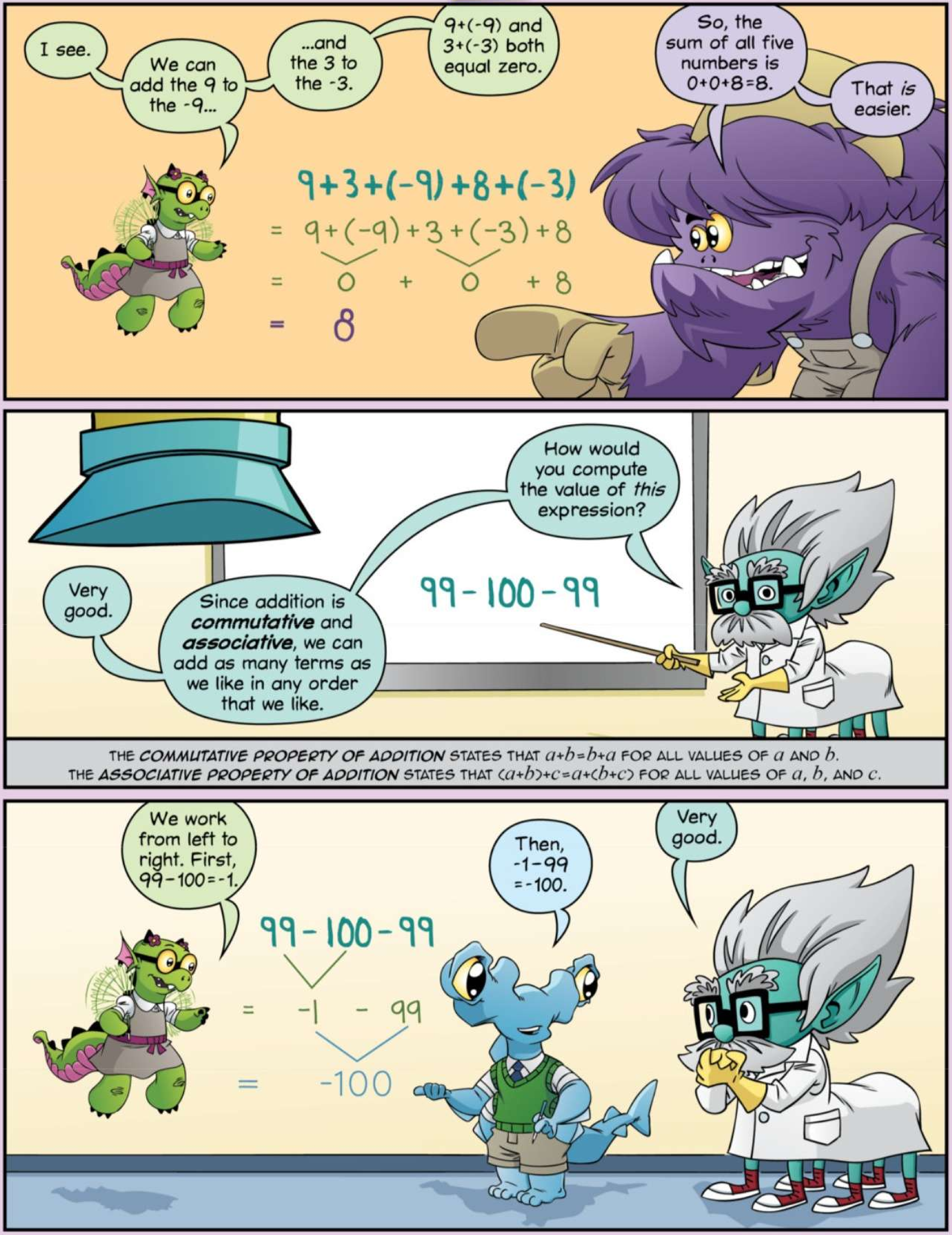
Aye, 'tis not always helpful to change subtraction to addition.
Let's try usin' some bigger numbers.
Find each o' these differences.
42-(-19)
-17-61
To subtract -19, we add its opposite.
42-(-19) equals 42+19=61.
42-(-19)
=42 + 19
=61
We can compute -17-61 with or without changing the subtraction to addition.
Either way, we start at -17 and move 61 units to the left.
-17-61
= -17+(-61)
61
-17
0
Arrr... Ye' be a smart bunch.
We land at -78.
-17-61
= -17+(-61)
61
-78
-17
0

