Contents: Chapter 7

Click the Play List tab in the top-left to view a recommended reading/practice sequence.

Next 14

What letter comes next in the sequence below?

J F M A M J

Sequences 20

What is the 100th term in the sequence below?

1, 4, 9, 16, 25, 36, ...

Subtractionacci 27

If your opponent chooses 47 as Picker, what number should you choose as Subtractor?

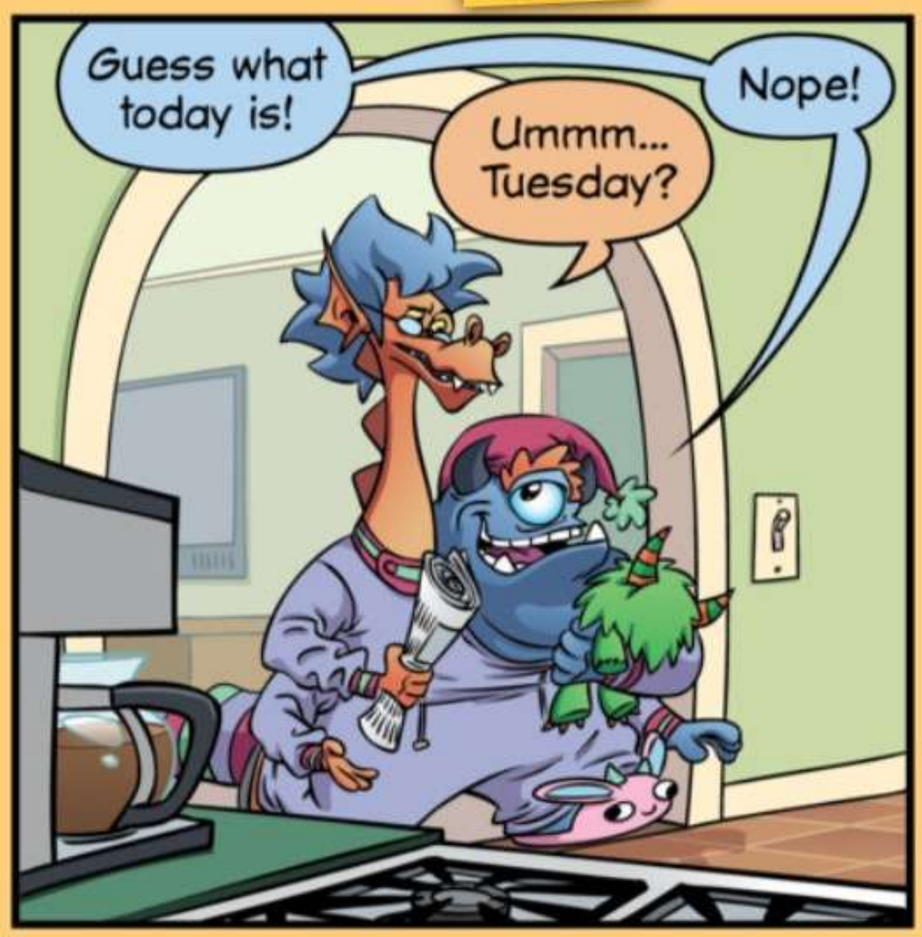
Arithmetic Sequences 28

What is the common difference of an arithmetic sequence with first term 10 and fifth term 54?

Triangular Numbers 35

What is the tenth triangular number?

Chapter 7:
Sequences



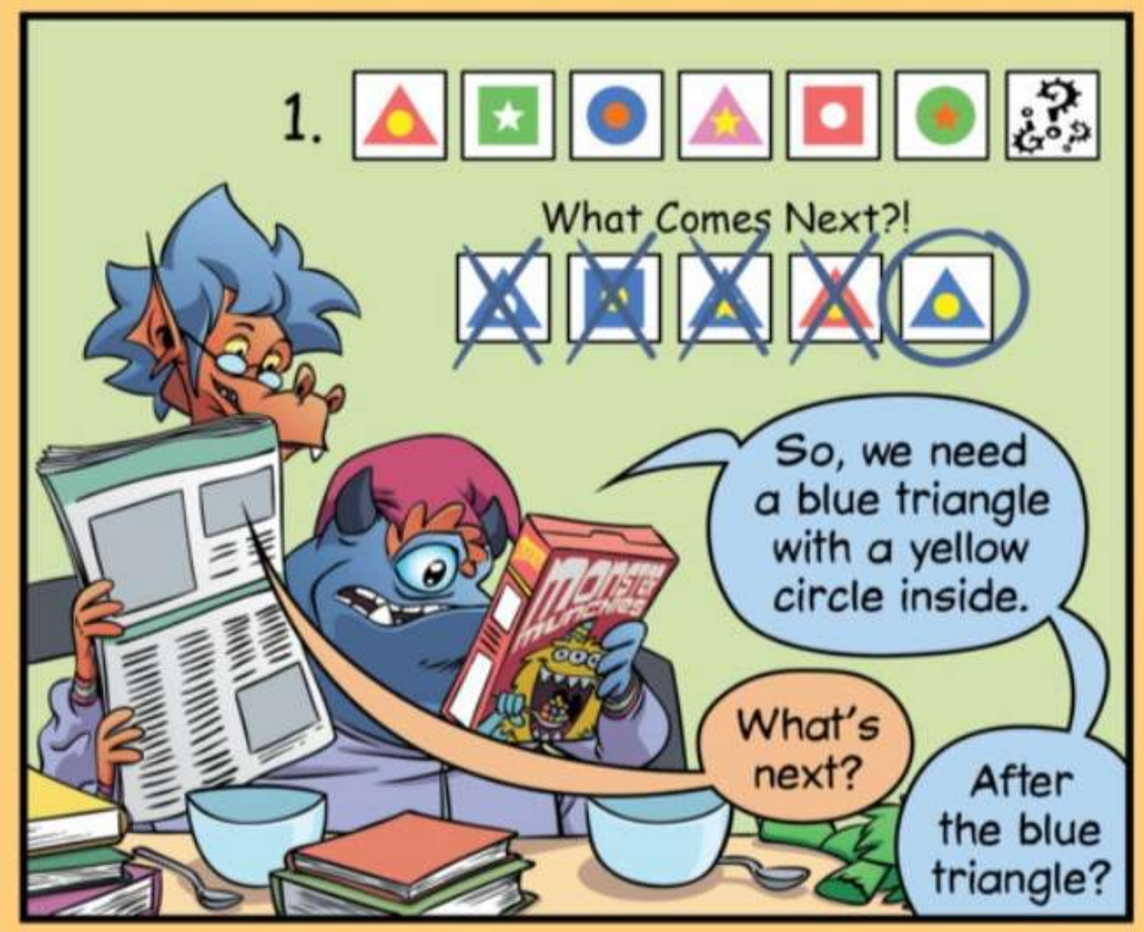
What comes next?

1.
What Comes Next?!
I can eliminate some wrong answers!
The big shapes repeat: triangle, square, circle, triangle, square, circle...
So, triangle is next. I can cross out the square.
Monster Munchies

1.
What Comes Next?!
The big-shape colors go: red, green, blue, pink, red, green...
So, next comes blue, not red.
Monster Munchies

1.
What Comes Next?!
The little shapes go: circle, star, circle, star, circle, star...
So, circle is next.
Monster Munchies

1.
What Comes Next?!
And the little-shape colors go: yellow, white, orange, yellow, white, orange...
So, yellow is next.
Monster Munchies


1.
What Comes Next?!
So, we need a blue triangle with a yellow circle inside.
What's next?
After the blue triangle?

2. 2 3 5 9 17 33
What Comes Next?!
44 50 51 65 67
I mean what's the next pattern?
Let's see. Oooh... numbers!
A pink square with a white star inside.
What comes next?

For these, I always try to figure out how to get from one number to the next.

2. 2 3 5 9 17 33 ?

What Comes Next?!

44 50 51 65 67

We add twice as much each time. +1, +2, +4, +8, +16, then +32.

So, next is 65.

2. 2 3 5 9 17 33 ?

+1 +2 +4 +8 +16 +32

What Comes Next?!

44 50 51 65 67

Cool.

I saw a different pattern.

Each number is one less than twice the number before it.

I got 65, too.

2. 2 3 5 9 17 33 ?

×2−1 ×2−1 ×2−1 ×2−1 ×2−1 ×2−1

What Comes Next?!

44 50 51 65 67

Let's try the next one!

New cereal day is the best!

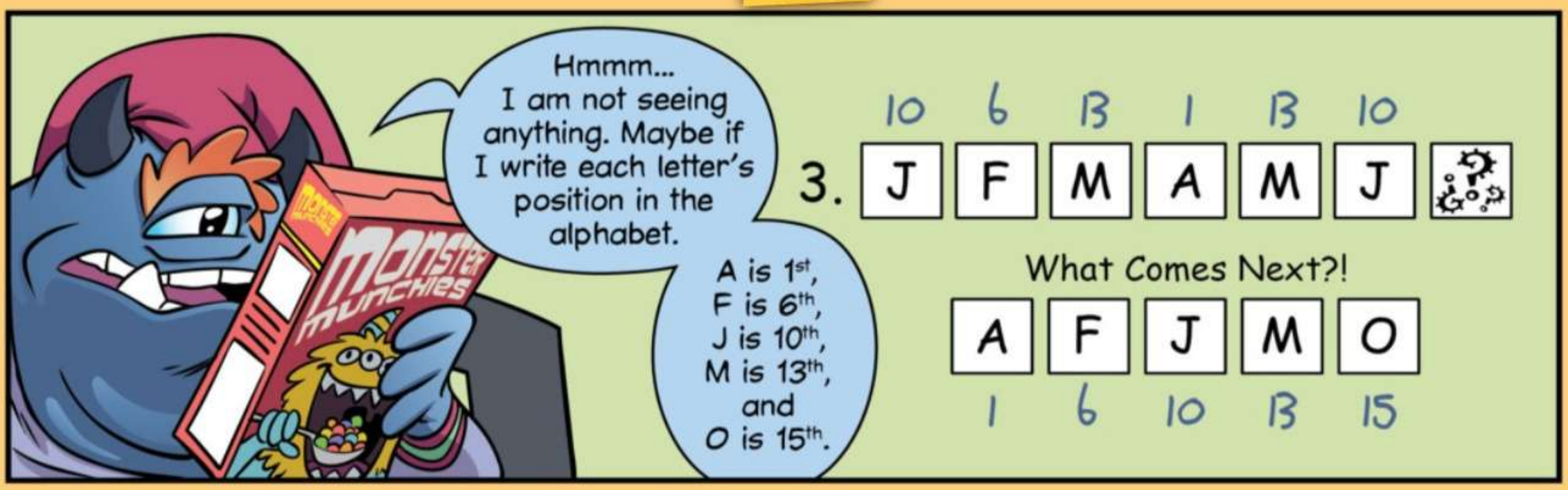
This one has letters!

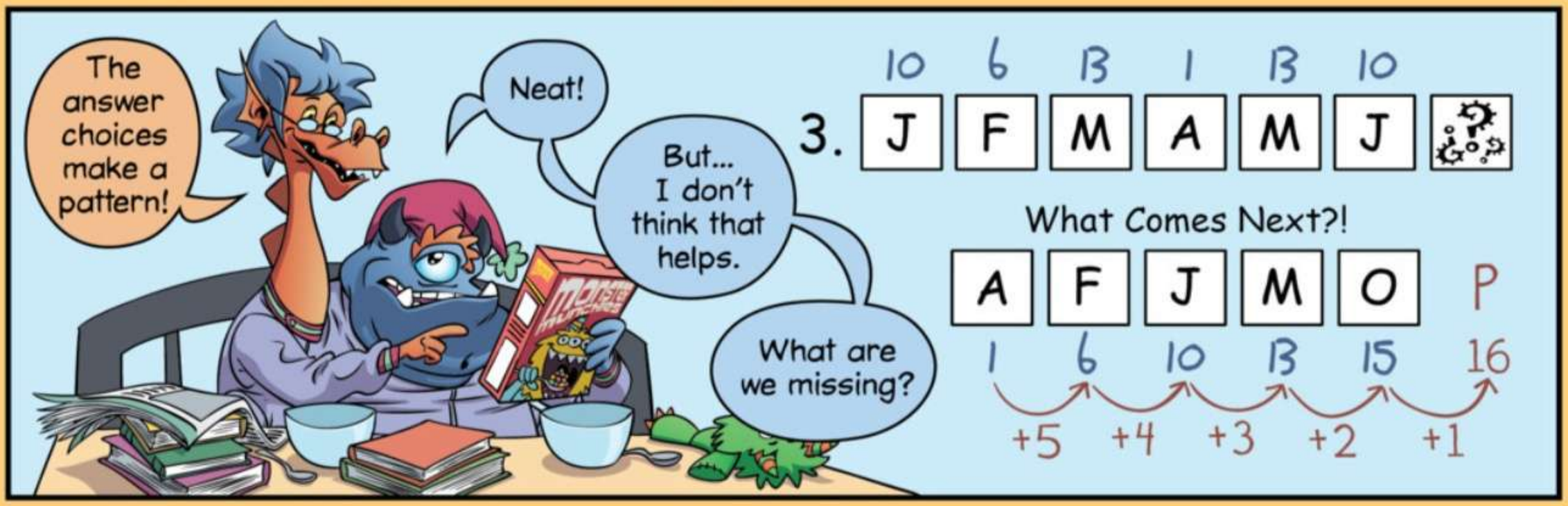
3. J F M A M J ?

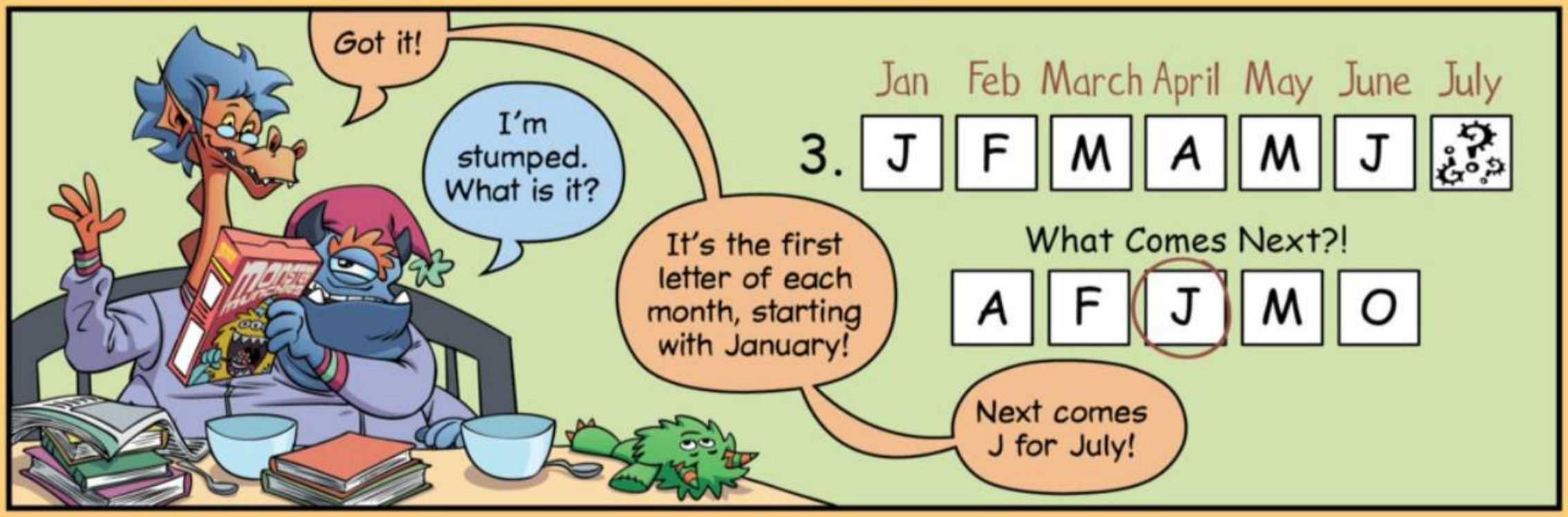
What Comes Next?!

