Definition 15. An angle is **acute** if it has less than 90°, and **obtuse** if it has more than 90°.

Definition 16. If $\angle BAC$ is a straight angle, and D is off the line BC, then $\angle BAD$ and $\angle DAC$ are called **supplementary angles**. They add to 180°.

Definition 17. When two lines AB and AC cross at a point A, they are **perpendicular** if $\angle BAC$ is a right angle.

Definition 18. Let A lie between B and C on the line BC, and also between D and E on the line DE. Then $\angle BAD$ and $\angle CAE$ are called **vertically-opposite angles**.

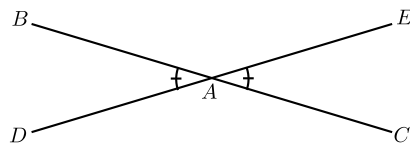

Figure 1.

Theorem 1 (Vertically-opposite Angles)**.**
Vertically opposite angles are equal in measure.

Proof. See Figure 1. The idea is to add the same supplementary angles to both, getting 180°. In detail,

$$\begin{aligned} |\angle BAD| + |\angle BAE| &= 180°, \\ |\angle CAE| + |\angle BAE| &= 180°, \end{aligned}$$

so subtracting gives:

$$\begin{aligned} |\angle BAD| - |\angle CAE| &= 0°, \\ |\angle BAD| &= |\angle CAE|. \end{aligned}$$

□

6.4 Congruent Triangles

Definition 19. Let A, B, C and A', B', C' be triples of non-collinear points. We say that the triangles ΔABC and $\Delta A'B'C'$ are **congruent** if all the sides and angles of one are equal to the corresponding sides and angles of the other, i.e. $|AB| = |A'B'|$, $|BC| = |B'C'|$, $|CA| = |C'A'|$, $|\angle ABC| = |\angle A'B'C'|$, $|\angle BCA| = |\angle B'C'A'|$, and $|\angle CAB| = |\angle C'A'B'|$. See Figure 2.

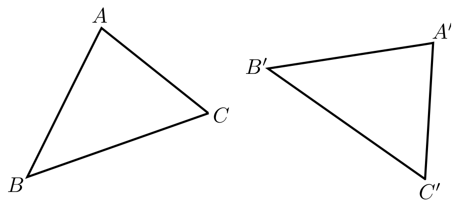

Figure 2.

Notation 3. Usually, we abbreviate the names of the angles in a triangle, by labelling them by the names of the vertices. For instance, we write $\angle A$ for $\angle CAB$.

Axiom 4 (SAS+ASA+SSS[12])**.**
If (1) $|AB| = |A'B'|$, $|AC| = |A'C'|$ *and* $|\angle A| = |\angle A'|$,
or
(2) $|BC| = |B'C'|$, $|\angle B| = |\angle B'|$, *and* $|\angle C| = |\angle C'|$,
or
(3) $|AB| = |A'B'|$, $|BC| = |B'C'|$, *and* $|CA| = |C'A'|$
then the triangles ΔABC *and* $\Delta A'B'C'$ *are congruent.*

Definition 20. A triangle is called **right-angled** if one of its angles is a right angle. The other two angles then add to $90°$, by Theorem 4, so are both acute angles. The side opposite the right angle is called the **hypotenuse**.

Definition 21. A triangle is called **isosceles** if two sides are equal[13]. It is **equilateral** if all three sides are equal. It is **scalene** if no two sides are equal.

Theorem 2 (Isosceles Triangles)**.**
(1) In an isosceles triangle the angles opposite the equal sides are equal.
(2) Conversely, If two angles are equal, then the triangle is isosceles.

Proof. (1) Suppose the triangle ΔABC has $AB = AC$ (as in Figure 3). Then ΔABC is congruent to ΔACB [SAS]
$\therefore \angle B = \angle C$.

[12] It would be possible to prove all the theorems using a weaker axiom (just SAS). We use this stronger version to shorten the course.

[13] The simple "equal" is preferred to "of equal length"

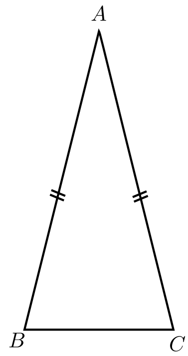

Figure 3.

(2) Suppose now that $\angle B = \angle C$. Then
ΔABC is congruent to ΔACB [ASA]
$\therefore |AB| = |AC|, \Delta ABC$ is isosceles. □

Acceptable Alternative Proof of (1). Let D be the midpoint of $[BC]$, and use SSS to show that the triangles ΔABD and ΔACD are congruent. (This proof is more complicated, but has the advantage that it yields the extra information that the angles $\angle ADB$ and $\angle ADC$ are equal, and hence both are right angles (since they add to a straight angle)). □

6.5 Parallels

Definition 22. Two lines l and m are **parallel** if they are either identical, or have no common point.

Notation 4. We write $l||m$ for "l is parallel to m".

Axiom 5 (Axiom of Parallels)**.** *Given any line l and a point P, there is exactly one line through P that is parallel to l.*

Definition 23. If l and m are lines, then a line n is called a **transversal** of l and m if it meets them both.

Definition 24. Given two lines AB and CD and a transversal BC of them, as in Figure 4, the angles $\angle ABC$ and $\angle BCD$ are called **alternate** angles.