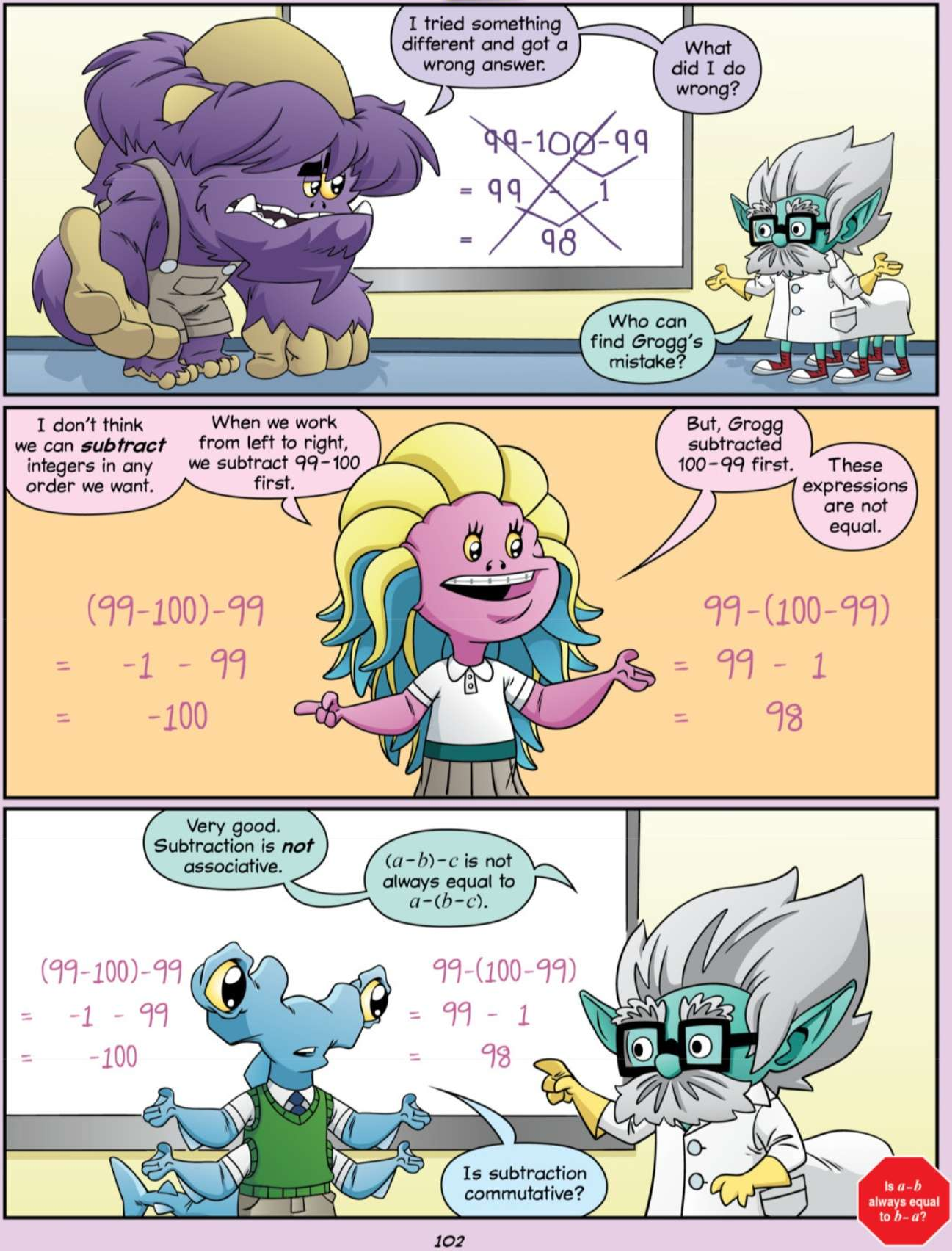


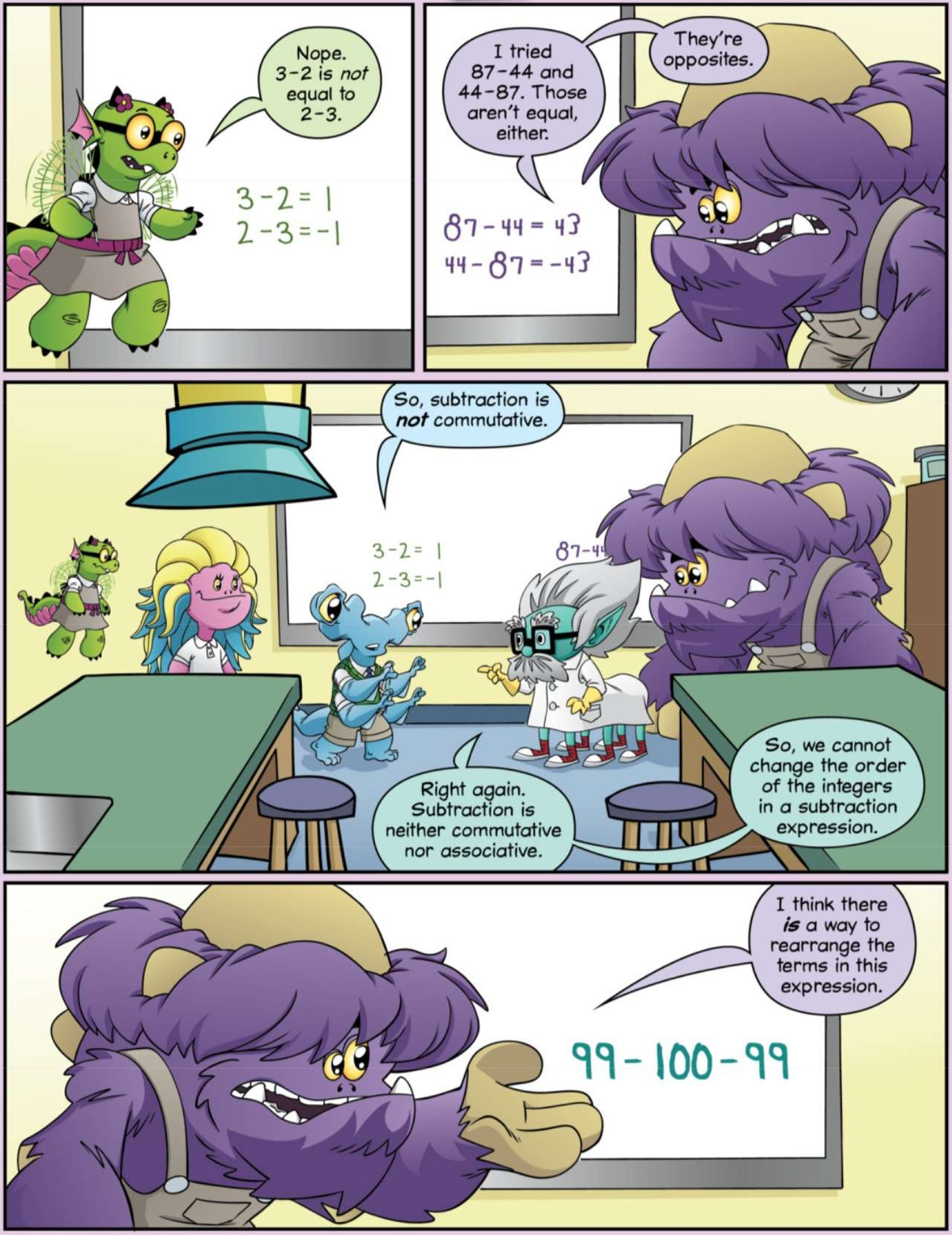