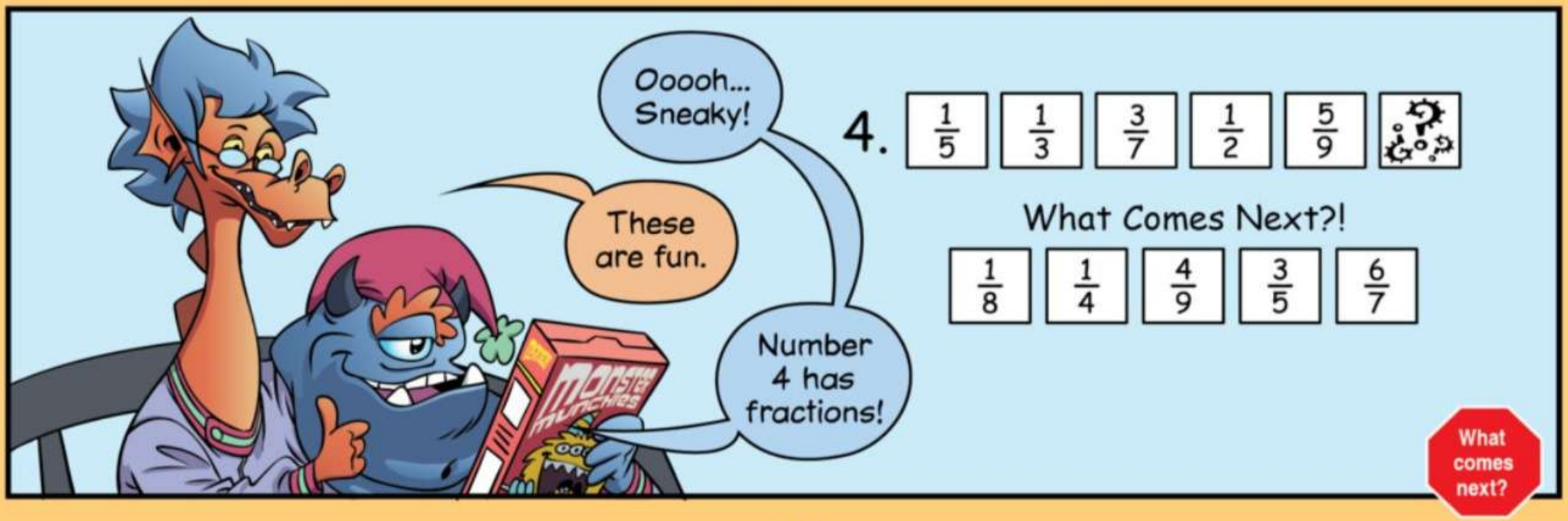
A F J M O

What comes next?

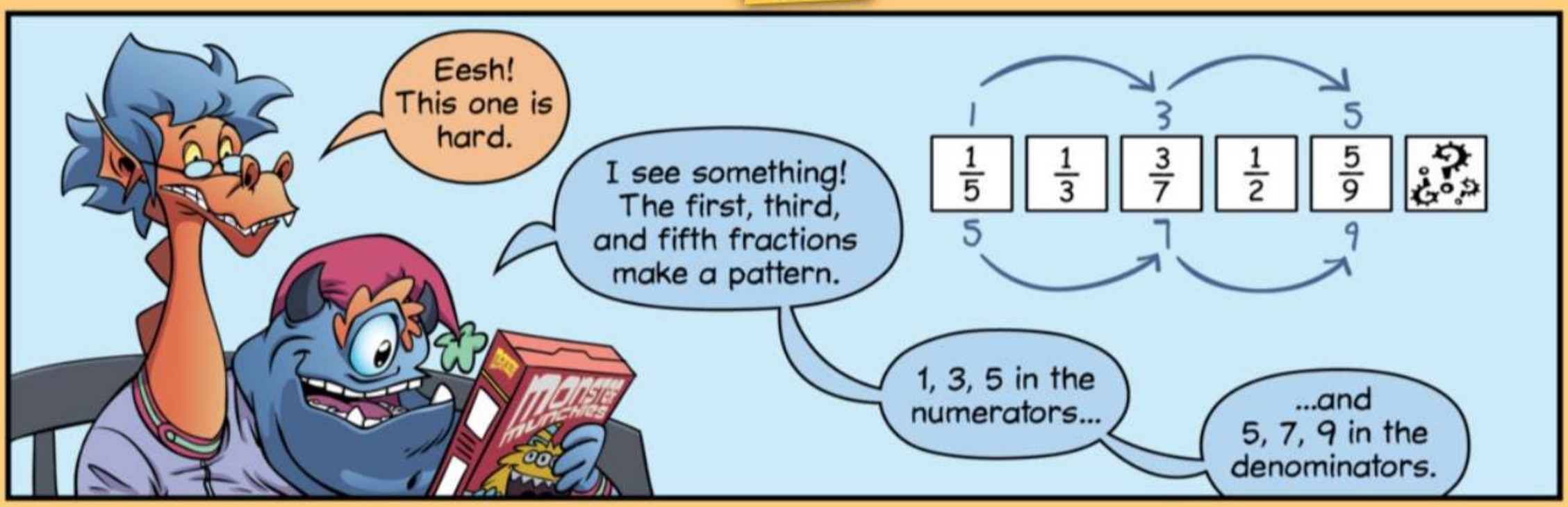


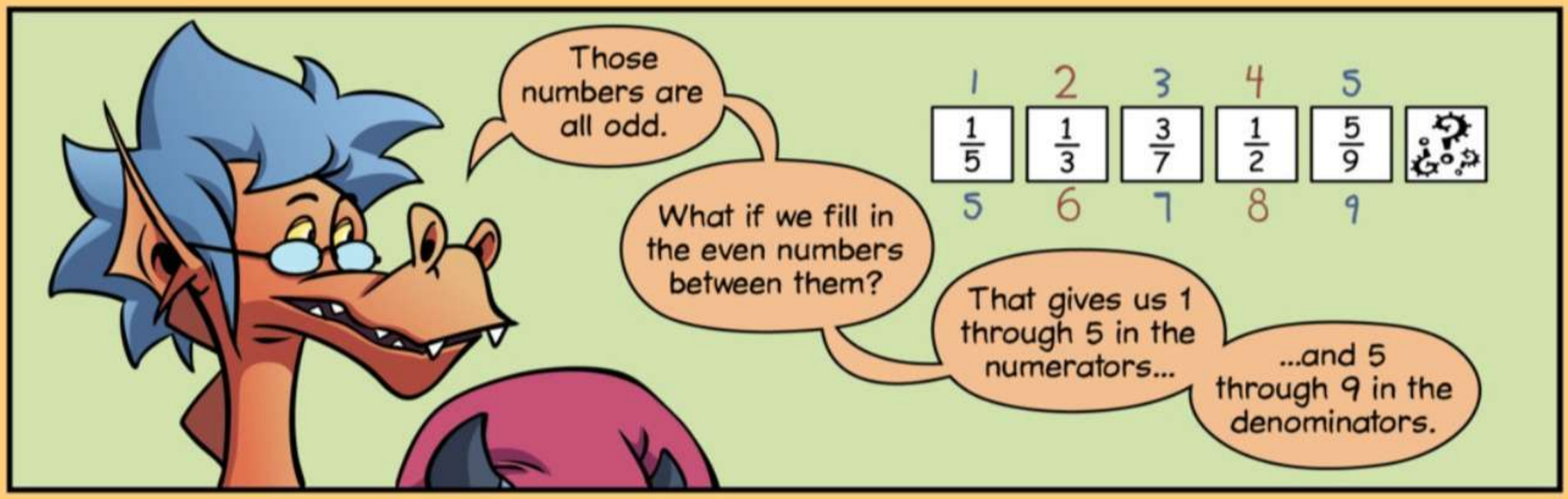


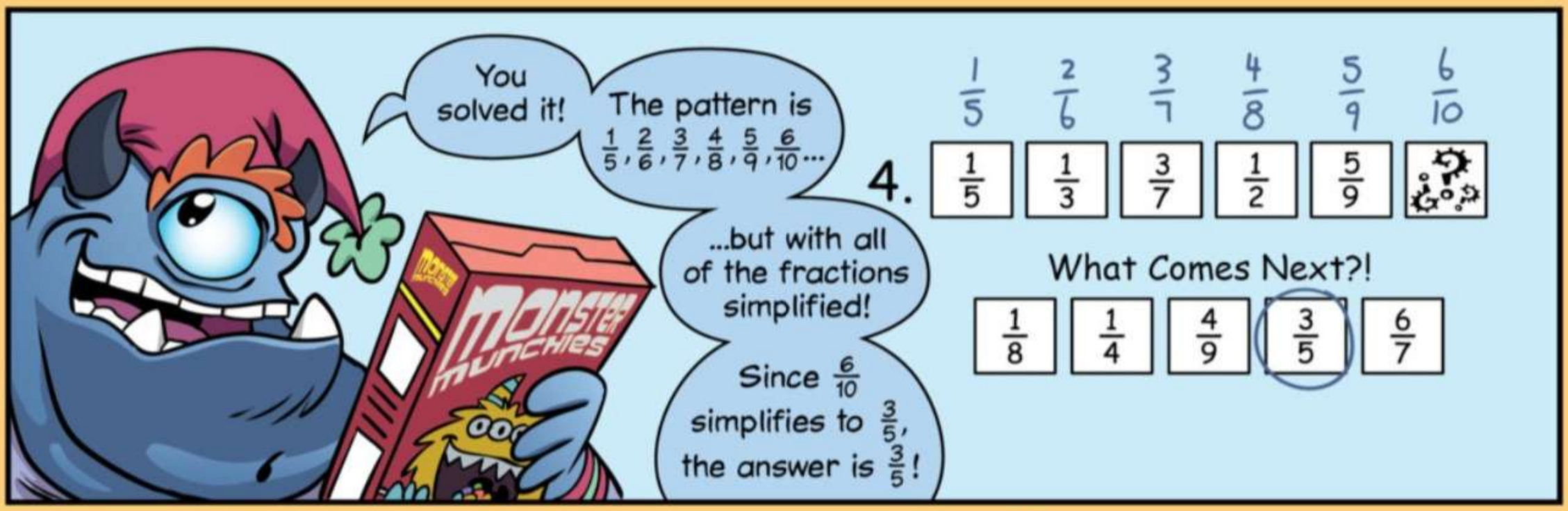


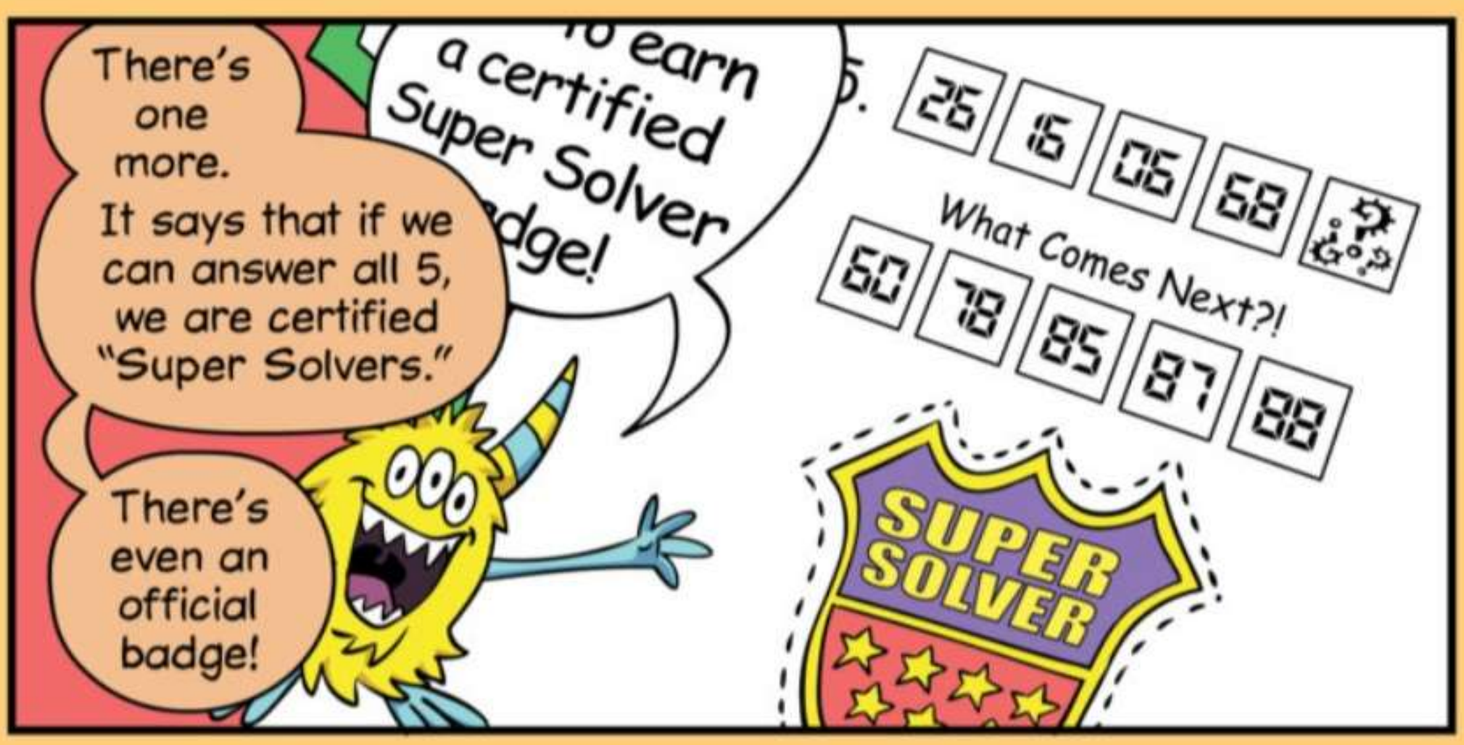


What comes next?


