Chapter 1

Exercise 1.1

1. (i) coefficient of $x^2 = 3$
(ii) coefficient of $x = -9$
(iii) independent term $= 5$.

2. (i) degree 2
(ii) degree 3
(iii) degree 4.

3. $\frac{-4}{x} = -4x^{-1}$, -1 is not a positive power;
$x^{\frac{3}{2}}$, $\frac{3}{2}$ is not an integer.

4. (i) $3x^2 - 6x + 7 + 5x^2 + 2x - 9 = 8x^2 - 4x - 2$
(ii) $x^3 - 4x^2 - 5x + 3x^3 + 6x^2 - x = 4x^3 + 2x^2 - 6x$
(iii) $x(x + 4) + 3x(2x - 3) = x^2 + 4x + 6x^2 - 9x$
$= 7x^2 - 5x$
(iv) $3(x^2 - 7) + 2x(3x - 1) - 7x + 2 = 3x^2 - 21 + 6x^2 - 2x - 7x + 2$
$= 9x^2 - 9x - 19$

5. (i) $3x^2(4x + 2) + 5x^2(2x - 5) = 12x^3 + 6x^2 + 10x^3 - 25x^2$
$= 22x^3 - 19x^2$
(ii) $x^3(x - 2) + 4x^3(2x - 6) = x^4 - 2x^3 + 8x^4 - 24x^3$
$= 9x^4 - 26x^3$
(iii) $x(x^3 + 4x^2 - 7x) + 3x^2(2x^2 - 3x + 4) = x^4 + 4x^3 - 7x^2 + 6x^4 - 9x^3 + 12x^2$
$= 7x^4 - 5x^3 + 5x^2$
(iv) $3x(x^2 - 7x + 1) + 2x^2(6x - 5) = 3x^3 - 21x^2 + 3x + 12x^3 - 10x^2$
$= 15x^3 - 31x^2 + 3x$

6. (i) $(x + 4)(2x + 5) = 2x^2 + 5x + 8x + 20 = 2x^2 + 13x + 20$
(ii) $(2x + 3)(x - 2) = 2x^2 - 4x + 3x - 6 = 2x^2 - x - 6$
(iii) $(3x - 2)(x + 3) = 3x^2 + 9x - 2x - 6 = 3x^2 + 7x - 6$
(iv) $(3x - 2)(4x - 1) = 12x^2 - 3x - 8x + 2 = 12x^2 - 11x + 2$
(v) $(3x - 1)(2x + 5) = 6x^2 + 15x - 2x - 5 = 6x^2 + 13x - 5$
(vi) $(4x + 1)(2x - 6) = 8x^2 - 24x + 2x - 6 = 8x^2 - 22x - 6$
(vii) $(x - 2)(x + 2) = x^2 + 2x - 2x - 4 = x^2 - 4$
(viii) $(2x + 5)(2x - 5) = 4x^2 - 10x + 10x - 25 = 4x^2 - 25$
(ix) $(ax - by)(ax + by) = a^2x^2 + abxy - abxy - b^2y^2 = a^2x^2 - b^2y^2$

7. (i) $(x + 2)^2 = x^2 + 4x + 4$
(ii) $(x - 3)^2 = x^2 - 6x + 9$
(iii) $(x + 5)^2 = x^2 + 10x + 25$
(iv) $(a + b)^2 = a^2 + 2ab + b^2$
(v) $(x - y)^2 = x^2 - 2xy + y^2$
(vi) $(a + 2b)^2 = a^2 + 4ab + 4b^2$
(vii) $(3x - y)^2 = 9x^2 - 6xy + y^2$
(viii) $(x - 5y)^2 = x^2 - 10xy + 25y^2$
(ix) $(2x + 3y)^2 = 4x^2 + 12xy + 9y^2$

8. (i) $\left(x+\frac{1}{2}\right)^2 = x^2 + 2(x)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 = x^2 + x + \frac{1}{4}$

(ii) $8\left(x-\frac{1}{4}\right)^2 = 8\left(x^2 - 2(x)\left(\frac{1}{4}\right) + \left(\frac{-1}{4}\right)^2\right) = 8\left(x^2 - \frac{x}{2} + \frac{1}{16}\right)$

$= 8x^2 - 4x + \frac{1}{2}$

(iii) $-(1-x)^2 = -(1-2x+x^2) = -1+2x-x^2$

9. (i) $x^2 + 5x + 25$; No, cannot be written in the form $(x+k)^2$

(ii) $9x^2 - 6x - 1$; No, cannot be written in the form $(x+k)^2$

(iii) $4 + 12x + 9x^2 = (2+3x)^2$; YES.

