Contents: Chapter 2

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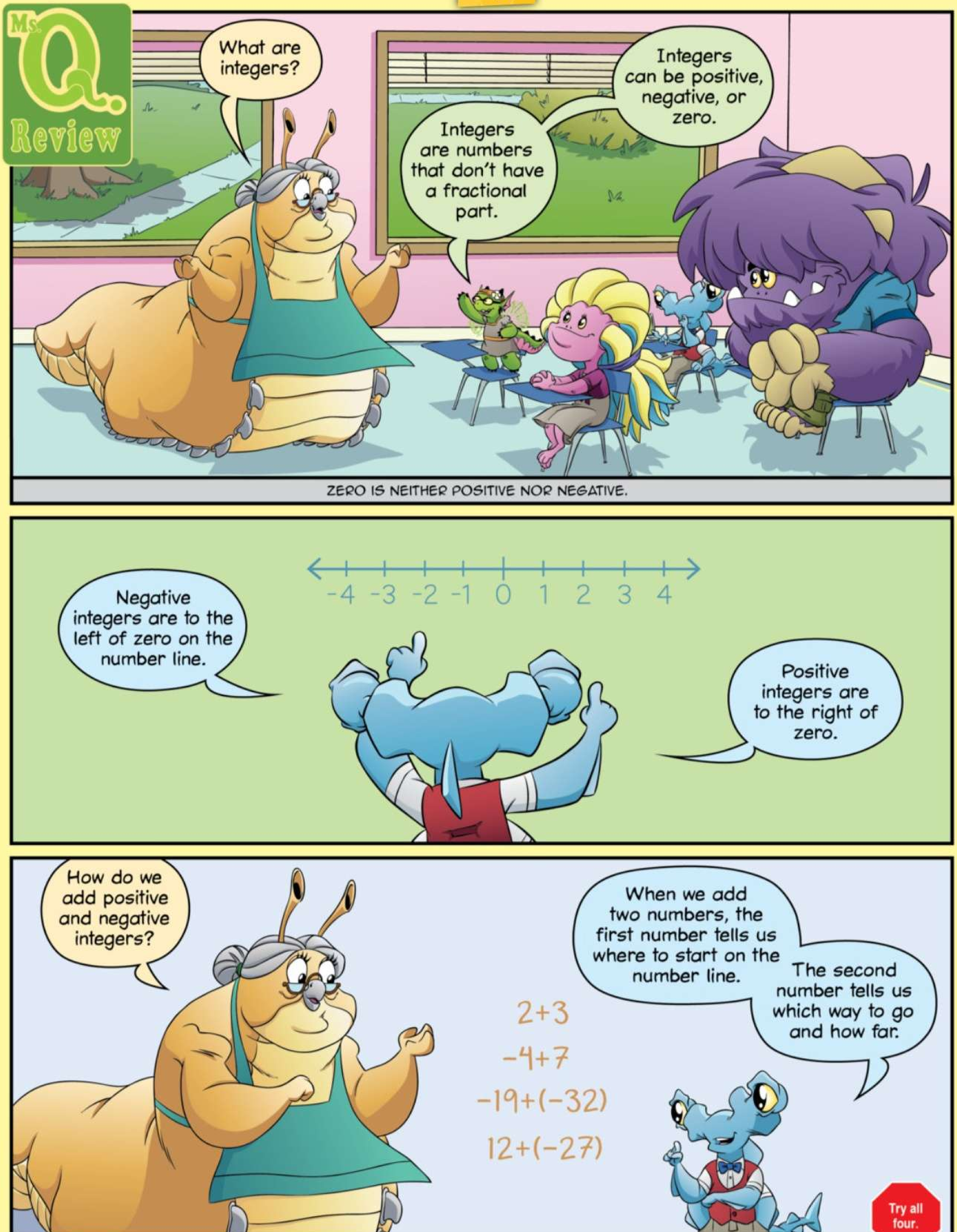
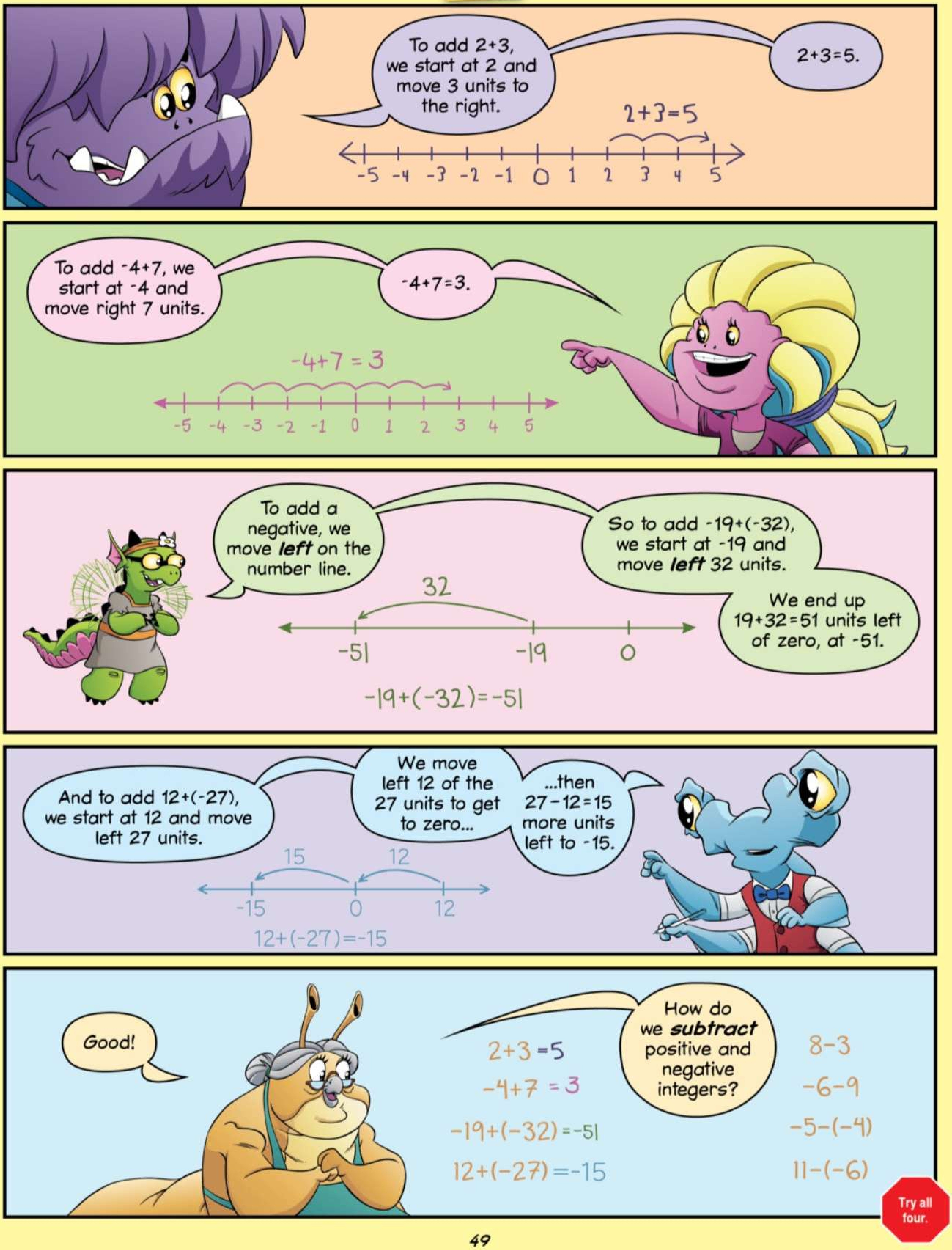
Chapter 2:
Integers


Ms. Q Review
What are integers?
Integers are numbers that don't have a fractional part.
Integers can be positive, negative, or zero.
ZERO IS NEITHER POSITIVE NOR NEGATIVE.
-4 -3 -2 -1 0 1 2 3 4
Negative integers are to the left of zero on the number line.
Positive integers are to the right of zero.
How do we add positive and negative integers?
When we add two numbers, the first number tells us where to start on the number line.
The second number tells us which way to go and how far.
2+3
-4+7
-19+(-32)
12+(-27)
Try all four.