Eesh! This one is hard.
I see something! The first, third, and fifth fractions make a pattern.
1 3 5
1/5 1/3 3/7 1/2 5/9 ?
5 7 9
1, 3, 5 in the numerators...
...and 5, 7, 9 in the denominators.


Those numbers are all odd.
What if we fill in the even numbers between them?
1 2 3 4 5
1/5 1/3 3/7 1/2 5/9 ?
5 6 7 8 9
That gives us 1 through 5 in the numerators...
...and 5 through 9 in the denominators.


You solved it!
The pattern is 1/5, 2/6, 3/7, 4/8, 5/9, 6/10...
...but with all of the fractions simplified!
Since 6/10 simplifies to 3/5, the answer is 3/5!
1/5 2/6 3/7 4/8 5/9 6/10
4.
1/5 1/3 3/7 1/2 5/9 ?
What Comes Next?!
1/8 1/4 4/9 3/5 6/7


There's one more.
It says that if we can answer all 5, we are certified "Super Solvers."
There's even an official badge!
To earn a certified Super Solver badge!
5.
26 16 06 68 ?
What Comes Next?!
60 78 85 87 88
SUPER SOLVER

Just one badge?

All of the numbers have two digits. Even the 06. Maybe that's a clue?

I don't see any patterns that involve addition or multiplication.

Me neither.

5. 26 16 06 68 ?

What Comes Next?!

60 78 85 87 88

What comes next?

There sure are a lot of 6's.

These look like digital clock numbers.

Maybe they're related to time?

I got it!

You just have to look at the numbers upside-down.

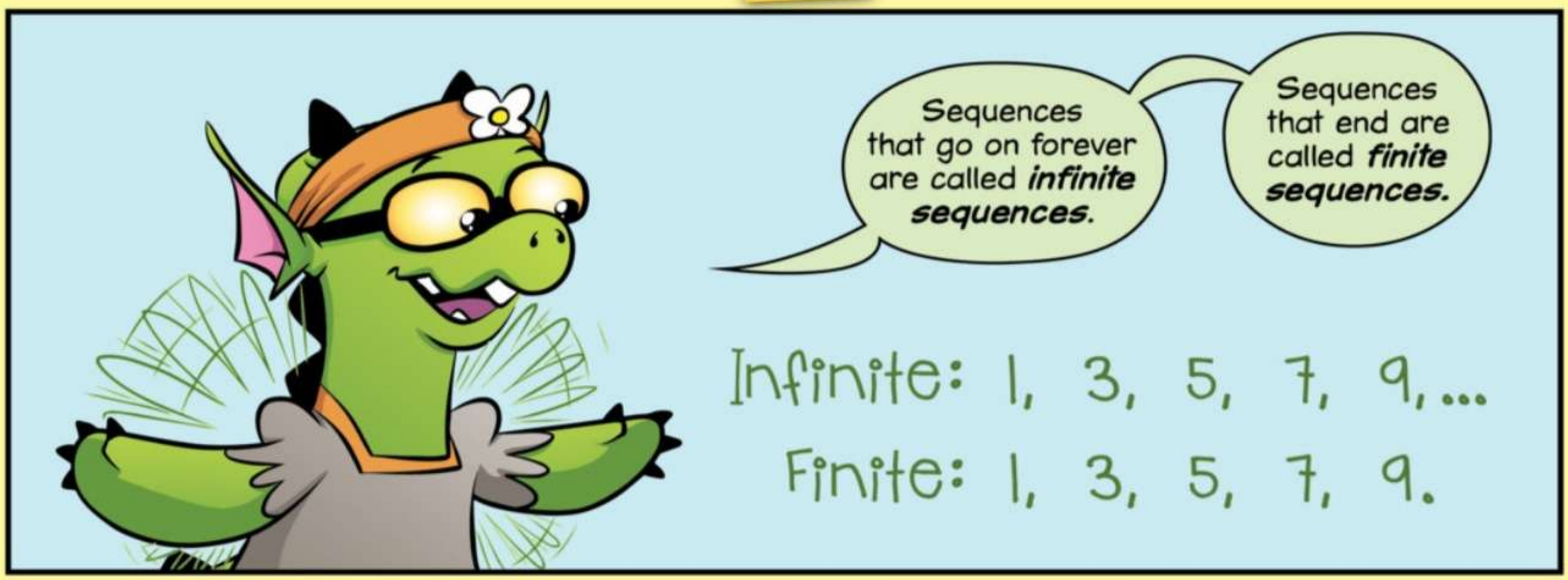
Really!? Let me see.

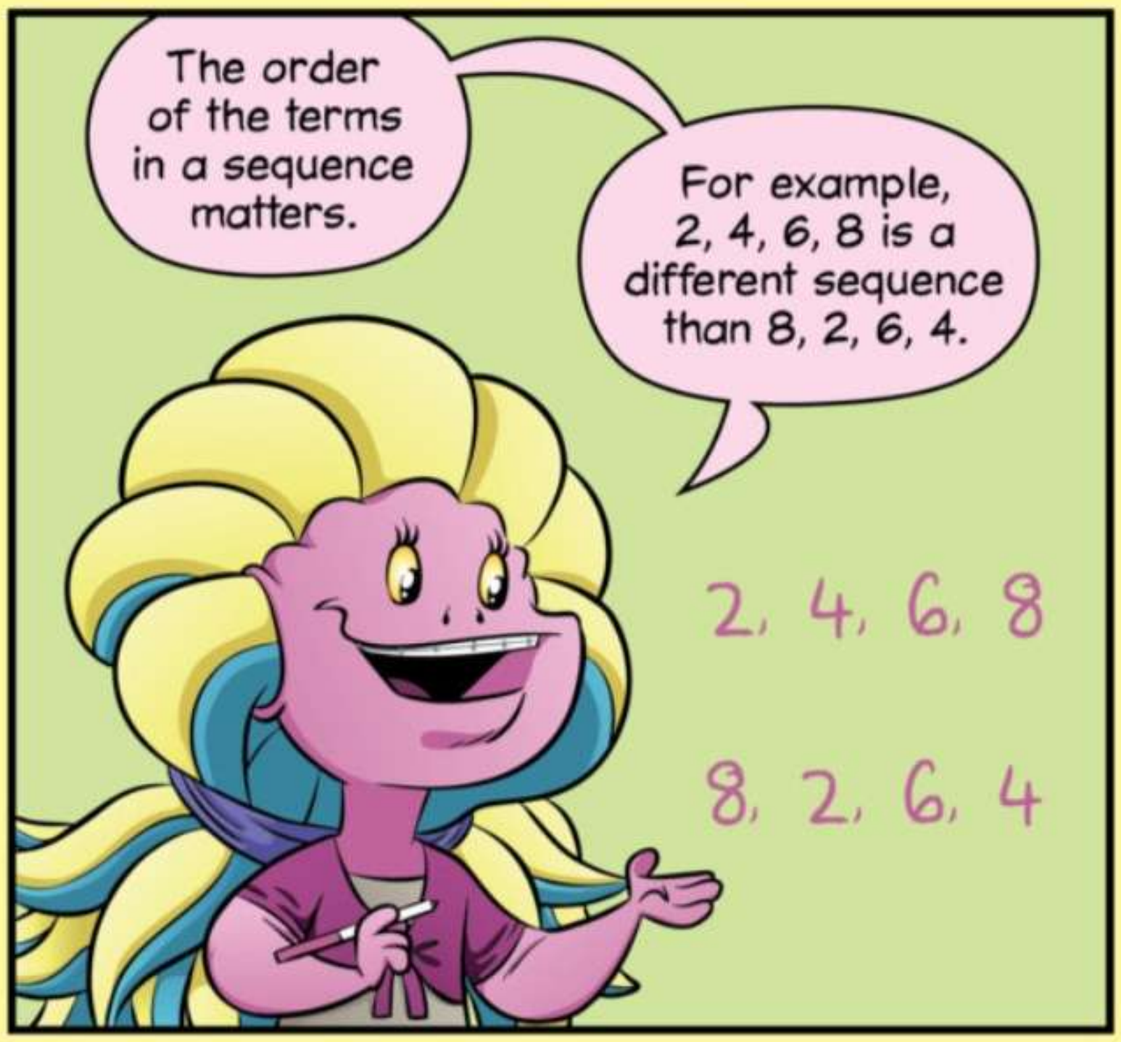
Grab!

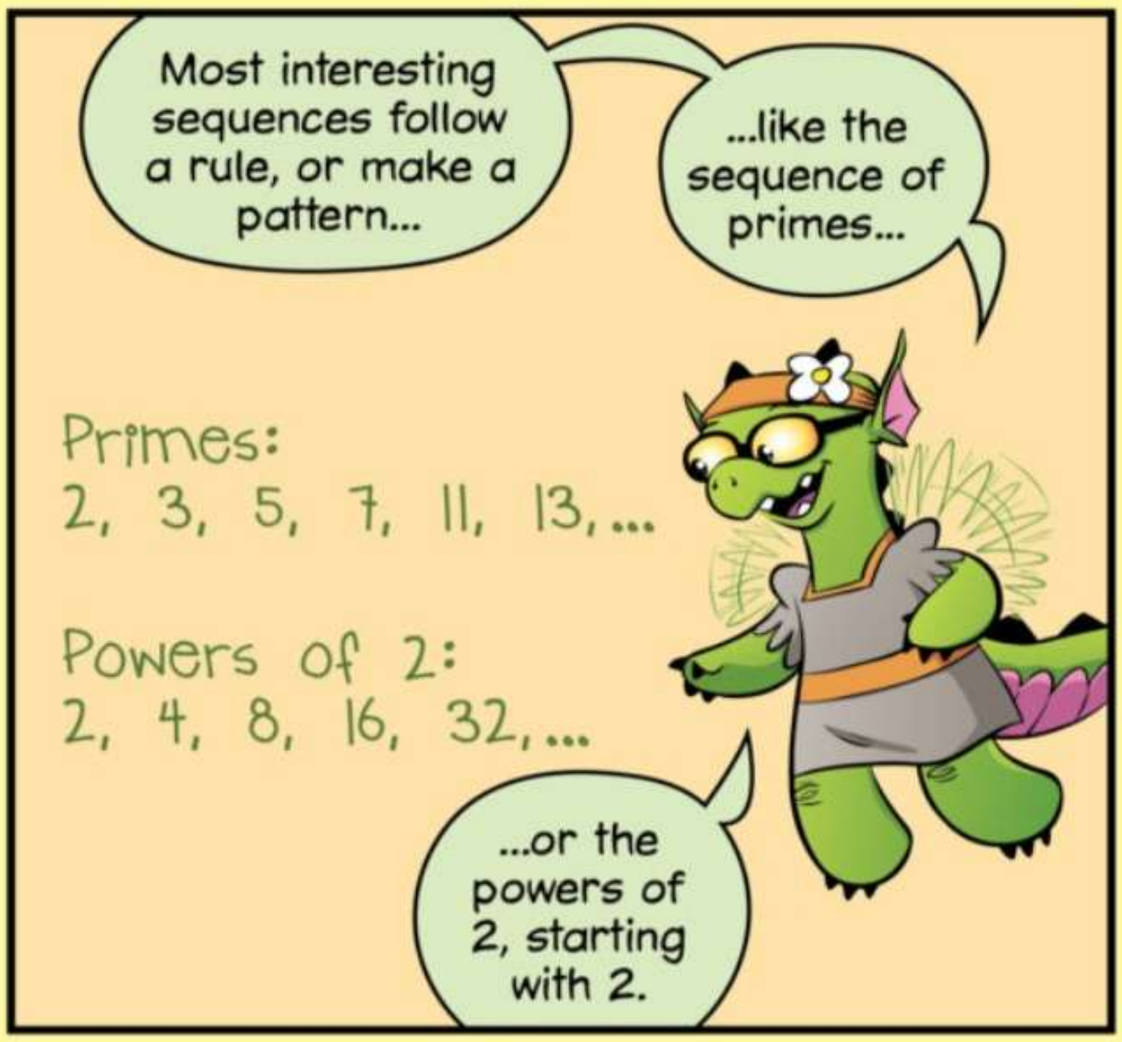
You're right!

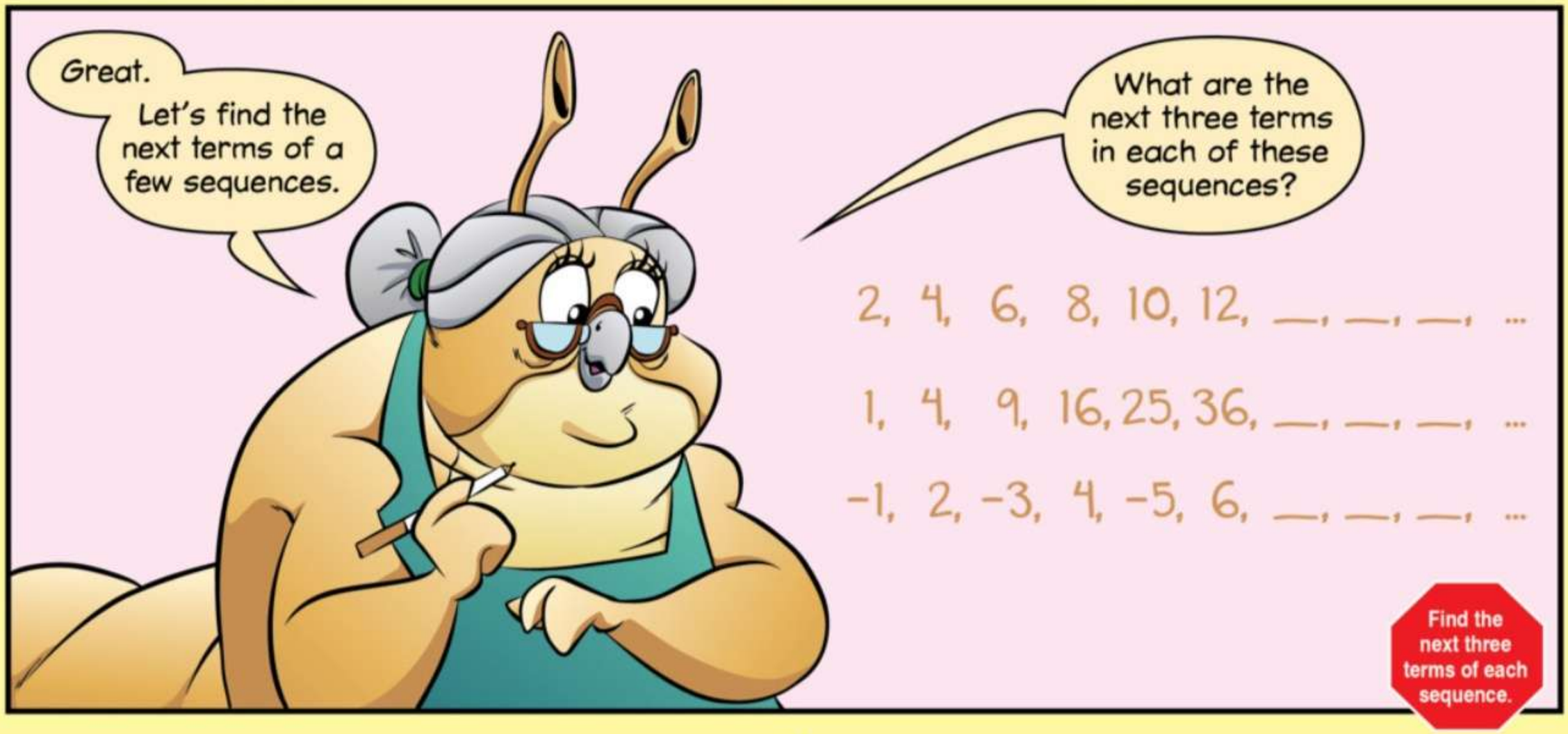
LOOK AT THE PATTERN UPSIDE-DOWN FOR THE ANSWER.

