Contents: Chapter 3

Click the Play List tab in the top-left to view a recommended reading/practice sequence.

Exponents 70
Given that $5^4=625$, what is 5^5?

Lizzie's Notes 75
How would you write $11\times11\times11\times11\times11$ as a power?

Order of Operations 76
How do you evaluate $(2+3^3)-5^2$?

Power Play 79
How can the number 81 be written as the sum of a power of 7 and a power of 2?

Perfect Squares 80
How can 2^8 be written as a perfect square?

Math Meet 85
Will the Little Monsters outscore the Bots?

Binary Island 93
What does the number 10,000 stand for on Binary Island?

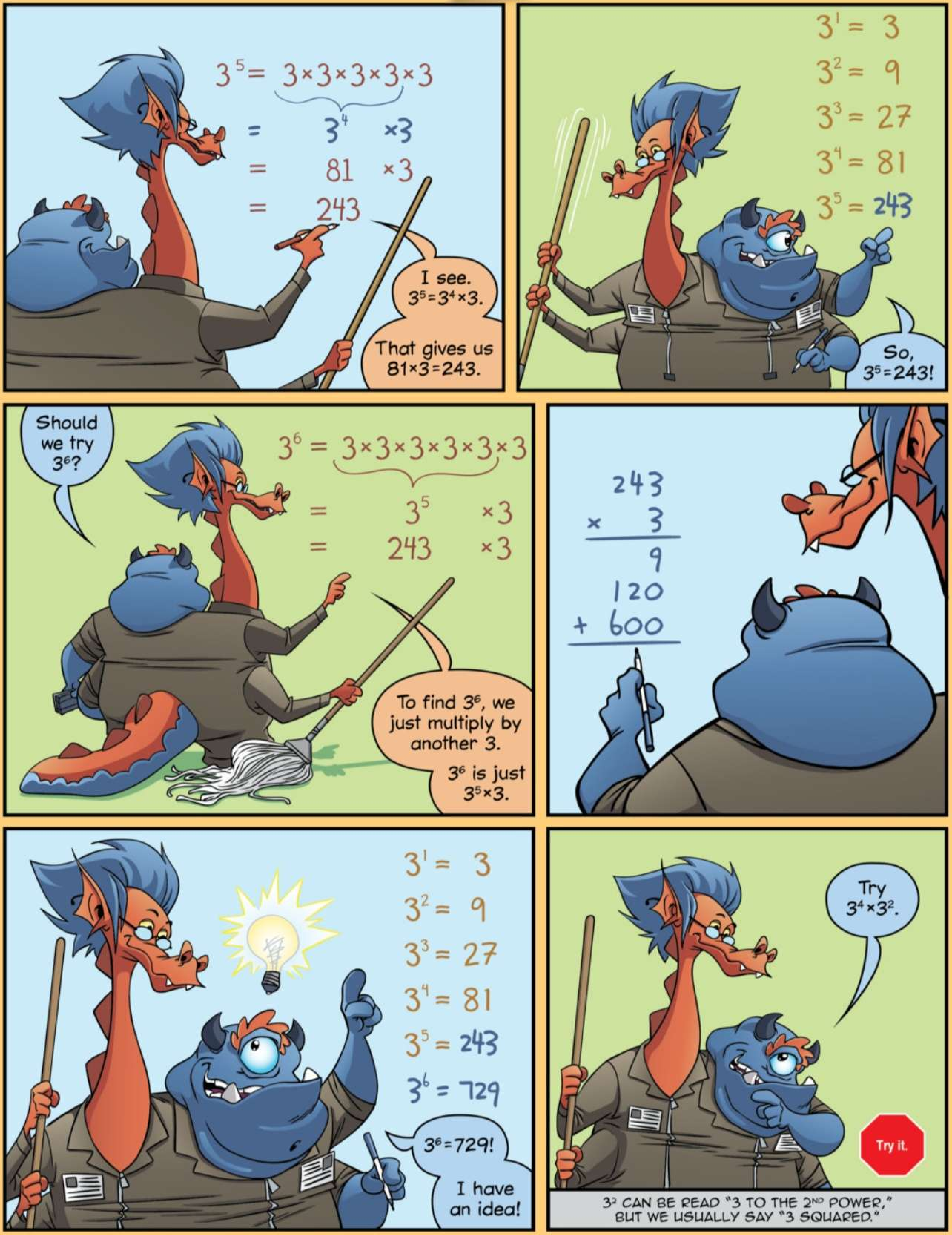
Base-2 98
How would you write the base-10 number 117 using base-2?

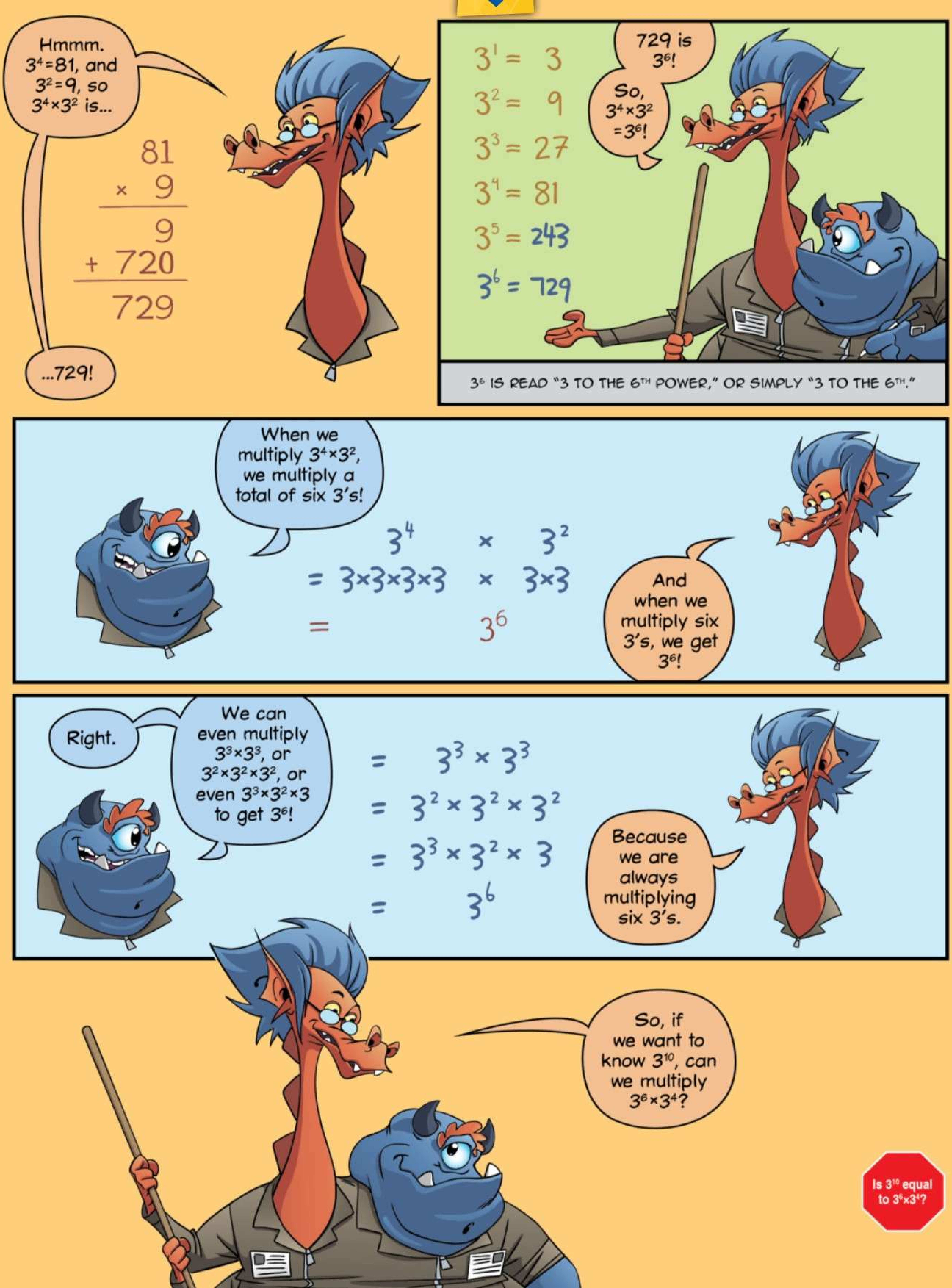
Grogg's Notes 107
Is the number 5 a happy number? How can you tell?

Congratulations!
You have reached the peak of
Mount Wanowanwanoh
Elevation:
10,110,010,111,011 feet
(110,110,100,010 meters)
Highest point on Binary Island

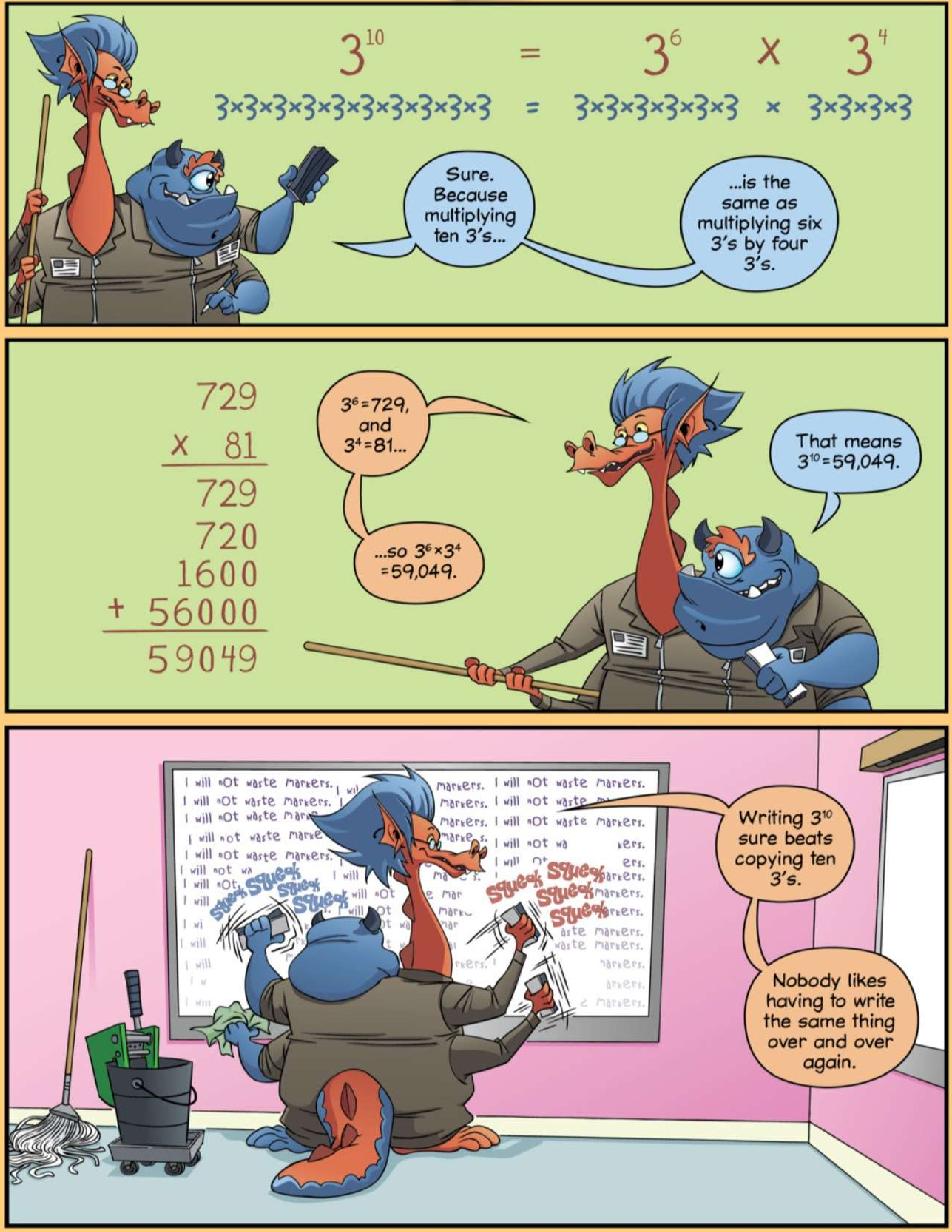

R&G
Exponents
53×38
= (50+
$3^1 = 3$
$3^2 = 9$
$3^3 = 27$
$3^4 = 81$
$3^5 =$
Look at these tiny numbers.
Why are they so small?
Those little numbers are called ***exponents***.
What are they for?
Exponents give us a shortcut for writing repeated multiplication.
Huh?
This 3 is the number you are multiplying.
3 is called the ***base***.
3^4
The 4 is the exponent. It tells you how many 3's to multiply.
3^4
3^4 IS READ "3 TO THE 4TH POWER," OR SIMPLY "3 TO THE 4TH."


So, 3^4 means $3×3×3×3$?
Exactly.
$3^4 = 3×3×3×3$
And since $3×3×3×3=81$, $3^4=81$!
That's right.
$3^4 = 3×3×3×3$
$= 9×9$
$= 81$
Ms. Q. left 3^5 blank. Let's see if we can figure out what 3^5 equals.
$3^1 = 3$
$3^2 = 9$
$3^3 = 27$
$3^4 = 81$
$3^5 =$
Try it.
3^5 IS READ "3 TO THE 5TH POWER."
We know that $3^5 =3×3×3×3×3$.
So, we need to multiply five 3's.
$3^5 = 3×3×3×3×3$
$3^5 = \underbrace{3×3×3×3}×3$
$= 3^4 × 3$
Since multiplying ***four*** 3's gives us $3^4=81$.
To multiply ***five*** 3's, we can multiply 3^4 by one more 3!


$3^5 = \underbrace{3 \times 3 \times 3 \times 3}_{} \times 3$
$= 3^4 \times 3$
$= 81 \times 3$
$= 243$
I see. $3^5 = 3^4 \times 3$.
That gives us $81 \times 3 = 243$.
$3^1 = 3$
$3^2 = 9$
$3^3 = 27$
$3^4 = 81$
$3^5 = 243$
So, $3^5 = 243$!
Should we try 3^6?
$3^6 = \underbrace{3 \times 3 \times 3 \times 3 \times 3}_{} \times 3$
$= 3^5 \times 3$
$= 243 \times 3$
To find 3^6, we just multiply by another 3.
3^6 is just $3^5 \times 3$.
243
× 3
9
120
+ 600
$3^1 = 3$
$3^2 = 9$
$3^3 = 27$
$3^4 = 81$
$3^5 = 243$
$3^6 = 729$
$3^6 = 729$!
I have an idea!
Try $3^4 \times 3^2$.
Try it.
3^2 CAN BE READ "3 TO THE 2ND POWER," BUT WE USUALLY SAY "3 SQUARED."



Hmmm. $3^4=81$, and $3^2=9$, so $3^4\times3^2$ is...
81×9
9
$+\ 720$
729
...729!
$3^1=3$
$3^2=9$
$3^3=27$
$3^4=81$
$3^5=243$
$3^6=729$
729 is 3^6!
So, $3^4\times3^2=3^6$!
3^6 IS READ "3 TO THE 6^{TH} POWER," OR SIMPLY "3 TO THE 6^{TH}."
When we multiply $3^4\times3^2$, we multiply a total of six 3's!
$3^4 \times 3^2$
$= 3\times3\times3\times3 \times 3\times3$
$= 3^6$
And when we multiply six 3's, we get 3^6!
Right.
We can even multiply $3^3\times3^3$, or $3^2\times3^2\times3^2$, or even $3^3\times3^2\times3$ to get 3^6!
$= 3^3\times3^3$
$= 3^2\times3^2\times3^2$
$= 3^3\times3^2\times3$
$= 3^6$
Because we are always multiplying six 3's.
So, if we want to know 3^{10}, can we multiply $3^6\times3^4$?
Is 3^{10} equal to $3^6\times3^4$?



$3^{10} = 3^6 \times 3^4$
$3\times3\times3\times3\times3\times3\times3\times3\times3\times3 = 3\times3\times3\times3\times3\times3 \times 3\times3\times3\times3$
Sure. Because multiplying ten 3's...
...is the same as multiplying six 3's by four 3's.
$3^6=729$, and $3^4=81$...
...so $3^6\times3^4$ $=59{,}049$.
That means $3^{10}=59{,}049$.
729 × 81 = 729 + 720 + 1600 + 56000 = 59049
I will not waste markers.
Squeak
Writing 3^{10} sure beats copying ten 3's.
Nobody likes having to write the same thing over and over again.

Lizzie

Powers

A power is a shortcut for writing repeated multiplication.
For example, we can write $5 \times 5 \times 5$ as a power: 5^3.

$$5 \times 5 \times 5 = 5^3$$

Exponent

Base

Every power has a base and an exponent.
The exponent tells us how many of the base we multiply.

5^3 is a power of 5 and is read, "five to the third power."