10. $px^2 + 4x + 1 = (ax+1)^2$

$px^2 + 4x + 1$ can be written in the form $(ax+1)^2 = a^2x^2 + 2ax + 1$

$\therefore 2a = 4 \Rightarrow a = 2$

$\therefore (ax+1)^2 = (2x+1)^2 = 4x^2 + 4x + 1$

$\therefore p = 4.$

11. $25x^2 + tx + 4 = (5x+2)^2$ or $25x^2 + tx + 4 = (5x-2)^2$

$= 25x^2 + 20x + 4$ $\qquad$ $= 25x^2 - 20x + 4$

$\Rightarrow t = 20$ $\qquad$ $\Rightarrow t = -20$

12. $9x^2 + 24x + s = (3x+a)^2 = 9x^2 + 6ax + a^2$

$\Rightarrow 6a = 24$

$a = 4.$

$\therefore 9x^2 + 24x + s = (3x+4)^2$

$= 9x^2 + 24x + 16$

$\Rightarrow s = 16.$

13. (i) $(x+2)(x^2+2x+6) = x^3 + 2x^2 + 6x + 2x^2 + 4x + 12$

$= x^3 + 4x^2 + 10x + 12$

(ii) $(x-4)(2x^2+3x-1) = 2x^3 + 3x^2 - x - 8x^2 - 12x + 4$

$= 2x^3 - 5x^2 - 13x + 4$

(iii) $(2x+3)(x^2-3x+2) = 2x^3 - 6x^2 + 4x + 3x^2 - 9x + 6$

$= 2x^3 - 3x^2 - 5x + 6$

(iv) $(3x-2)(2x^2-4x+2) = 6x^3 - 12x^2 + 6x - 4x^2 + 8x - 4$

$= 6x^3 - 16x^2 + 14x - 4.$

14. $(x+y)(x^2-xy+y^2) = x^3 - \cancel{x^2y} + \cancel{xy^2} + \cancel{x^2y} - \cancel{xy^2} + y^3$

$= x^3 + y^3$

15. $(x-y)(x^2+xy+y^2) = x^3 + \cancel{x^2y} + \cancel{xy^2} - \cancel{x^2y} - \cancel{xy^2} - y^3$

$= x^3 - y^3$

16. $(2x-3)(3x^2-2x+4) = 6x^3 - 4x^2 + 8x - 9x^2 + 6x - 12$

$= 6x^3 - 13x^2 + 14x - 12$

$\Rightarrow$ coefficient of $x = 14.$

17. $(x+3)(x-4)(2x+1) = (x+3)(2x^2 + x - 8x - 4)$

$= (x+3)(2x^2 - 7x - 4)$

$= 2x^3 - 7x^2 - 4x + 6x^2 - 21x - 12$

$= 2x^3 - x^2 - 25x - 12.$

18. $(x^2-3x-2)(2x^2-4x+1) = 2x^4 - 4x^3 + x^2 - 6x^3 + 12x^2 - 3x$

$- 4x^2 + 8x - 2$

$= 2x^4 - 10x^3 + 9x^2 + 5x - 2.$

19. $(3x^2 + 5x - 1)(2x^2 - 6x - 5)$; x^2 coefficients include $-15x^2 - 30x^2 - 2x^2$

$$= -47x^2$$

coefficient of $x^2 = -47$.

20. (i) $\dfrac{3x+6}{3} = x + 2$

(ii) $\dfrac{x^2+2x}{x} = x + 2$

(iii) $\dfrac{3x^3 - 6x^2}{3x} = \dfrac{3x^2(x-2)}{3x} = x(x-2) = x^2 - 2x$

(iv) $\dfrac{15x^2y - 10xy^2}{5xy} = 3x - 2y$

21. (i) $\dfrac{6x^2y + 9xy^2 - 3xy}{3xy} = x + 3y - 1$

(ii) $\dfrac{6x^4 - 9x^3 + 12x^2}{3x^2} = \dfrac{3x^2(x^2 - 3x + 4)}{3x^2}$

$$= x^2 - 3x + 4$$

22. (i) $\dfrac{\overset{4}{\cancel{12}}a^{\cancel{2}}\cancel{b}}{\cancel{3}\cancel{a}\cancel{b}} = 4a$