Ms. Q.
Sequences
What is a sequence?
Sequins are those shiny things Winnie has on her backpack!
I have them on one of my vests!
Sequence, Grogg!
Not sequins.
Oh.
A sequence is a list of terms, usually numbers.
For example, 1, 3, 5, 7, 9 is a sequence of 1-digit odd numbers.
1, 3, 5, 7, 9
Some sequences go on forever.
For example, the sequence of positive multiples of 7 begins 7, 14, 21, 28...
...and goes on and on and on and on.
7, 14, 21, 28, ...
THREE DOTS AT THE END OF A SEQUENCE LIKE THE ONE ABOVE MEAN THAT THE PATTERN CONTINUES FOREVER.

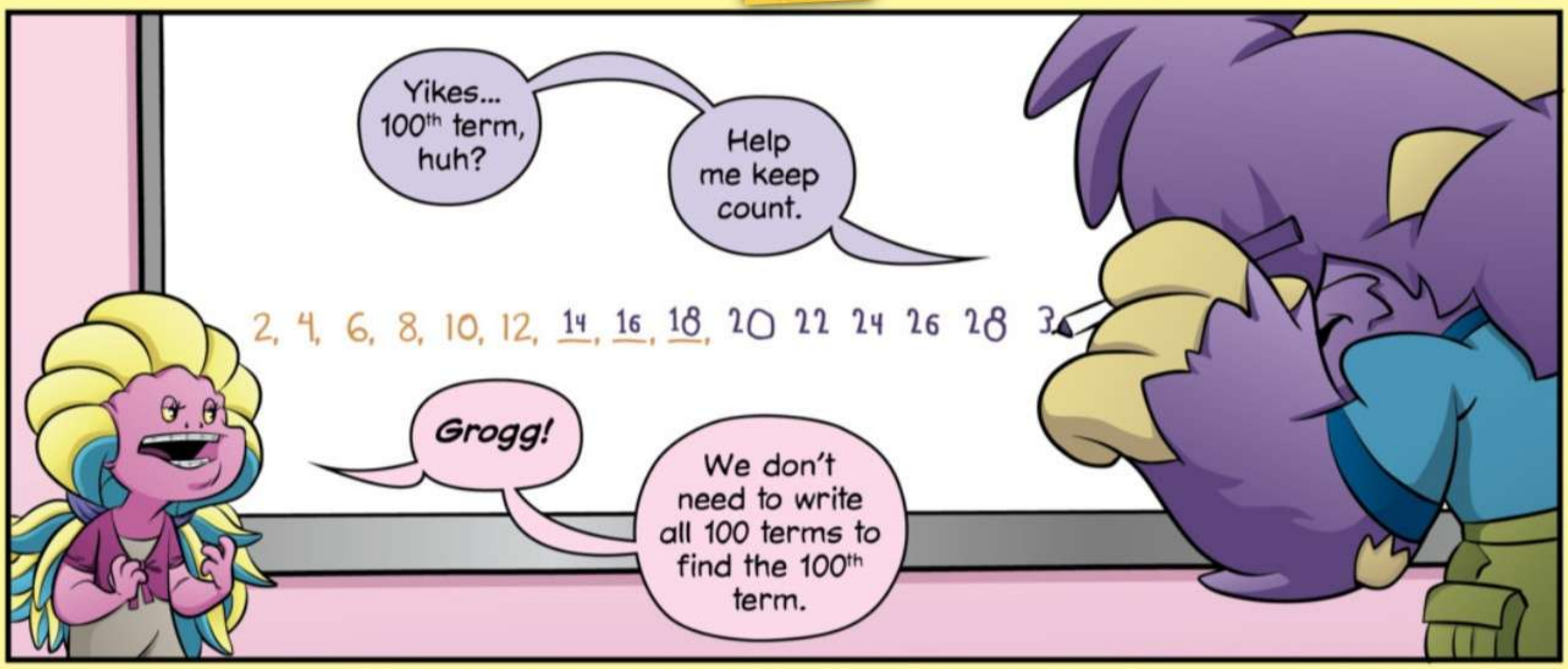

Sequences that go on forever are called ***infinite sequences.***
Sequences that end are called ***finite sequences.***
Infinite: 1, 3, 5, 7, 9, ...
Finite: 1, 3, 5, 7, 9.

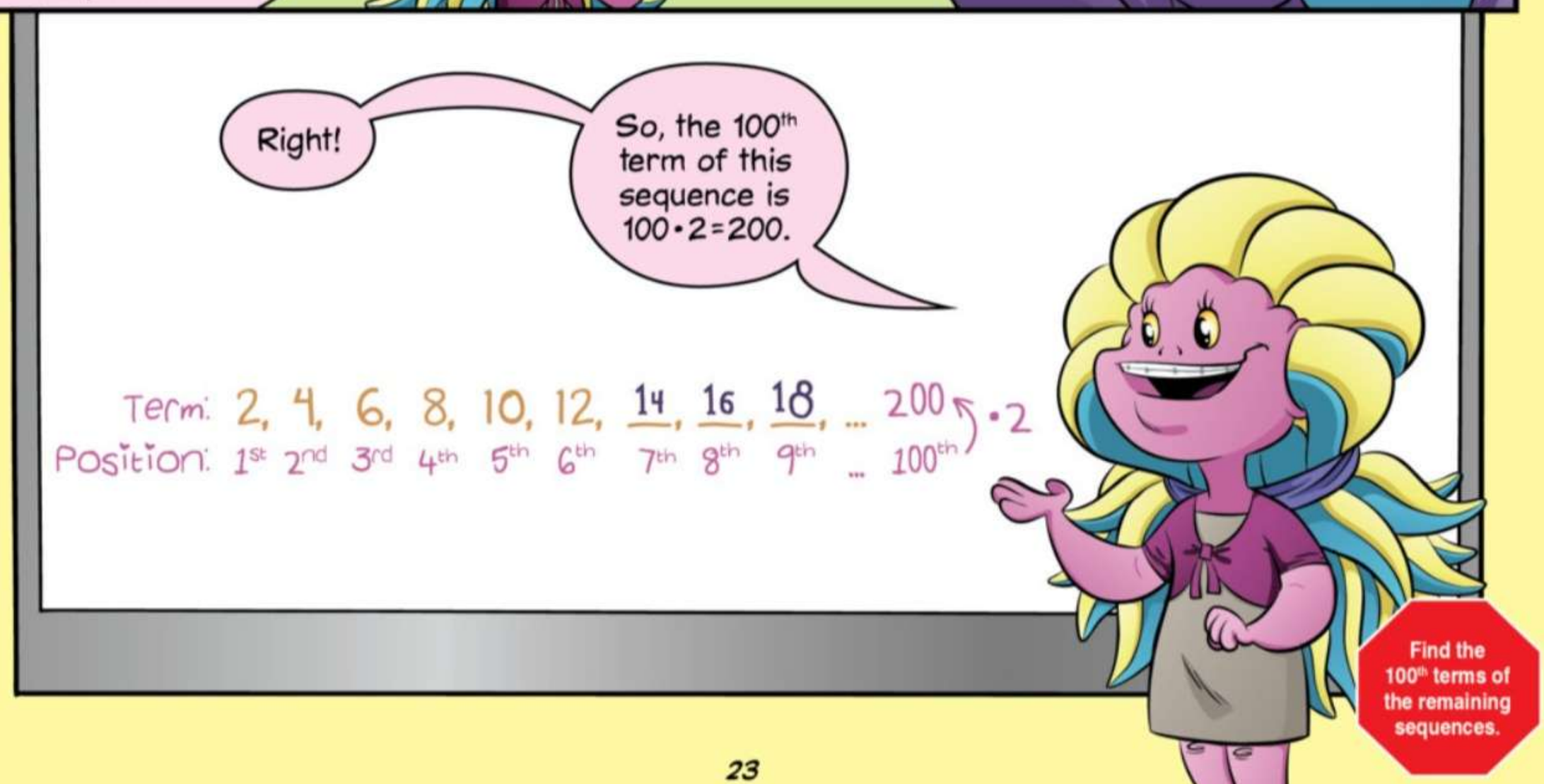

The order of the terms in a sequence matters.
For example, 2, 4, 6, 8 is a different sequence than 8, 2, 6, 4.
2, 4, 6, 8
8, 2, 6, 4


Most interesting sequences follow a rule, or make a pattern...
...like the sequence of primes...
Primes:
2, 3, 5, 7, 11, 13, ...
Powers of 2:
2, 4, 8, 16, 32, ...
...or the powers of 2, starting with 2.

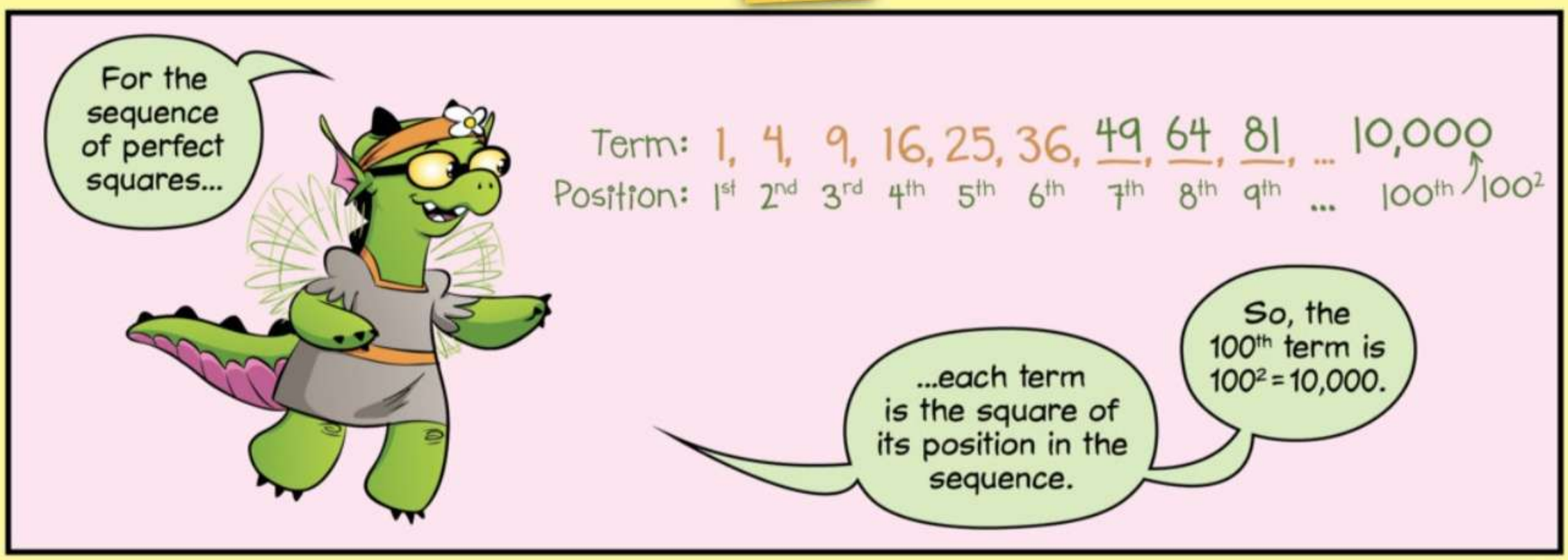

Great.
Let's find the next terms of a few sequences.
What are the next three terms in each of these sequences?
2, 4, 6, 8, 10, 12, __, __, __, ...
1, 4, 9, 16, 25, 36, __, __, __, ...
-1, 2, -3, 4, -5, 6, __, __, __, ...
Find the next three terms of each sequence.

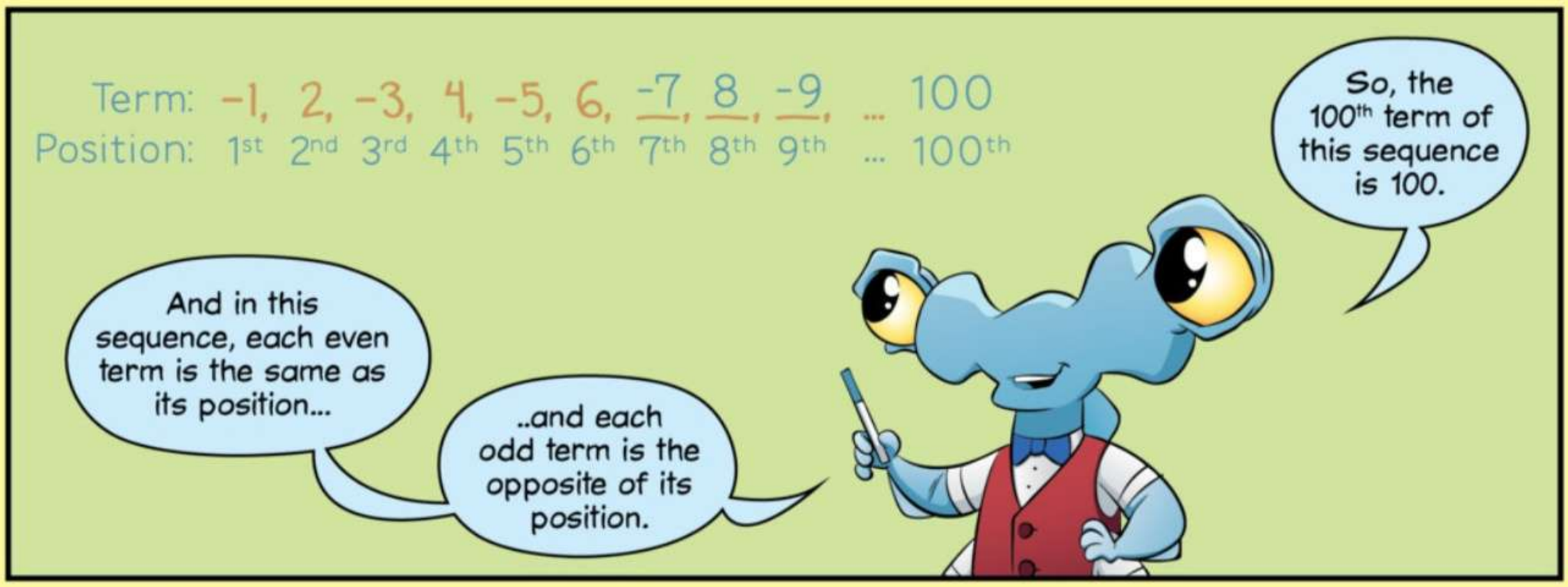
In the first sequence, we just add 2 to get the next term.
2, 4, 6, 8, 10, 12, 14, 16, 18, ...
12+2 =14...
14+2 =16...
And 16+2 =18.
The second sequence is the list of positive perfect squares.
1, 4, 9, 16, 25, 36, 49, 64, 81, ...
Next, we have $7^2=49$, $8^2=64$, and $9^2=81$.
The third sequence is just 1, 2, 3, 4, 5, 6,... but the numbers alternate between positive and negative.
-1, 2, -3, 4, -5, 6, -7, 8, -9, ...
So, it continues with -7, 8, and -9.
Very good, little monsters.
Now, can you find the 100th term in each of these sequences?
2, 4, 6, 8, 10, 12, 14, 16, 18, ...
1, 4, 9, 16, 25, 36, 49, 64, 81, ...
-1, 2, -3, 4, -5, 6, -7, 8, -9, ...
Find the 100th term of each sequence.