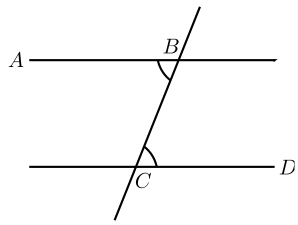

Figure 4.

Theorem 3 (Alternate Angles)**.** *Suppose that A and D are on opposite sides of the line BC.*
(1) If $|\angle ABC| = |\angle BCD|$, then $AB||CD$. In other words, if a transversal makes equal alternate angles on two lines, then the lines are parallel.
(2) Conversely, if $AB||CD$, then $|\angle ABC| = |\angle BCD|$. In other words, if two lines are parallel, then any transversal will make equal alternate angles with them.

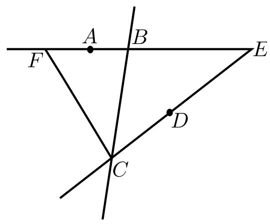

Figure 5.

Proof. (1) Suppose $|\angle ABC| = |\angle BCD|$. If the lines AB and CD do not meet, then they are parallel, by definition, and we are done. Otherwise, they meet at some point, say E. Let us assume that E is on the same side of BC as D.[14] Take F on EB, on the same side of BC as A, with $|BF| = |CE|$ (see Figure 5). [Ruler Axiom]

[14]Fuller detail: There are three cases:
1°: E lies on BC. Then (using Axiom 1) we must have $E = B = C$, and $AB = CD$.
2°: E lies on the same side of BC as D. In that case, take F on EB, on the same side of BC as A, with $|BF| = |CE|$. [Ruler Axiom]
Then ΔBCE is congruent to ΔCBF. [SAS]
Thus

$$|\angle BCF| = |\angle CBE| = 180° - |\angle ABC| = 180° - |\angle BCD|,$$

Then ΔBCE is congruent to ΔCBF. [SAS]
Thus

$$|\angle BCF| = |\angle CBE| = 180° - |\angle ABC| = 180° - |\angle BCD|,$$

so that F lies on DC. [Ruler Axiom]
Thus AB and CD both pass through E and F, and hence coincide, [Axiom 1]
Hence AB and CD are parallel. [Definition of parallel]

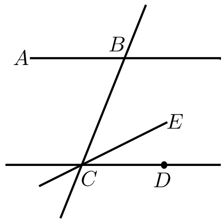

Figure 6.

(2) To prove the converse, suppose $AB||CD$. Pick a point E on the same side of BC as D with $|\angle BCE| = |\angle ABC|$. (See Figure 6.) By Part (1), the line CE is parallel to AB. By Axiom 5, there is only one line through C parallel to AB, so $CE = CD$. Thus $|\angle BCD| = |\angle BCE| = |\angle ABC|$. □

Theorem 4 (Angle Sum 180)**.** *The angles in any triangle add to* 180°*.*

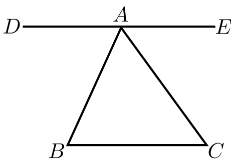

Figure 7.

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so that F lies on DC. [Protractor Axiom]
Thus AB and CD both pass through E and F, and hence coincide. [Axiom 1]
3°: E lies on the same side of BC as A. Similar to the previous case.
Thus, in all three cases, $AB = CD$, so the lines are parallel.

Proof. Let ΔABC be given. Take a segment $[DE]$ passing through A, parallel to BC, with D on the opposite side of AB from C, and E on the opposite side of AC from B (as in Figure 7). [Axiom of Parallels]
Then AB is a transversal of DE and BC, so by the Alternate Angles Theorem,

$$|\angle ABC| = |\angle DAB|.$$

Similarly, AC is a transversal of DE and BC, so

$$|\angle ACB| = |\angle CAE|.$$

Thus, using the Protractor Axiom to add the angles,

$$\begin{aligned} & |\angle ABC| + |\angle ACB| + |\angle BAC| \\ = & \ |\angle DAB| + |\angle CAE| + |\angle BAC| \\ = & \ |\angle DAE| = 180^\circ, \end{aligned}$$

since $\angle DAE$ is a straight angle. □

Definition 25. Given two lines AB and CD, and a transversal AE of them, as in Figure 8(a), the angles $\angle EAB$ and $\angle ACD$ are called **corresponding** angles[15].

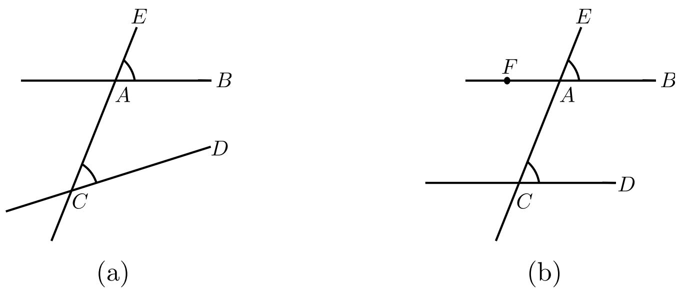

Figure 8.

Theorem 5 (Corresponding Angles)**.** *Two lines are parallel if and only if for any transversal, corresponding angles are equal.*

[15]with respect to the two lines and the given transversal.