More examples:

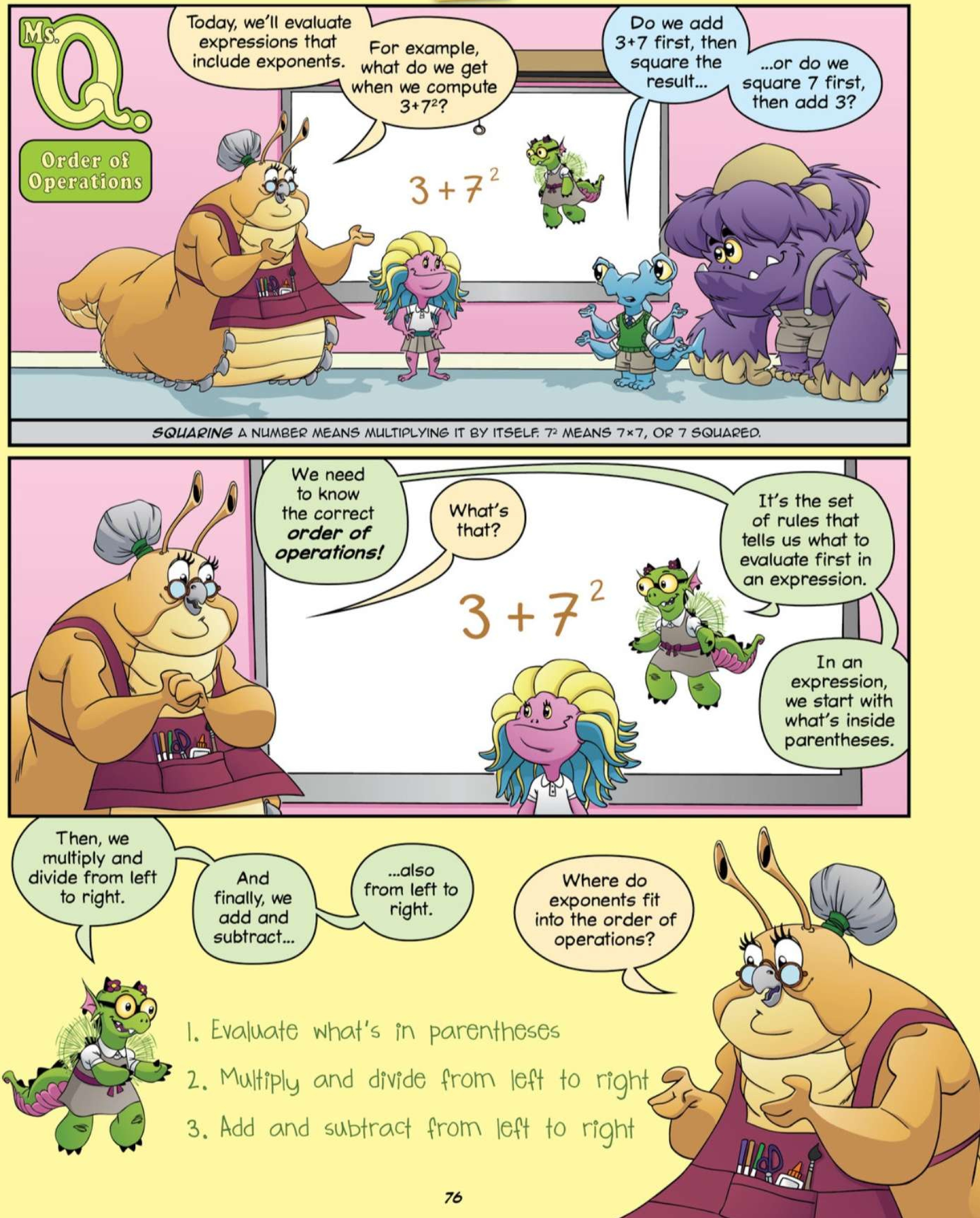
$7 \times 7 \times 7 \times 7 = 7^4$ "seven to the fourth power"

$3 \times 3 = 3^2$ "three to the second power," or "three squared"

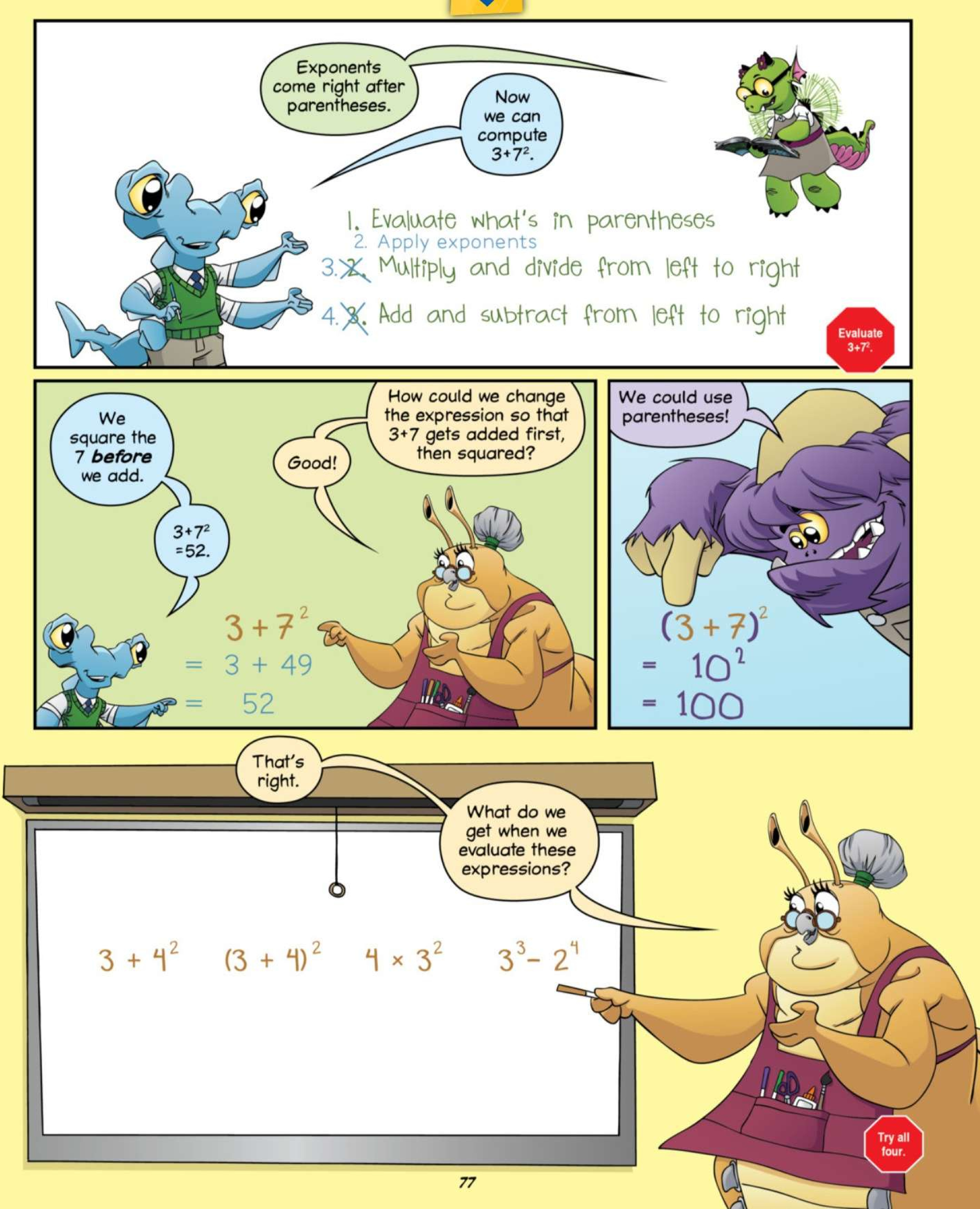
$2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6$ "two to the sixth power"

Powers of 2:

$$\begin{aligned}
2 &= 2^1 \\
2\times2 &= 2^2 = 2^1\times2 = 4 \\
2\times2\times2 &= 2^3 = 2^2\times2 = 8 \\
2\times2\times2\times2 &= 2^4 = 2^3\times2 = 16 \\
2\times2\times2\times2\times2 &= 2^5 = 2^4\times2 = 32 \\
2\times2\times2\times2\times2\times2 &= 2^6 = 2^5\times2 = 64 \\
2\times2\times2\times2\times2\times2\times2 &= 2^7 = 2^6\times2 = 128 \\
2\times2\times2\times2\times2\times2\times2\times2 &= 2^8 = 2^7\times2 = 256 \\
2\times2\times2\times2\times2\times2\times2\times2\times2 &= 2^9 = 2^8\times2 = 512 \\
2\times2\times2\times2\times2\times2\times2\times2\times2\times2 &= 2^{10} = 2^9\times2 = 1{,}024
\end{aligned}$$



Ms. Q.
Order of Operations
Today, we'll evaluate expressions that include exponents.
For example, what do we get when we compute $3+7^2$?
Do we add 3+7 first, then square the result...
...or do we square 7 first, then add 3?
$3+7^2$
SQUARING A NUMBER MEANS MULTIPLYING IT BY ITSELF. 7^2 MEANS 7×7, OR 7 SQUARED.
We need to know the correct ***order of operations!***
What's that?
It's the set of rules that tells us what to evaluate first in an expression.
$3+7^2$
In an expression, we start with what's inside parentheses.
Then, we multiply and divide from left to right.
And finally, we add and subtract...
...also from left to right.
Where do exponents fit into the order of operations?
1. Evaluate what's in parentheses
2. Multiply and divide from left to right
3. Add and subtract from left to right


Exponents come right after parentheses.
Now we can compute 3+7².
1. Evaluate what's in parentheses
2. Apply exponents
3. ~~2.~~ Multiply and divide from left to right
4. ~~3.~~ Add and subtract from left to right
Evaluate 3+7².
We square the 7 *before* we add.
3+7² =52.
3 + 7²
= 3 + 49
= 52
Good!
How could we change the expression so that 3+7 gets added first, then squared?
We could use parentheses!
(3 + 7)²
= 10²
= 100
That's right.
What do we get when we evaluate these expressions?
3 + 4²   (3 + 4)²   4 × 3²   3³ − 2⁴
Try all four.


We apply exponents before we add, so $3+4^2=3+16=19$.
$3 + 4^2$
$= 3 + 16$
$= 19$
In this problem, we start with what's inside the parentheses. $3+4=7$.
Then, 7 squared is 49.
$(3 + 4)^2$
$= 7^2$
$= 49$
We apply exponents before we multiply, so $4\times3^2=4\times9=36$.
4×3^2
$= 4 \times 9$
$= 36$
We apply the exponents first in this problem. $3^3=27$, and $2^4=16$.
Then, we subtract.
$3^3 - 2^4$
$= 27 - 16$
$= 11$
So, 3^3-2^4 $=27-16$ $=11$.
Doing things in the right order can make a big difference!
Remember, honey,
underwear first,
Mom

RECESS

Power Play

Power Play is a game for two or more players. You will need a deck of cards numbered 0 through 9 (you can make your own, or print a set from www.BeastAcademy.com).
Each round, two cards are turned over from the top of the deck to create a two-digit target number.
Every player then attempts to write the target number as a sum of two or more powers.

For example, if the cards turned are a 5 then a 2, players attempt to write a sum of powers that is equal to 52.
There are a few rules:

1. You may not use powers of 1.
   $6^2 + 5^2 + 1^3 = 52$ is not allowed.
2. All exponents must be greater than 1.
   $7^2 + 3^1 = 52$ is not allowed.
3. You may not use the same base more than once.
   $2^5 + 2^4 + 2^2 = 52$ is not allowed.

5 2

There are three ways to write 52 as a sum of two or more powers using the rules above. They are listed upside-down at the bottom of this page.

Sometimes, the target number is impossible to write as a sum of powers. In this case, the goal is to write a sum of powers that is as close as possible to the target.

Scoring

Each game lasts 5 rounds. The goal is to have the ***lowest*** score after the final round.
Score each turn as follows:

- If your expression equals the target number, you get 0 points.
- If your expression does not equal the target number, your score is the difference between the value of your expression and the target number.

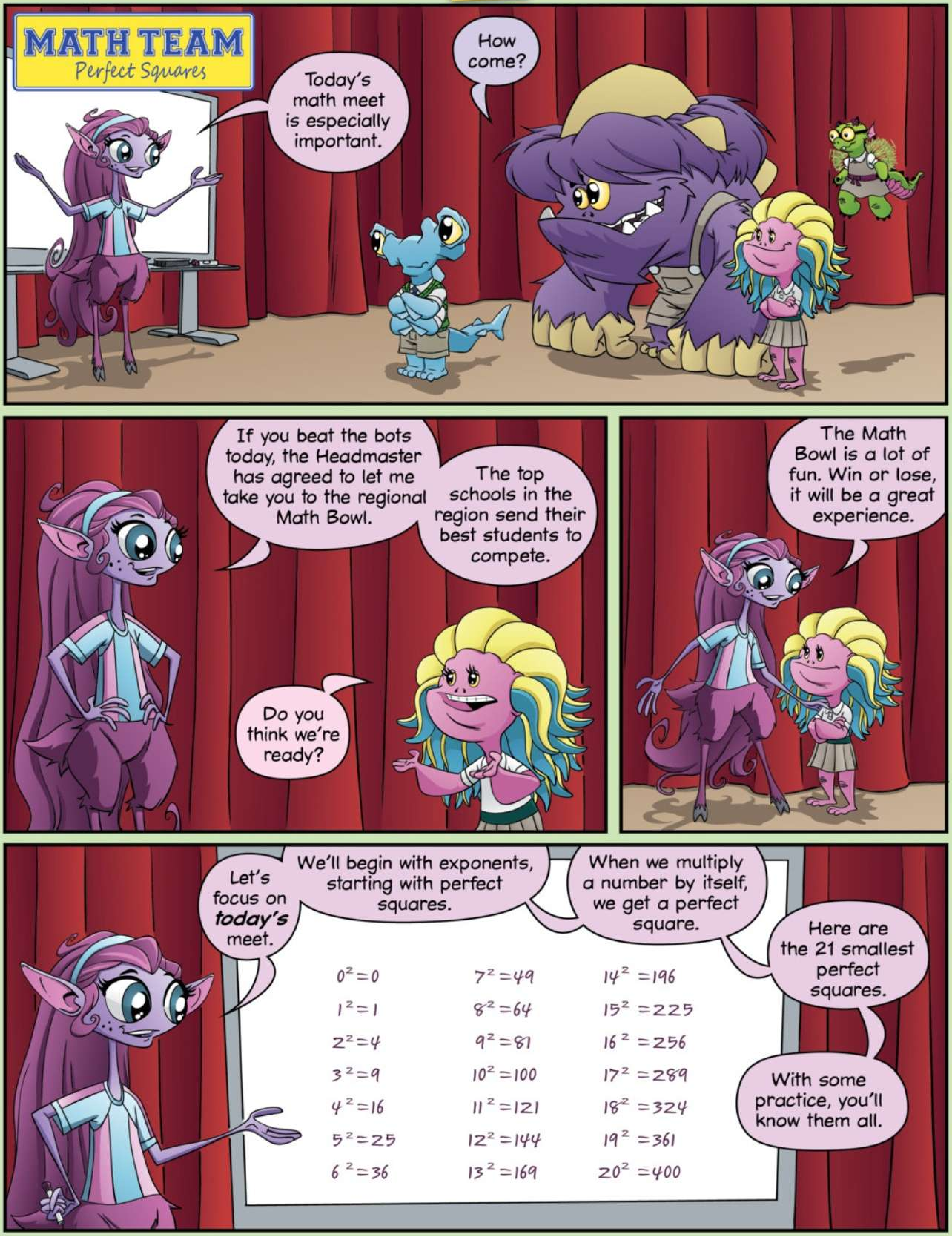
Add your score for each of the five rounds to get your total score. The winner is the player who scores the fewest points.