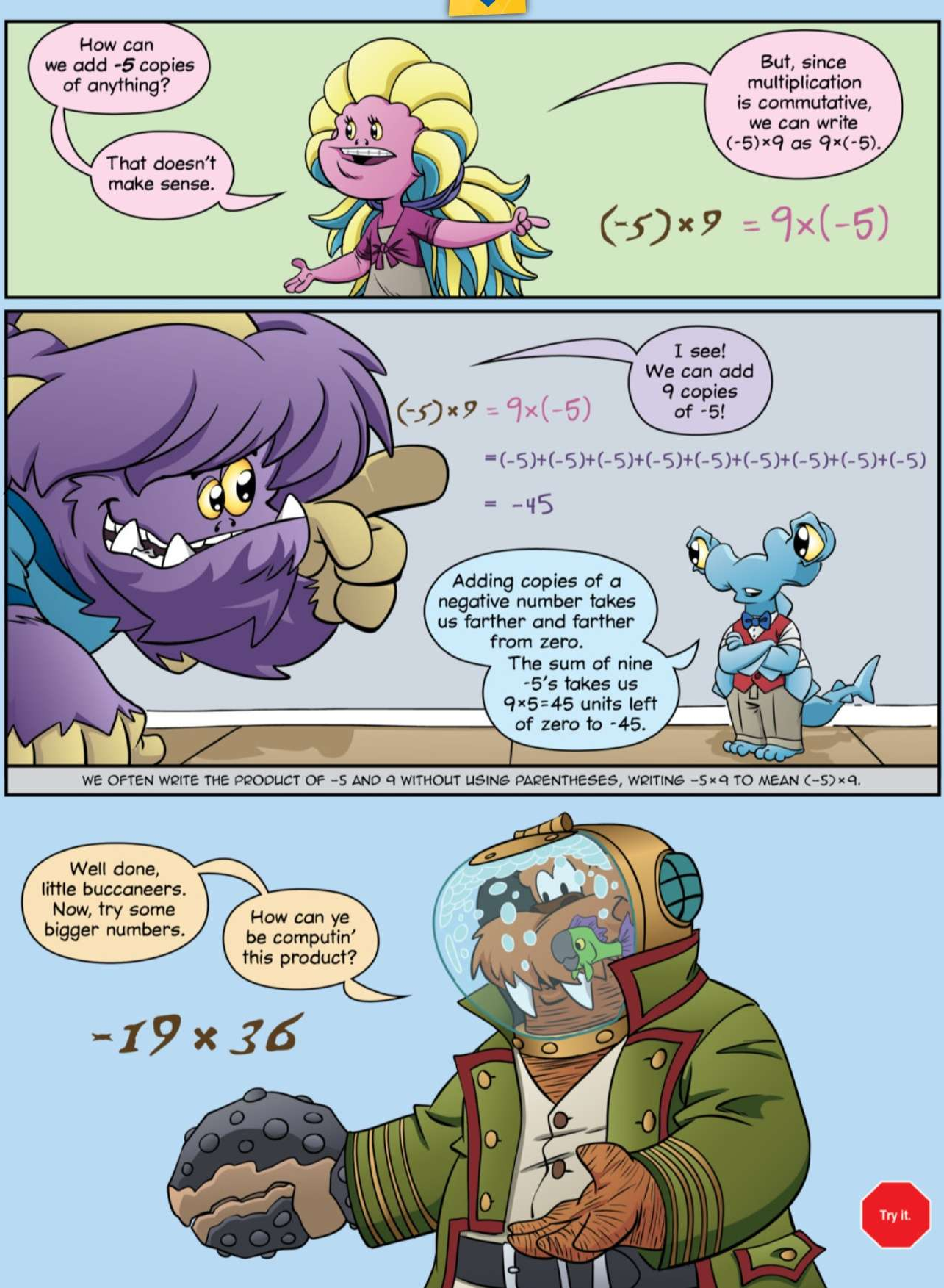
To add 2+3, we start at 2 and move 3 units to the right.
2+3=5.
2+3=5
-5 -4 -3 -2 -1 0 1 2 3 4 5
To add -4+7, we start at -4 and move right 7 units.
-4+7=3.
-4+7 = 3
-5 -4 -3 -2 -1 0 1 2 3 4 5
To add a negative, we move ***left*** on the number line.
So to add -19+(-32), we start at -19 and move ***left*** 32 units.
We end up 19+32=51 units left of zero, at -51.
32
-51 -19 0
-19+(-32)=-51
And to add 12+(-27), we start at 12 and move left 27 units.
We move left 12 of the 27 units to get to zero...
...then 27-12=15 more units left to -15.
15 12
-15 0 12
12+(-27)=-15
Good!
How do we ***subtract*** positive and negative integers?
2+3 =5
-4+7 = 3
-19+(-32)=-51
12+(-27)=-15
8-3
-6-9
-5-(-4)
11-(-6)
Try all four.

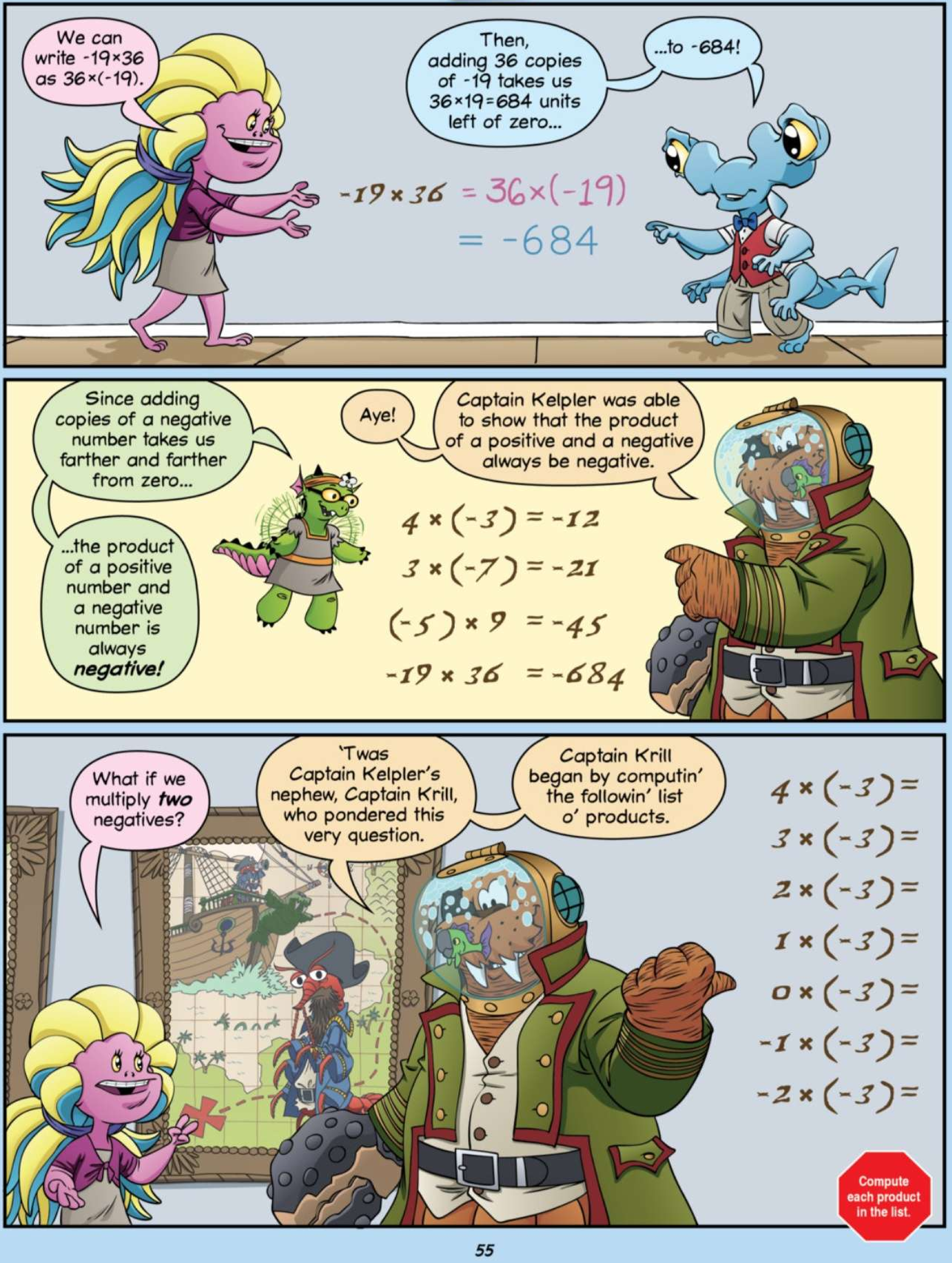
To subtract an integer, we add its opposite.
What does Lizzie mean by ***opposite?***
Two integers are opposites if their sum is zero.
For example, the opposite of ***positive*** 3 is ***negative*** 3, because 3+(-3)=0.
We can rewrite all subtraction as addition.
8−3 = 8+(−3)
−6−9 = −6+(−9)
−5−(−4) = −5+4
11−(−6) = 11+6
Subtracting a ***negative*** 6 is the same as adding a ***positive*** 6.
8−3 = 8+(−3)
= 5
Sometimes, it's best to leave subtraction as subtraction. 8−3=5.
−6−9 = −6+(−9)
= −15
-6−9 =-15.
Who's the new student?
−5−(−4) = −5+4
= −1
Other times, changing subtraction to addition is really helpful. -5−(-4) =-1.
11−(−6) = 11+6
= 17
And 11−(-6) =17.