Proof. See Figure 8(b). We first assume that the corresponding angles $\angle EAB$ and $\angle ACD$ are equal. Let F be a point on AB such that F and B are on opposite sides of AE. Then we have
$|\angle EAB| = |\angle FAC|$ [Vertically opposite angles]
Hence the alternate angles $\angle FAC$ and $\angle ACD$ are equal and therefore the lines $FA = AB$ and CD are parallel.

For the converse, let us assume that the lines AB and CD are parallel. Then the alternate angles $\angle FAC$ and $\angle ACD$ are equal. Since
$|\angle EAB| = |\angle FAC|$ [Vertically opposite angles]
we have that the corresponding angles $\angle EAB$ and $\angle ACD$ are equal. □

Definition 26. In Figure 9, the angle α is called an **exterior angle** of the triangle, and the angles β and γ are called (corresponding) **interior opposite angles**.[16]

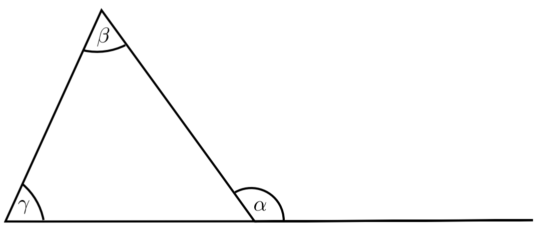

Figure 9.

Theorem 6 (Exterior Angle)**.** *Each exterior angle of a triangle is equal to the sum of the interior opposite angles.*

Proof. See Figure 10. In the triangle ΔABC let α be an exterior angle at A. Then
$|\alpha| + |\angle A| = 180°$ [Supplementary angles]
and
$|\angle B| + |\angle C| + |\angle A| = 180°$. [Angle sum 180°]
Subtracting the two equations yields $|\alpha| = |\angle B| + |\angle C|$. □

[16]The phrase **interior remote angles** is sometimes used instead of **interior opposite angles**.

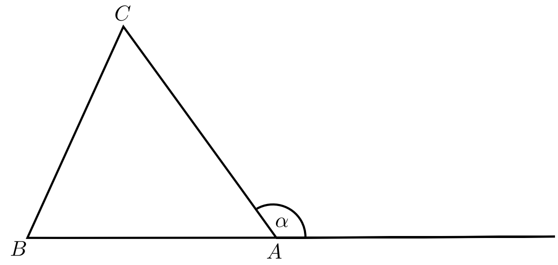

Figure 10.

Theorem 7.
(1) In ΔABC, suppose that $|AC| > |AB|$. Then $|\angle ABC| > |\angle ACB|$. In other words, the angle opposite the greater of two sides is greater than the angle opposite the lesser side.
(2) Conversely, if $|\angle ABC| > |\angle ACB|$, then $|AC| > |AB|$. In other words, the side opposite the greater of two angles is greater than the side opposite the lesser angle.

Proof.
(1) Suppose that $|AC| > |AB|$. Then take the point D on the segment $[AC]$ with
$|AD| = |AB|$. [Ruler Axiom]

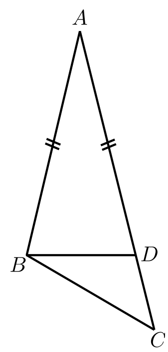

Figure 11.

See Figure 11. Then ΔABD is isosceles, so

$$\begin{array}{rcll} |\angle ACB| & < & |\angle ADB| & \text{[Exterior Angle]} \\ & = & |\angle ABD| & \text{[Isosceles Triangle]} \\ & < & |\angle ABC|. & \end{array}$$

Thus $|\angle ACB| < |\angle ABC|$, as required.

(2)(This is a Proof by Contradiction!)
Suppose that $|\angle ABC| > |\angle ACB|$. See Figure 12.

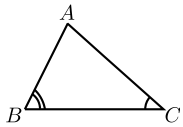

Figure 12.

If it could happen that $|AC| \leq |AB|$, then
either Case 1°: $|AC| = |AB|$, in which case ΔABC is isosceles, and then $|\angle ABC| = |\angle ACB|$, which contradicts our assumption,
or Case 2°: $|AC| < |AB|$, in which case Part (1) tells us that $|\angle ABC| < |\angle ACB|$, which also contradicts our assumption. Thus it cannot happen, and we conclude that $|AC| > |AB|$. □

Theorem 8 (Triangle Inequality)**.**
Two sides of a triangle are together greater than the third.

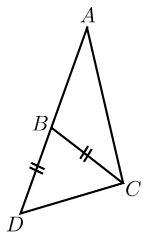

Figure 13.

Proof. Let ΔABC be an arbitrary triangle. We choose the point D on AB such that B lies in $[AD]$ and $|BD| = |BC|$ (as in Figure 13). In particular

$$|AD| = |AB| + |BD| = |AB| + |BC|.$$

Since B lies in the angle $\angle ACD$[17] we have

$$|\angle BCD| < |\angle ACD|.$$

[17] B lies in a segment whose endpoints are on the arms of $\angle ACD$. Since this angle is $< 180°$ its inside is convex.