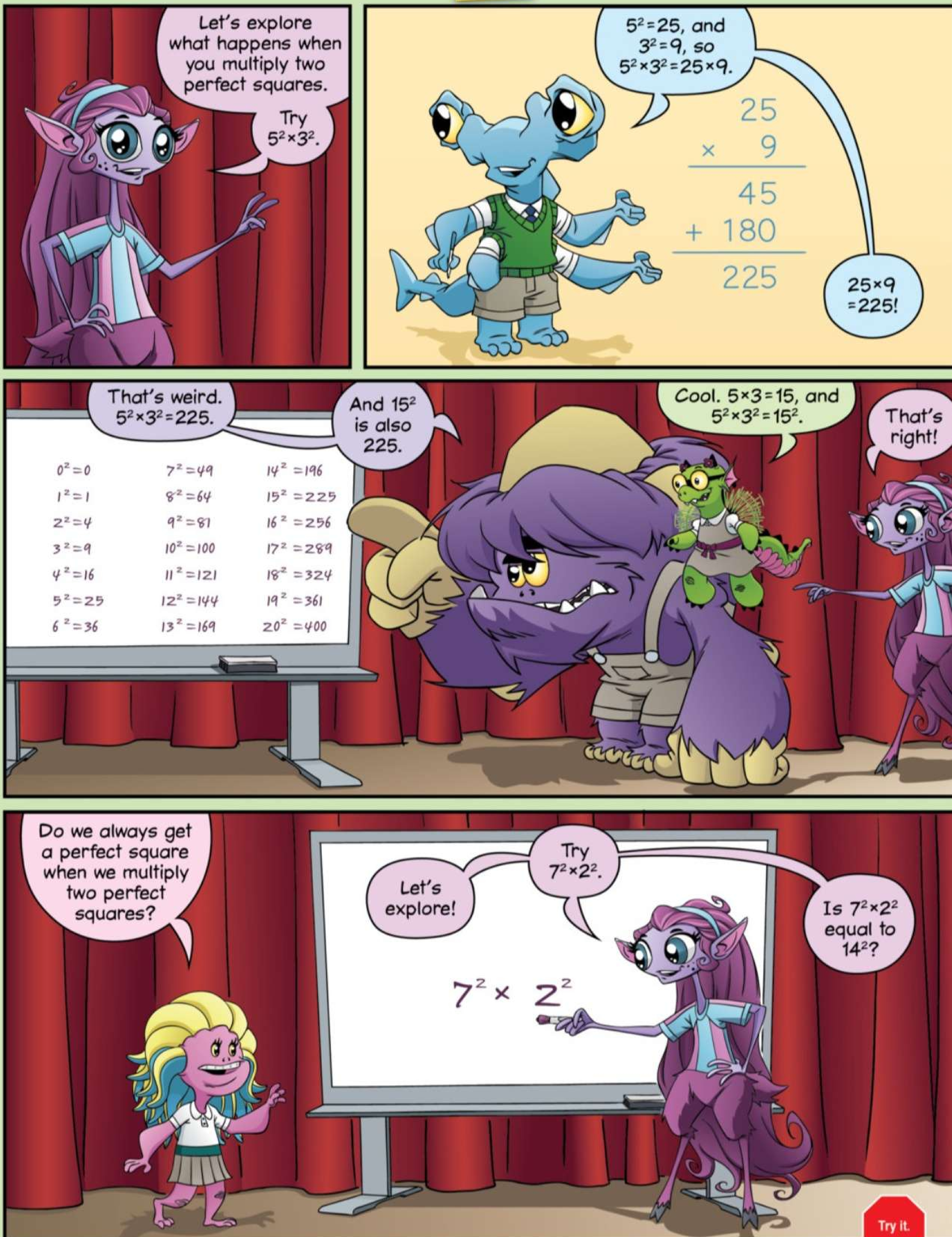
Variations

You may set a time limit for each round.
To make the game easier, you may eliminate rule 3 above.

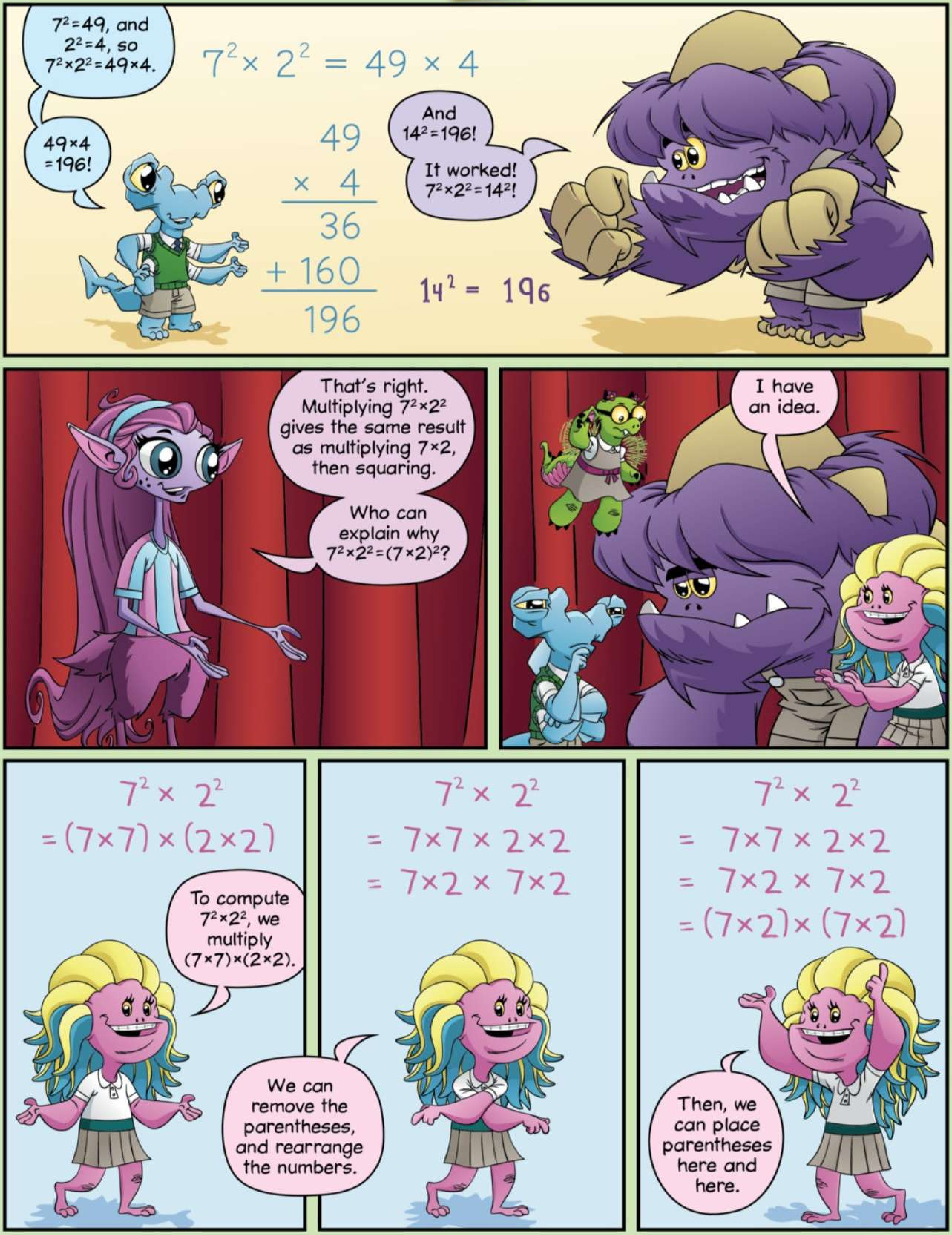
Find a partner and play!

$6^2 + 2^4 = 52$, $6^2 + 4^2 = 52$, $5^2 + 3^3 = 52$


MATH TEAM
Perfect Squares
Today's math meet is especially important.
How come?
If you beat the bots today, the Headmaster has agreed to let me take you to the regional Math Bowl.
The top schools in the region send their best students to compete.
Do you think we're ready?
The Math Bowl is a lot of fun. Win or lose, it will be a great experience.
Let's focus on *today's* meet.
We'll begin with exponents, starting with perfect squares.
When we multiply a number by itself, we get a perfect square.
Here are the 21 smallest perfect squares.
$0^2=0$
$1^2=1$
$2^2=4$
$3^2=9$
$4^2=16$
$5^2=25$
$6^2=36$
$7^2=49$
$8^2=64$
$9^2=81$
$10^2=100$
$11^2=121$
$12^2=144$
$13^2=169$
$14^2=196$
$15^2=225$
$16^2=256$
$17^2=289$
$18^2=324$
$19^2=361$
$20^2=400$
With some practice, you'll know them all.

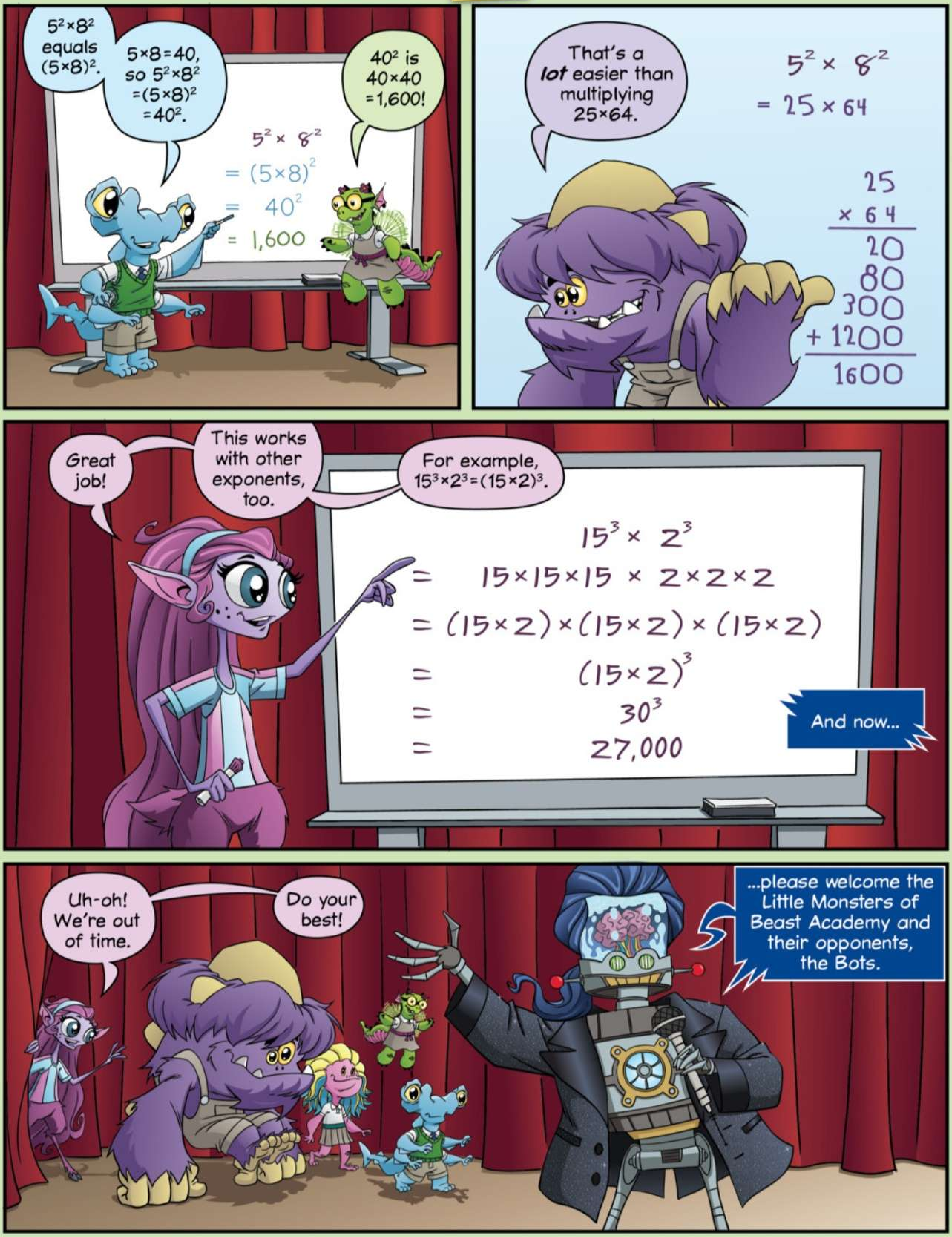


Let's explore what happens when you multiply two perfect squares.
Try $5^2 \times 3^2$.
$5^2 = 25$, and $3^2 = 9$, so $5^2 \times 3^2 = 25 \times 9$.
25
× 9
45
+ 180
225
$25 \times 9 = 225$!
That's weird. $5^2 \times 3^2 = 225$.
And 15^2 is also 225.
$0^2=0$
$1^2=1$
$2^2=4$
$3^2=9$
$4^2=16$
$5^2=25$
$6^2=36$
$7^2=49$
$8^2=64$
$9^2=81$
$10^2=100$
$11^2=121$
$12^2=144$
$13^2=169$
$14^2=196$
$15^2=225$
$16^2=256$
$17^2=289$
$18^2=324$
$19^2=361$
$20^2=400$
Cool. $5 \times 3 = 15$, and $5^2 \times 3^2 = 15^2$.
That's right!
Do we always get a perfect square when we multiply two perfect squares?
Let's explore!
Try $7^2 \times 2^2$.
Is $7^2 \times 2^2$ equal to 14^2?
$7^2 \times 2^2$
Try it.



7²=49, and 2²=4, so 7²×2²=49×4.
$7^2 \times 2^2 = 49 \times 4$
49×4 =196!
$49 \times 4 = 36 + 160 = 196$
And 14²=196!
It worked! 7²×2²=14²!
$14^2 = 196$
That's right. Multiplying 7²×2² gives the same result as multiplying 7×2, then squaring.
Who can explain why 7²×2²=(7×2)²?
I have an idea.
$7^2 \times 2^2 = (7\times7)\times(2\times2)$
To compute 7²×2², we multiply (7×7)×(2×2).
$7^2 \times 2^2 = 7\times7\times2\times2 = 7\times2\times7\times2$
We can remove the parentheses, and rearrange the numbers.
$7^2 \times 2^2 = 7\times7\times2\times2 = 7\times2\times7\times2 = (7\times2)\times(7\times2)$
Then, we can place parentheses here and here.

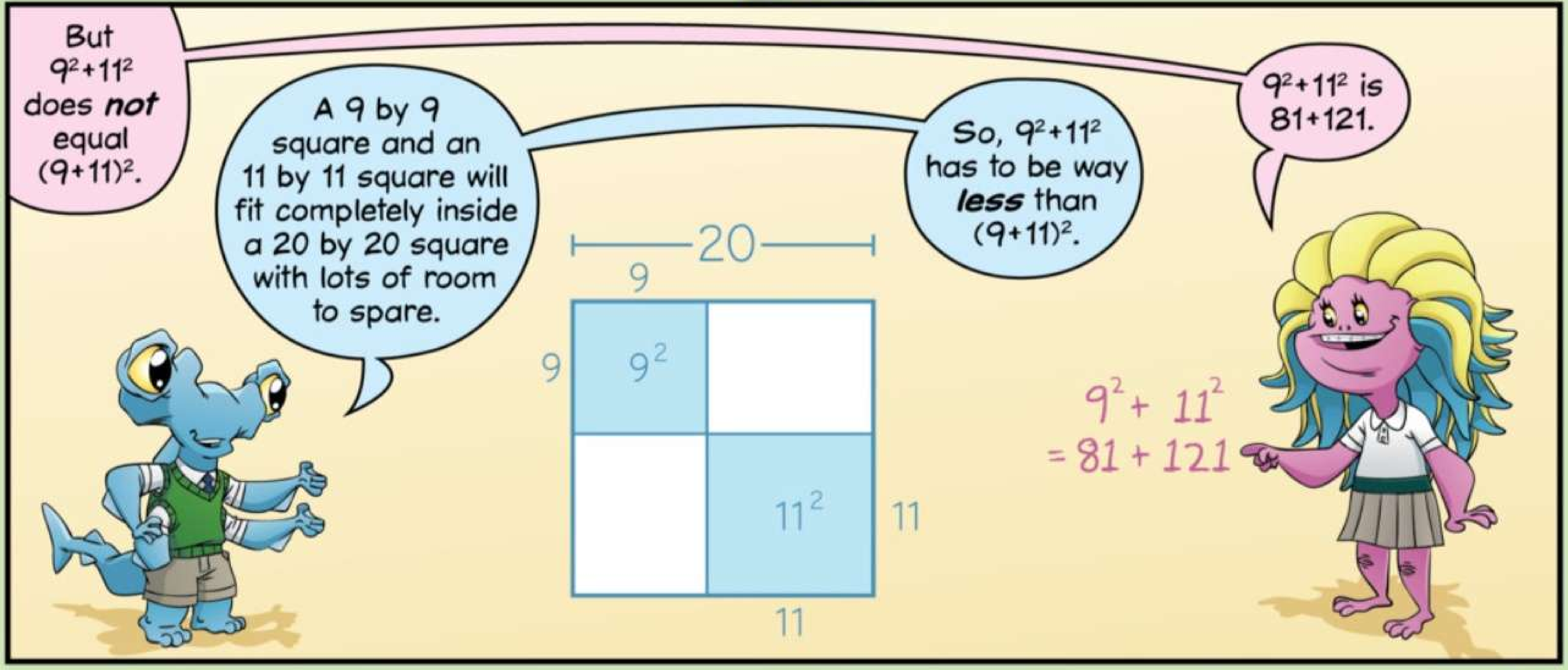

Since we are multiplying (7×2) by itself, (7×2)×(7×2) is the same as (7×2)²!
So, 7²×2² =(7×2)²!
7² × 2²
= 7×7 × 2×2
= 7×2 × 7×2
= (7×2) × (7×2)
= (7×2)²
Great job, Winnie.
This works for 5²×3² and 7²×2². Does it always work?
Let's use variables to stand for two numbers: a and b.
Yep.
We can use the same steps that Winnie used, only we'll use variables instead of numbers.
first number: a
second number: b
a² × b²
= (a×a) × (b×b)
It doesn't matter what numbers a and b are...
...a²×b² always equals (a×b)².
a² × b²
= a×a × b×b
= a×b × a×b
= (a×b) × (a×b)
= (a × b)²
5² × 8²
How can knowing that a²×b²=(a×b)² help you compute 5²×8²?
What is 5²×8²?