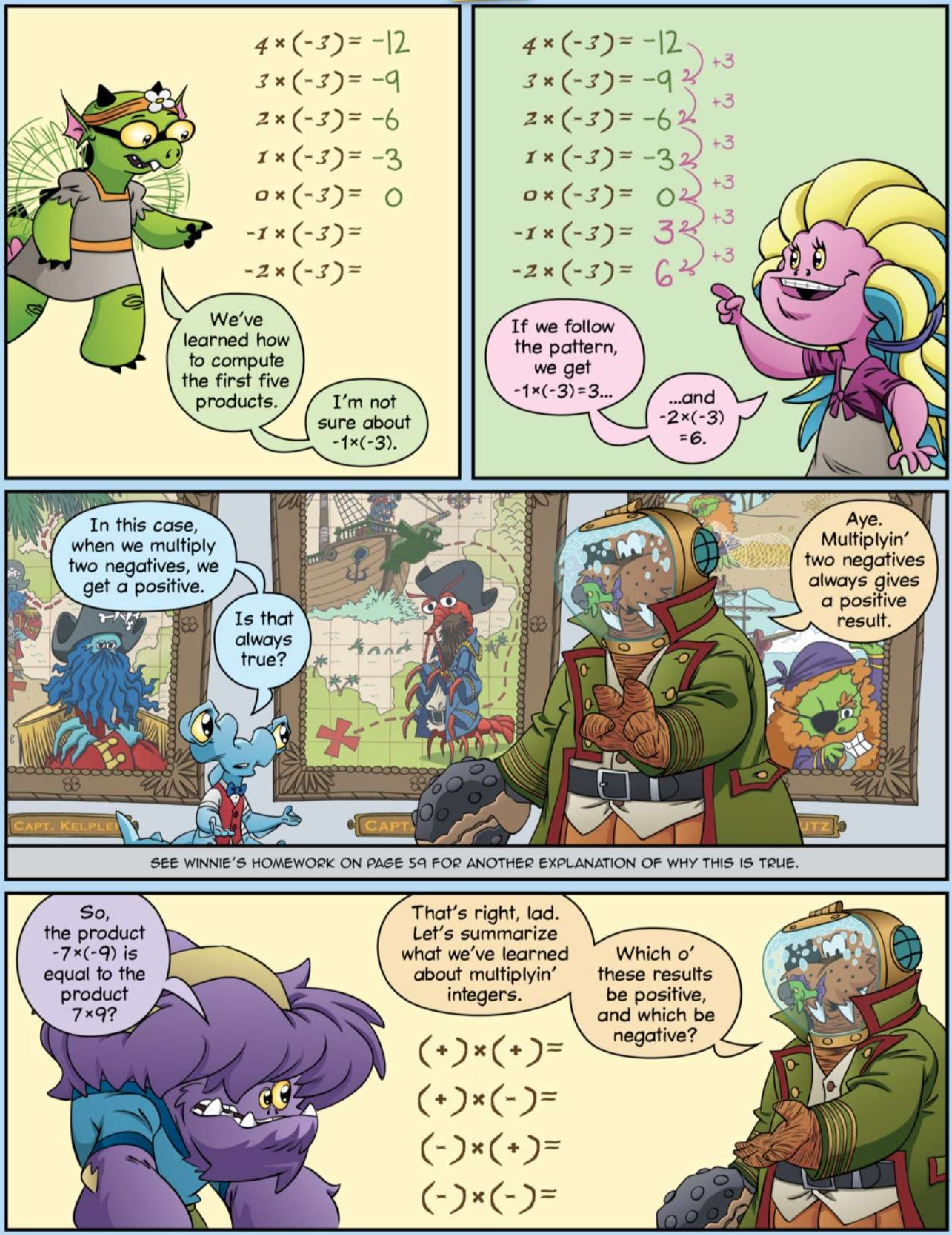

Lunch
Max! What are you doing here?
My mom enrolled me in Beast Academy after your team won last year's Math Relays.
She thought it would be good for me to have classmates who love math as much as I do.
Why haven't we seen you in math class?
I take all of my math classes at the Upper School.
Whoa.
Are you going to join the math team?
Not yet...
...but I'm hoping Beast Academy can enter a team for the World Math Olympiad qualifier this winter.
World Math Olympiad?
Yeah, it's awesome.
I gotta go. See you soon!


Workshop
MULTIPLYING INTEGERS
Captain Kraken, who are all these pirates?
Arrr... These be some o' the greatest beast captains in pirate history.
Here be Captain Kelpler...
...Kelpler charted the entire coastline o' Beast Island.
He was also one o' the island's earliest experts in integer operations.
CAPT. KELPLER
'Twas Kelpler who first pondered the result o' multiplyin' a positive by a negative.
For example, Kelpler correctly deduced the product 4×(-3).
Whoa. How did he do that?

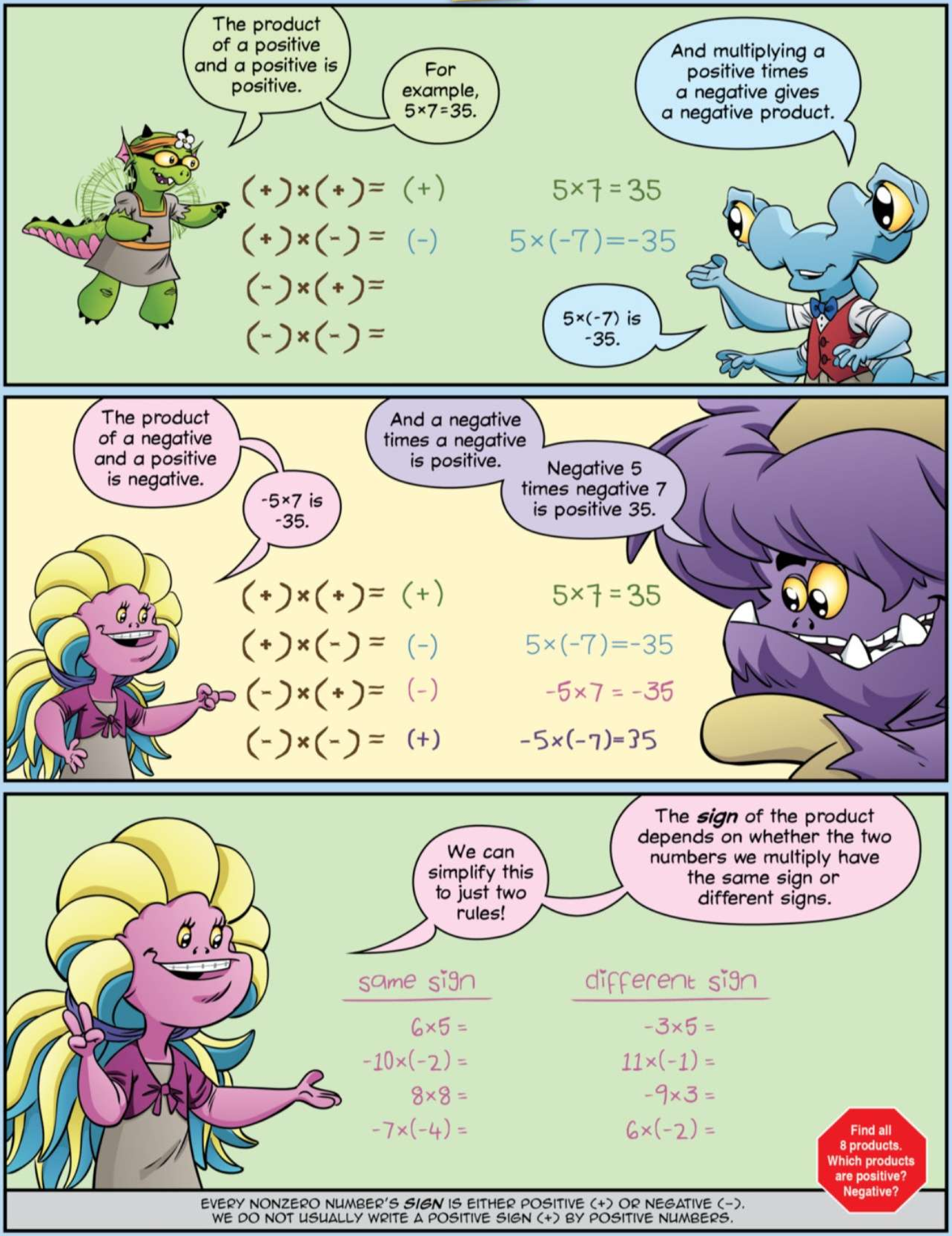

4×3 = 3+3+3+3
= 12
To multiply 4 times **positive** 3, we can add 4 copies of 3.
4×(-3) = (-3)+(-3)+(-3)+(-3)
= -12
So to multiply 4 times **negative** 3, we should add 4 copies of -3!
4×(-3) = -12!
Aye. Excellent figurin', little monsters.
After convincin' his crew that 4×(-3) be -12, Kelpler set out to compute even more products.
Try each o' these.
3×(-7)
(-5)×9
3×(-7) = (-7)+(-7)+(-7)
= -21
To multiply 3 times -7, we add 3 copies of -7.
3×(-7) = -21.
To multiply -5 times 9, we add -5 copies of 9...
Wait, ummm...
Scratch Scratch
(-5)×9 =
How could you multiply (-5)×9?


How can we add -5 copies of anything?
That doesn't make sense.
But, since multiplication is commutative, we can write (-5)×9 as 9×(-5).
(-5)×9 = 9×(-5)
I see! We can add 9 copies of -5!
(-5)×9 = 9×(-5)
=(-5)+(-5)+(-5)+(-5)+(-5)+(-5)+(-5)+(-5)+(-5)
= -45
Adding copies of a negative number takes us farther and farther from zero.
The sum of nine -5's takes us 9×5=45 units left of zero to -45.
WE OFTEN WRITE THE PRODUCT OF -5 AND 9 WITHOUT USING PARENTHESES, WRITING -5×9 TO MEAN (-5)×9.
Well done, little buccaneers. Now, try some bigger numbers.
How can ye be computin' this product?
-19×36
Try it.