Because of $|BD| = |BC|$ and the Theorem about Isosceles Triangles we have $|\angle BCD| = |\angle BDC|$, hence $|\angle ADC| = |\angle BDC| < |\angle ACD|$. By the previous theorem applied to ΔADC we have

$$|AC| < |AD| = |AB| + |BC|.$$

□

6.6 Perpendicular Lines

Proposition 1. [18] *Two lines perpendicular to the same line are parallel to one another.*

Proof. This is a special case of the Alternate Angles Theorem. □

Proposition 2. *There is a unique line perpendicular to a given line and passing though a given point. This applies to a point on or off the line.*

Definition 27. The **perpendicular bisector** of a segment $[AB]$ is the line through the midpoint of $[AB]$, perpendicular to AB.

6.7 Quadrilaterals and Parallelograms

Definition 28. A closed chain of line segments laid end-to-end, not crossing anywhere, and not making a straight angle at any endpoint encloses a piece of the plane called a **polygon**. The segments are called the **sides** or edges of the polygon, and the endpoints where they meet are called its **vertices**. Sides that meet are called **adjacent sides**, and the ends of a side are called **adjacent vertices**. The angles at adjacent vertices are called **adjacent angles**. A polygon is called **convex** if it contains the whole segment connecting any two of its points.

Definition 29. A **quadrilateral** is a polygon with four vertices.

Two sides of a quadrilateral that are not adjacent are called **opposite sides**. Similarly, two angles of a quadrilateral that are not adjacent are called **opposite angles**.

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[18] In this document, a proposition is a useful or interesting statement that could be proved at this point, but whose proof is not stipulated as an essential part of the programme. Teachers are free to deal with them as they see fit. For instance, they might be just mentioned, or discussed without formal proof, or used to give practice in reasoning for HLC students. It is desirable that they be mentioned, at least.

Definition 30. A **rectangle** is a quadrilateral having right angles at all four vertices.

Definition 31. A **rhombus** is a quadrilateral having all four sides equal.

Definition 32. A **square** is a rectangular rhombus.

Definition 33. A polygon is **equilateral** if all its sides are equal, and **regular** if all its sides and angles are equal.

Definition 34. A **parallelogram** is a quadrilateral for which both pairs of opposite sides are parallel.

Proposition 3. *Each rectangle is a parallelogram.*

Theorem 9. *In a parallelogram, opposite sides are equal, and opposite angles are equal.*

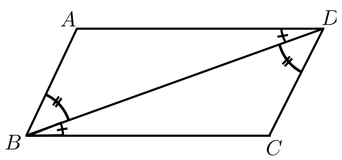

Figure 14.

Proof. See Figure 14. Idea: Use Alternate Angle Theorem, then ASA to show that a diagonal divides the parallelogram into two congruent triangles. This gives opposite sides and (one pair of) opposite angles equal.

In more detail, let $ABCD$ be a given parallelogram, $AB||CD$ and $AD||BC$. Then

$|\angle ABD| = |\angle BDC|$ [Alternate Angle Theorem]
$|\angle ADB| = |\angle DBC|$ [Alternate Angle Theorem]
ΔDAB is congruent to ΔBCD. [ASA]

$$\therefore |AB| = |CD|,\ |AD| = |CB|,\ \text{and } |\angle DAB| = |\angle BCD|.$$

□

Remark 1. Sometimes it happens that the converse of a true statement is false. For example, it is true that if a quadrilateral is a rhombus, then its diagonals are perpendicular. But it is not true that a quadrilateral whose diagonals are perpendicular is always a rhombus.

It may also happen that a statement admits several valid converses. Theorem 9 has two:

Converse 1 to Theorem 9: *If the opposite angles of a convex quadrilateral are equal, then it is a parallelogram.*

Proof. First, one deduces from Theorem 4 that the angle sum in the quadrilateral is 360°. It follows that adjacent angles add to 180°. Theorem 3 then yields the result. □

Converse 2 to Theorem 9: *If the opposite sides of a convex quadrilateral are equal, then it is a parallelogram.*

Proof. Drawing a diagonal, and using SSS, one sees that opposite angles are equal. □

Corollary 1. *A diagonal divides a parallelogram into two congruent triangles.*

Remark 2. The converse is false: It may happen that a diagonal divides a convex quadrilateral into two congruent triangles, even though the quadrilateral is not a parallelogram.

Proposition 4. *A quadrilateral in which one pair of opposite sides is equal and parallel, is a parallelogram.*

Proposition 5. *Each rhombus is a parallelogram.*

Theorem 10. *The diagonals of a parallelogram bisect one another.*

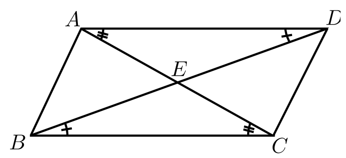

Figure 15.

Proof. See Figure 15. Idea: Use Alternate Angles and ASA to establish congruence of ΔADE and ΔCBE.

In detail: Let AC cut BD in E. Then

$$\begin{aligned} |\angle EAD| &= |\angle ECB| \text{ and} \\ |\angle EDA| &= |\angle EBC| && \text{[Alternate Angle Theorem]} \\ |AD| &= |BC|. && \text{[Theorem 9]} \end{aligned}$$

$\therefore \Delta ADE$ is congruent to ΔCBE. [ASA] □

Proposition 6 (Converse)**.** *If the diagonals of a quadrilateral bisect one another, then the quadrilateral is a parallelogram.*

Proof. Use SAS and Vertically Opposite Angles to establish congruence of ΔABE and ΔCDE. Then use Alternate Angles. □

6.8 Ratios and Similarity

Definition 35. If the three angles of one triangle are equal, respectively, to those of another, then the two triangles are said to be **similar**.