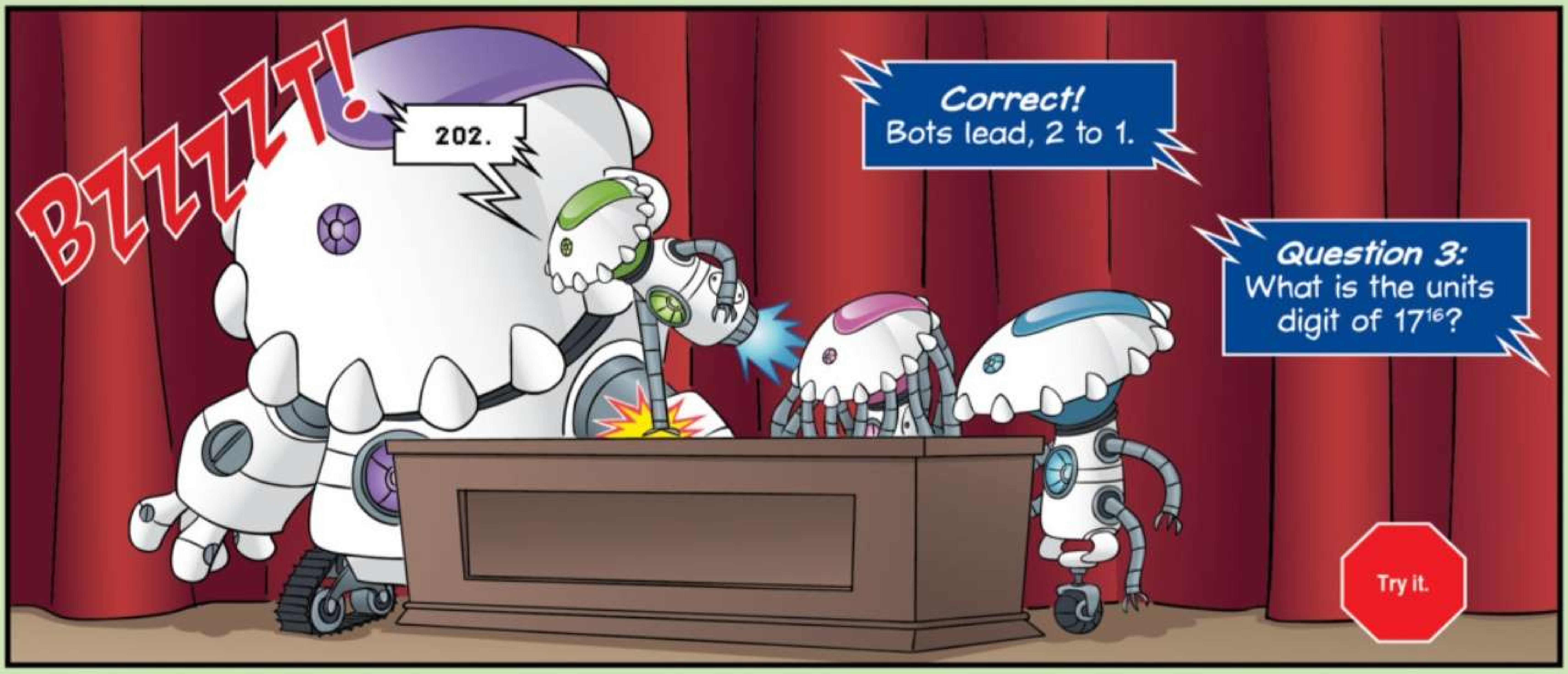


$5^2 \times 8^2$ equals $(5\times8)^2$.
$5\times8=40$, so $5^2\times8^2 = (5\times8)^2 = 40^2$.
40^2 is $40\times40 = 1{,}600$!
$5^2 \times 8^2 = (5\times8)^2 = 40^2 = 1{,}600$
That's a ***lot*** easier than multiplying 25×64.
$5^2 \times 8^2 = 25 \times 64$
25 × 64 = 20 + 80 + 300 + 1200 = 1600
Great job!
This works with other exponents, too.
For example, $15^3\times2^3=(15\times2)^3$.
$15^3 \times 2^3 = 15\times15\times15 \times 2\times2\times2 = (15\times2)\times(15\times2)\times(15\times2) = (15\times2)^3 = 30^3 = 27{,}000$
And now...
Uh-oh! We're out of time.
Do your best!
...please welcome the Little Monsters of Beast Academy and their opponents, the Bots.

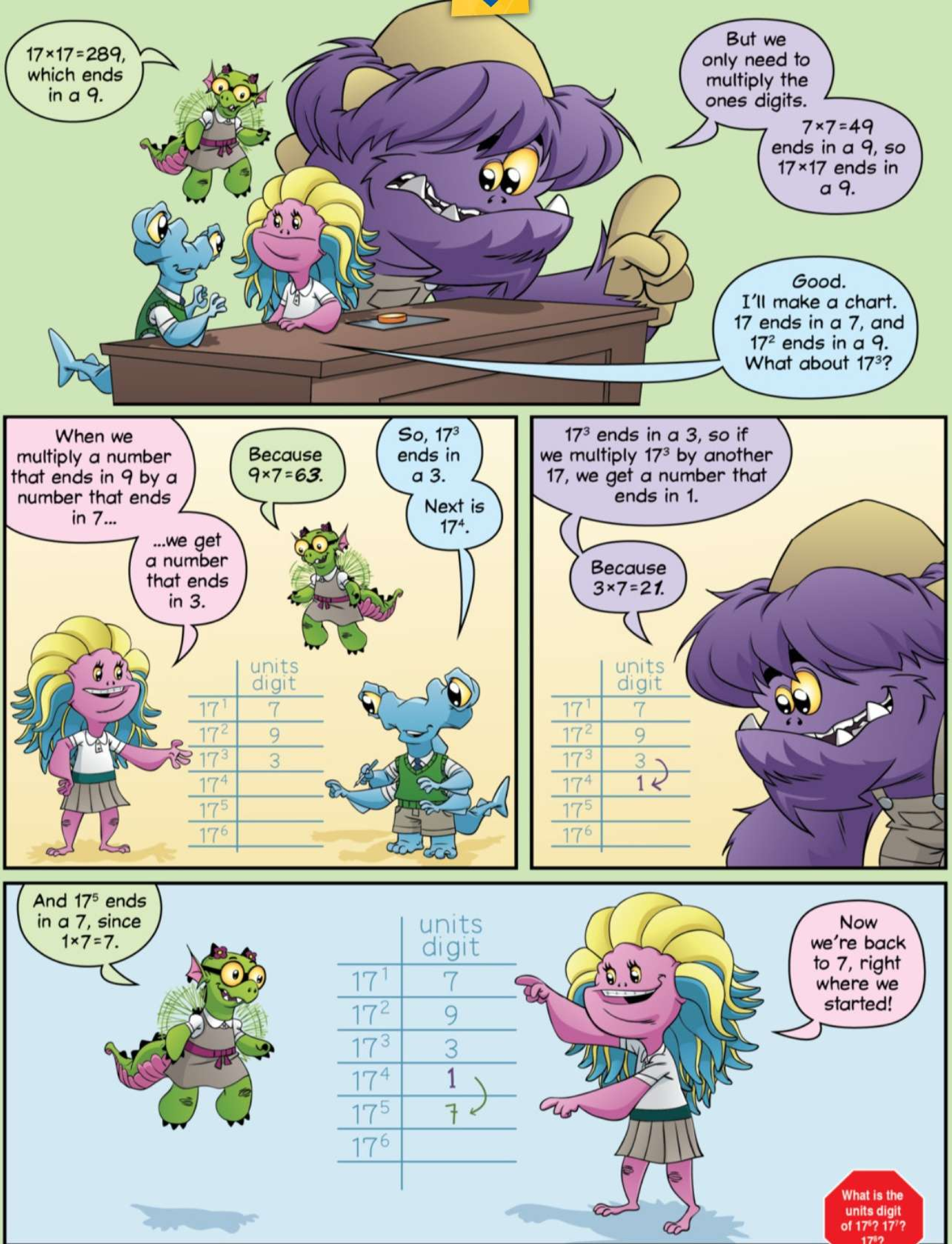

Exponents are the focus of today's math meet. I will ask 6 questions. The first five are each worth one point, and the final question is worth two. The team with the most points wins the meet. Is everyone ready for the first question?
Question 1: Compute 2^8.
Try it.
Ding!
BZZZZT!
256.
256.
We'll have to go to the judges on that one.
The judges award both teams a point!
How did you get that so fast, Lizzie?
I learned all of the powers of 2 up to $2^{10} = 1{,}024$.

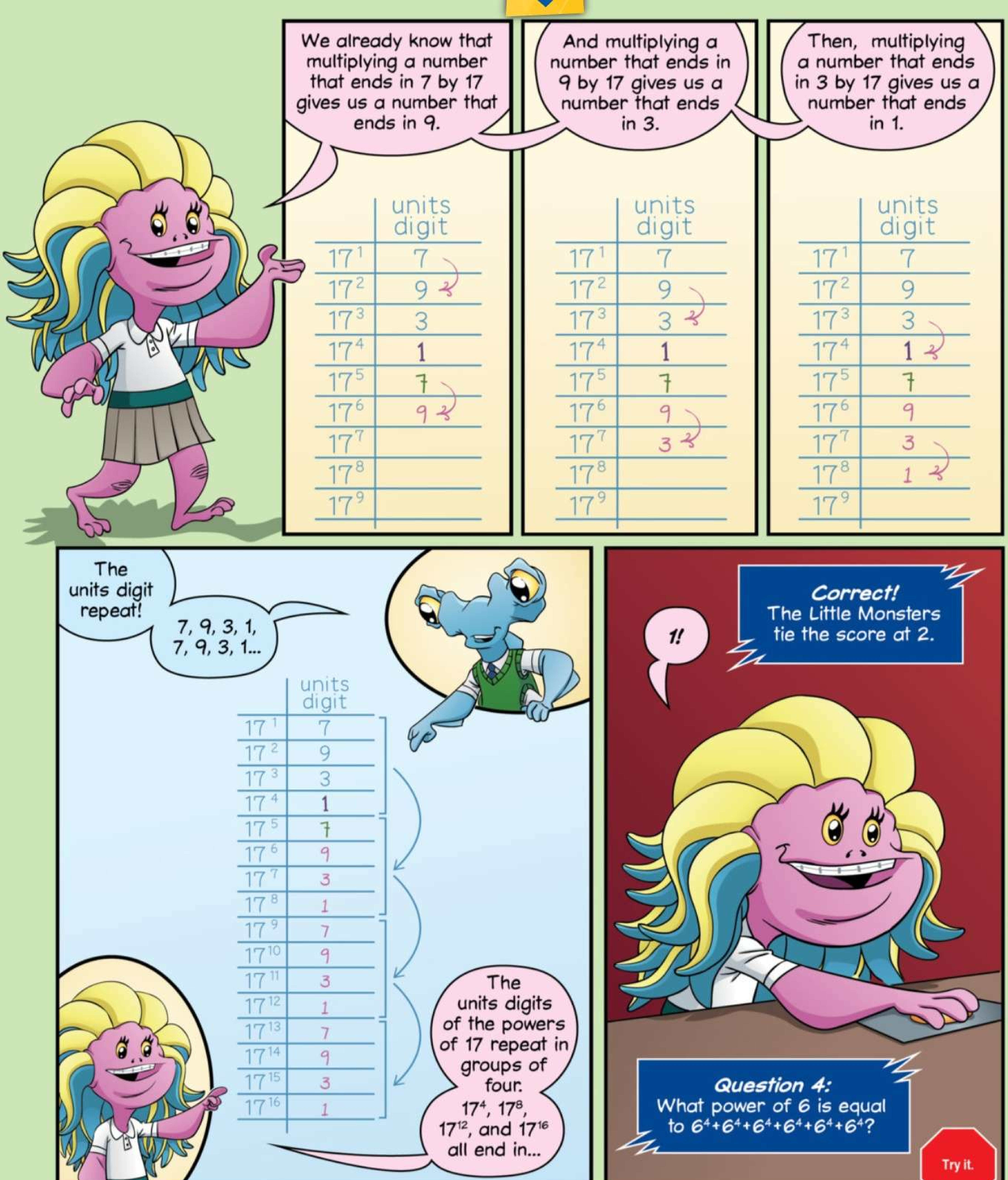
To find 2^8, you can keep doubling until you've multiplied eight 2's!
2 ×2 4 ×2 8 ×2 16 ×2 32 ×2 64 ×2 128 ×2 256
Or, you can pair up the 2's to get 4's...
...then pair the 4's to get 16's...
...then square 16 to get 256!
$2\times2\times2\times2\times2\times2\times2\times2$
$= 4\times4\times4\times4$
$= 16\times16$
$= 256$
Cool! $2^8=4^4=16^2=256$!
Question 2: Compute 9^2+11^2.
I know! 9^2+11^2 is the same as $(9+11)^2$!
Ding!
Wait!
Is Grogg correct?
400!
I'm sorry, that answer is incorrect.
Grogg! That only works if we are *multiplying* numbers that have the same exponents!
$9^2\times11^2$ equals $(9\times11)^2$ because multiplying $(9\times9)\times(11\times11)$ is the same as multiplying $(9\times11)\times(9\times11)$.
$9^2\times11^2$
$= 9\times9\times11\times11$
$=(9\times11)\times(9\times11)$
$= (9\times11)^2$



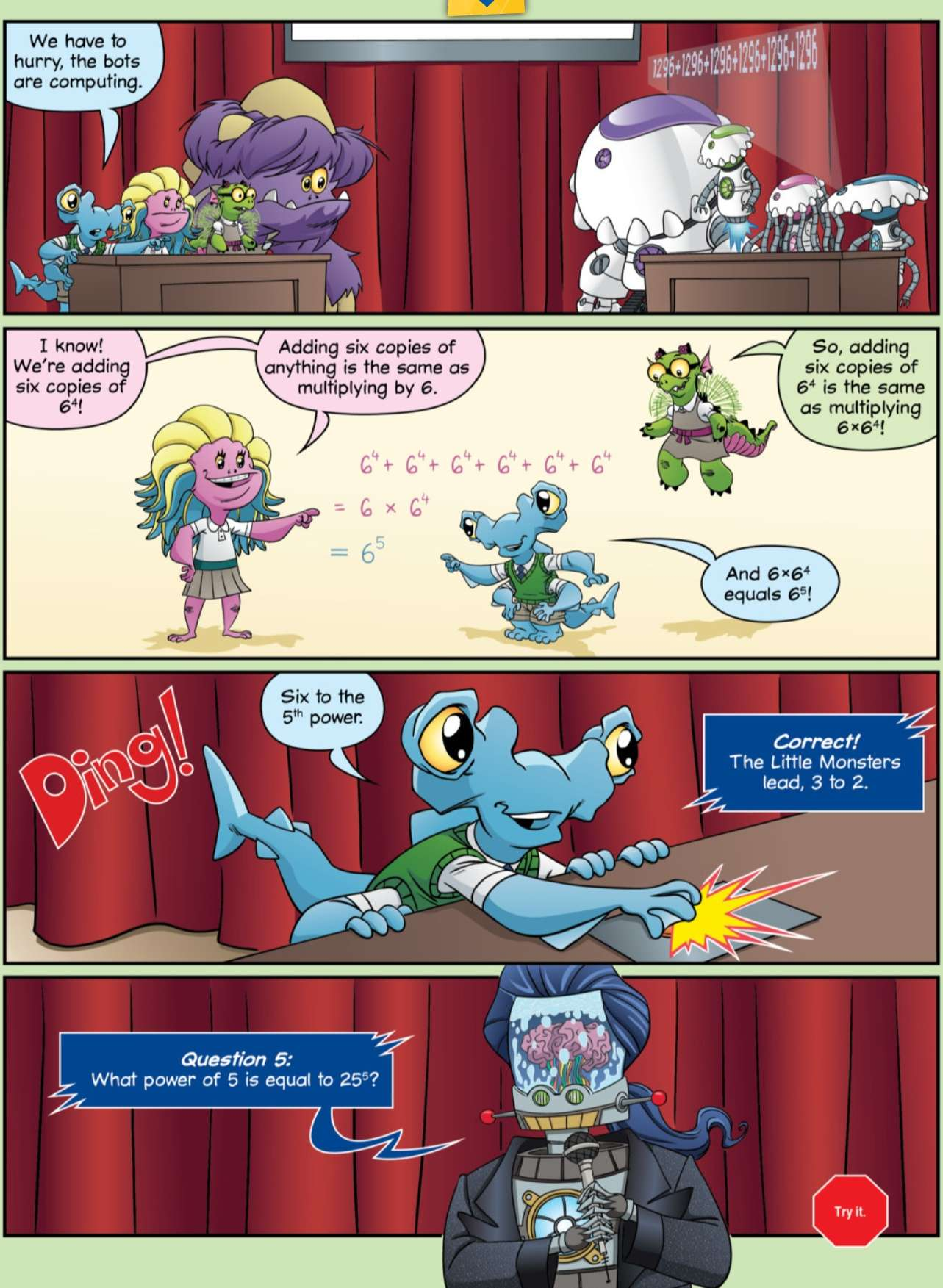


*SEE PAGE 57. NOW, WE HAVE A SHORT WAY TO WRITE THE PRODUCT OF NINETY-NINE 99'S: 99^{99}.