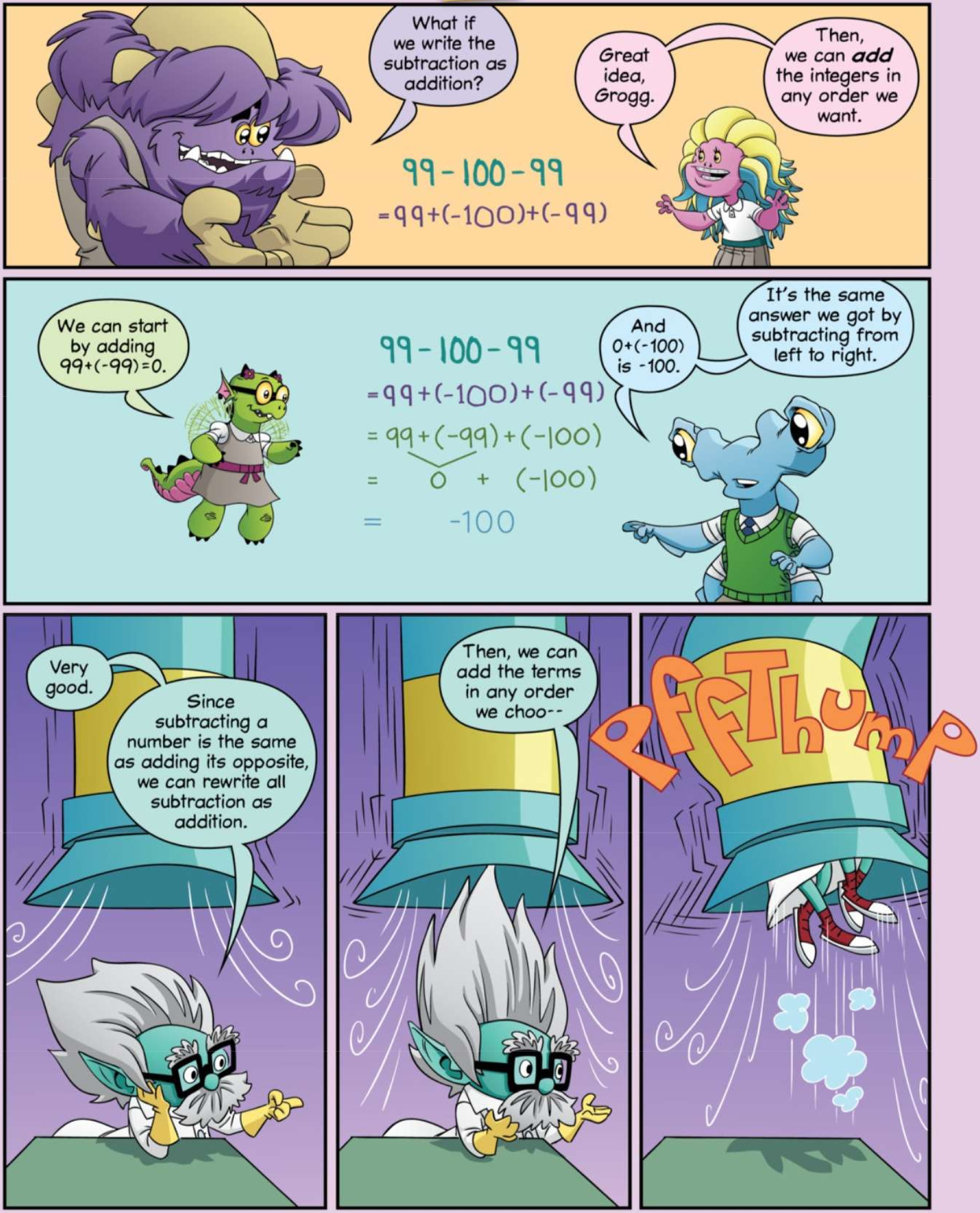
How can changing the order of the addition make the problem easier?