Remark 3. Obviously, two right-angled triangles are similar if they have a common angle other than the right angle.

(The angles sum to 180°, so the third angles must agree as well.)

Theorem 11. *If three parallel lines cut off equal segments on some transversal line, then they will cut off equal segments on any other transversal.*

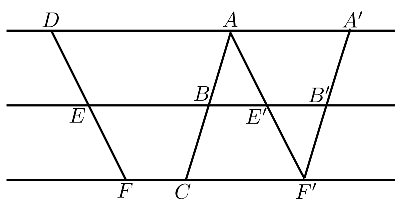

Figure 16.

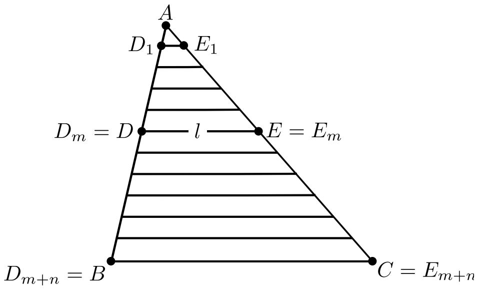

Figure 17.

equally spaced along $[AB]$, i.e. the segments

$$[D_0D_1], [D_1D_2], \ldots [D_iD_{i+1}], \ldots [D_{m+n-1}D_{m+n}]$$

have equal length.

Draw lines $D_1E_1, D_2E_2, \ldots$ parallel to BC with $E_1, E_2, \ldots$ on $[AC]$.

Then all the segments

$$[AE_1], [E_1E_2], [E_2E_3], \ldots, [E_{m+n-1}C]$$

have the same length, [Theorem 11]
and $E_m = E$ is the point where l cuts $[AC]$. [Axiom of Parallels]
Hence E divides $[AC]$ in the ratio $m : n$. □

Proposition 7. *If two triangles ΔABC and $\Delta A'B'C'$ have*

$$|\angle A| = |\angle A'|, \ \textit{and} \ \frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|},$$

then they are similar.

Proof. Suppose $|A'B'| \leq |AB|$. If equal, use SAS. Otherwise, note that then $|A'B'| < |AB|$ and $|A'C'| < |AC|$. Pick B'' on $[AB$ and C'' on $[AC$ with $|A'B'| = |AB''|$ and $|A'C'| = |AC''|$. [Ruler Axiom] Then by SAS, $\Delta A'B'C'$ is congruent to $\Delta AB''C''$.

Draw $[B''D$ parallel to BC [Axiom of Parallels], and let it cut AC at D. Now the last theorem and the hypothesis tell us that D and C'' divide $[AC]$ in the same ratio, and hence $D = C''$.

Thus

$$\begin{aligned} |\angle B| &= |\angle AB''C''| \text{ [Corresponding Angles]} \\ &= |\angle B'|, \end{aligned}$$

and

$$|\angle C| = |\angle AC''B''| = |\angle C'|,$$

so ΔABC is similar to $\Delta A'B'C'$. [Definition of similar]

□

Remark 5. The **Converse to Theorem 12** is true:
Let ΔABC be a triangle. If a line l cuts the sides AB and AC in the same ratio, then it is parallel to BC.

Proof. This is immediate from Proposition 7 and Theorem 5. □

Theorem 13. *If two triangles ΔABC and $\Delta A'B'C'$ are similar, then their sides are proportional, in order:*

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|}.$$

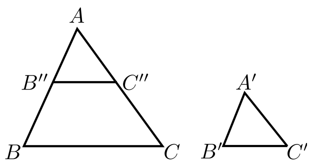

Figure 18.

Proof. We may suppose $|A'B'| \le |AB|$. Pick B'' on $[AB]$ with $|AB''| = |A'B'|$, and C'' on $[AC]$ with $|AC''| = |A'C'|$. Refer to Figure 18. Then

$$\begin{array}{rcll}
\Delta AB''C'' & \text{is congruent to} & \Delta A'B'C' & \text{[SAS]} \\
\therefore |\angle AB''C''| & = & |\angle ABC| & \\
\therefore B''C'' & || & BC & \text{[Corresponding Angles]} \\
\therefore \dfrac{|A'B'|}{|A'C'|} & = & \dfrac{|AB''|}{|AC''|} & \text{[Choice of } B'', C''] \\
& = & \dfrac{|AB|}{|AC|} & \text{[Theorem 12]} \\
\dfrac{|AC|}{|A'C'|} & = & \dfrac{|AB|}{|A'B'|} & \text{[Re-arrange]}
\end{array}$$

Similarly, $\dfrac{|BC|}{|B'C'|} = \dfrac{|AB|}{|A'B'|}$ □

Proposition 8 (Converse). *If*

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|},$$

then the two triangles ΔABC and $\Delta A'B'C'$ are similar.