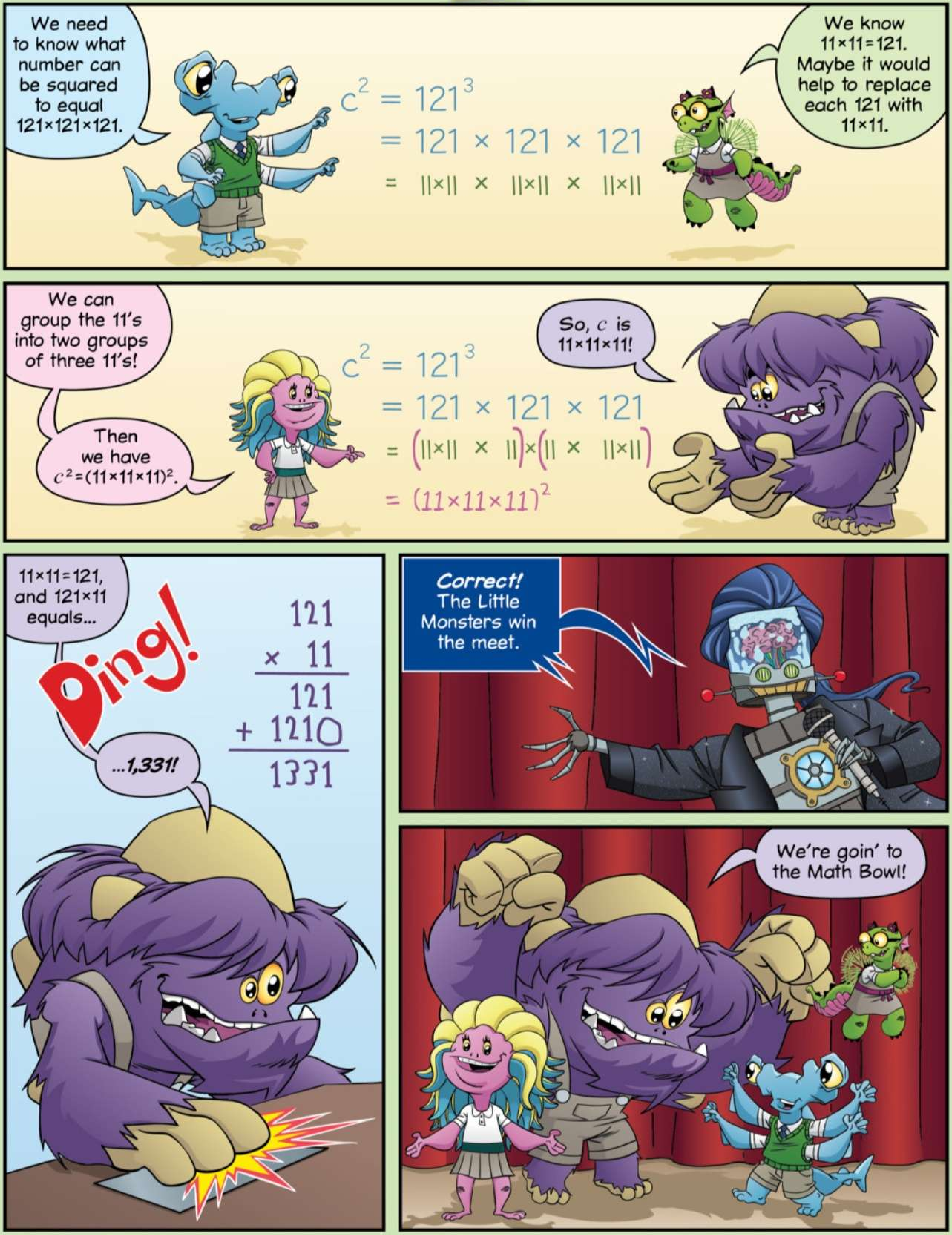

17×17=289, which ends in a 9.
But we only need to multiply the ones digits.
7×7=49 ends in a 9, so 17×17 ends in a 9.
Good. I'll make a chart. 17 ends in a 7, and 17^2 ends in a 9. What about 17^3?
When we multiply a number that ends in 9 by a number that ends in 7...
...we get a number that ends in 3.
Because 9×7=63.
So, 17^3 ends in a 3.
Next is 17^4.
units digit
17^1 7
17^2 9
17^3 3
17^4
17^5
17^6
17^3 ends in a 3, so if we multiply 17^3 by another 17, we get a number that ends in 1.
Because 3×7=21.
units digit
17^1 7
17^2 9
17^3 3
17^4 1
17^5
17^6
And 17^5 ends in a 7, since 1×7=7.
units digit
17^1 7
17^2 9
17^3 3
17^4 1
17^5 7
17^6
Now we're back to 7, right where we started!
What is the units digit of 17^6? 17^7? 17^8?



THE ACTUAL VALUE OF 17^{16} IS 48,661,191,875,666,868,481! TO GIVE YOU AN IDEA OF HOW BIG THAT NUMBER IS, THE DISTANCE FROM THE SUN TO THE NEAREST NEIGHBORING STAR IS ABOUT 48,661,191,875,666,868,481 MILLIMETERS. (THE SIDE LENGTH OF THIS TINY SQUARE ➔▪ IS 1 MILLIMETER.)

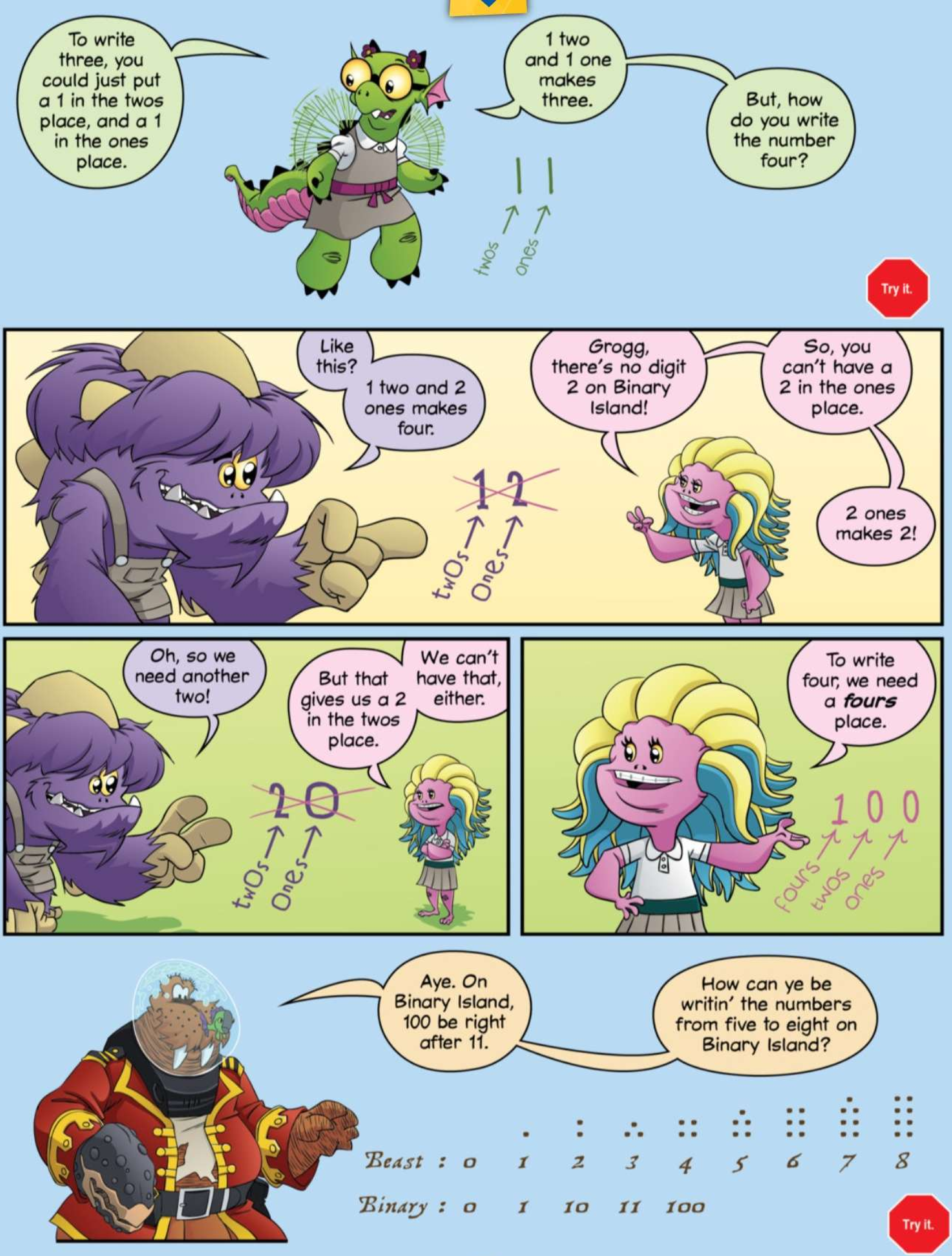

We have to hurry, the bots are computing.
1296+1296+1296+1296+1296+1296
I know! We're adding six copies of 6^4!
Adding six copies of anything is the same as multiplying by 6.
So, adding six copies of 6^4 is the same as multiplying 6×6^4!
$6^4 + 6^4 + 6^4 + 6^4 + 6^4 + 6^4$
$= 6 \times 6^4$
$= 6^5$
And 6×6^4 equals 6^5!
Ding!
Six to the 5th power.
Correct! The Little Monsters lead, 3 to 2.
Question 5: What power of 5 is equal to 25^5?
Try it.

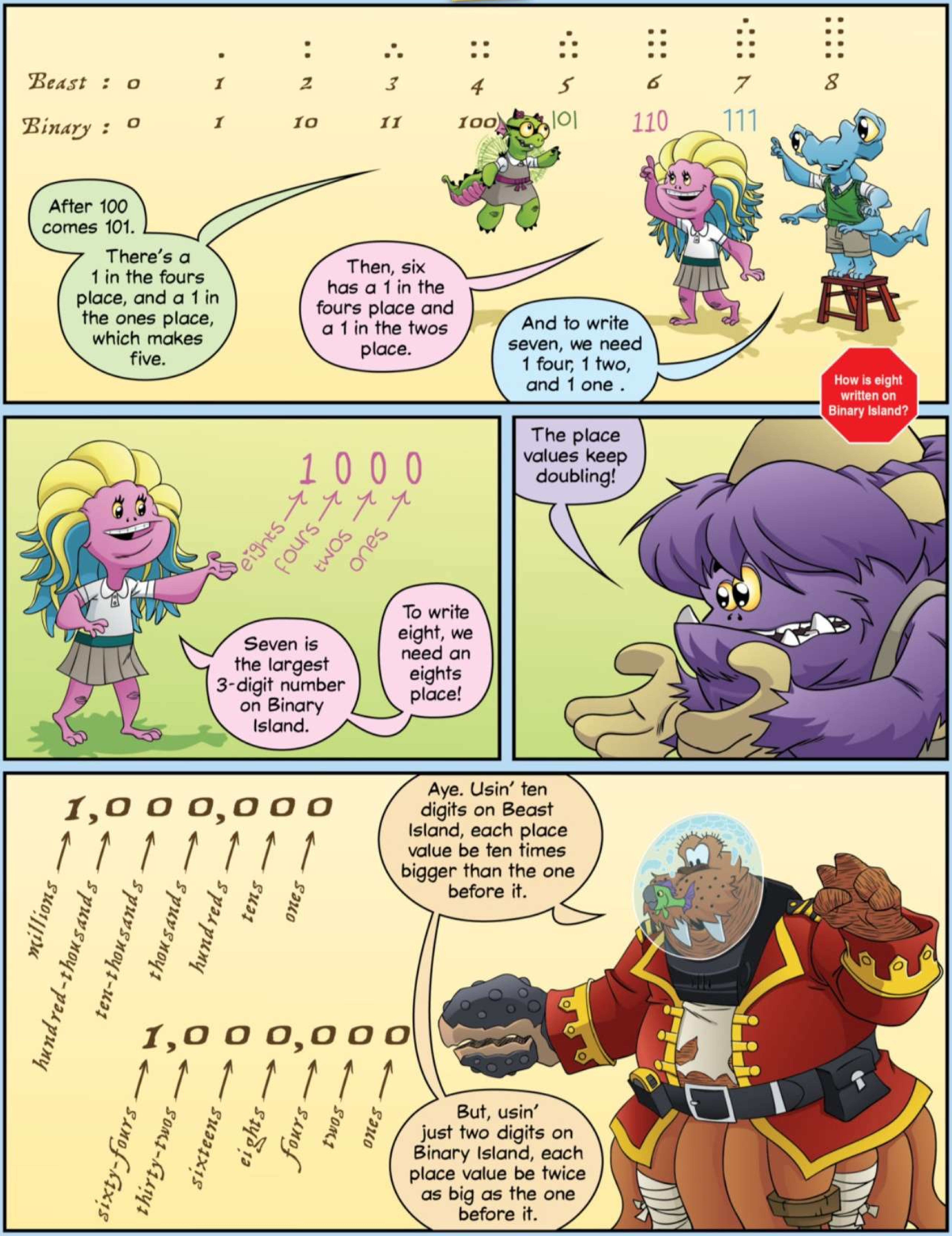
25⁵ means we multiply five 25's.
We need to know how many 5's must be multiplied to equal the product of five 25's.
25⁵ = 25 × 25 × 25 × 25 × 25
Multiplying two 5's gives us 25!
25⁵ = 25 × 25 × 25 × 25 × 25
= 5×5 × 5×5 × 5×5 × 5×5 × 5×5
Since 5×5=25, we can replace each 25 with 5×5.
So, 25 to the 5th power equals 5 to the --
BZZZZZT!
5¹⁰
Five to the tenth power is correct! The bots have tied the score with just one question remaining.
25⁵ = 9,765,625
5¹ = 5
5² = 25
5³ = 125
5⁴ = 625
5⁵ = 3,125
5⁶ = 15,625
5⁷ = 78,125
5⁸ = 390,625
5⁹ = 1,953,125
5¹⁰ = 9,765,625
The last question will determine today's math meet winner.
Question 6:
If c² = 121³, what is c?


We need to know what number can be squared to equal 121×121×121.
$c^2 = 121^3$
$= 121 \times 121 \times 121$
$= 11\times11 \times 11\times11 \times 11\times11$
We know 11×11=121. Maybe it would help to replace each 121 with 11×11.
We can group the 11's into two groups of three 11's!
$c^2 = 121^3$
$= 121 \times 121 \times 121$
$= (11\times11 \times 11)\times(11 \times 11\times11)$
$= (11\times11\times11)^2$
Then we have $c^2=(11\times11\times11)^2$.
So, c is 11×11×11!
11×11=121, and 121×11 equals...
Ding!
...1,331!
121
× 11
121
+ 1210
1331
Correct! The Little Monsters win the meet.
We're goin' to the Math Bowl!