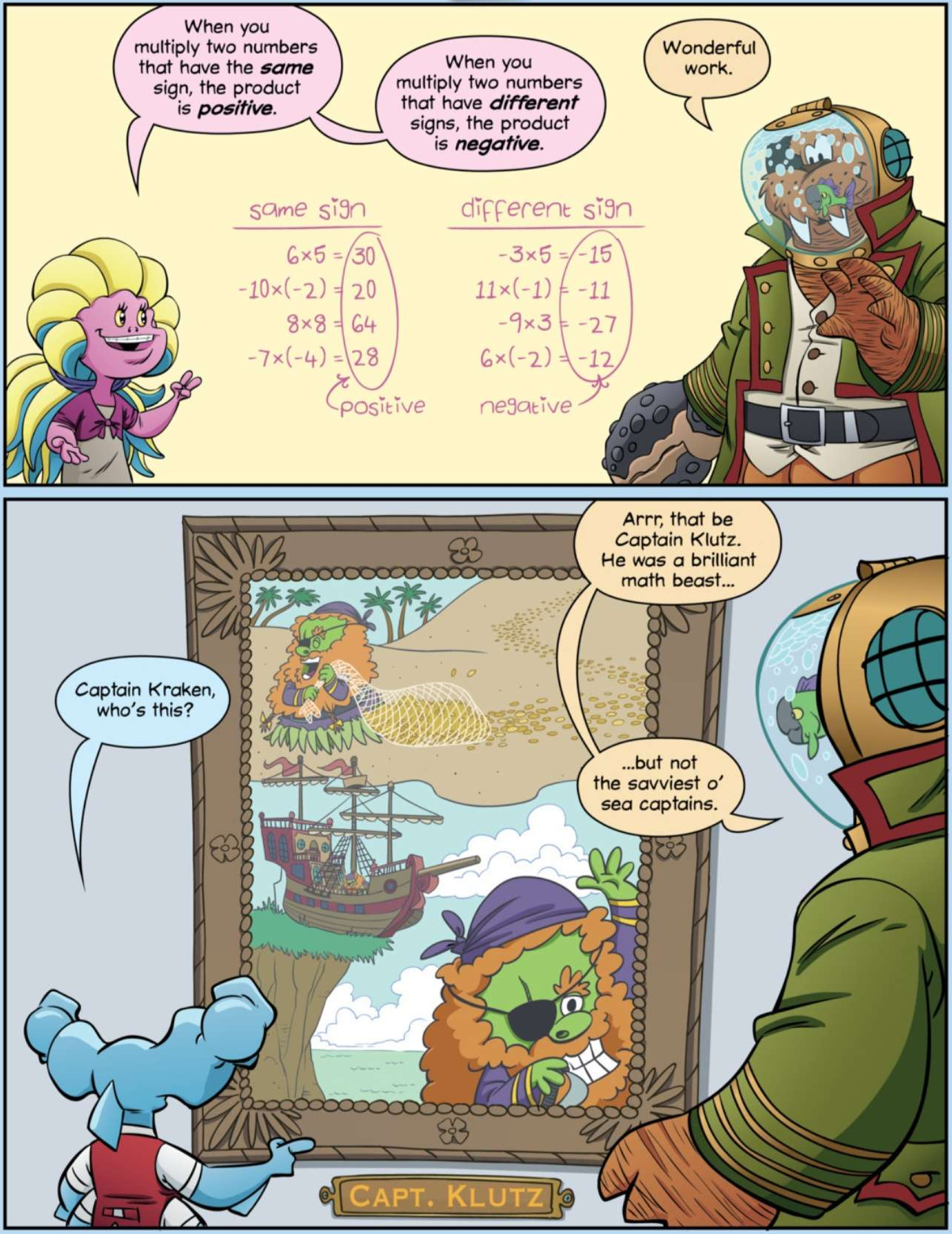

We can write -19×36 as 36×(-19).
Then, adding 36 copies of -19 takes us 36×19=684 units left of zero...
...to -684!
-19 × 36 = 36×(-19)
= -684
Since adding copies of a negative number takes us farther and farther from zero...
...the product of a positive number and a negative number is always ***negative!***
Aye!
Captain Kelpler was able to show that the product of a positive and a negative always be negative.
4 × (-3) = -12
3 × (-7) = -21
(-5) × 9 = -45
-19 × 36 = -684
What if we multiply ***two*** negatives?
'Twas Captain Kelpler's nephew, Captain Krill, who pondered this very question.
Captain Krill began by computin' the followin' list o' products.
4 × (-3) =
3 × (-3) =
2 × (-3) =
1 × (-3) =
0 × (-3) =
-1 × (-3) =
-2 × (-3) =
Compute each product in the list.



$4 \times (-3) = -12$
$3 \times (-3) = -9$
$2 \times (-3) = -6$
$1 \times (-3) = -3$
$0 \times (-3) = 0$
$-1 \times (-3) =$
$-2 \times (-3) =$
We've learned how to compute the first five products.
I'm not sure about -1×(-3).
$4 \times (-3) = -12$
$3 \times (-3) = -9$
$2 \times (-3) = -6$
$1 \times (-3) = -3$
$0 \times (-3) = 0$
$-1 \times (-3) = 3$
$-2 \times (-3) = 6$
+3
+3
+3
+3
+3
+3
If we follow the pattern, we get -1×(-3)=3...
...and -2×(-3) =6.
In this case, when we multiply two negatives, we get a positive.
Is that always true?
Aye. Multiplyin' two negatives always gives a positive result.
CAPT. KELPLE
CAPT.
UTZ
SEE WINNIE'S HOMEWORK ON PAGE 59 FOR ANOTHER EXPLANATION OF WHY THIS IS TRUE.
So, the product -7×(-9) is equal to the product 7×9?
That's right, lad. Let's summarize what we've learned about multiplyin' integers.
Which o' these results be positive, and which be negative?
(+)×(+)=
(+)×(−)=
(−)×(+)=
(−)×(−)=



The product of a positive and a positive is positive.
For example, 5×7=35.
And multiplying a positive times a negative gives a negative product.
(+)×(+)= (+)
(+)×(-)= (-)
(-)×(+)=
(-)×(-)=
5×7=35
5×(-7)=-35
5×(-7) is -35.
The product of a negative and a positive is negative.
-5×7 is -35.
And a negative times a negative is positive.
Negative 5 times negative 7 is positive 35.
(+)×(+)= (+)
(+)×(-)= (-)
(-)×(+)= (-)
(-)×(-)= (+)
5×7=35
5×(-7)=-35
-5×7 = -35
-5×(-7)=35
We can simplify this to just two rules!
The *sign* of the product depends on whether the two numbers we multiply have the same sign or different signs.
same sign
6×5 =
-10×(-2) =
8×8 =
-7×(-4) =
different sign
-3×5 =
11×(-1) =
-9×3 =
6×(-2) =
Find all 8 products. Which products are positive? Negative?
EVERY NONZERO NUMBER'S *SIGN* IS EITHER POSITIVE (+) OR NEGATIVE (–).
WE DO NOT USUALLY WRITE A POSITIVE SIGN (+) BY POSITIVE NUMBERS.


When you multiply two numbers that have the **same** sign, the product is **positive**.
When you multiply two numbers that have **different** signs, the product is **negative**.
Wonderful work.
same sign
6×5 = 30
-10×(-2) = 20
8×8 = 64
-7×(-4) = 28
positive
different sign
-3×5 = -15
11×(-1) = -11
-9×3 = -27
6×(-2) = -12
negative
Captain Kraken, who's this?
Arrr, that be Captain Klutz. He was a brilliant math beast...
...but not the savviest o' sea captains.
CAPT. KLUTZ

PRACTICE | Solve each.

1. Compute: $5\times(-4+4)$

$= 5\times 0$

$= 0$

1. 0

2. Use the distributive property to rewrite the expression $5\times(-4+4)$ as the sum of two products.

2. $5\times(-4) + 5\times 4$

3. Use your answers to problems 1 and 2 to compute the value of $5\times(-4)$. Explain your reasoning.

$5\times(-4+4)=0$

$5\times(-4) + 5\times 4 = 0$

$\underline{5\times(-4)} + 20 = 0$

Two numbers that have a sum of zero are opposites.
So, $5\times(-4)$ is the opposite of $5\times 4 = 20$.
That means $5\times(-4) = \boxed{-20}$

4. Use the reasoning from problem 3 above to show that $-4\times(-5)=20$.

We have:

$-4\times(-5+5)$

$= -4\times 0$

$= 0$

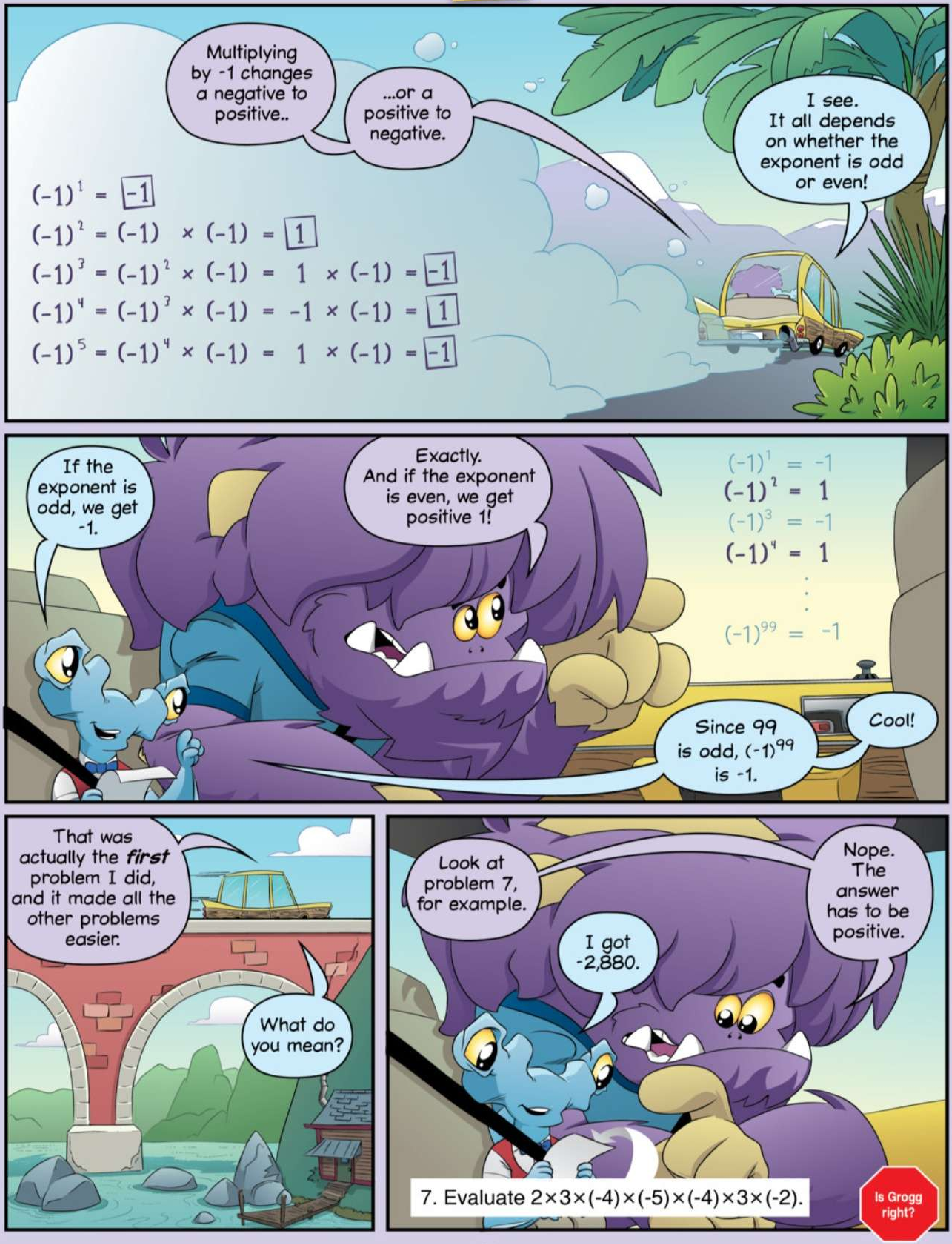
so,

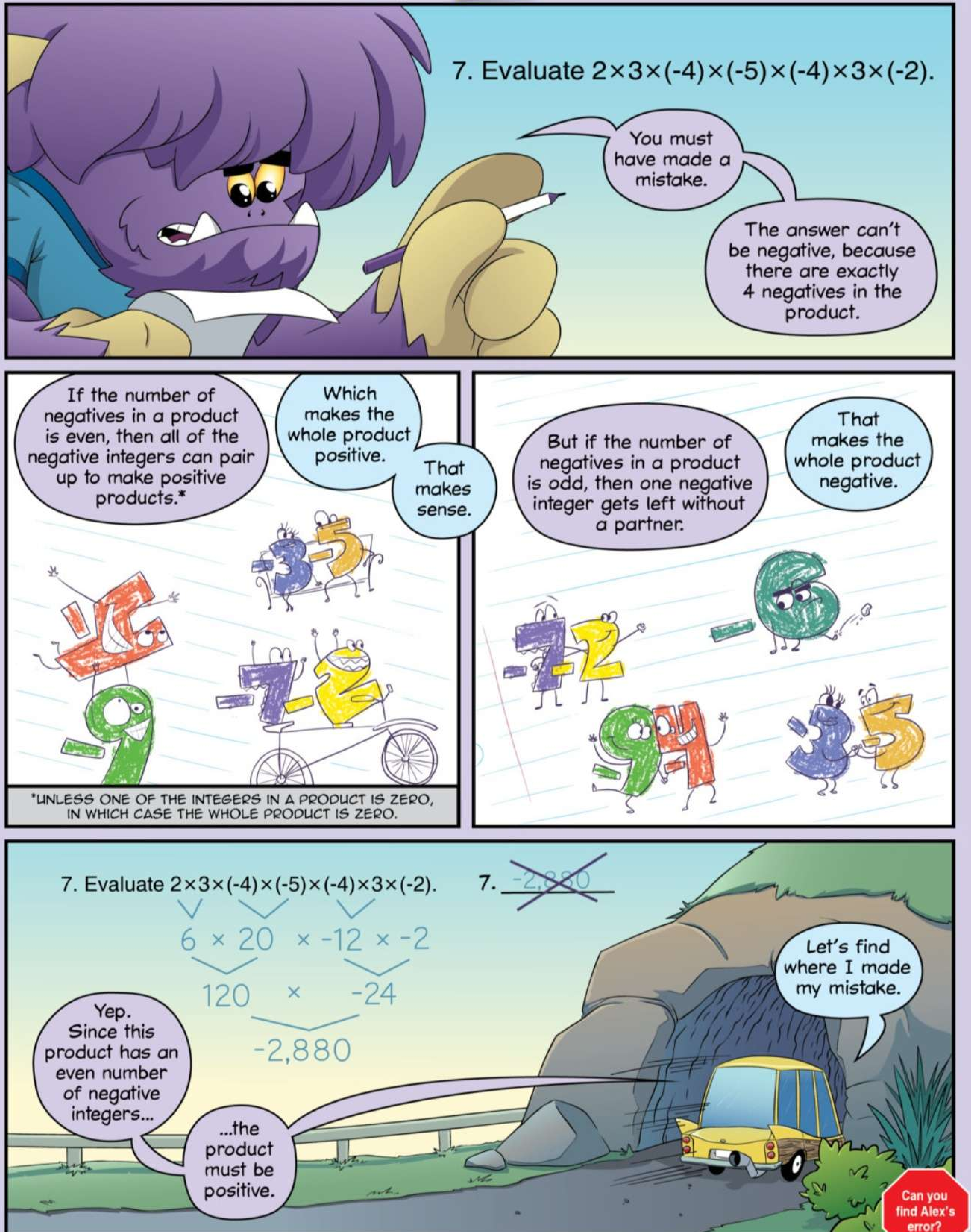
$-4\times(-5+5) = 0$

$-4\times(-5) + (-4)\times 5 = 0$

Two numbers that have a sum of zero are opposites.
So, $-4\times(-5)$ and $(-4)\times 5$ are opposites.
We showed in problem 3 that $5\times(-4) = -20$, so $(-4)\times 5 = -20$.
$-4\times(-5)$ is the opposite of $(-4)\times 5$, so $-4\times(-5) = \boxed{20}$


CAR POOL
Have you done your math homework yet?
Yep. It should be right he...
...uhh...
...uh-oh.
I finished all but the last problem.
10. Compute $(-1)^{99}$.
That one was fun.
How did you solve it?
First I tried to multiply ninety-nine -1's.
That doesn't sound like fun.
It wasn't, but once I got to $(-1)^5$, I noticed a pattern that made it easy.
What is $(-1)^{99}$?


Multiplying by -1 changes a negative to positive..
...or a positive to negative.
$(-1)^1 = \boxed{-1}$
$(-1)^2 = (-1) \times (-1) = \boxed{1}$
$(-1)^3 = (-1)^2 \times (-1) = 1 \times (-1) = \boxed{-1}$
$(-1)^4 = (-1)^3 \times (-1) = -1 \times (-1) = \boxed{1}$
$(-1)^5 = (-1)^4 \times (-1) = 1 \times (-1) = \boxed{-1}$
I see. It all depends on whether the exponent is odd or even!
If the exponent is odd, we get -1.
Exactly. And if the exponent is even, we get positive 1!
$(-1)^1 = -1$
$(-1)^2 = 1$
$(-1)^3 = -1$
$(-1)^4 = 1$
$\vdots$
$(-1)^{99} = -1$
Since 99 is odd, $(-1)^{99}$ is -1.
Cool!
That was actually the ***first*** problem I did, and it made all the other problems easier.
What do you mean?
Look at problem 7, for example.
I got -2,880.
Nope. The answer has to be positive.
7. Evaluate $2\times3\times(-4)\times(-5)\times(-4)\times3\times(-2)$.
Is Grogg right?



7. Evaluate 2×3×(-4)×(-5)×(-4)×3×(-2).
You must have made a mistake.
The answer can't be negative, because there are exactly 4 negatives in the product.
If the number of negatives in a product is even, then all of the negative integers can pair up to make positive products.*
Which makes the whole product positive.
That makes sense.
*UNLESS ONE OF THE INTEGERS IN A PRODUCT IS ZERO, IN WHICH CASE THE WHOLE PRODUCT IS ZERO.
But if the number of negatives in a product is odd, then one negative integer gets left without a partner.
That makes the whole product negative.
7. Evaluate 2×3×(-4)×(-5)×(-4)×3×(-2).
6 × 20 × -12 × -2
120 × -24
-2,880
7. ~~-2,880~~
Let's find where I made my mistake.
Yep. Since this product has an even number of negative integers...
...the product must be positive.
Can you find Alex's error?