Proof. Refer to Figure 18. If $|A'B'| = |AB|$, then by SSS the two triangles are congruent, and therefore similar. Otherwise, assuming $|A'B'| < |AB|$, choose B'' on AB and C'' on AC with $|AB''| = |A'B'|$ and $|AC''| = |A'C'|$. Then by Proposition 7, $\Delta AB''C''$ is similar to ΔABC, so

$$|B''C''| = |AB''| \cdot \frac{|BC|}{|AB|} = |A'B'| \cdot \frac{|BC|}{|AB|} = |B'C'|.$$

Thus by SSS, $\Delta A'B'C'$ is congruent to $\Delta AB''C''$, and hence similar to ΔABC. □

6.9 Pythagoras

Theorem 14 (Pythagoras). *In a right-angle triangle the square of the hypotenuse is the sum of the squares of the other two sides.*

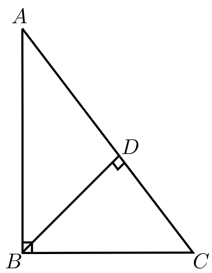

Figure 19.

Proof. Let ΔABC have a right angle at B. Draw the perpendicular BD from the vertex B to the hypotenuse AC (shown in Figure 19).

The right-angle triangles ΔABC and ΔADB have a common angle at A.
$\therefore$ ΔABC is similar to ΔADB.

$$\therefore \frac{|AC|}{|AB|} = \frac{|AB|}{|AD|},$$

so

$$|AB|^2 = |AC| \cdot |AD|.$$

Similarly, ΔABC is similar to ΔBDC.

$$\therefore \frac{|AC|}{|BC|} = \frac{|BC|}{|DC|},$$

so

$$|BC|^2 = |AC| \cdot |DC|.$$

Thus

$$\begin{aligned} |AB|^2 + |BC|^2 &= |AC| \cdot |AD| + |AC| \cdot |DC| \\ &= |AC| \left(|AD| + |DC| \right) \\ &= |AC| \cdot |AC| \\ &= |AC|^2. \end{aligned}$$

□

Theorem 15 (Converse to Pythagoras)**.** *If the square of one side of a triangle is the sum of the squares of the other two, then the angle opposite the first side is a right angle.*

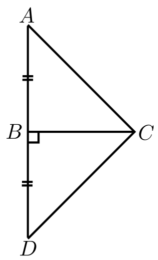

Figure 20.

Proof. (Idea: Construct a second triangle on the other side of $[BC]$, and use Pythagoras and SSS to show it congruent to the original.)
In detail: We wish to show that $|\angle ABC| = 90°$.

Draw $BD \perp BC$ and make $|BD| = |AB|$ (as shown in Figure 20).

Then

$$\begin{aligned} |DC| &= \sqrt{|DC|^2} & \\ &= \sqrt{|BD|^2 + |BC|^2} & \text{[Pythagoras]} \\ &= \sqrt{|AB|^2 + |BC|^2} & [\ |AB| = |BD|\ ] \\ &= \sqrt{|AC|^2} & \text{[Hypothesis]} \\ &= |AC|. \end{aligned}$$

$\therefore \Delta ABC$ is congruent to ΔDBC. [SSS]
$\therefore |\angle ABC| = |\angle DBC| = 90°$. □

Proposition 9 (RHS)**.** *If two right angled triangles have hypotenuse and another side equal in length, respectively, then they are congruent.*

Proof. Suppose ΔABC and $\Delta A'B'C'$ are right-angle triangles, with the right angles at B and B', and have hypotenuses of the same length, $|AC| = |A'C'|$, and also have $|AB| = |A'B'|$. Then by using Pythagoras' Theorem, we obtain $|BC| = |B'C'|$, so by SSS, the triangles are congruent. □

Proposition 10. *Each point on the perpendicular bisector of a segment* $[AB]$ *is equidistant from the ends.*

Proposition 11. *The perpendiculars from a point on an angle bisector to the arms of the angle have equal length.*

6.10 Area

Definition 37. If one side of a triangle is chosen as the base, then the opposite vertex is the **apex** corresponding to that base. The corresponding **height** is the length of the perpendicular from the apex to the base. This perpendicular segment is called an **altitude** of the triangle.

Theorem 16. *For a triangle, base times height does not depend on the choice of base.*

Proof. Let AD and BE be altitudes (shown in Figure 21). Then ΔBCE and ΔACD are right-angled triangles that share the angle C, hence they are similar. Thus

$$\frac{|AD|}{|BE|} = \frac{|AC|}{|BC|}.$$

Re-arrange to yield the result. □

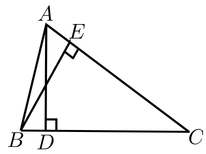

Figure 21.

Definition 38. The **area** of a triangle is half the base by the height.

Notation 5. We denote the area by "area of ΔABC"[19].

Proposition 12. *Congruent triangles have equal areas.*

Remark 6. This is another example of a proposition whose converse is false. It may happen that two triangles have equal area, but are not congruent.

Proposition 13. *If a triangle ΔABC is cut into two by a line AD from A to a point D on the segment $[BC]$, then the areas add up properly:*

$$\text{area of } \Delta ABC = \text{area of } \Delta ABD + \text{area of } \Delta ADC.$$

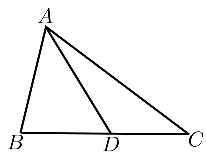

Figure 22.

Proof. See Figure 22. All three triangles have the same height, say h, so it comes down to

$$\frac{|BC| \times h}{2} = \frac{|BD| \times h}{2} + \frac{|DC| \times h}{2},$$

which is obvious, since

$$|BC| = |BD| + |DC|.$$

□

[19] $|\Delta ABC|$ will also be accepted.