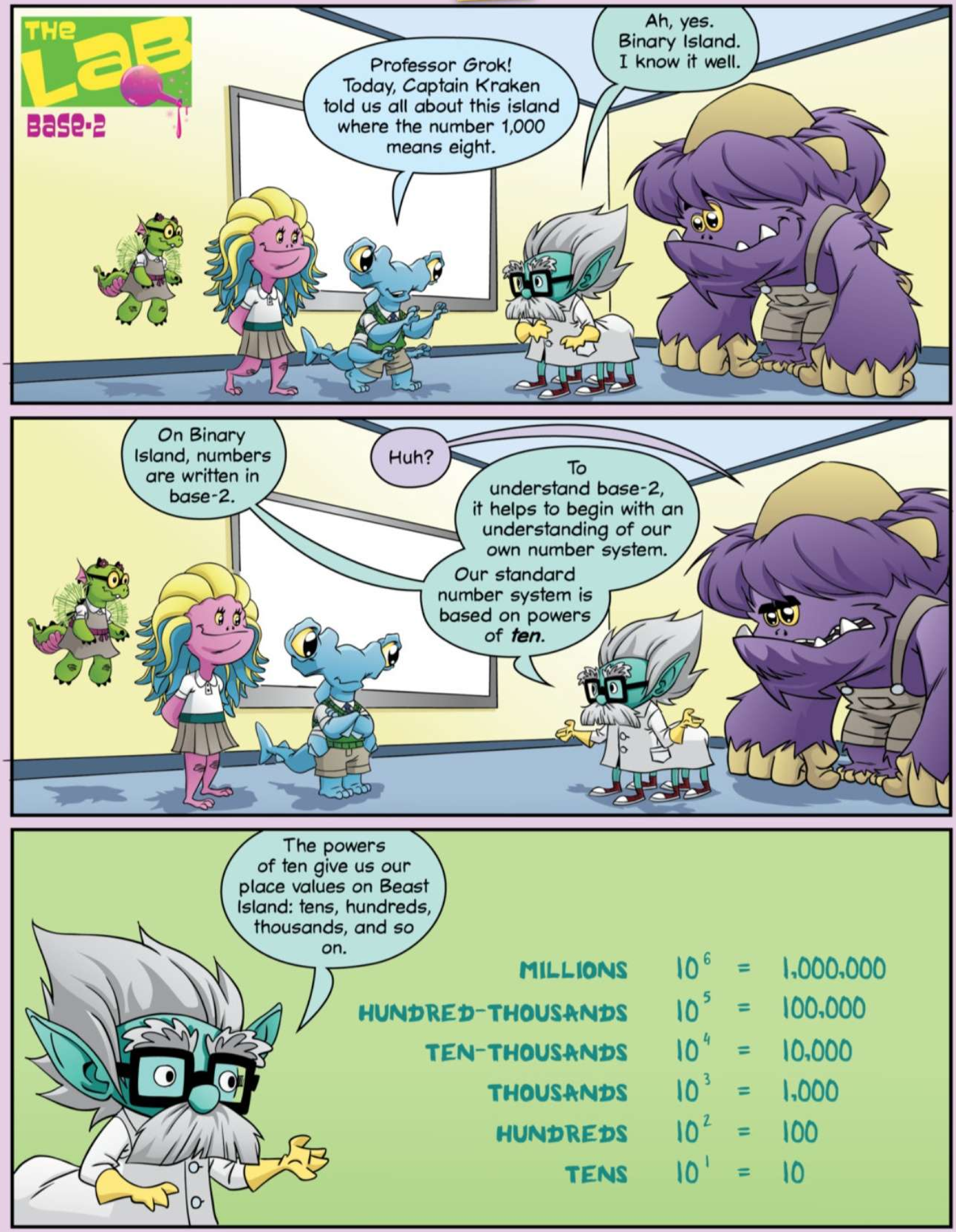

WorkShop
BINARY ISLAND
Captain Kraken, is math the same everywhere?
Arrr... What do ya mean, lad?
Great question!
I mean, in all of your pirate adventures, have you ever met monsters that do math differently?
There be one island where numbers look very different than they do here on Beast Island.
'Tis called Binary Island.
What's so different about math on Binary Island?
On Beast Island, there be ***ten*** digits... 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.
But on Binary Island, there be only ***two*** digits! 0 and 1.
Weird!
Whoa.
How does ***that*** work?

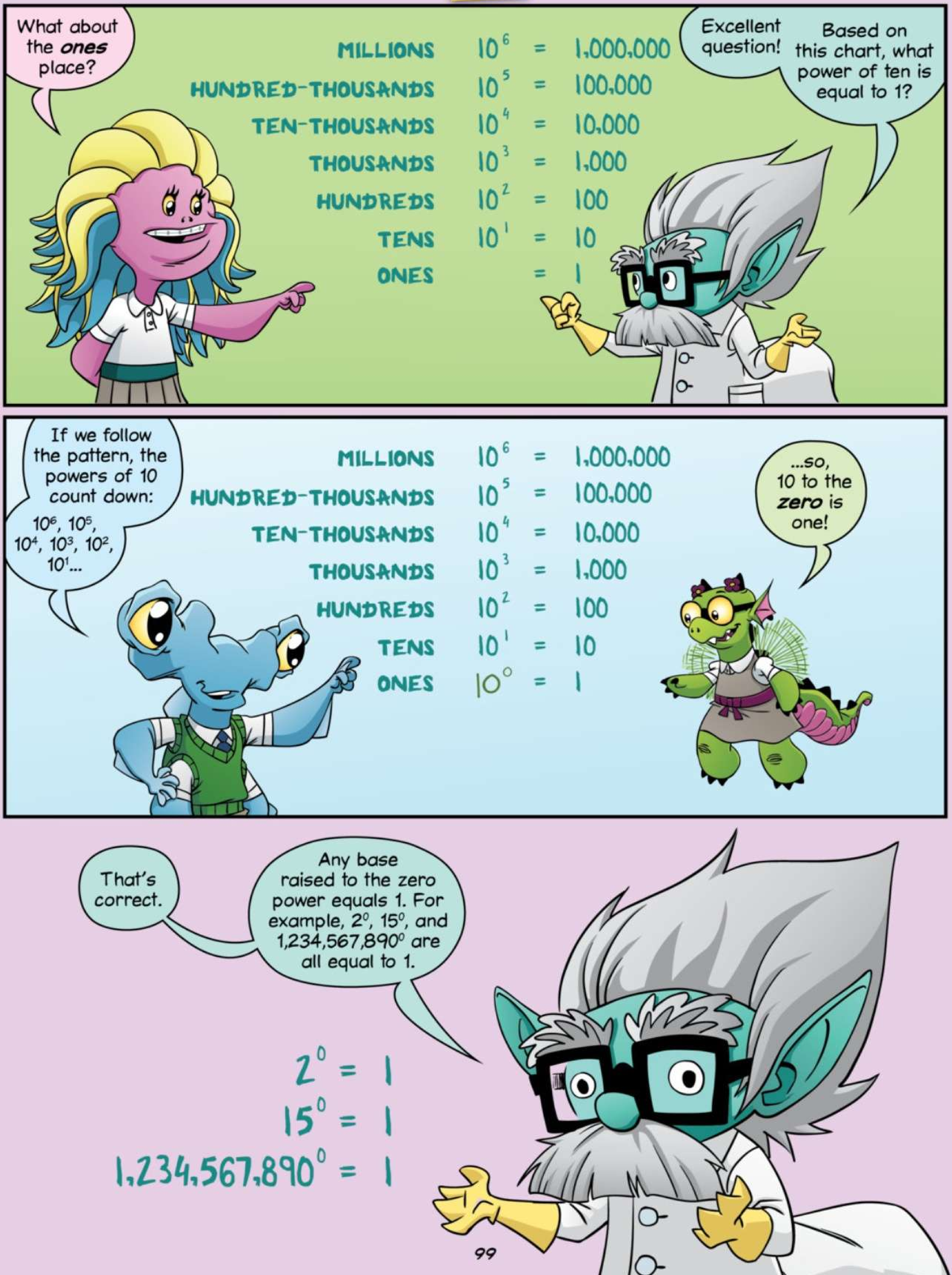

0 1 2 3 4 5 6 7 8 9 10 ...
On Beast Island, we can write the numbers zero through nine usin' only one o' the ten digits.
But, we be needin' a **tens** place to write the number ten.
With just two digits on Binary Island, they can only write the numbers zero and one with a single digit.
Do they need a **twos** place to write the number two?
Aye.
There be no digit 2 on Binary Island. So, to write two, they be needin' a **twos** place.
zero one two
0 1 10
Whoa. So, on Binary Island, this number means "two"?
I get it.
On Beast Island, this number has 1 ten and 0 ones.
Beast Island
10 =
tens
ones
But on Binary Island, the 1 is in the **twos** place. So, this number has 1 two and 0 ones.
Binary Island
10 =
twos
ones
How do they write the number three on Binary Island?
Try it.

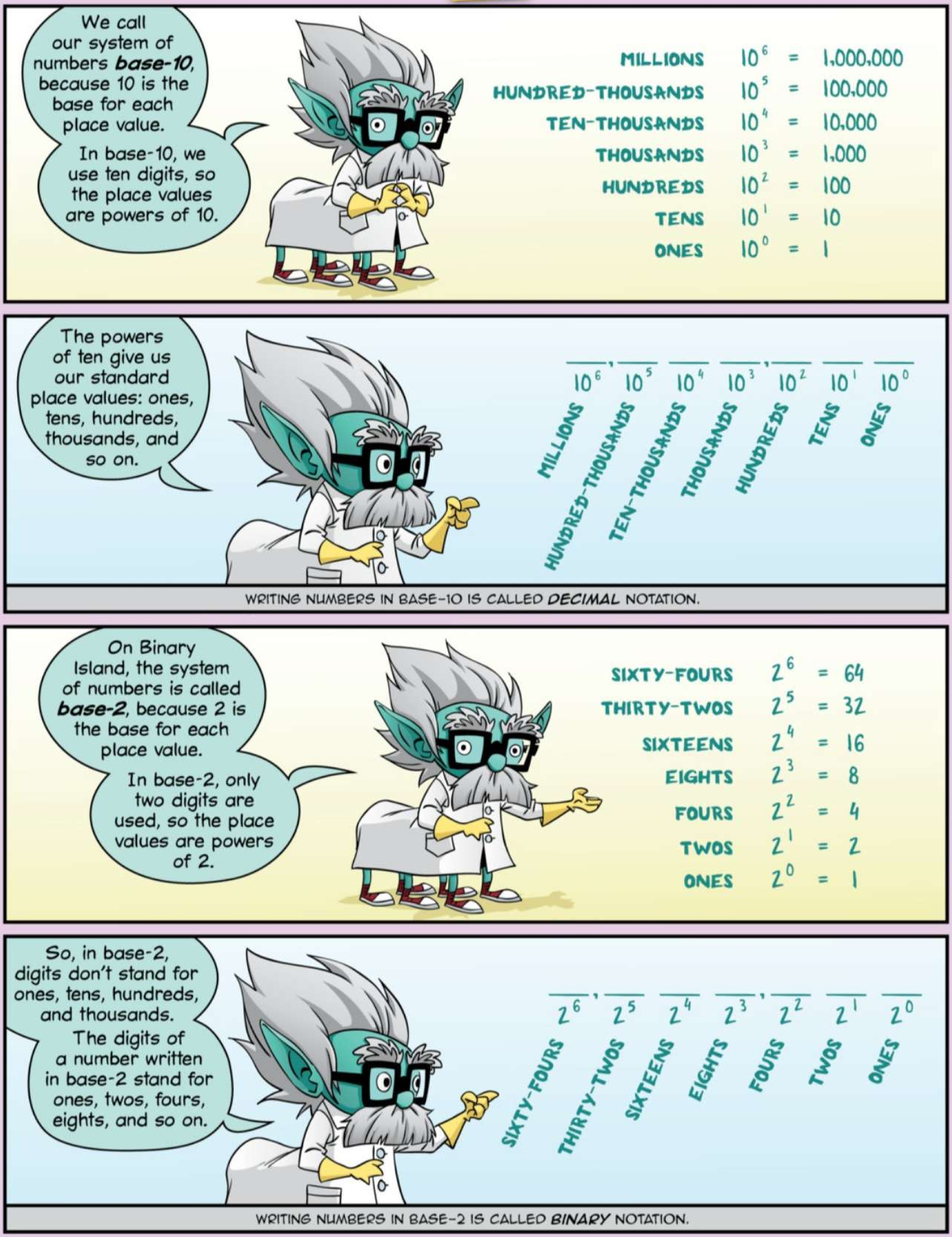

To write three, you could just put a 1 in the twos place, and a 1 in the ones place.
1 two and 1 one makes three.
But, how do you write the number four?
1 1
twos ones
Try it.
Like this?
1 two and 2 ones makes four.
Grogg, there's no digit 2 on Binary Island!
So, you can't have a 2 in the ones place.
2 ones makes 2!
~~1 2~~
twos ones
Oh, so we need another two!
But that gives us a 2 in the twos place.
We can't have that, either.
~~2 0~~
twos ones
To write four, we need a ***fours*** place.
1 0 0
fours twos ones
Aye. On Binary Island, 100 be right after 11.
How can ye be writin' the numbers from five to eight on Binary Island?
Beast : 0 1 2 3 4 5 6 7 8
Binary : 0 1 10 11 100
Try it.

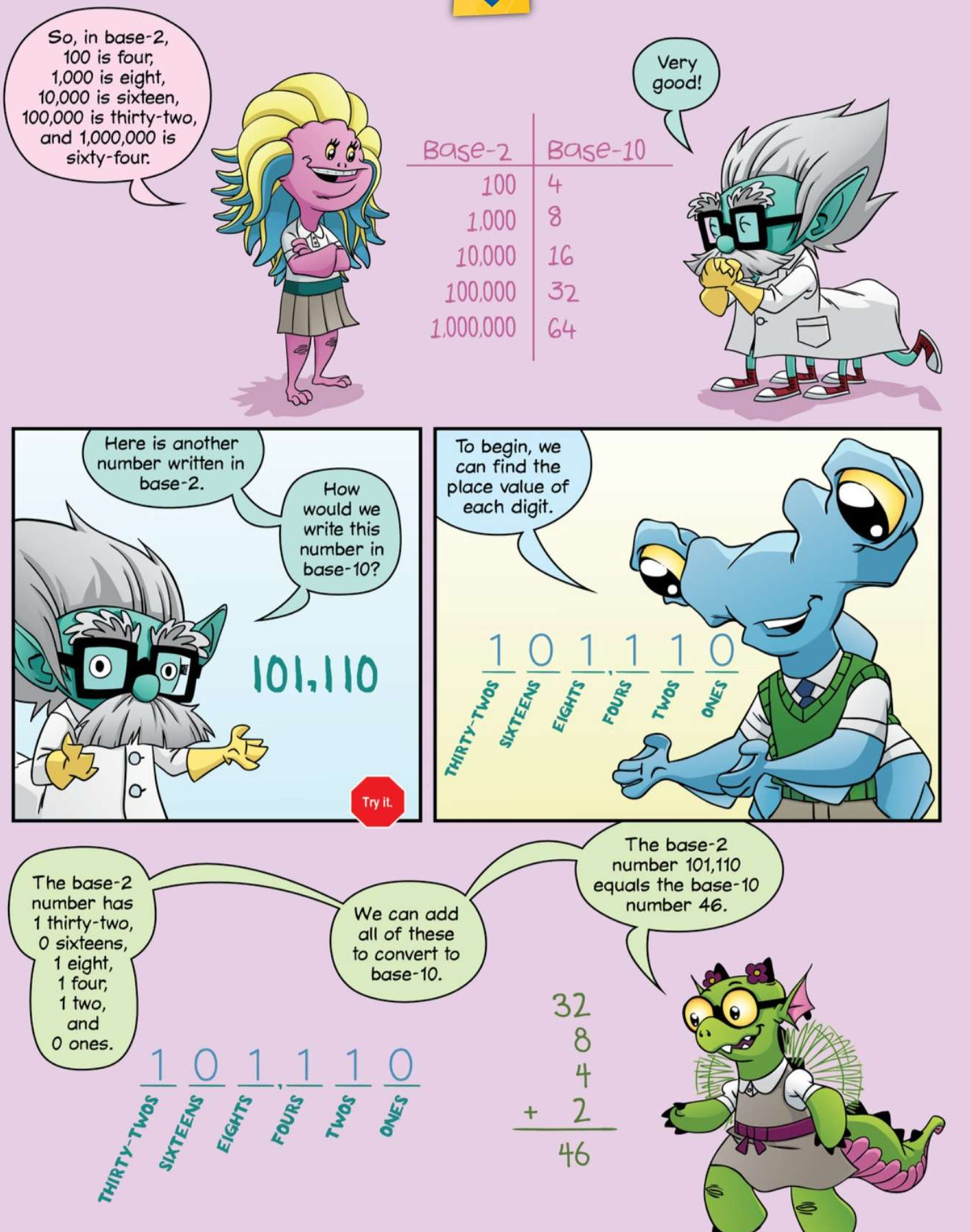

Beast : 0 1 2 3 4 5 6 7 8
Binary : 0 1 10 11 100 101 110 111
After 100 comes 101.
There's a 1 in the fours place, and a 1 in the ones place, which makes five.
Then, six has a 1 in the fours place and a 1 in the twos place.
And to write seven, we need 1 four, 1 two, and 1 one .
How is eight written on Binary Island?
1 0 0 0
eights fours twos ones
Seven is the largest 3-digit number on Binary Island.
To write eight, we need an eights place!
The place values keep doubling!
1,000,000
millions hundred-thousands ten-thousands thousands hundreds tens ones
Aye. Usin' ten digits on Beast Island, each place value be ten times bigger than the one before it.
1,000,000
sixty-fours thirty-twos sixteens eights fours twos ones
But, usin' just two digits on Binary Island, each place value be twice as big as the one before it.