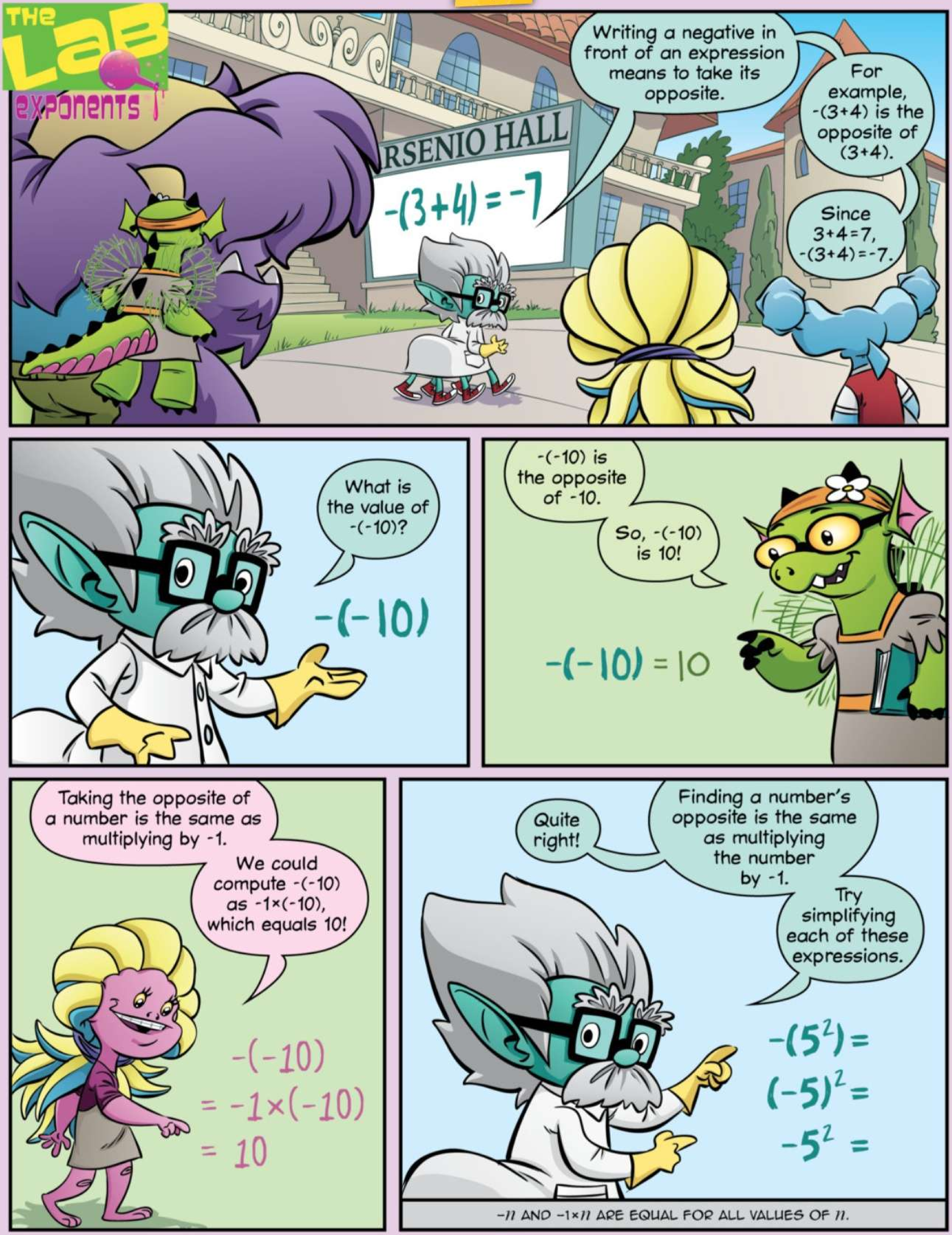
7. Evaluate 2×3×(-4)×(-5)×(-4)×3×(-2).
7. ~~-2,880~~
6 × 20 × -12 × -2
120 × -24
-2,880
It's an easy mistake to make. You can avoid it by doing what I do.
Right here, when I multiplied -12 times -2, I should have gotten ***positive*** 24.
What's that?
I always multiply all of the numbers without keeping track of signs...
...then count the negatives to figure out whether the result is positive or negative.
Cool!
You're pretty smart, Grogg.
Unfortunately, not when it comes to keeping track of my homework.


R&G
Dividing Integers
Looks like the big purple monster lost his homework again.
That's the third time this week.
It's only *Tuesday!*
Fitness Trail
What's the homework this time?
Multiplying integers.
I know the rules for multiplying integers!
BEAST ACADEMY
Fitness Trail
Two numbers with the same sign have a positive product...
...and two numbers with different signs have a negative product.
ZERO IS NEITHER POSITIVE NOR NEGATIVE. THE PRODUCT OF ANY NUMBER AND ZERO IS ZERO.
I have a question.
I already told you...
...this track suit does ***not*** make you look lumpy.
BEAST ACADEMY
Fitness Trail
No, not that.
What are the rules for *dividing* integers?

We can use the rules for multiplication to figure out the rules for division.
How?
Division problems can be solved using multiplication.
For example, we can answer 12÷3 by asking, "What number times 3 equals 12?"
Since 4×3=12, we know that 12÷3=4.
4×3=12, so 12÷3=4
Oh, right. So to answer 12÷(-3), we ask, "What number times -3 equals 12?"
-4×(-3)=12, so 12÷(-3)=-4
Since -4×(-3)=12, 12÷(-3)=-4.
CHIN-UP
Yep.
Is a positive divided by a negative always negative?
It has to be. A negative times a negative is positive.
So a positive divided by a negative is negative.
(-)×(-)=(+), so (+)÷(-)=(-)
What about a negative divided by a positive?
-12÷3=
What is -12÷3?

-4×3=-12, so -12÷3=-4
Since -4×3=-12, -12÷3=-4.
You got it.
Since a negative times a positive is negative...
...a negative divided by a positive is negative.
(-)×(+)=(-), so (-)÷(+)=(-)
So, two numbers with different signs have a negative quotient.
(+)÷(-)=(-)
(-)÷(+)=(-)
Right. What do we get when we divide two negatives?
What's -12÷(-3)?
4×(-3)=-12, so -12÷(-3)=4
Since 4×(-3)=-12, -12÷(-3)=4.
Yep. Since a positive times a negative is negative...
...a negative divided by a negative is positive.
(+)×(-)=(-), so (-)÷(-)=(+)

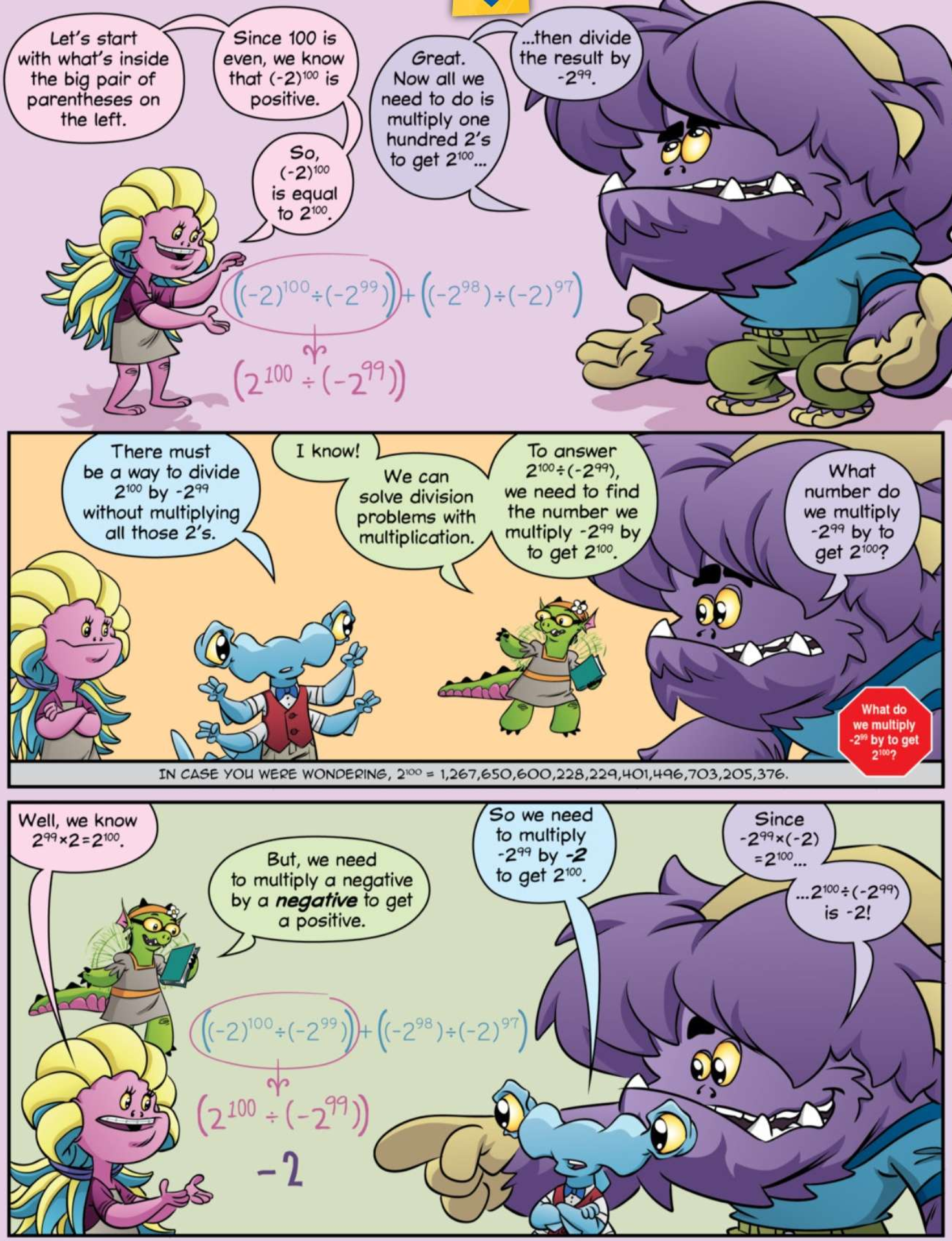
And we already know that dividing two positives gives a positive quotient...
...so we have all our rules.
That makes things easy to remember.
(+)÷(−)=(−)
(−)÷(+)=(−)
(−)÷(−)=(+)
(+)÷(+)=(+)
What do you mean?
Two numbers with ***different*** signs have a ***negative*** quotient...
(+)÷(−)=(−)
(−)÷(+)=(−)
different = negative
(−)÷(−)=(+)
(+)÷(+)=(+)
same = positive
CUSTODIAN
...and two numbers with the ***same*** sign have a ***positive*** quotient.
The sign rules for division are the exact same as the rules we use for multiplication.
We have the same sign!
You mean we're both positive?
December
No, we're both Capricorns!