If a figure can be cut up into nonoverlapping triangles (i.e. triangles that either don't meet, or meet only along an edge), then its area is taken to be the sum of the area of the triangles[20].

If figures of equal areas are added to (or subtracted from) figures of equal areas, then the resulting figures also have equal areas[21].

Proposition 14. *The area of a rectangle having sides of length a and b is ab.*

Proof. Cut it into two triangles by a diagonal. Each has area $\frac{1}{2}ab$. □

Theorem 17. *A diagonal of a parallelogram bisects the area.*

Proof. A diagonal cuts the parallelogram into two congruent triangles, by Corollary 1. □

Definition 39. Let the side AB of a parallelogram $ABCD$ be chosen as a base (Figure 23). Then the **height** of the parallelogram **corresponding to that base** is the height of the triangle ΔABC.

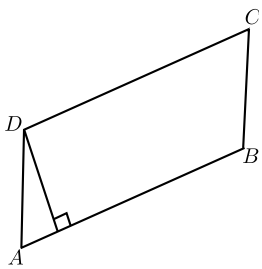

Figure 23.

Proposition 15. *This height is the same as the height of the triangle ΔABD, and as the length of the perpendicular segment from D onto AB.*

[20] If students ask, this does not lead to any ambiguity. In the case of a convex quadrilateral, $ABCD$, one can show that

$$\text{area of } \Delta ABC + \text{area of } \Delta CDA = \text{area of } \Delta ABD + \text{area of } \Delta BCD.$$

In the general case, one proves the result by showing that there is a common refinement of any two given triangulations.

[21] Follows from the previous footnote.

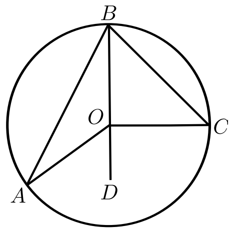

Figure 24.

In detail, for the given figure, Figure 24, we wish to show that $|\angle AOC| = 2|\angle ABC|$.

Join B to O and continue the line to D. Then

$$\begin{array}{rcll} |OA| &=& |OB|. & \text{[Definition of circle]} \\ \therefore |\angle BAO| &=& |\angle ABO|. & \text{[Isosceles triangle]} \\ \therefore |\angle AOD| &=& |\angle BAO| + |\angle ABO| & \text{[Exterior Angle]} \\ &=& 2\cdot|\angle ABO|. & \end{array}$$

Similarly,

$$|\angle COD| = 2\cdot|\angle CBO|.$$

Thus

$$\begin{array}{rcl} |\angle AOC| &=& |\angle AOD| + |\angle COD| \\ &=& 2\cdot|\angle ABO| + 2\cdot|\angle CBO| \\ &=& 2\cdot|\angle ABC|. \end{array}$$

□

Corollary 2. *All angles at points of the circle, standing on the same arc, are equal. In symbols, if A, A', B and C lie on a circle, and both A and A' are on the same side of the line BC, then $\angle BAC = \angle BA'C$.*

Proof. Each is half the angle subtended at the centre. □

Remark 7. The converse is true, but one has to careful about sides of BC:

Converse to Corollary 2: *If points A and A' lie on the same side of the line BC, and if $|\angle BAC| = |\angle BA'C|$, then the four points A, A', B and C lie on a circle.*

Proof. Consider the circle s through A, B and C. If A' lies outside the circle, then take A'' to be the point where the segment $[A'B]$ meets s. We then have

$$|\angle BA'C| = |\angle BAC| = |\angle BA''C|,$$

Definition 42. The line l is called a **tangent** to the circle s when $l \cap s$ has exactly one point. The point is called the **point of contact** of the tangent.

Theorem 20.
(1) Each tangent is perpendicular to the radius that goes to the point of contact.
(2) If P lies on the circle s, and a line l through P is perpendicular to the radius to P, then l is tangent to s.

Proof. (1) This proof is a proof by contradiction.

Suppose the point of contact is P and the tangent l is not perpendicular to OP.

Let the perpendicular to the tangent from the centre O meet it at Q. Pick R on PQ, on the other side of Q from P, with $|QR| = |PQ|$ (as in Figure 25).

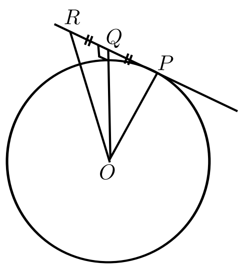

Figure 25.

Then ΔOQR is congruent to ΔOQP. [SAS]

$$\therefore |OR| = |OP|,$$

so R is a second point where l meets the circle. This contradicts the given fact that l is a tangent.

Thus l must be perpendicular to OP, as required.

(2) (Idea: Use Pythagoras. This shows directly that each other point on l is further from O than P, and hence is not on the circle.)

In detail: Let Q be any point on l, other than P. See Figure 26. Then

$$\begin{aligned} |OQ|^2 &= |OP|^2 + |PQ|^2 && \text{[Pythagoras]} \\ &> |OP|^2. \\ \therefore |OQ| &> |OP|. \end{aligned}$$

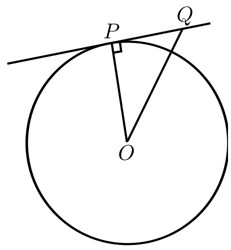

Figure 26.