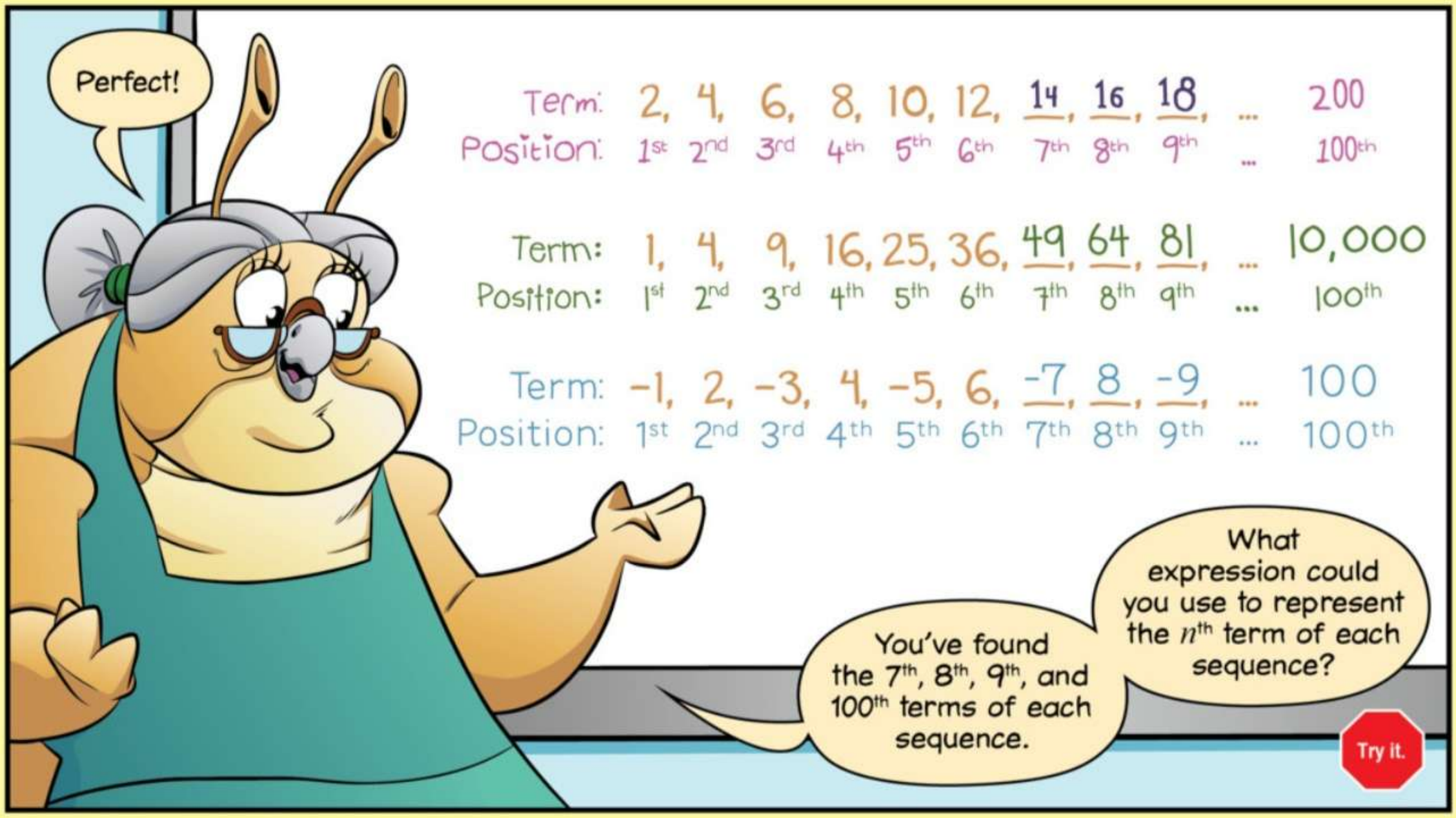




Find the 100th terms of the remaining sequences.


For the sequence of perfect squares...
Term: 1, 4, 9, 16, 25, 36, 49, 64, 81, ... 10,000
Position: 1st 2nd 3rd 4th 5th 6th 7th 8th 9th ... 100th
100²
...each term is the square of its position in the sequence.
So, the 100th term is 100² = 10,000.


Term: −1, 2, −3, 4, −5, 6, −7, 8, −9, ... 100
Position: 1st 2nd 3rd 4th 5th 6th 7th 8th 9th ... 100th
And in this sequence, each even term is the same as its position...
..and each odd term is the opposite of its position.
So, the 100th term of this sequence is 100.


Perfect!
Term: 2, 4, 6, 8, 10, 12, 14, 16, 18, ... 200
Position: 1st 2nd 3rd 4th 5th 6th 7th 8th 9th ... 100th
Term: 1, 4, 9, 16, 25, 36, 49, 64, 81, ... 10,000
Position: 1st 2nd 3rd 4th 5th 6th 7th 8th 9th ... 100th
Term: −1, 2, −3, 4, −5, 6, −7, 8, −9, ... 100
Position: 1st 2nd 3rd 4th 5th 6th 7th 8th 9th ... 100th
You've found the 7th, 8th, 9th, and 100th terms of each sequence.
What expression could you use to represent the nth term of each sequence?
Try it.

Term:	2,	4,	6,	8,	10,	12,	14,	16,	18,	...	2n
Position:	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	...	nth

In the first sequence, each term is twice its position in the sequence.

So, the nth term is $2n$.

Term:	1,	4,	9,	16,	25,	36,	49,	64,	81,	...	n^2
Position:	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	...	nth

In the second sequence, each term is the square of its position in the sequence.

So, the nth term is n^2.

Term:	-1,	2,	-3,	4,	-5,	6,	-7,	8,	-9,	...	
Position:	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	...	nth

In this sequence, sometimes the nth term is n...

...other times, it's $-n$.

Term:	-1,	2,	-3,	4,	-5,	6,	-7,	8,	-9,	...	
Position:	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	...	nth

How do we write an expression that works for both?

The terms in odd positions are negative...

...and the terms in even positions are positive.

We need to write an expression that is negative for odd values of n, but positive for even values of n.

Can you find an expression that works?

We can use the powers of -1!

Odd powers of -1 are negative...

...and the even powers are positive.

$(-1)^{99} = -1$

$(-1)^{100} = 1$

Excellent observation, Grogg!

How can we use the powers of -1 to make the odd terms negative, and the even terms positive?

Term:	-1,	2,	-3,	4,	-5,	6,	-7,	8,	-9,	...	$n \cdot (-1)^n$
Position:	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	...	n^{th}

So, the nth term of this sequence is n times $(-1)^n$.

If we multiply n by $(-1)^n$, the odd terms will all be negative...

...and the even terms will all be positive.

Why the sad face Grogg?

I really thought this class was going to be all about ***sequins!***

I even made this sweet cape!

Nice stitch work!

RECESS

Subtractionacci

Subtractionacci is a pencil-and-paper game for two players.

Find a partner and play!

Play

In a game of Subtractionacci, players take turns as the Picker and the Subtractor.

The Picker selects a two-digit number to be used as the first term of a Subtractionacci sequence.

The Subtractor selects a smaller number that is the second term of the sequence.

The sequence is completed by the Subtractor according to the following rules:

To find the next term in the sequence, take the two previous terms and subtract the smaller one from the larger one.
The sequence ends when a term in the sequence is larger than the one before it. Three examples are shown below:

Picker chooses: 41
Subtractor chooses: 23
The third term is 41−23 = 18
The fourth term is 23−18= 5
The last term is 18−5= 13

Picker chooses: 75
Subtractor chooses: 48
The third term is 75−48 = 27
The fourth term is 48−27 = 21
The fifth term is 27−21 = 6
The last term is 21−6 = 15

Picker chooses: 98
Subtractor chooses: 66
The third term is 98−66 = 32
The last term is 66−32 = 34

The last term is added to the Picker's score. In the three examples above, the Picker scores 13, 15, and 34 points.
The goal of the Subtractor is to choose a number that will give the Picker as few points as possible.

Players take turns as the Picker and the Subtractor. Pickers may not choose the same number twice.

Winning

The first player to score 100 points wins.

Strategies to Consider:

- As Picker, is it generally better to choose a large or a small number?
- As Subtractor, is it better to choose a number that is more than or less than half of the Picker's number?
- As Subtractor, is it better to choose a number that is more than or less than two thirds of the Picker's number?
- If the Picker chooses 89, what number can the Subtractor choose that gives the Picker just 1 point?
- The name of the game, Subtractionacci, is a reference to the Fibonacci sequence. Learn more about the Fibonacci sequence online or in the Practice book. What does this game have to do with the Fibonacci sequence?

MATH TEAM
Arithmetic Sequences

One of the first things you ever learned in math was a sequence...

...when you learned to recite the positive integers.

Later on, you learned how to skip-count.

Starting at a number, we always add the same amount to get the next term.

For example, the terms of this sequence start at 50 and count by 7's.

A sequence where we always add the same amount to get from one term to the next is called an ***arithmetic sequence***.

How could you find the 100th term of this arithmetic sequence?

50, 57, 64, 71, 78, ...

What is the 100th term?

We add 7 to get from each term to the next.

We just need to know how many 7's we add to get the 100th term.

To get the 100th term, can we just add one hundred 7's?

I don't think that's right.

+7 +7 +7 +7

50, 57, 64, 71, 78, ...

Term: 1st 2nd 3rd 4th 5th

50, 57, 64, 71, 78, ...

$50+(1\cdot7)=$
$50+(2\cdot7)=$
$50+(3\cdot7)=$
$50+(4\cdot7)=$

To get the ***second*** term, we only add ***one*** 7.

To get the ***third*** term, we add ***two*** 7's.

To get the ***fourth*** term, we add ***three*** 7's.

Term: 1st 2nd 3rd 4th 5th 100th n^{th}

50, 57, 64, 71, 78, ... 743

$50+(1\cdot7)=$
$50+(2\cdot7)=$
$50+(3\cdot7)=$
$50+(4\cdot7)=$
$50+(99\cdot7)=$

Good. What is the nth term of this sequence?

To get the ***100th*** term, we have to add ***99*** 7's.

$50+(99\cdot7)$
$=50+693$
$=743.$

Try it.

To get the nth term, we have to add $(n-1)$ 7's.

Term:	1st	2nd	3rd	4th	5th		100th	n^{th}
	50,	57,	64,	71,	78,	...	743	$50+7(n-1)$
	$50+(1\cdot7)=$	$50+(2\cdot7)=$	$50+(3\cdot7)=$	$50+(4\cdot7)=$			$50+(99\cdot7)=$	

So, the nth term is $50+7(n-1)$, which simplifies to $7n+43$.*

*TO SIMPLIFY $50+7(n-1)$, DISTRIBUTE THE 7 TO GET $50+7n-7$, THEN COMBINE LIKE TERMS TO GET $7n+43$.

Well done.

$$50 \xrightarrow{+7} 57 \xrightarrow{+7} 64 \xrightarrow{+7} 71 \xrightarrow{+7} 78, \ldots$$

The amount you add is called the ***common difference*** of the sequence.

This sequence has a common difference of 7.

Find the common difference for each of these arithmetic sequences.

199, 298, 397, 496, ...

13, 5, −3, −11, −19, ...

$\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{5}{12}, \frac{1}{2}, \ldots$

Try it.

We can find the common difference in an arithmetic sequence by subtracting any term from the next term.

The common difference is **99**.

$$199 \xrightarrow{+99} 298 \xrightarrow{+99} 397 \xrightarrow{+99} 496, \ldots$$

In this sequence, the numbers are decreasing!

Each number is 8 less than the one before it.

We can add -8 to get the next term, so the common difference is -8.

$$13 \xrightarrow{-8} 5 \xrightarrow{-8} -3 \xrightarrow{-8} -11 \xrightarrow{-8} -19, \ldots$$

This one looked hard at first...

...until I changed all of the fractions into twelfths.

The common difference is $\frac{1}{12}$.

$$\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{5}{12}, \frac{1}{2}, \ldots$$

$$\frac{2}{12}, \frac{3}{12}, \frac{4}{12}, \frac{5}{12}, \frac{6}{12}, \ldots$$

$$+\frac{1}{12} \quad +\frac{1}{12} \quad +\frac{1}{12} \quad +\frac{1}{12}$$

Perfect. Use these common differences to compute the 21st term of each sequence.

$$199 \xrightarrow{+99} 298 \xrightarrow{+99} 397 \xrightarrow{+99} 496, \ldots$$

$$13 \xrightarrow{-8} 5 \xrightarrow{-8} -3 \xrightarrow{-8} -11 \xrightarrow{-8} -19, \ldots$$

$$\frac{1}{6} \xrightarrow{+\frac{1}{12}} \frac{1}{4} \xrightarrow{+\frac{1}{12}} \frac{1}{3} \xrightarrow{+\frac{1}{12}} \frac{5}{12} \xrightarrow{+\frac{1}{12}} \frac{1}{2}, \ldots$$

Try it.