I see.
We can add the 9 to the -9...
...and the 3 to the -3.
9+(-9) and 3+(-3) both equal zero.
9+3+(-9)+8+(-3)
= 9+(-9)+3+(-3)+8
= 0 + 0 + 8
= 8
So, the sum of all five numbers is 0+0+8=8.
That *is* easier.
How would you compute the value of *this* expression?
99-100-99
Since addition is ***commutative*** and ***associative***, we can add as many terms as we like in any order that we like.
Very good.
THE ***COMMUTATIVE PROPERTY OF ADDITION*** STATES THAT $a+b=b+a$ FOR ALL VALUES OF a AND b.
THE ***ASSOCIATIVE PROPERTY OF ADDITION*** STATES THAT $(a+b)+c=a+(b+c)$ FOR ALL VALUES OF a, b, AND c.
We work from left to right. First, 99-100=-1.
99-100-99
= -1 - 99
= -100
Then, -1-99 =-100.
Very good.

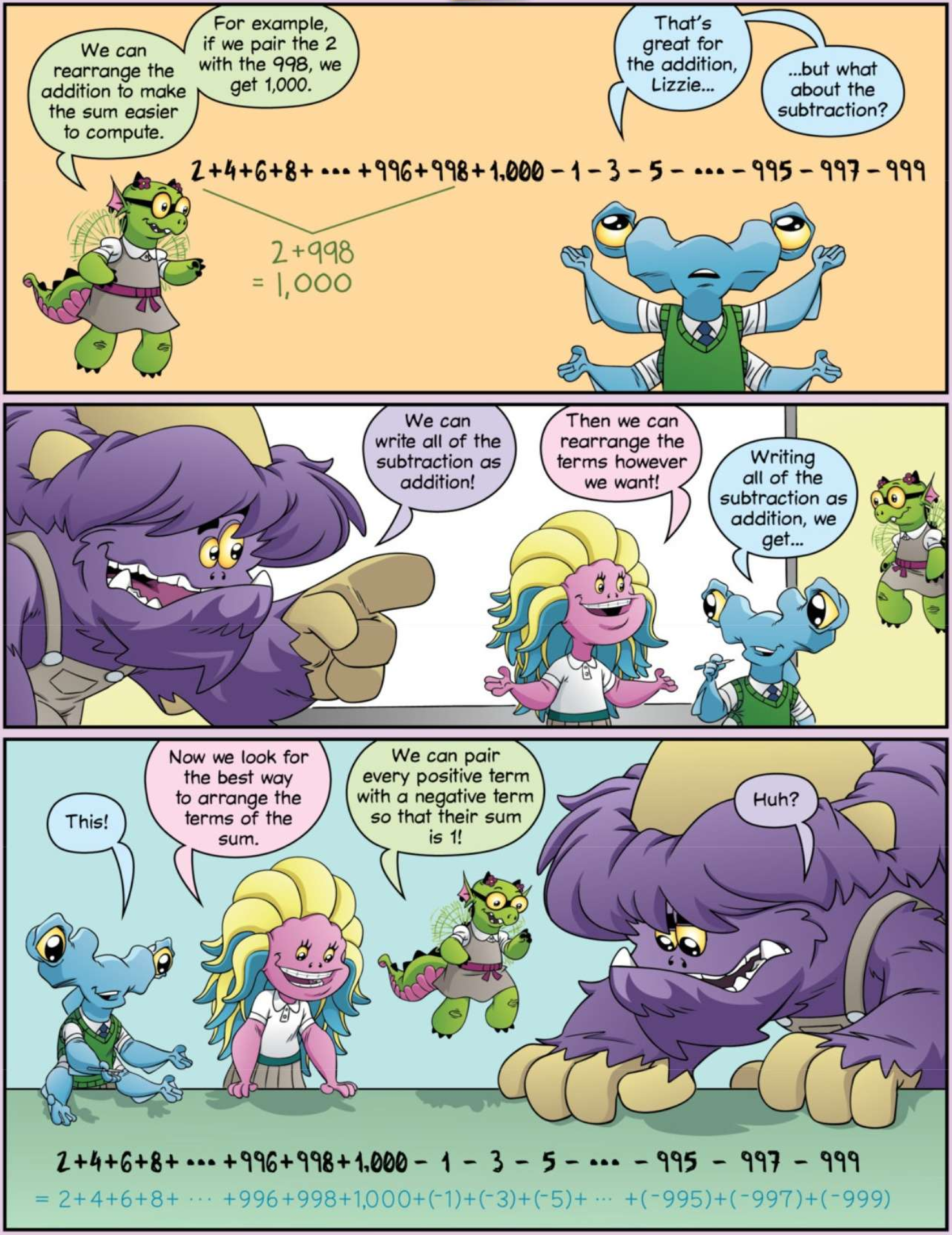

I tried something different and got a wrong answer.
What did I do wrong?
99-100-99
= 99 - 1
= 98
Who can find Grogg's mistake?
I don't think we can ***subtract*** integers in any order we want.
When we work from left to right, we subtract 99−100 first.
But, Grogg subtracted 100−99 first.
These expressions are not equal.
(99-100)-99
= -1 - 99
= -100
99-(100-99)
= 99 - 1
= 98
Very good. Subtraction is ***not*** associative.
$(a-b)-c$ is not always equal to $a-(b-c)$.
(99-100)-99
= -1 - 99
= -100
99-(100-99)
= 99 - 1
= 98
Is subtraction commutative?
Is $a-b$ always equal to $b-a$?