(ii) $\dfrac{\overset{4}{\cancel{12}}a^{\cancel{2}}b\cancel{c}}{\cancel{3}\cancel{a}\cancel{c}} = 4ab$

(iii) $\dfrac{\overset{2}{\cancel{4}}\cancel{x}y^{\cancel{2}}z}{\cancel{2}\cancel{x}\cancel{y}} = 2yz$

(iv) $\dfrac{3xy}{2} \cdot \dfrac{4}{6x^2} = \dfrac{\cancel{12}\cancel{x}y}{\cancel{12}x^2} = \dfrac{y}{x}$

23. (i) $\dfrac{2x^2 + 5x - 3}{2x - 1} = \dfrac{\cancel{(2x-1)}(x+3)}{\cancel{2x-1}} = x + 3$

(ii) $\dfrac{2x^2 - 2x - 12}{x - 3} = \dfrac{2(x^2 - x - 6)}{x - 3} = \dfrac{2\cancel{(x-3)}(x+2)}{\cancel{x-3}}$

$$= 2(x+2) = 2x + 4$$

(iii) $\dfrac{8x^2 + 8x - 6}{4x - 2} = \dfrac{2(4x^2 + 4x - 3)}{2(2x-1)} = \dfrac{\cancel{2}\cancel{(2x-1)}(2x+3)}{\cancel{2}\cancel{(2x-1)}}$

$$= 2x + 3$$

24. (i)

$$\begin{array}{r|l} & x^2 - 7x + 12 \\ \hline x - 1 & x^3 - 8x^2 + 19x - 12 \\ & \underline{x^3 - x^2} \\ & \quad -7x^2 + 19x - 12 \\ & \quad \underline{-7x^2 + 7x} \\ & \qquad\qquad 12x - 12 \\ & \qquad\qquad \underline{12x - 12} \end{array}$$

(ii)

$$\begin{array}{r|l} & x^2 - 1 \\ \hline 2x - 1 & 2x^3 - x^2 - 2x + 1 \\ & \underline{2x^3 - x^2} \\ & \qquad\quad -2x + 1 \\ & \qquad\quad \underline{-2x + 1} \end{array}$$

(iii)

$$\begin{array}{r|l} & x^2 - 1 \\ \hline 3x - 4 & 3x^3 - 4x^2 - 3x + 4 \\ & \underline{3x^3 - 4x^2} \\ & \qquad\quad -3x + 4 \\ & \qquad\quad \underline{-3x + 4} \end{array}$$

(iv)
$$
\begin{array}{r|l}
 & 4x^2 + 5x - 6 \\ \hline
x - 3 & 4x^3 - 7x^2 - 21x + 18 \\
 & \underline{4x^3 - 12x^2} \\
 & \qquad +5x^2 - 21x + 18 \\
 & \qquad \underline{+5x^2 - 15x} \\
 & \qquad\qquad -6x + 18 \\
 & \qquad\qquad \underline{-6x + 18}
\end{array}
$$

(v)
$$
\begin{array}{r|l}
 & x^2 - 5x + 3 \\ \hline
x + 5 & x^3 \qquad\quad -22x + 15 \\
 & \underline{x^3 + 5x^2} \\
 & \quad -5x^2 - 22x + 15 \\
 & \quad \underline{-5x^2 - 25x} \\
 & \qquad\quad +3x + 15 \\
 & \qquad\quad \underline{+3x + 15}
\end{array}
$$

(vi)
$$
\begin{array}{r|l}
 & 2x^2 + 3x + 6 \\ \hline
x - 2 & 2x^3 - x^2 \qquad\quad -12 \\
 & \underline{2x^3 - 4x^2} \\
 & \qquad 3x^2 \qquad\quad -12 \\
 & \qquad \underline{3x^2 - 6x} \\
 & \qquad\qquad 6x - 12 \\
 & \qquad\qquad 6x - 12
\end{array}
$$

25. (i)
$$
\begin{array}{r|l}
 & x - 2 \\ \hline
x^2 + 2 & x^3 - 2x^2 + 2x - 4 \\
 & \underline{x^3 \qquad\quad + 2x} \\
 & \quad - 2x^2 \qquad - 4 \\
 & \quad \underline{- 2x^2 \qquad - 4}
\end{array}
$$