To get the 21st term of this sequence, we start at 199, and add twenty 99's...

...199+(20•99)
=199+1,980
=**2,179.**

+99 +99 +99

199, 298, 397, 496,... 2,179

In this sequence, we add twenty -8's to get the 21st term.

The first term is 13, so...

...the 21st term is
13+20(-8)
=13+(-160)
=**-147.**

-8 -8 -8 -8

13, 5, -3, -11, -19,... -147

And in this one, we add twenty twelfths to one sixth to get the 21st term.

$\frac{1}{6}+\left(20\cdot\frac{1}{12}\right)$
$=\frac{1}{6}+\frac{20}{12}$
$=\frac{1}{6}+\frac{10}{6}$
$=\frac{11}{6}$...

...or $1\frac{5}{6}$.

$+\frac{1}{12}$ $+\frac{1}{12}$ $+\frac{1}{12}$ $+\frac{1}{12}$

$\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{5}{12}, \frac{1}{2}, \ldots$ $1\frac{5}{6}$

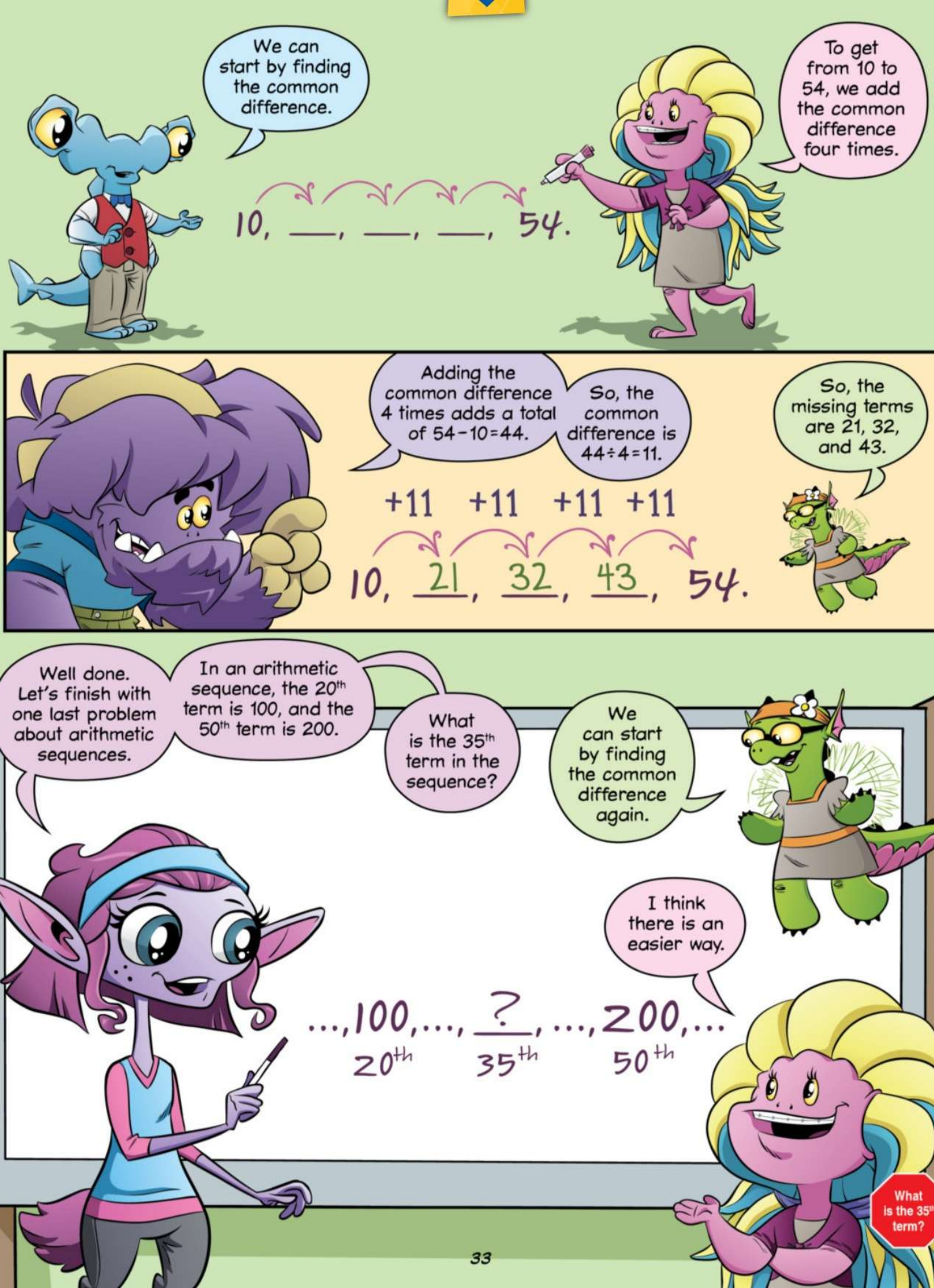
Great work.

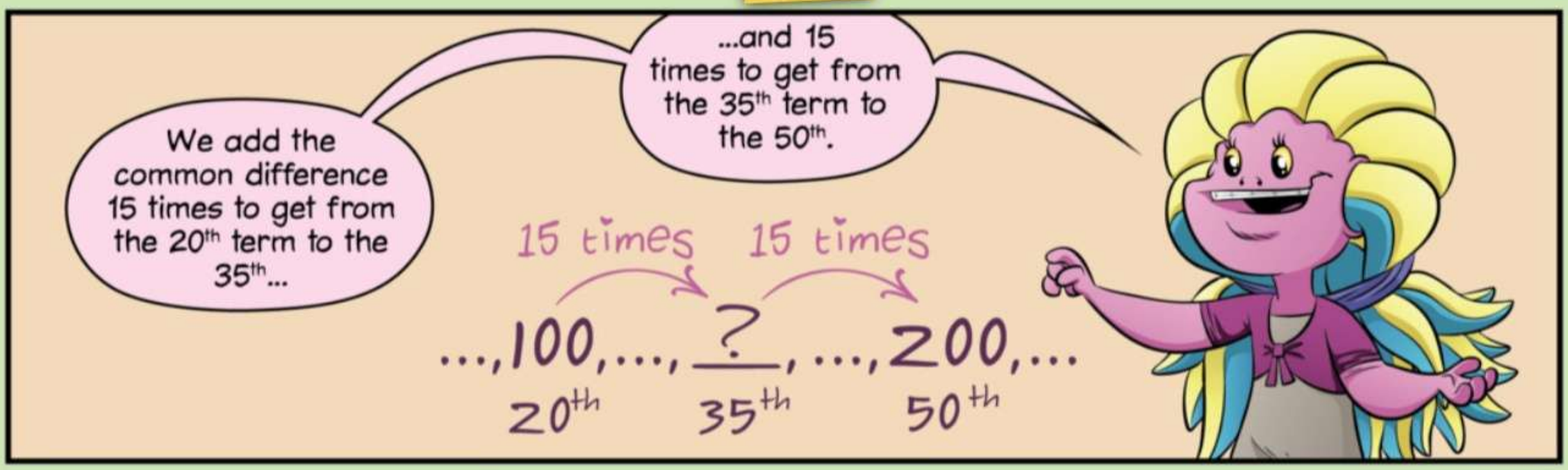
In this next problem, you need to find some missing terms in the ***middle*** of an arithmetic sequence.

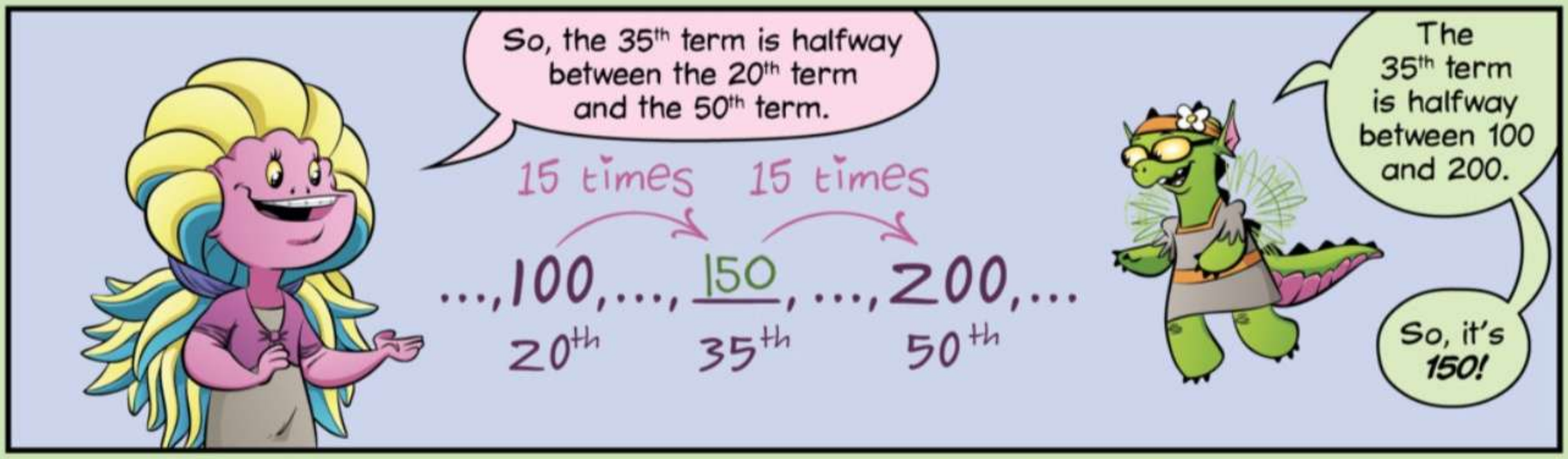
How would you begin?

10, ___, ___, ___, 54.

Try it.

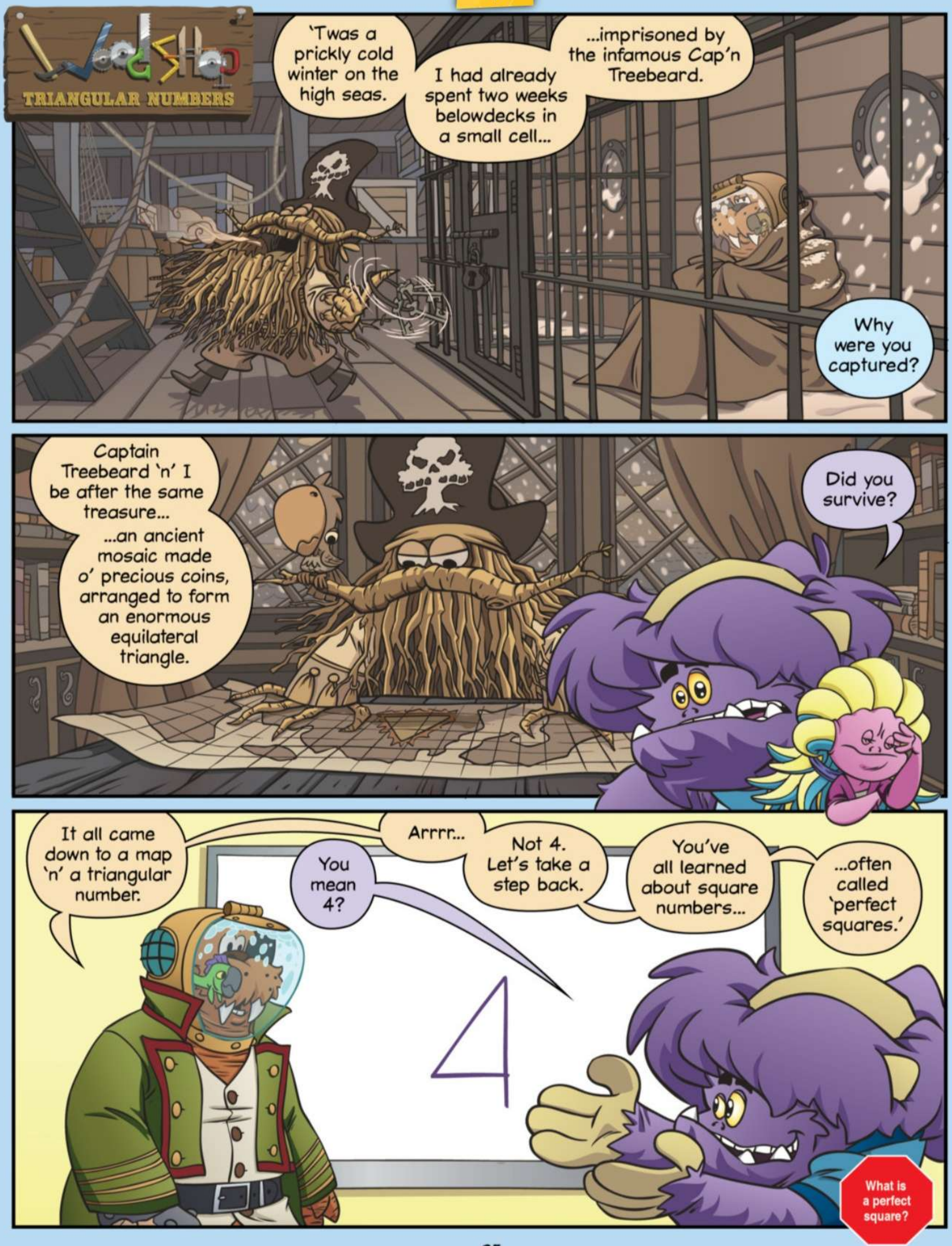

We can start by finding the common difference.
10, ___, ___, ___, 54.
To get from 10 to 54, we add the common difference four times.
Adding the common difference 4 times adds a total of 54−10=44.
So, the common difference is 44÷4=11.
+11 +11 +11 +11
10, 21, 32, 43, 54.
So, the missing terms are 21, 32, and 43.
Well done. Let's finish with one last problem about arithmetic sequences.
In an arithmetic sequence, the 20th term is 100, and the 50th term is 200.
What is the 35th term in the sequence?
We can start by finding the common difference again.
I think there is an easier way.
..., 100, ..., ?, ..., 200, ...
20th 35th 50th
What is the 35th term?


We add the common difference 15 times to get from the 20th term to the 35th...
...and 15 times to get from the 35th term to the 50th.
15 times
15 times
...,100,...,?,...,200,...
20th
35th
50th


So, the 35th term is halfway between the 20th term and the 50th term.
15 times
15 times
...,100,...,150,...,200,...
20th
35th
50th
The 35th term is halfway between 100 and 200.
So, it's 150!

We've really come a long way with sequences since our toddler years!
Imagine what we'll be doing with sequences when we're senior citizens!

Knit 1, purl 2...
...knit 1, purl 2...
Read 'em and weep!
Step 1, and 2, and 3...
Yahtzee!