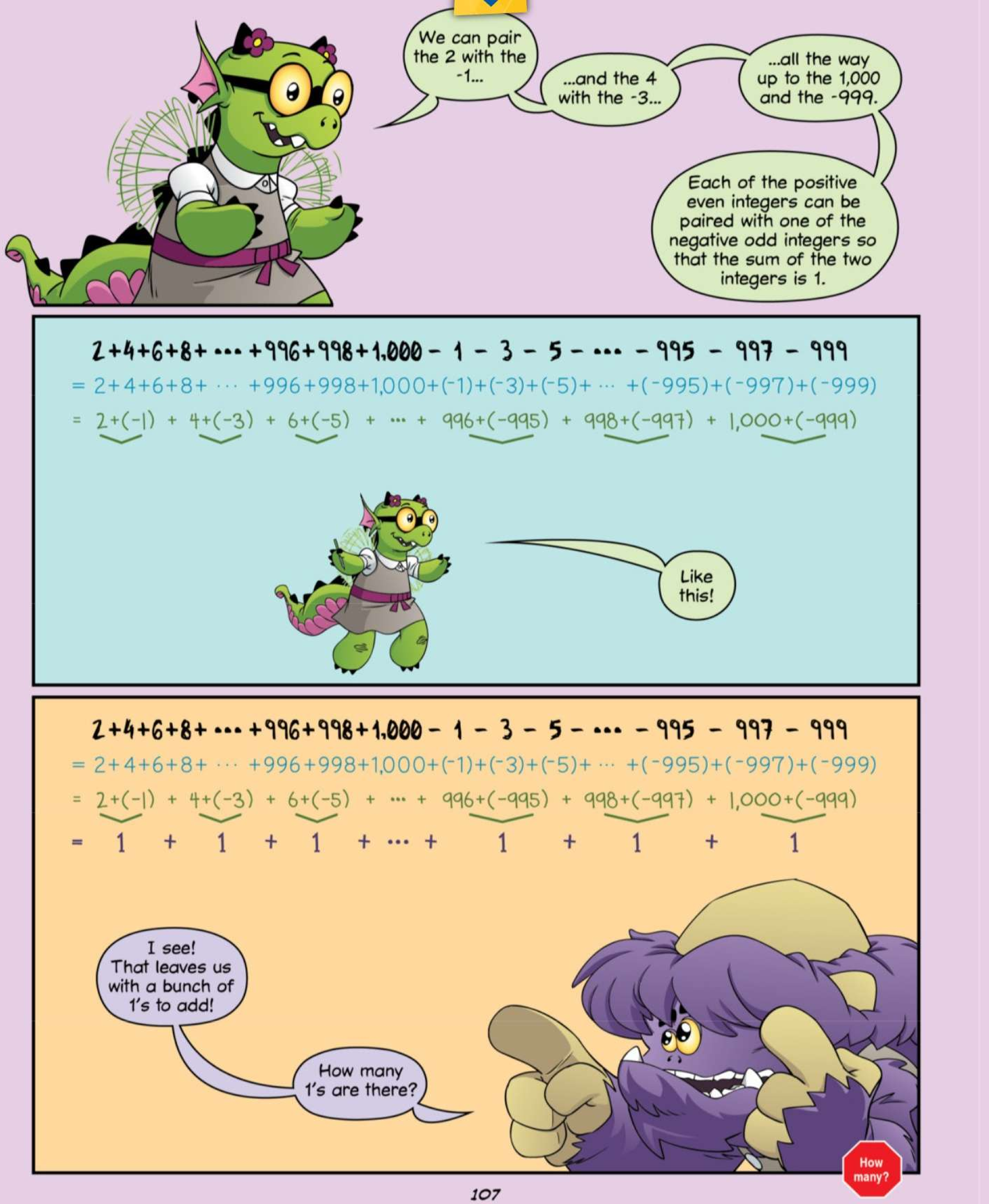
Nope. 3-2 is *not* equal to 2-3.
3-2=1
2-3=-1
I tried 87-44 and 44-87. Those aren't equal, either.
They're opposites.
87-44=43
44-87=-43
So, subtraction is ***not*** commutative.
3-2=1
2-3=-1
Right again. Subtraction is neither commutative nor associative.
So, we cannot change the order of the integers in a subtraction expression.
I think there ***is*** a way to rearrange the terms in this expression.
99-100-99


What if we write the subtraction as addition?
99 - 100 - 99
= 99 + (-100) + (-99)
Great idea, Grogg.
Then, we can ***add*** the integers in any order we want.
We can start by adding 99+(-99)=0.
99 - 100 - 99
= 99 + (-100) + (-99)
= 99 + (-99) + (-100)
= 0 + (-100)
= -100
And 0+(-100) is -100.
It's the same answer we got by subtracting from left to right.
Very good.
Since subtracting a number is the same as adding its opposite, we can rewrite all subtraction as addition.
Then, we can add the terms in any order we choo--
PFFTHUMP

Bwah Hah Hah! Professor Grok is gone! I've abducted your educator! It's time for something diabolically difficult!
A summing of several integers is executed easily.
But simplifying a more substantial expression can be excruciating.
To save your schoolteacher, you must complete a clever computation.
Add each even integer from 2 to 1,000.
From that sum, subtract each odd integer from 1 to 999.
The result is the room number of your professor's imprisonment.
Compute quickly, or your educator will endure disquieting discomfort ***indefinitely!***
PFFTHUMP
2+4+6+8+ ··· +996+998+1,000 − 1 − 3 − 5 − ··· − 995 − 997 − 999
That is a ***lot*** of numbers to add and subtract.
There must be a good way to do this.
Can you compute the value of Clod's expression?