So, on Binary Island, 10,000 means sixteen.
Happy Sweet 10,000
And 100,000 means thirty-two.
MONSTER GULP
1qt. (100,000oz.)
And 1,000,000 means sixty-four!
1,000,000 CRAYONS
Aye. On Binary Island, 1,000,000 be the number sixty-four.
Imagine me disappointment upon voyagin' to Binary Island in search o' the famed treasure of 1,000,000 pearls...
...only to discover that on Binary Island, 1,000,000 means sixty-four.

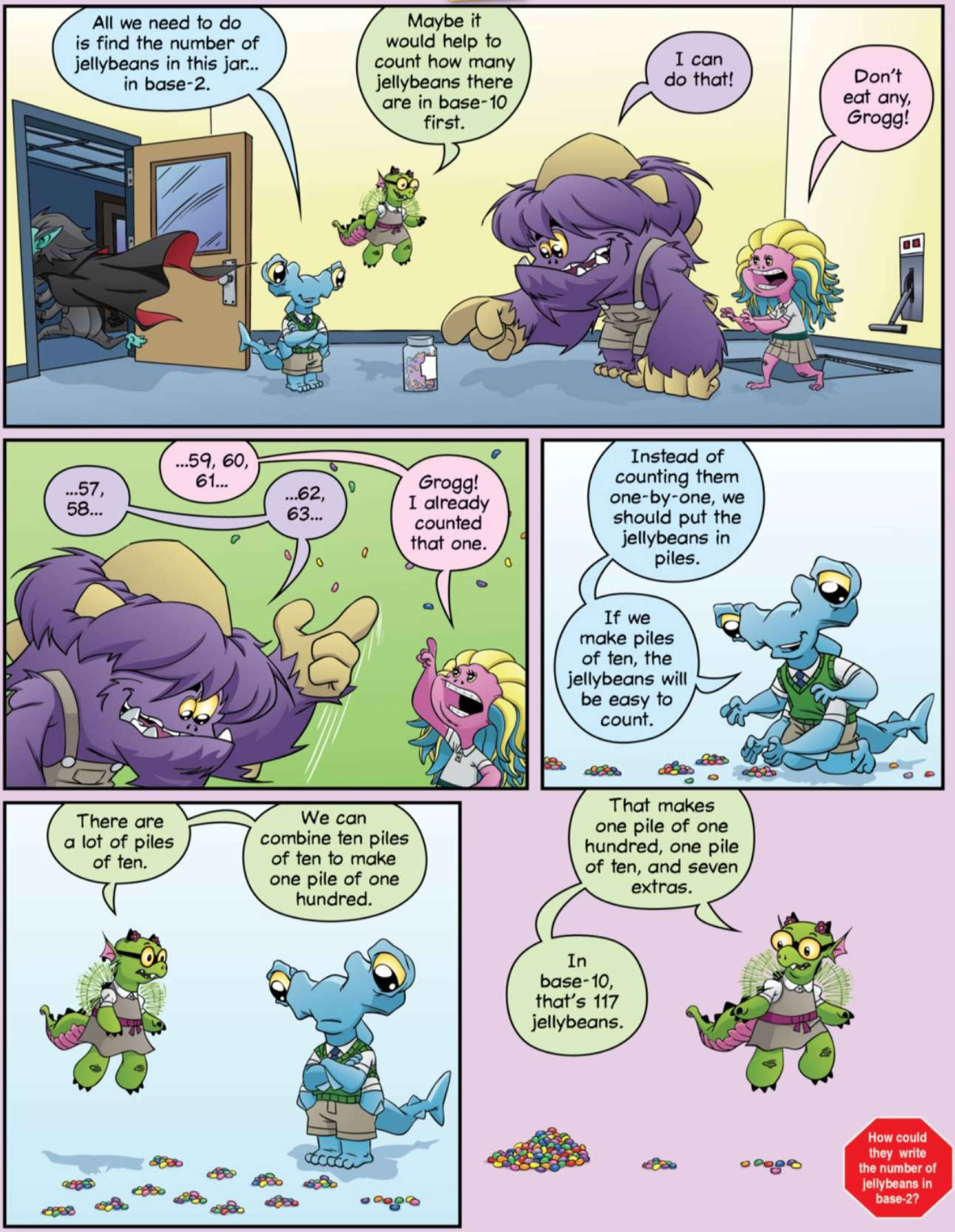

THE LAB
BASE-2
Professor Grok! Today, Captain Kraken told us all about this island where the number 1,000 means eight.
Ah, yes. Binary Island. I know it well.
On Binary Island, numbers are written in base-2.
Huh?
To understand base-2, it helps to begin with an understanding of our own number system.
Our standard number system is based on powers of *ten*.
The powers of ten give us our place values on Beast Island: tens, hundreds, thousands, and so on.
MILLIONS 10^6 = 1,000,000
HUNDRED-THOUSANDS 10^5 = 100,000
TEN-THOUSANDS 10^4 = 10,000
THOUSANDS 10^3 = 1,000
HUNDREDS 10^2 = 100
TENS 10^1 = 10

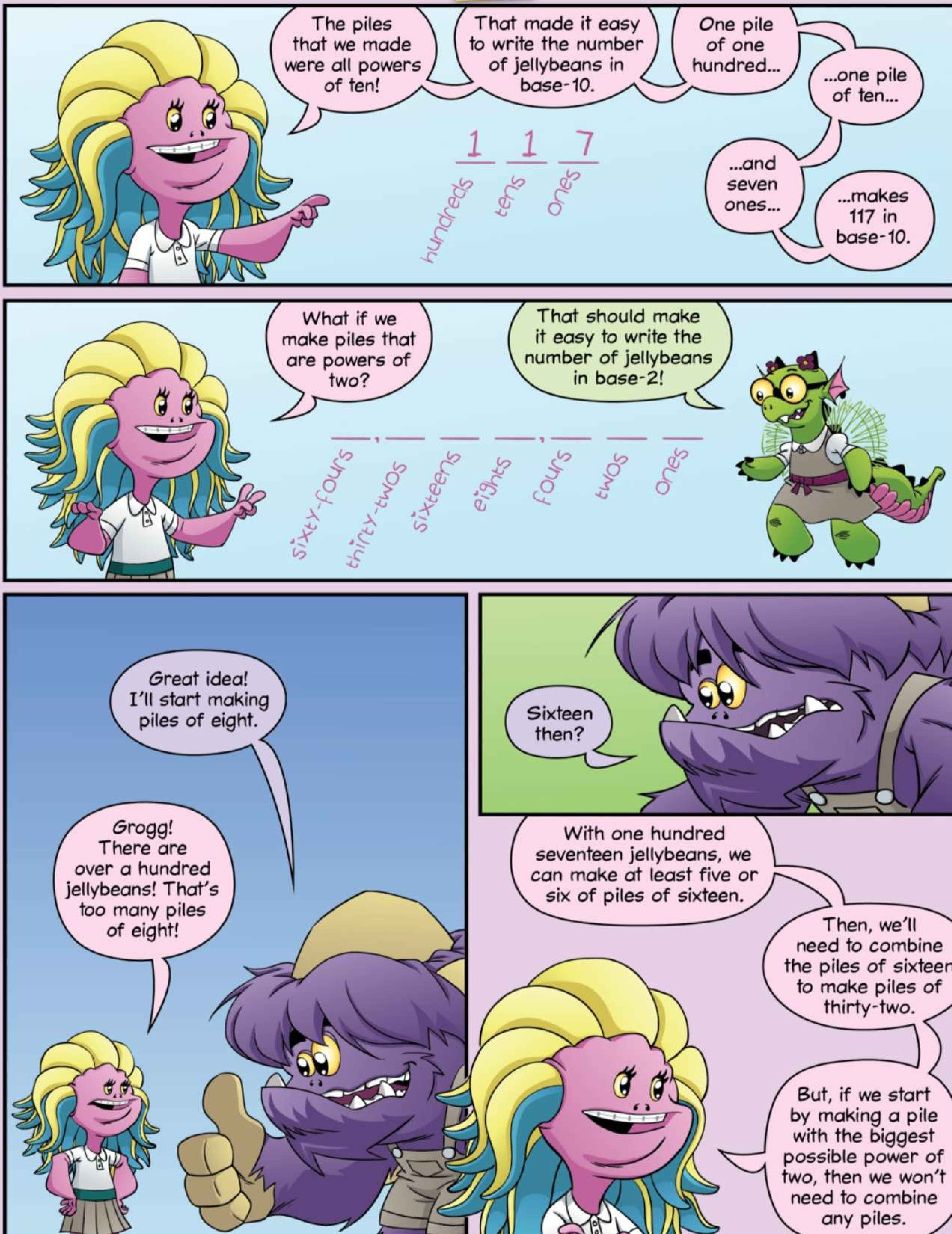

What about the *ones* place?
MILLIONS 10^6 = 1,000,000
HUNDRED-THOUSANDS 10^5 = 100,000
TEN-THOUSANDS 10^4 = 10,000
THOUSANDS 10^3 = 1,000
HUNDREDS 10^2 = 100
TENS 10^1 = 10
ONES = 1
Excellent question!
Based on this chart, what power of ten is equal to 1?
If we follow the pattern, the powers of 10 count down:
10^6, 10^5, 10^4, 10^3, 10^2, 10^1...
MILLIONS 10^6 = 1,000,000
HUNDRED-THOUSANDS 10^5 = 100,000
TEN-THOUSANDS 10^4 = 10,000
THOUSANDS 10^3 = 1,000
HUNDREDS 10^2 = 100
TENS 10^1 = 10
ONES 10^0 = 1
...so, 10 to the *zero* is one!
That's correct.
Any base raised to the zero power equals 1. For example, 2^0, 15^0, and $1{,}234{,}567{,}890^0$ are all equal to 1.
$2^0 = 1$
$15^0 = 1$
$1{,}234{,}567{,}890^0 = 1$

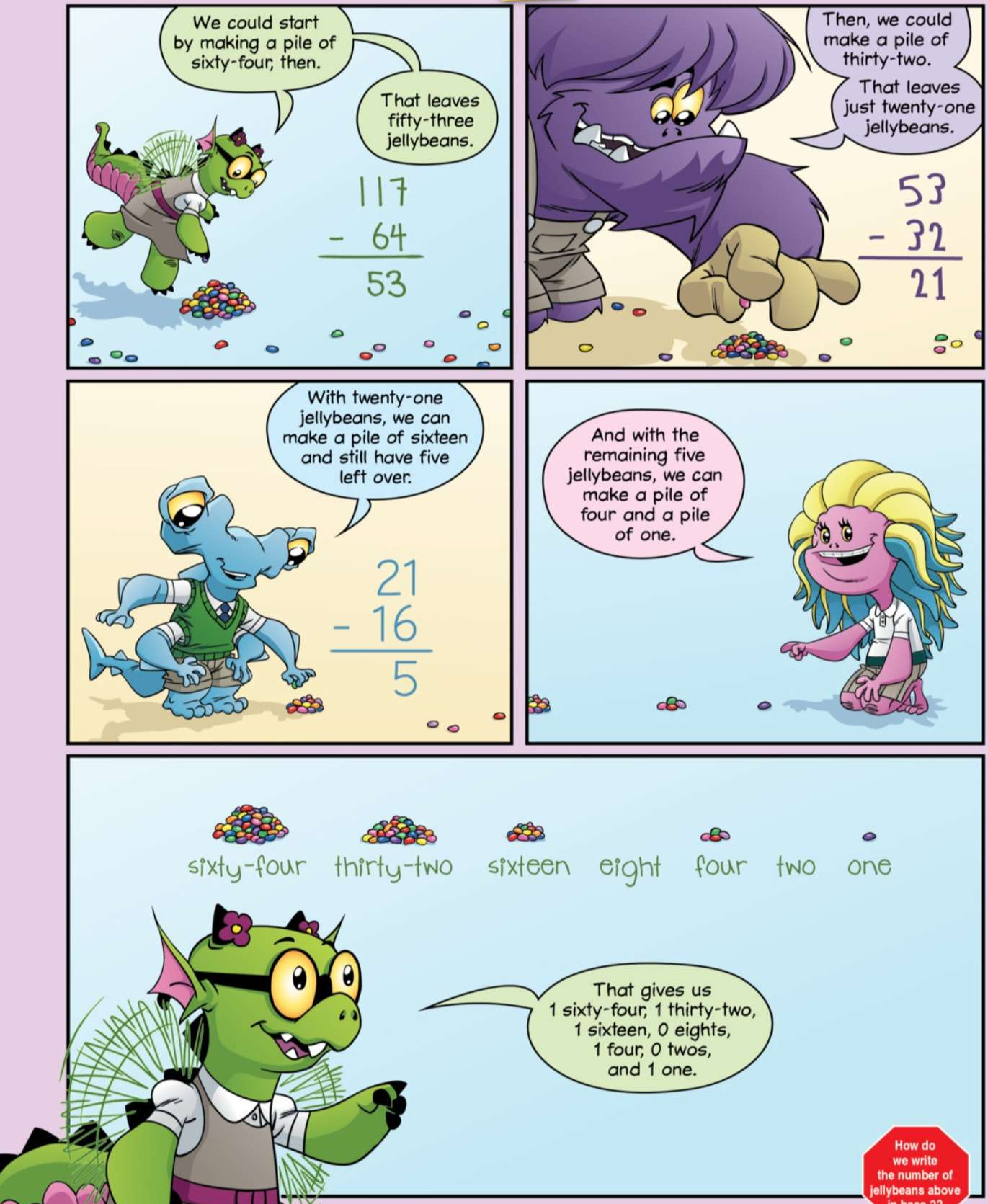

We call our system of numbers **base-10**, because 10 is the base for each place value.
In base-10, we use ten digits, so the place values are powers of 10.
MILLIONS 10^6 = 1,000,000
HUNDRED-THOUSANDS 10^5 = 100,000
TEN-THOUSANDS 10^4 = 10,000
THOUSANDS 10^3 = 1,000
HUNDREDS 10^2 = 100
TENS 10^1 = 10
ONES 10^0 = 1
The powers of ten give us our standard place values: ones, tens, hundreds, thousands, and so on.
10^6 , 10^5 10^4 10^3 , 10^2 10^1 10^0
MILLIONS
HUNDRED-THOUSANDS
TEN-THOUSANDS
THOUSANDS
HUNDREDS
TENS
ONES
WRITING NUMBERS IN BASE-10 IS CALLED *DECIMAL* NOTATION.
On Binary Island, the system of numbers is called **base-2**, because 2 is the base for each place value.
In base-2, only two digits are used, so the place values are powers of 2.
SIXTY-FOURS 2^6 = 64
THIRTY-TWOS 2^5 = 32
SIXTEENS 2^4 = 16
EIGHTS 2^3 = 8
FOURS 2^2 = 4
TWOS 2^1 = 2
ONES 2^0 = 1
So, in base-2, digits don't stand for ones, tens, hundreds, and thousands.
The digits of a number written in base-2 stand for ones, twos, fours, eights, and so on.
2^6 , 2^5 2^4 2^3 , 2^2 2^1 2^0
SIXTY-FOURS
THIRTY-TWOS
SIXTEENS
EIGHTS
FOURS
TWOS
ONES
WRITING NUMBERS IN BASE-2 IS CALLED *BINARY* NOTATION.