TRIANGULAR NUMBERS
'Twas a prickly cold winter on the high seas.
I had already spent two weeks belowdecks in a small cell...
...imprisoned by the infamous Cap'n Treebeard.
Why were you captured?
Captain Treebeard 'n' I be after the same treasure...
...an ancient mosaic made o' precious coins, arranged to form an enormous equilateral triangle.
Did you survive?
It all came down to a map 'n' a triangular number.
Arrrr...
You mean 4?
Not 4. Let's take a step back.
You've all learned about square numbers...
...often called 'perfect squares.'
4
What is a perfect square?

When you multiply an integer by itself, the result is a perfect square.

Aye, 'n' why do we call a perfect square a perfect square?

The area of a square that has integer side lengths is a perfect square.

We can arrange a perfect square number of coins into a square.

$1^2=1$ $2^2=4$ $3^2=9$ $4^2=16$ $5^2=25$

So, can a triangular number of coins make a triangle?

Now yer catchin' on, lad!

The same way we can arrange a square number o' coins into a square...

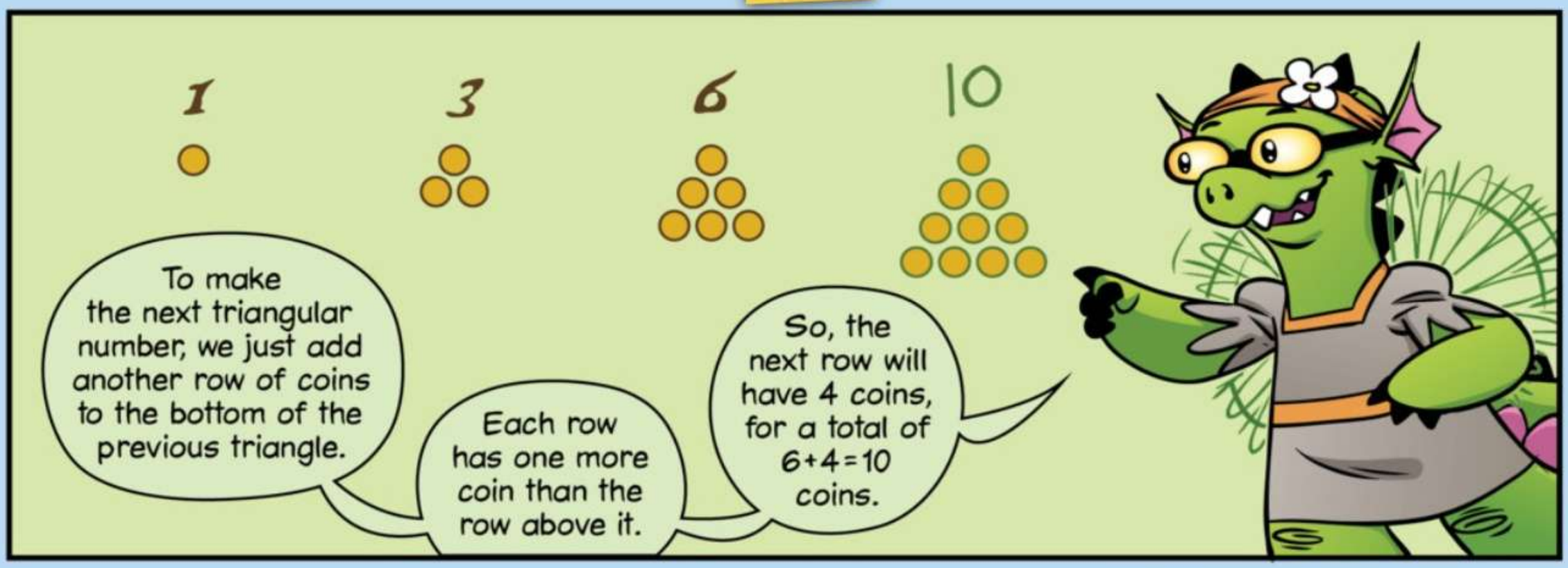
...we can arrange a triangular number o' coins into an equilateral triangle.

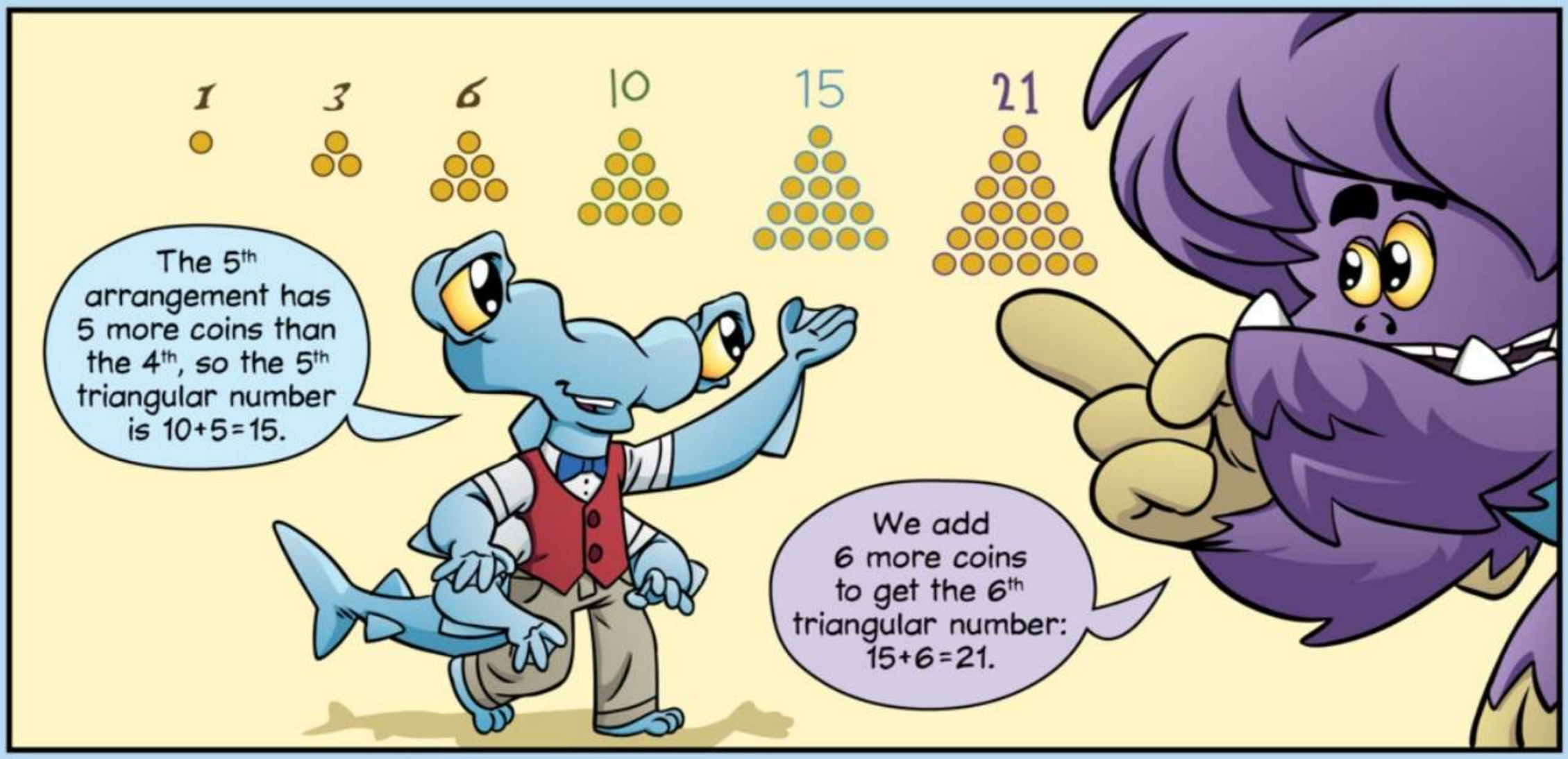
The first three triangular numbers be 1, 3, 'n' 6.

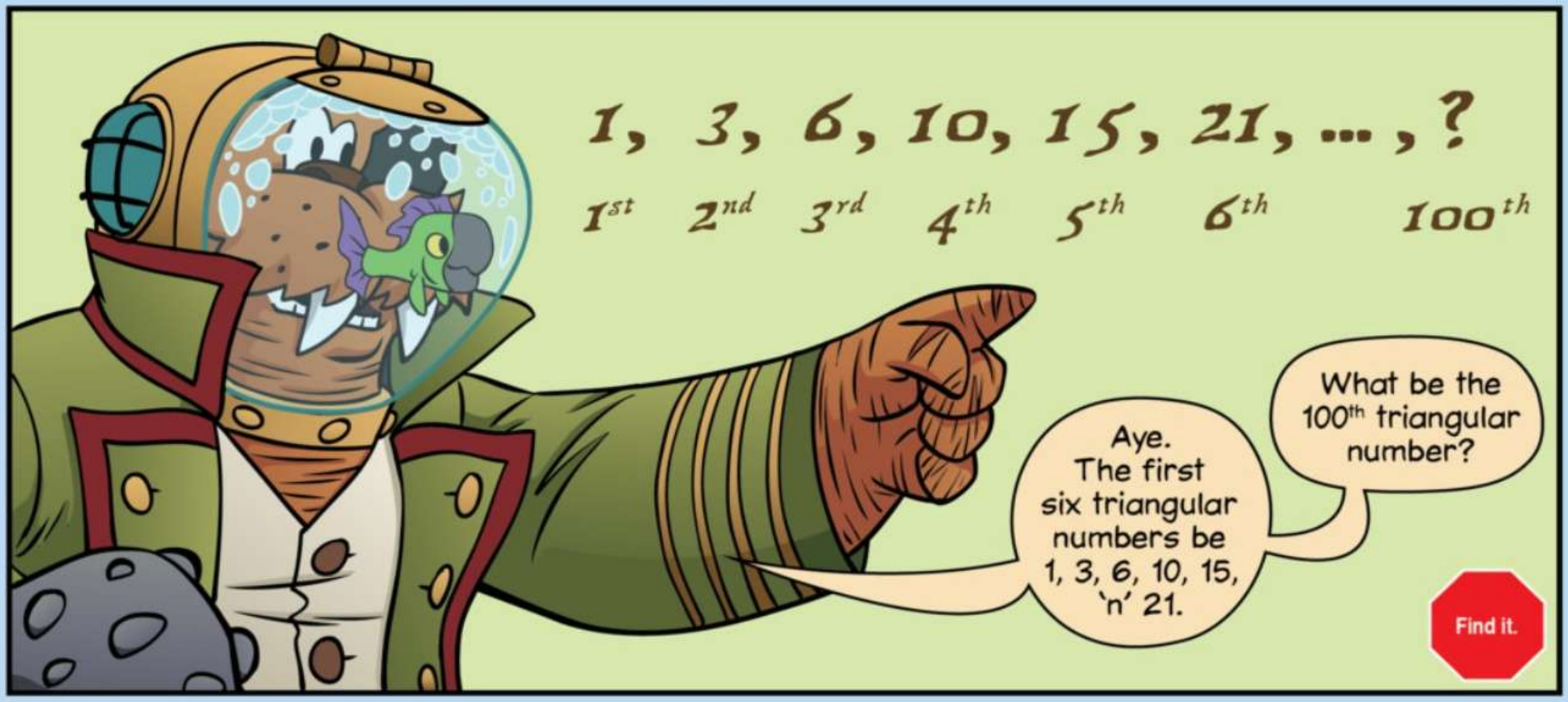
Who can be namin' the next three triangular numbers?

1 3 6

What are the next three?


1
3
6
10
To make the next triangular number, we just add another row of coins to the bottom of the previous triangle.
Each row has one more coin than the row above it.
So, the next row will have 4 coins, for a total of 6+4=10 coins.


1
3
6
10
15
21
The 5th arrangement has 5 more coins than the 4th, so the 5th triangular number is 10+5=15.
We add 6 more coins to get the 6th triangular number: 15+6=21.


1, 3, 6, 10, 15, 21, ... , ?
1st 2nd 3rd 4th 5th 6th 100th
Aye. The first six triangular numbers be 1, 3, 6, 10, 15, 'n' 21.
What be the 100th triangular number?
Find it.

The fourth triangular number is 1+2+3+4=10.

The fifth triangular number is 1+2+3+4+5=15.

The sixth triangular number is 1+2+3+4+5+6=21.

The 100th triangular number is the sum of every positive integer from 1 to 100.

$$1+2+3+\cdots+98+99+100$$

I know an easy way to add the list of numbers 1 through 100!

To begin, we add ***two*** copies of the list...

...with one list written backwards.

That way, all 100 pairs of numbers have the same sum... 101.

$$\begin{array}{c} 1+2+3+\cdots+98+99+100 \\ 100+99+98+\cdots+3+2+1 \\ \hline 101+101+101+\cdots+101+101+101 \end{array}$$

LIZZIE INTRODUCES THIS METHOD IN HER NOTES ON PAGE 45 OF BEAST ACADEMY GUIDE 4B.

$$\begin{array}{c} 1+2+3+\cdots+98+99+100 \\ 100+99+98+\cdots+3+2+1 \\ \hline \underbrace{101+101+101+\cdots+101+101+101}_{100\cdot 101=10{,}100} \end{array}$$

But, that counts every number twice.

So, the sum of all 100 pairs is 100•101=10,100.