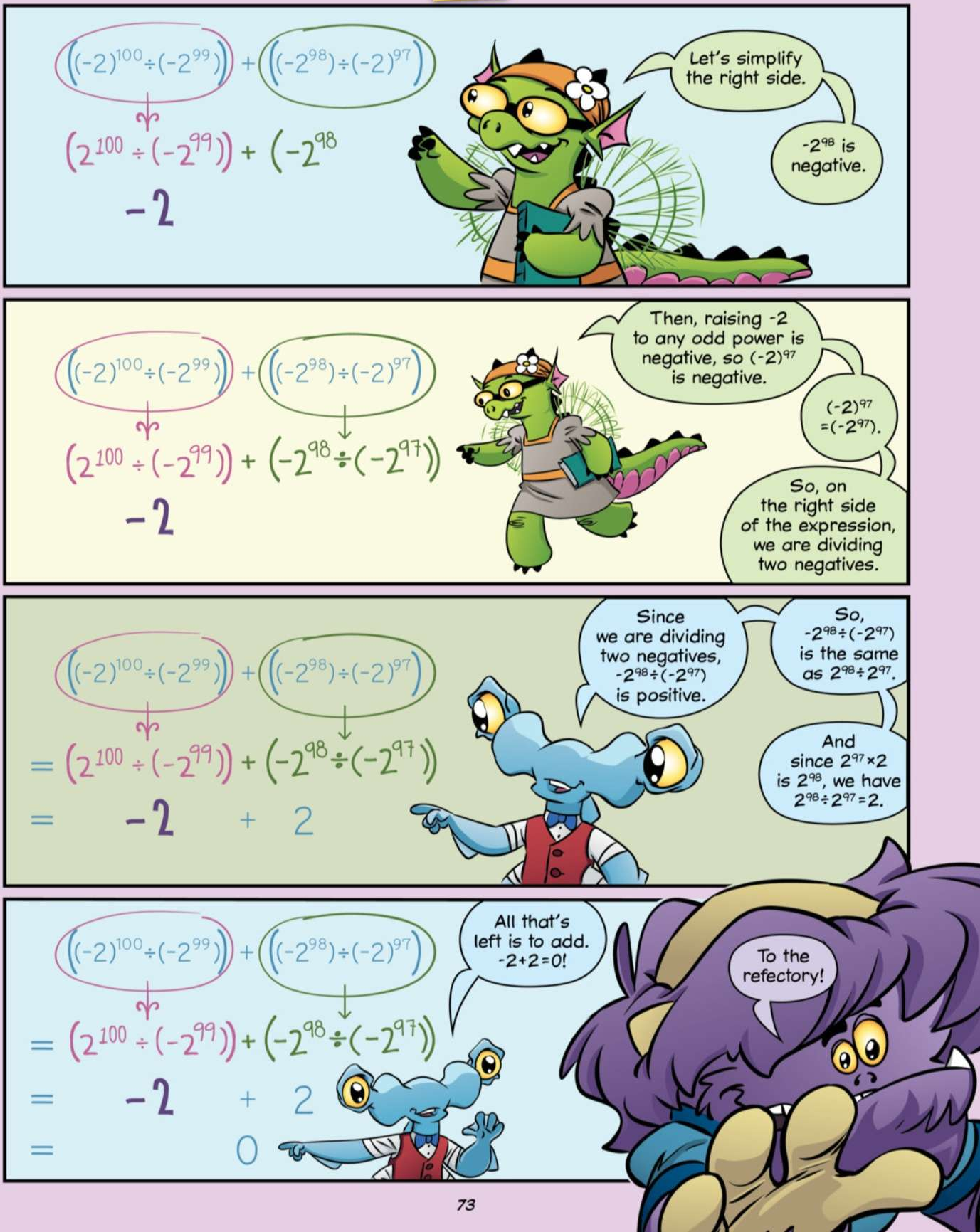

THE LAB
EXPONENTS
ARSENIO HALL
-(3+4) = -7
Writing a negative in front of an expression means to take its opposite.
For example, -(3+4) is the opposite of (3+4).
Since 3+4=7, -(3+4)=-7.
What is the value of -(-10)?
-(-10)
-(-10) is the opposite of -10.
So, -(-10) is 10!
-(-10) = 10
Taking the opposite of a number is the same as multiplying by -1.
We could compute -(-10) as -1×(-10), which equals 10!
-(-10)
= -1×(-10)
= 10
Quite right!
Finding a number's opposite is the same as multiplying the number by -1.
Try simplifying each of these expressions.
-(5²) =
(-5)² =
-5² =
-n AND -1×n ARE EQUAL FOR ALL VALUES OF n.

$-(5^2) = -(25) = -25$
HALL
$-(5^2)$ is the opposite of 5^2. Since $5^2 = 25$, $-(5^2) = -25$.
$(-5)^2 = (-5) \times (-5) = 25$
To simplify $(-5)^2$, we square -5. Since $(-5) \times (-5) = 25$, $(-5)^2 = 25$.
$-5^2 =$
I'm not sure what -5^2 means. Do we square -5, or find the opposite of 5^2?
HALL
ARSENIO HALL
The expression -5^2 means $-(5^2)$.
So, to compute -5^2, we square the 5 first, then take its opposite.
$-5^2 = -(5^2)$
$= -(25)$
$= -25$
Excellent!
Try a few more.
$-3^4 =$
$(-3)^4 =$
$-(-3^5) =$
Try all 3.


$-3^4 = -(3\times3\times3\times3) = -(81) = -81$
To simplify -3^4, we start with $3\times3\times3\times3=81$. Then, we take its opposite. $-3^4=-81$.
WE COULD ALSO WRITE $-3^4=-1\times3^4=-1\times81=-81$.
To simplify $(-3)^4$...
$(-3)^4 = (-3)\times(-3)\times(-3)\times(-3) = 9 \times 9 = 81$
...we multiply four (-3)'s: $(-3)\times(-3)\times(-3)\times(-3)$. That gives us $9\times9=81$.
That makes sense. When we multiply an even number of negatives, the result is positive.
Any ***even*** power of a negative is positive!
What about this last one?
We start inside the parentheses with -3^5.
We know $3^4=81$. So, $3^5=3^4\times3 = 81\times3=243$.
Since $3^5=243$, that means $-3^5=-243$.
$-(-3^5) = -(-243)$
$-(-3^5) = -(-243) = 243$
And the opposite of -243 is 243. So, -(-243) = 243.
Exquisite work, little monsters.
Wait, watch out--
KRASH!
OOF!

Bwah Hah Hah! Professor Grok is gone! I've abducted your educator! It's time for something much more diabolically difficult!
Evaluating such elementary exponential expressions is unconditionally unremarkable.
But simplification of exceedingly elaborate expressions requires ***exceptional expertise.***
Compute the value of this expression and relay your response to the register in the refectory.
Refectory?
Muzzle it! I'm monologuing.
Answer correctly, and your esteemed educator will be emancipated.
But answer incorrectly, and he will be ***perpetually impounded*** in ***imponderable peril!***
$\left((-2)^{100} \div (-2^{99})\right) + \left((-2^{98}) \div (-2)^{97}\right)$
Wow, this looks complicated!
How should we start?
rub rub
?
$\left((-2)^{100} \div (-2^{99})\right) + \left((-2^{98}) \div (-2)^{97}\right)$
Can you simplify this expression?