So, in base-2, 100 is four, 1,000 is eight, 10,000 is sixteen, 100,000 is thirty-two, and 1,000,000 is sixty-four.
Very good!
Base-2 | Base-10
100 | 4
1,000 | 8
10,000 | 16
100,000 | 32
1,000,000 | 64
Here is another number written in base-2.
How would we write this number in base-10?
101,110
Try it.
To begin, we can find the place value of each digit.
1 0 1, 1 1 0
THIRTY-TWOS SIXTEENS EIGHTS FOURS TWOS ONES
The base-2 number has 1 thirty-two, 0 sixteens, 1 eight, 1 four, 1 two, and 0 ones.
We can add all of these to convert to base-10.
The base-2 number 101,110 equals the base-10 number 46.
1 0 1, 1 1 0
THIRTY-TWOS SIXTEENS EIGHTS FOURS TWOS ONES
32
8
4
+ 2
46

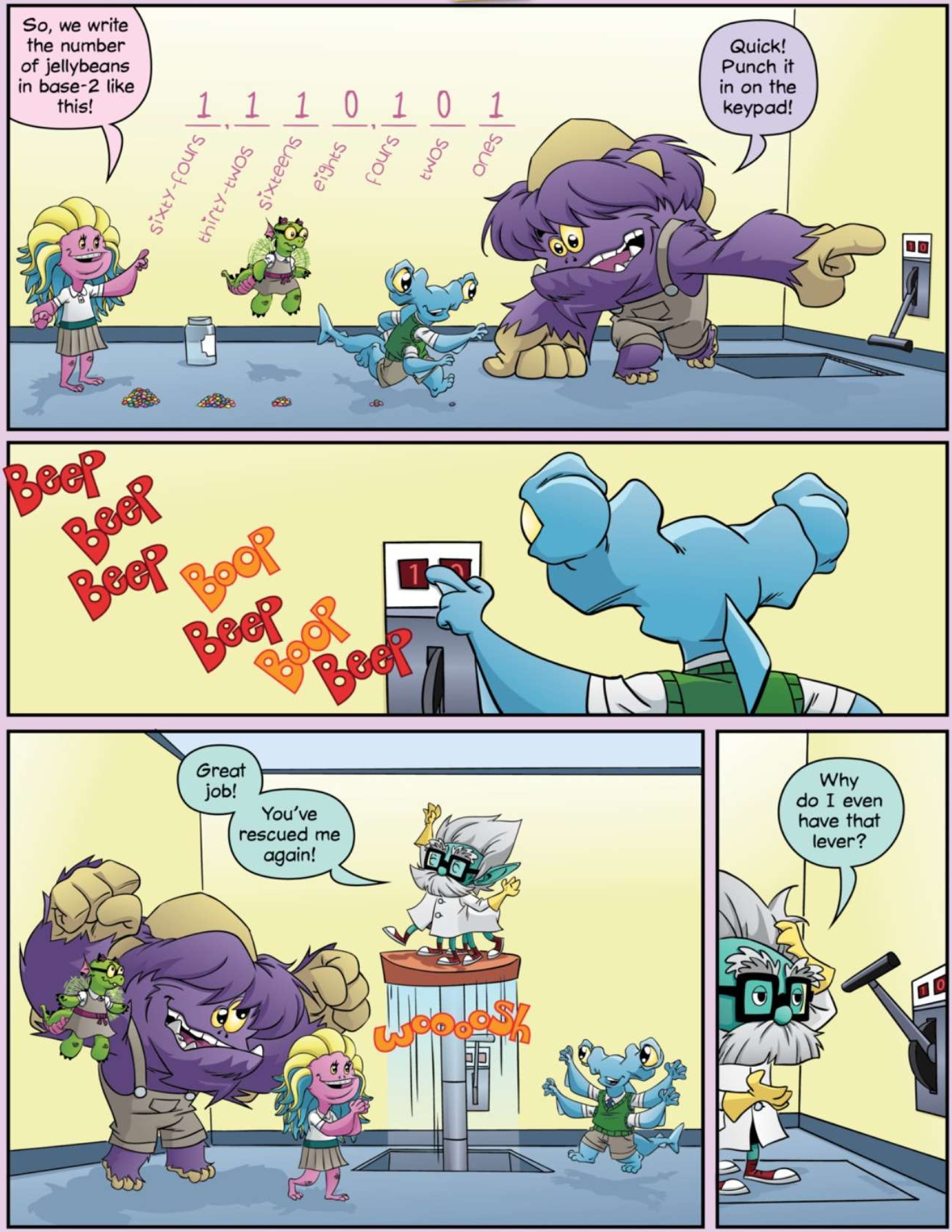

That's right. If you understand place value, it is not difficult to convert any number written in base-2 to base-10.
Hmmm... I don't remember this lever being here.
10
WoOoOoSh
Bwah Hah Hah! Professor Grok is gone! I've abducted your educator! It's time for something diabolically difficult!
The combination code required to retrieve your professor is the number of jellybeans in this jar.
But this keypad only has *two* digits!
Undoubtedly!
The combination code is in ***base-2!***
While base-10 tallying is terribly trivial....
...counting a collection of items in base-2 can be ***delightfully difficult.***
Key in the base-2 code correctly, or...
...your ***Professor's Imprisonment*** will be ***Permanent!***


All we need to do is find the number of jellybeans in this jar... in base-2.
Maybe it would help to count how many jellybeans there are in base-10 first.
I can do that!
Don't eat any, Grogg!
...57, 58...
...59, 60, 61...
...62, 63...
Grogg! I already counted that one.
Instead of counting them one-by-one, we should put the jellybeans in piles.
If we make piles of ten, the jellybeans will be easy to count.
There are a lot of piles of ten.
We can combine ten piles of ten to make one pile of one hundred.
That makes one pile of one hundred, one pile of ten, and seven extras.
In base-10, that's 117 jellybeans.
How could they write the number of jellybeans in base-2?


The piles that we made were all powers of ten!
That made it easy to write the number of jellybeans in base-10.
One pile of one hundred...
...one pile of ten...
...and seven ones...
...makes 117 in base-10.
1 1 7
hundreds tens ones
What if we make piles that are powers of two?
That should make it easy to write the number of jellybeans in base-2!
sixty-fours thirty-twos sixteens eights fours twos ones
Great idea! I'll start making piles of eight.
Grogg! There are over a hundred jellybeans! That's too many piles of eight!
Sixteen then?
With one hundred seventeen jellybeans, we can make at least five or six of piles of sixteen.
Then, we'll need to combine the piles of sixteen to make piles of thirty-two.
But, if we start by making a pile with the biggest possible power of two, then we won't need to combine any piles.



We could start by making a pile of sixty-four, then.
That leaves fifty-three jellybeans.
117
- 64
53
Then, we could make a pile of thirty-two.
That leaves just twenty-one jellybeans.
53
- 32
21
With twenty-one jellybeans, we can make a pile of sixteen and still have five left over.
21
- 16
5
And with the remaining five jellybeans, we can make a pile of four and a pile of one.
sixty-four thirty-two sixteen eight four two one
That gives us 1 sixty-four, 1 thirty-two, 1 sixteen, 0 eights, 1 four, 0 twos, and 1 one.
How do we write the number of jellybeans above in base-2?



So, we write the number of jellybeans in base-2 like this!
1 1 1 0 1 0 1
sixty-fours thirty-twos sixteens eights fours twos ones
Quick! Punch it in on the keypad!
Beep Beep Beep Boop Beep Boop Beep
Great job!
You've rescued me again!
Wooooosh
Why do I even have that lever?

Grogg

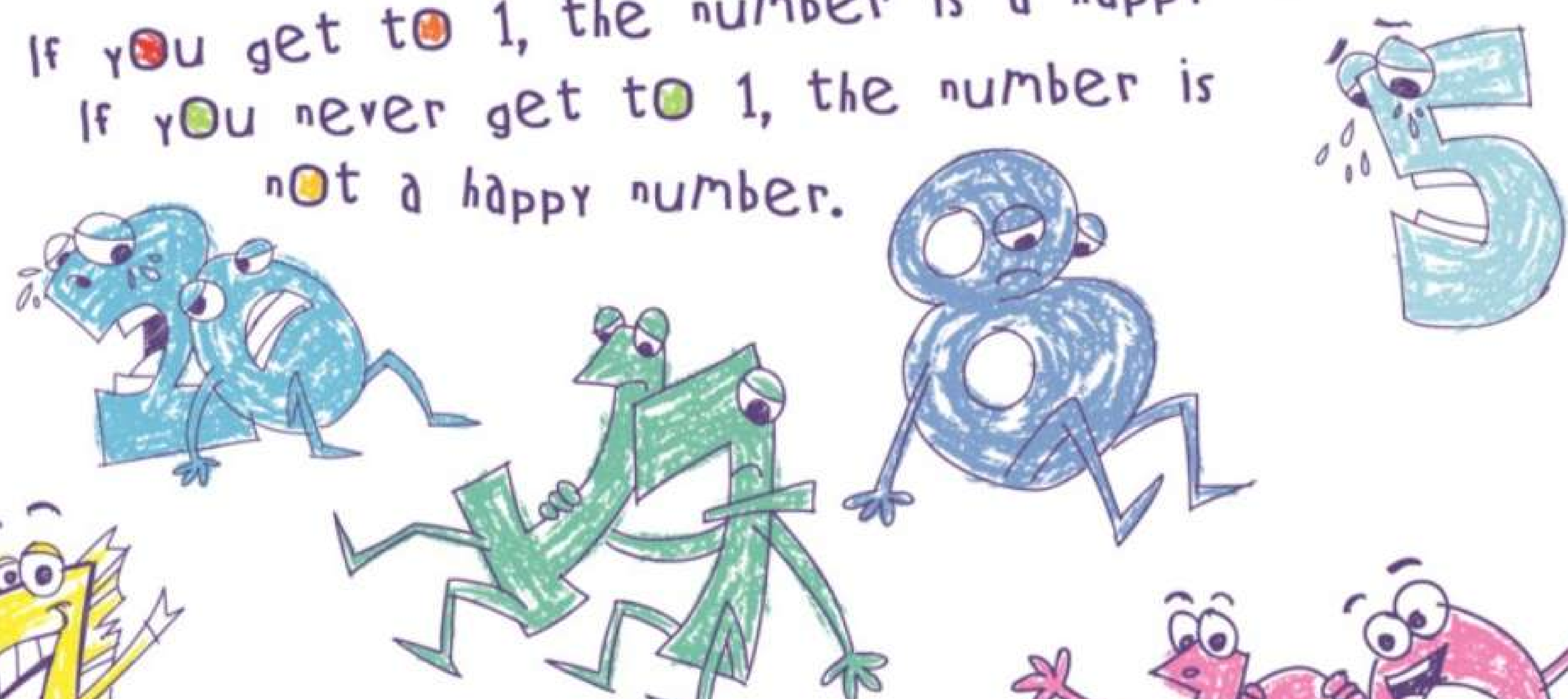
Happy numbers

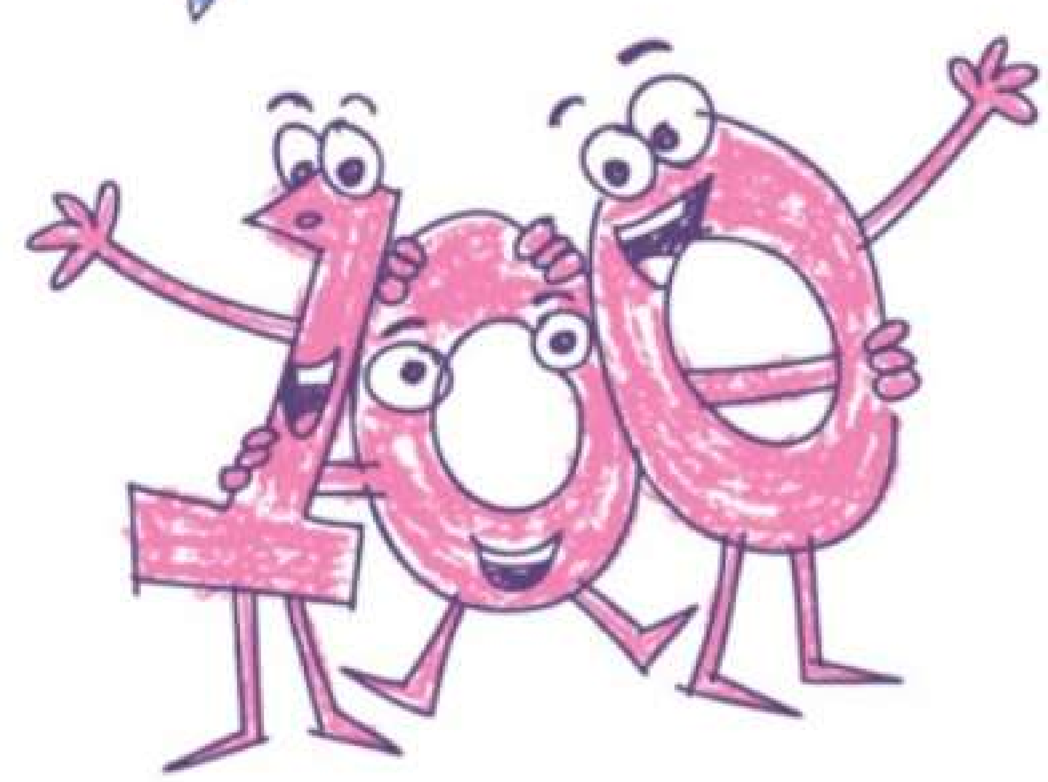
To find out if a number is "happy":

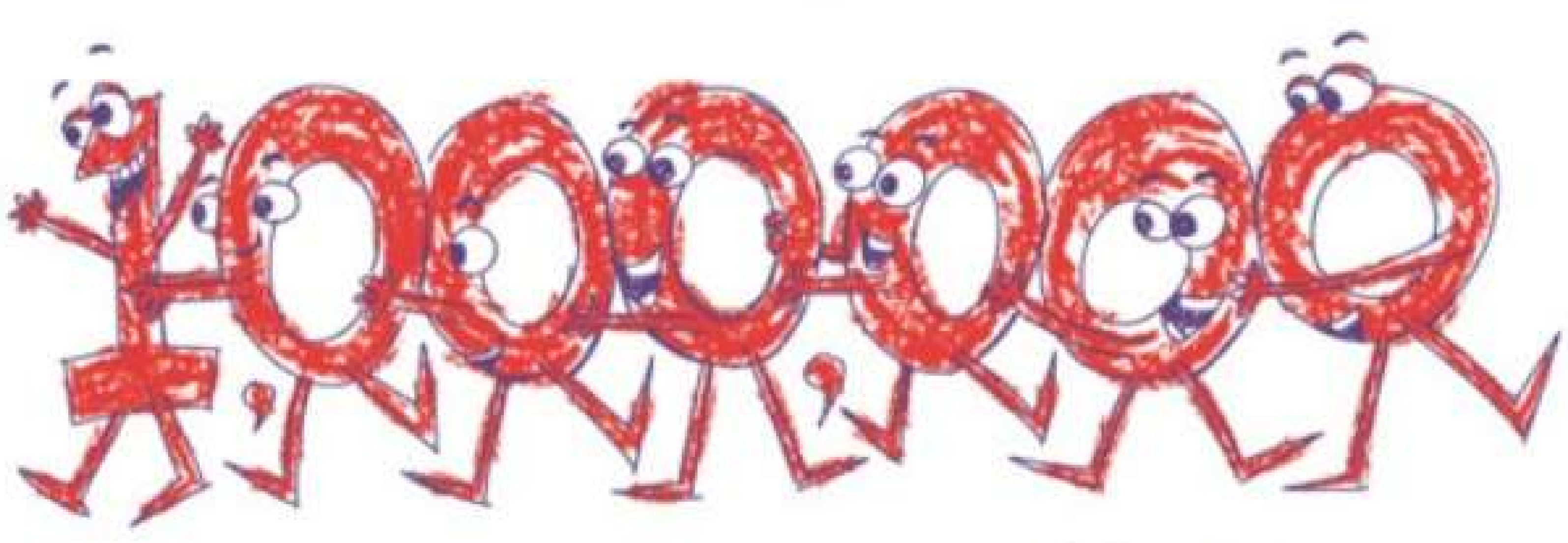
1. Square the digits of the number. Add the squares to get a new number.

2. Repeat step 1 with the new number until one of two things happens:

If you get to 1, the number is a happy number!
If you never get to 1, the number is not a happy number.







17
$1^2+7^2=50$
$5^2+0^2=25$
$2^2+5^2=29$
$2^2+9^2=85$
$8^2+5^2=89$
$8^2+9^2=145$
$1^2+4^2+5^2=42$
$4^2+2^2=20$
$2^2+0^2=4$
$4^2=16$
$1^2+6^2=37$
$3^2+7^2=58$
$8^2+5^2=89$
$8^2+9^2=145$
$1^2+4^2+5^2=42$
$4^2+2^2=20$
$2^2+0^2=4$
$4^2=16$
$1^2+6^2=37$
$3^2+7^2=58$
$8^2+5^2=89$
$8^2+9^2=145$
$1^2+4^2+5^2=42$
$4^2+2^2=20$
$2^2+0^2=4$
$4^2=16$
$1^2+6^2=37$
$3^2+7^2=58$
$5^2+8^2=89$

7
$7^2=49$
$4^2+9^2=97$
$9^2+7^2=130$
$1^2+3^2+0^2=10$
$1^2+0^2=1$
7 is Happy!

1,000
$1^2+0^2+0^2+0^2=1$
1,000 is happy!
(so is 1,000,000!)

23
$2^2+3^2=13$
$1^2+3^2=10$
$1^2+0^2=1$
23 is happy!
(32 must be happy, too!)

is not happy