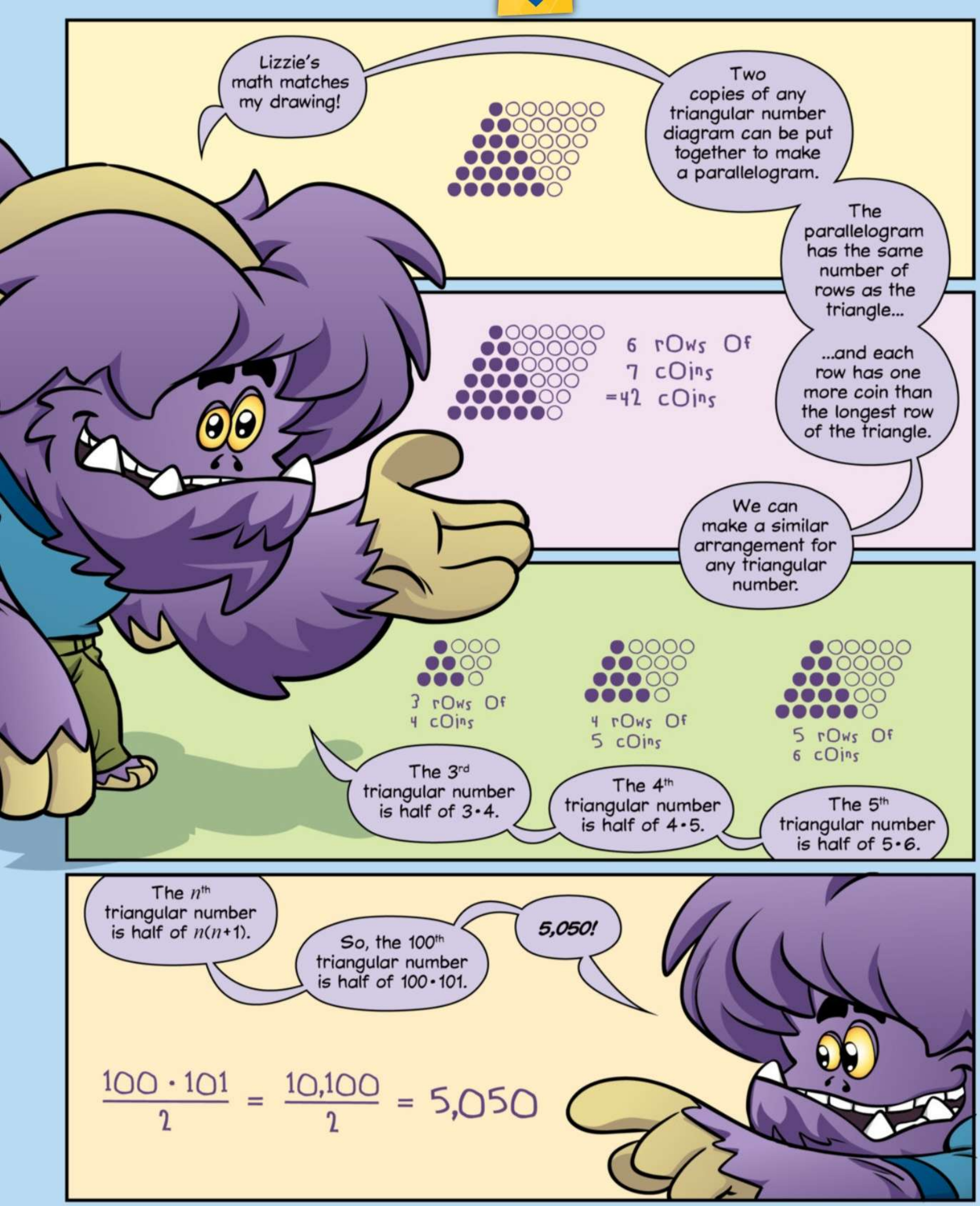
$$1,\ 3,\ 6,\ 10,\ 15,\ 21,\ \ldots,\ 5{,}050$$

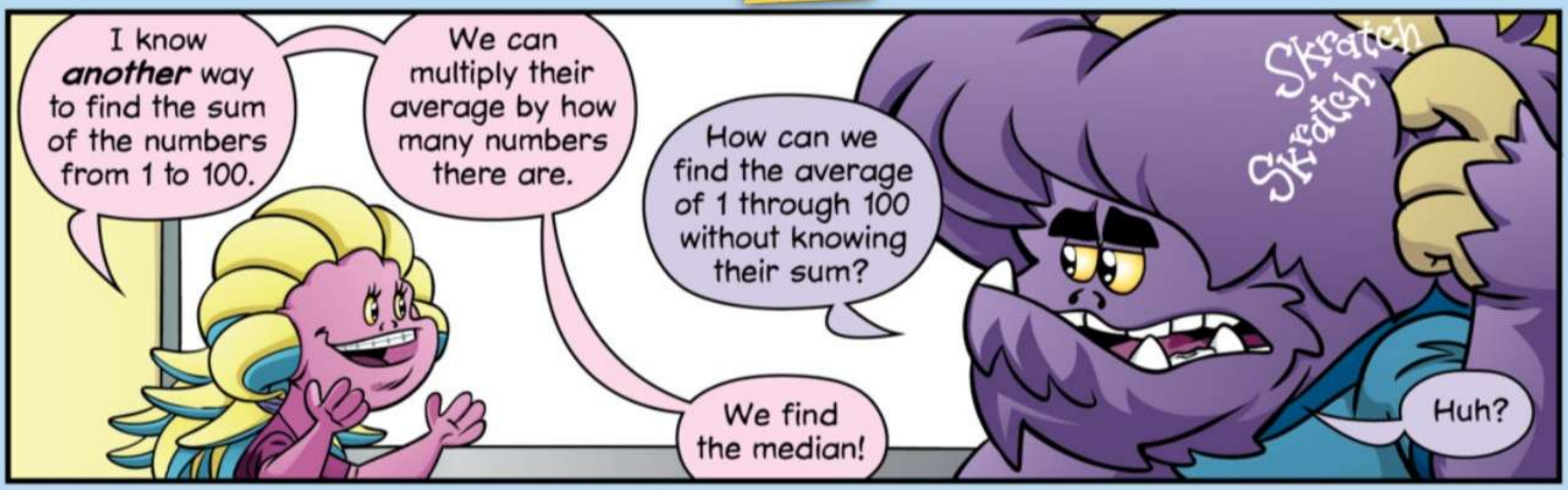
1st 2nd 3rd 4th 5th 6th 100th

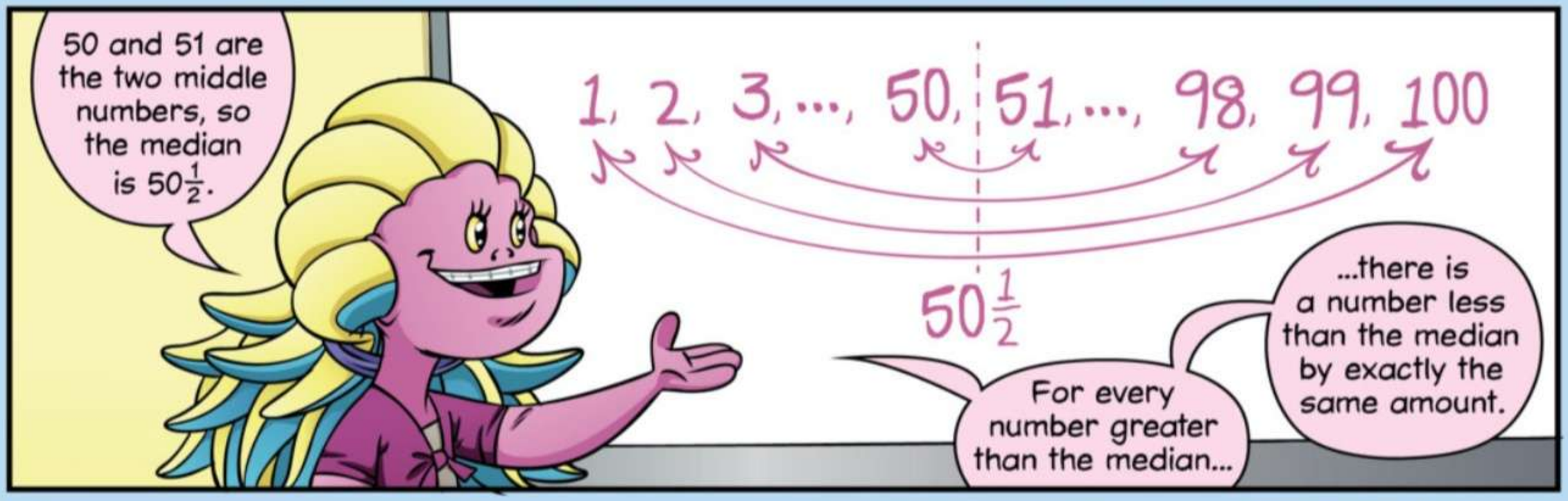
If the sum of ***two*** copies of the list is 10,100...

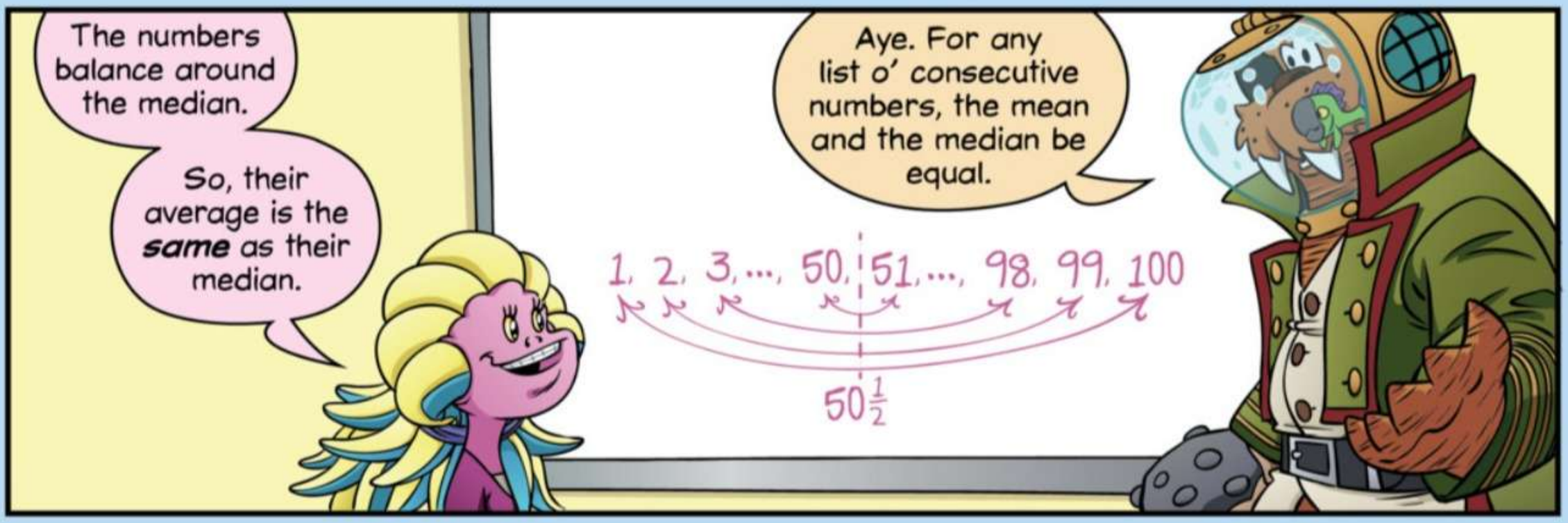
...then the sum of 1 copy of the list is 10,100÷2=5,050.

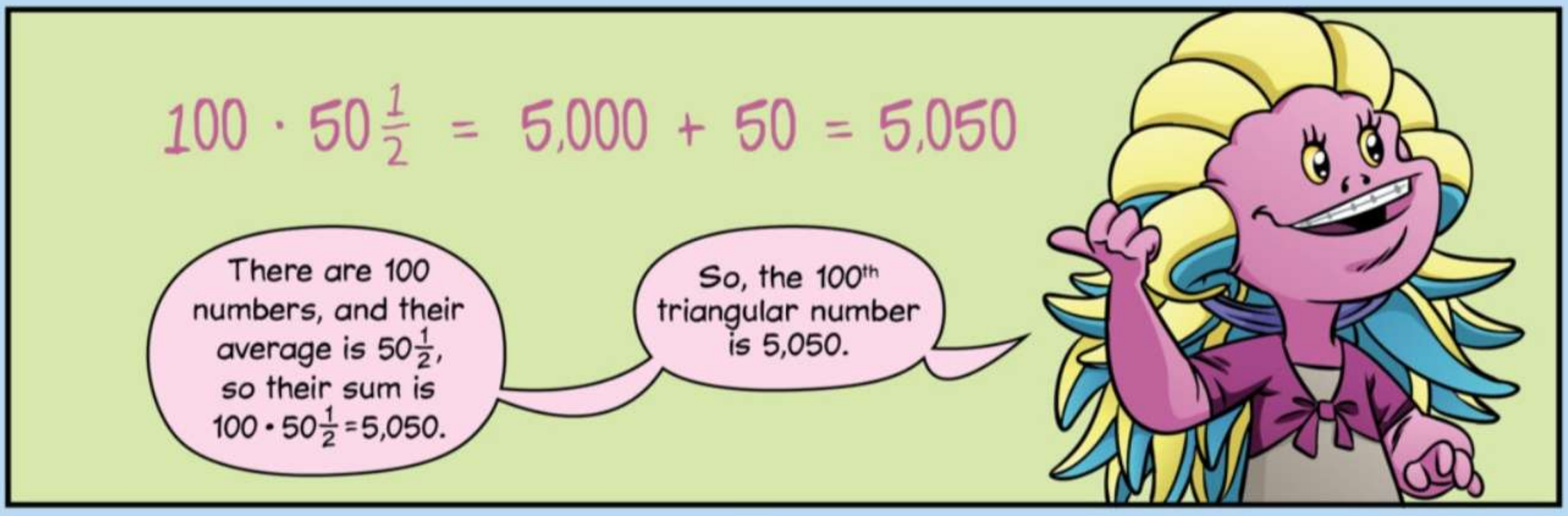
So, the 100th triangular number is 5,050.


Lizzie's math matches my drawing!
Two copies of any triangular number diagram can be put together to make a parallelogram.
The parallelogram has the same number of rows as the triangle...
...and each row has one more coin than the longest row of the triangle.
6 rows of 7 coins =42 coins
We can make a similar arrangement for any triangular number.
3 rows of 4 coins
4 rows of 5 coins
5 rows of 6 coins
The 3rd triangular number is half of 3•4.
The 4th triangular number is half of 4•5.
The 5th triangular number is half of 5•6.
The nth triangular number is half of $n(n+1)$.
So, the 100th triangular number is half of 100•101.
5,050!
$\frac{100 \cdot 101}{2} = \frac{10{,}100}{2} = 5{,}050$


I know **another** way to find the sum of the numbers from 1 to 100.
We can multiply their average by how many numbers there are.
How can we find the average of 1 through 100 without knowing their sum?
We find the median!
Skratch Skratch
Huh?


50 and 51 are the two middle numbers, so the median is $50\frac{1}{2}$.
1, 2, 3, ..., 50, 51, ..., 98, 99, 100
$50\frac{1}{2}$
For every number greater than the median...
...there is a number less than the median by exactly the same amount.


The numbers balance around the median.
So, their average is the **same** as their median.
Aye. For any list o' consecutive numbers, the mean and the median be equal.
1, 2, 3, ..., 50, 51, ..., 98, 99, 100
$50\frac{1}{2}$


$100 \cdot 50\frac{1}{2} = 5{,}000 + 50 = 5{,}050$
There are 100 numbers, and their average is $50\frac{1}{2}$, so their sum is $100 \cdot 50\frac{1}{2} = 5{,}050$.
So, the 100th triangular number is 5,050.

So, what do triangular numbers have to do with your escape?

Cap'n Treebeard possessed a map to the treasure.

But only knowin' the number o' coins in the 100-row treasure triangle be givin' the correct location on the map.

None o' his crew be clever enough to complete the computation.

So, you gave him the location and he let you go?

'Course not!

I gave Cap'n Treebeard the wrong location...

...'n' sent him right into a trap.

Rescued by me crew, we set sail to find the treasure ourselves!

With the help o' Mr. Wriggles, I sent a plan to me crew.

Splooosh