Let's start with what's inside the big pair of parentheses on the left.
Since 100 is even, we know that (-2)^100 is positive.
So, (-2)^100 is equal to 2^100.
Great. Now all we need to do is multiply one hundred 2's to get 2^100...
...then divide the result by -2^99.
$((-2)^{100} \div (-2^{99})) + ((-2^{98}) \div (-2)^{97})$
$(2^{100} \div (-2^{99}))$
There must be a way to divide 2^100 by -2^99 without multiplying all those 2's.
I know!
We can solve division problems with multiplication.
To answer 2^100÷(-2^99), we need to find the number we multiply -2^99 by to get 2^100.
What number do we multiply -2^99 by to get 2^100?
What do we multiply -2^99 by to get 2^100?
IN CASE YOU WERE WONDERING, 2^100 = 1,267,650,600,228,229,401,496,703,205,376.
Well, we know 2^99×2=2^100.
But, we need to multiply a negative by a *negative* to get a positive.
So we need to multiply -2^99 by **-2** to get 2^100.
Since -2^99×(-2) =2^100...
...2^100÷(-2^99) is -2!
$((-2)^{100} \div (-2^{99})) + ((-2^{98}) \div (-2)^{97})$
$(2^{100} \div (-2^{99}))$
-2



$((-2)^{100} \div (-2^{99})) + ((-2^{98}) \div (-2)^{97})$
$(2^{100} \div (-2^{99})) + (-2^{98}$
-2
Let's simplify the right side.
-2^{98} is negative.
$((-2)^{100} \div (-2^{99})) + ((-2^{98}) \div (-2)^{97})$
$(2^{100} \div (-2^{99})) + (-2^{98} \div (-2^{97}))$
-2
Then, raising -2 to any odd power is negative, so $(-2)^{97}$ is negative.
$(-2)^{97} = (-2^{97})$.
So, on the right side of the expression, we are dividing two negatives.
$((-2)^{100} \div (-2^{99})) + ((-2^{98}) \div (-2)^{97})$
$= (2^{100} \div (-2^{99})) + (-2^{98} \div (-2^{97}))$
$= -2 + 2$
Since we are dividing two negatives, $-2^{98} \div (-2^{97})$ is positive.
So, $-2^{98} \div (-2^{97})$ is the same as $2^{98} \div 2^{97}$.
And since $2^{97} \times 2$ is 2^{98}, we have $2^{98} \div 2^{97} = 2$.
$((-2)^{100} \div (-2^{99})) + ((-2^{98}) \div (-2)^{97})$
$= (2^{100} \div (-2^{99})) + (-2^{98} \div (-2^{97}))$
$= -2 + 2$
$= 0$
All that's left is to add. $-2+2=0$!
To the refectory!


What's a refectory?
I thought one of you knew.
A refectory is "a room used for serving meals."
To the cafeteria!
MENU
CORN DOGS & TOTS
Yum!
MENU
CORN DOGS & TOTS
Okay, I guess I just push 0 on the register?
CLEAR
NO SALE
7 8 9
VOUCHER
4 5 6
SUB TOTAL
1 2 3
CASH TENDER
TOTAL
KLIK
Wooooosh
You've rescued me again, little monsters!
ding!
And just in time for muffins!

Integer Operations Summary:

Lizzie

Multiplying Positives and Negatives:

Same Sign: Positive

$5 \times 7 = 35$

$(-3) \times (-6) = 18$

$8 \times 4 = 32$

$(-7) \times (-2) = 14$

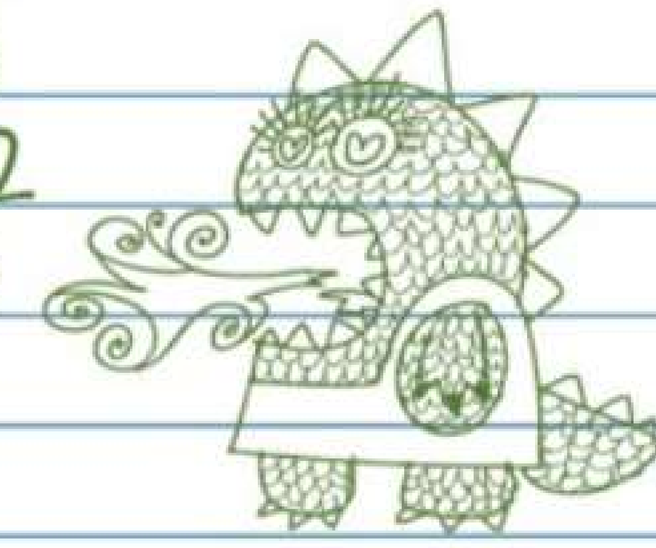
Different Sign: Negative

$-5 \times 7 = -35$

$3 \times (-6) = -18$

$-8 \times 4 = -32$

$7 \times (-2) = -14$



Dividing Positives and Negatives:

Since division is the same as multiplying by the reciprocal, the rules for dividing positives and negatives are the same as the rules for multiplying positives and negatives.

Same Sign: Positive

$35 \div 7 = 5$

$(-18) \div (-6) = 3$

Different Sign: Negative

$-32 \div 4 = -8$

$14 \div (-2) = -7$

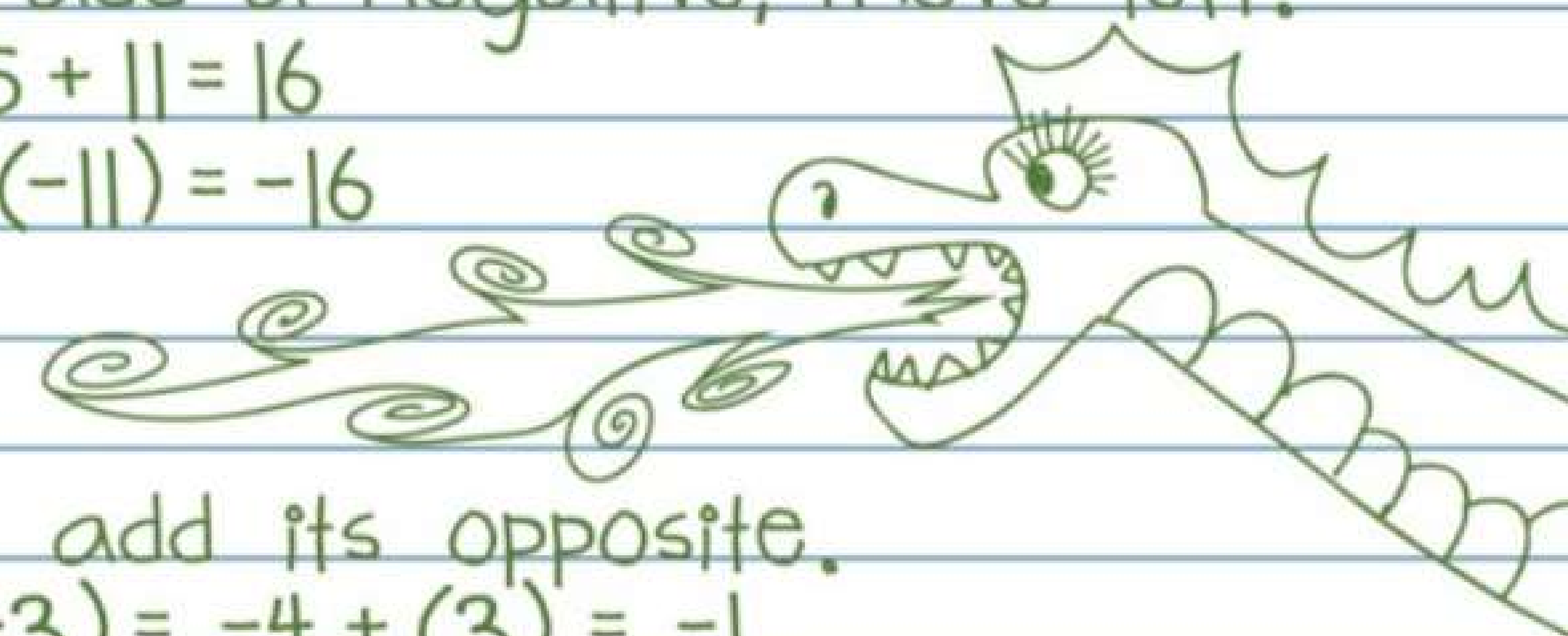
*** SUPER IMPORTANT! ***
Do NOT try to use the rules for multiplication and division for addition and subtraction.

Adding Positives and Negatives:

On the number line, start at the first number.
To add a positive, move right. To add a negative, move left.

$7 + (-9) = -2$ $\quad$ $5 + 11 = 16$

$-7 + 9 = 2$ $\quad$ $-5 + (-11) = -16$



Subtracting Positives and Negatives:

To subtract a number, you can add its opposite.

$6 - (-7) = 6 + (7) = 13$ $\quad$ $-4 - (-3) = -4 + (3) = -1$

$-5 - 9 = -5 + (-9) = -14$ $\quad$ $7 - 18 = 7 + (-18) = -11$

(Sometimes, it's better to leave subtraction as subtraction.)

Exponents:

-5^4 means $-(5^4)$, which is $-(5 \times 5 \times 5 \times 5) = -(625) = -625$.

$(-5)^4$ is $(-5) \times (-5) \times (-5) \times (-5) = 625$.