We can rearrange the addition to make the sum easier to compute.
For example, if we pair the 2 with the 998, we get 1,000.
That's great for the addition, Lizzie...
...but what about the subtraction?
2+4+6+8+ ••• +996+998+1,000 − 1 − 3 − 5 − ••• − 995 − 997 − 999
2+998 = 1,000
We can write all of the subtraction as addition!
Then we can rearrange the terms however we want!
Writing all of the subtraction as addition, we get...
This!
Now we look for the best way to arrange the terms of the sum.
We can pair every positive term with a negative term so that their sum is 1!
Huh?
2+4+6+8+ ••• +996+998+1,000 − 1 − 3 − 5 − ••• − 995 − 997 − 999
= 2+4+6+8+ ⋯ +996+998+1,000+(⁻1)+(⁻3)+(⁻5)+ ⋯ +(⁻995)+(⁻997)+(⁻999)


We can pair the 2 with the -1...
...and the 4 with the -3...
...all the way up to the 1,000 and the -999.
Each of the positive even integers can be paired with one of the negative odd integers so that the sum of the two integers is 1.
2+4+6+8+ ··· +996+998+1,000 − 1 − 3 − 5 − ··· − 995 − 997 − 999
= 2+4+6+8+ ··· +996+998+1,000+(−1)+(−3)+(−5)+ ··· +(−995)+(−997)+(−999)
= 2+(−1) + 4+(−3) + 6+(−5) + ··· + 996+(−995) + 998+(−997) + 1,000+(−999)
Like this!
2+4+6+8+ ··· +996+998+1,000 − 1 − 3 − 5 − ··· − 995 − 997 − 999
= 2+4+6+8+ ··· +996+998+1,000+(−1)+(−3)+(−5)+ ··· +(−995)+(−997)+(−999)
= 2+(−1) + 4+(−3) + 6+(−5) + ··· + 996+(−995) + 998+(−997) + 1,000+(−999)
= 1 + 1 + 1 + ··· + 1 + 1 + 1
I see! That leaves us with a bunch of 1's to add!
How many 1's are there?
How many?

2+4+6+8+ ··· +996+998+1,000 − 1 − 3 − 5 − ··· − 995 − 997 − 999
= 2+4+6+8+ ··· +996+998+1,000+(−1)+(−3)+(−5)+ ··· +(−995)+(−997)+(−999)
= 2+(−1) + 4+(−3) + 6+(−5) + ··· + 996+(−995) + 998+(−997) + 1,000+(−999)
= 1 + 1 + 1 + ··· + 1 + 1 +
= 500
There are 1,000 integers from 1 to 1,000.
Half of them are even, and half of them are odd.
So, there are 500 even numbers and 500 odd numbers.
That makes 500 pairs...
...so the sum is 500.
To room 500!
We have a pool!?
Come on in! The water's great!

Writing Expressions without Negatives

Lizzie

Some expressions are a lot clearer if you write them without using negatives. Here are some ways to make your math a lot easier to read:

Subtracting a negative is the same as adding a positive.

$$a-(^{-}b)=a+b$$

Examples:

$5-(-9)=5+9$ $7-(^{-}2)=7+2$ $6-(^{-}3)=6+3$

Adding a negative is the same as subtracting a positive.

$$a+(^{-}b)=a-b$$

Examples:

$7+(^{-}4)=7-4$ $10+(^{-}3)=10-3$ $5+(^{-}1)=5-1$

A negative plus a positive can be written as a difference between two positives.

$$\star\ {}^{-}a+b=b+(-a)=b-a$$

Examples:

$-2+6=6-2$ $-3+4=4-3$ $-11+5=5-11$

A difference of two negatives can be written as a difference of two positives.

$$\star\ {}^{-}a-(-b)=-a+b=b-a$$

Examples:

$-2-(-7)=7-2$ $-9-(-10)=10-9$ $-6-(-4)=4-6$