$\therefore Q$ is not on the circle. [Definition of circle]
$\therefore P$ is the only point of l on the circle.
$\therefore l$ is a tangent. [Definition of tangent]
□

Corollary 6. *If two circles share a common tangent line at one point, then the two centres and that point are collinear.*

Proof. By part (1) of the theorem, both centres lie on the line passing through the point and perpendicular to the common tangent. □

The circles described in Corollary 6 are shown in Figure 27.

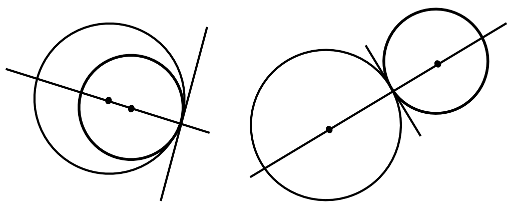

Figure 27.

Remark 9. Any two distinct circles will intersect in 0, 1, or 2 points.

If they have two points in common, then the common chord joining those two points is perpendicular to the line joining the centres.

If they have just one point of intersection, then they are said to be *touching* and this point is referred to as their *point of contact*. The centres and the point of contact are collinear, and the circles have a common tangent at that point.

Theorem 21.
(1) The perpendicular from the centre to a chord bisects the chord.
(2) The perpendicular bisector of a chord passes through the centre.

Proof. (1) (Idea: Two right-angled triangles with two pairs of sides equal.) See Figure 28.

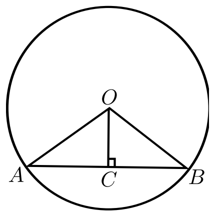

Figure 28.

In detail:

$$\begin{aligned} |OA| &= |OB| && \text{[Definition of circle]} \\ |OC| &= |OC| && \\ |AC| &= \sqrt{|OA|^2 - |OC|^2} && \text{[Pythagoras]} \\ &= \sqrt{|OB|^2 - |OC|^2} && \\ &= |CB|. && \text{[Pythagoras]} \end{aligned}$$

$\therefore \Delta OAC$ is congruent to ΔOBC. [SSS]
$\therefore |AC| = |CB|$.

(2) This uses the Ruler Axiom, which has the consequence that a segment has exactly one midpoint.
Let C be the foot of the perpendicular from O on AB.
By Part (1), $|AC| = |CB|$, so C is the midpoint of $[AB]$.
Thus CO is the perpendicular bisector of AB.
Hence the perpendicular bisector of AB passes through O. □

6.12 Special Triangle Points

Proposition 17. *If a circle passes through three non-collinear points A, B, and C, then its centre lies on the perpendicular bisector of each side of the triangle ΔABC.*

Definition 43. The **circumcircle** of a triangle ΔABC is the circle that passes through its vertices (see Figure 29). Its centre is the **circumcentre** of the triangle, and its radius is the **circumradius**.

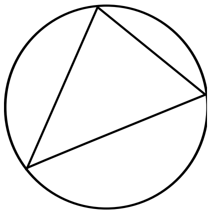

Figure 29.

Proposition 18. *If a circle lies inside the triangle ΔABC and is tangent to each of its sides, then its centre lies on the bisector of each of the angles $\angle A$, $\angle B$, and $\angle C$.*

Definition 44. The **incircle** of a triangle is the circle that lies inside the triangle and is tangent to each side (see Figure 30). Its centre is the **incentre**, and its radius is the **inradius**.

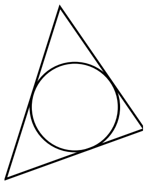

Figure 30.

Proposition 19. *The lines joining the vertices of a triangle to the centre of the opposite sides meet in one point.*

Definition 45. A line joining a vertex of a triangle to the midpoint of the opposite side is called a **median** of the triangle. The point where the three medians meet is called the **centroid**.

Proposition 20. *The perpendiculars from the vertices of a triangle to the opposite sides meet in one point.*

Definition 46. The point where the perpendiculars from the vertices to the opposite sides meet is called the **orthocentre** (see Figure 31).

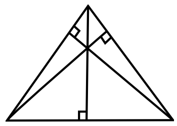

Figure 31.

7 Constructions to Study

The instruments that may be used are:

straight-edge: This may be used (together with a pencil) to draw a straight line passing through two marked points.

compass: This instrument allows you to draw a circle with a given centre, passing through a given point. It also allows you to take a given segment $[AB]$, and draw a circle centred at a given point C having radius $|AB|$.

ruler: This is a straight-edge marked with numbers. It allows you measure the length of segments, and to mark a point B on a given ray with vertex A, such that the length $|AB|$ is a given positive number. It can also be employed by sliding it along a set square, or by other methods of sliding, while keeping one or two points on one or two curves.

protractor: This allows you to measure angles, and mark points C such that the angle $\angle BAC$ made with a given ray $[AB$ has a given number of degrees. It can also be employed by sliding it along a line until some line on the protractor lies over a given point.

set-squares: You may use these to draw right angles, and angles of $30°$, $60°$, and $45°$. It can also be used by sliding it along a ruler until some coincidence occurs.

The prescribed constructions are:

1. Bisector of a given angle, using only compass and straight edge.

2. Perpendicular bisector of a segment, using only compass and straight edge.

3. Line perpendicular to a given line l, passing through a given point not on l.