(ii)
$$
\begin{array}{r|l}
 & x - 3 \\ \hline
x^2 - 6x + 9 & x^3 - 9x^2 + 27x - 27 \\
 & \underline{x^3 - 6x^2 + 9x} \\
 & \quad -3x^2 + 18x - 27 \\
 & \quad \underline{-3x^2 + 18x - 27}
\end{array}
$$

(iii)
$$
\begin{array}{r|l}
 & 3x - 1 \\ \hline
x^2 + x - 2 & 3x^3 + 2x^2 - 7x + 2 \\
 & \underline{3x^3 + 3x^2 - 6x} \\
 & \qquad -x^2 - x + 2 \\
 & \qquad \underline{-x^2 - x + 2}
\end{array}
$$

(iv)
$$
\begin{array}{r|l}
 & x + 2 \\ \hline
5x^2 + 4x - 1 & 5x^3 + 14x^2 + 7x - 2 \\
 & \underline{5x^3 + 4x^2 - x} \\
 & \quad +10x^2 + 8x - 2 \\
 & \quad \underline{+10x^2 + 8x - 2}
\end{array}
$$

26. (i)
$$
\begin{array}{r|l}
 & x^2 + 2x + 4 \\ \hline
x - 2 & x^3 \qquad\qquad - 8 \\
 & \underline{x^3 - 2x^2} \\
 & \quad 2x^2 \qquad - 8 \\
 & \quad \underline{2x^2 - 4x} \\
 & \qquad\quad 4x - 8 \\
 & \qquad\quad 4x - 8
\end{array}
$$

(ii)

$$
\begin{array}{r|l}
 & 4x^2 + 6xy + 9y^2 \\
\hline
2x - 3y & 8x^3 \qquad\qquad\qquad -27y^3 \\
 & 8x^3 - 12x^2y \\
 & \quad 12x^2y \qquad\quad -27y^3 \\
 & \quad \underline{12x^2y - 18xy^2} \\
 & \qquad\quad 18xy^2 - 27y^3 \\
 & \qquad\quad 18xy^2 - 27y^3
\end{array}
$$

Exercise 1.2

1. x cm length of smaller side,
$(x + 4)$ cm length of longer side.
(i) $A(x) = x(x + 4) = (x^2 + 4x)$ cm^2
(ii) $P(x) = 2[x + (x + 4)] = (4x + 8)$ cm

2. (i) Area = length × width

$$\Rightarrow \text{width} = \frac{\text{Area}}{\text{length}} = \frac{6x^2 + 4x - 2}{3x - 1}$$

$$= \frac{(3x - 1)(2x + 2)}{3x - 1} = 2x + 2$$

(ii) Perimeter = 2(length + width)
$= 2((3x - 1) + (2x + 2))$
$P(x) = 10x + 2$

3. (a) $V(x) = (2x + 3)(x)(x + 1)$
$= (2x + 3)(x^2 + x)$
$= 2x^3 + 2x^2 + 3x^2 + 3x$
$= 2x^3 + 5x^2 + 3x$

(b) $S(x) = (x)(2x + 3) + 2(x)(x + 1) + 2(2x + 3)(x + 1)$
$= 2x^2 + 3x + 2x^2 + 2x + 4x^2 + 4x + 6x + 6$
$= 8x^2 + 15x + 6$

(c) (i) $V(5) = 2(5)^3 + 5(5)^2 + 3(5) = 390$ cm^3
(ii) $S(5) = 8(5)^2 + 15(5) + 6 = 281$ cm^2

4. $f(x) = 2x^3 - x^2 - 5x - 4$
(a) $f(0) = 2(0)^3 - (0)^2 - 5(0) - 4 = -4$
(b) $f(1) = 2(1)^3 - (1)^2 - 5(1) - 4 = -8$
(c) $f(-2) = 2(-2)^3 - (-2)^2 - 5(-2) - 4 = -14$
(d) $f(3a) = 2(3a)^3 - (3a)^2 - 5(3a) - 4 = 54a^3 - 9a^2 - 15a - 4$

5. $f(x) = x^2 - 3x + 6$
(a) $f(0) = (0)^2 - 3(0) + 6 = 6$
(b) $f(-5) = (-5)^2 - 3(-5) + 6 = 46$
(c) $f\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^2 - 3\left(-\frac{1}{2}\right) + 6 = \frac{31}{4} = 7.75$
(d) $f\left(\frac{a}{4}\right) = \left(\frac{a}{4}\right)^2 - 3\left(\frac{a}{4}\right) + 6 = \frac{a^2}{16} - \frac{3a}{4} + 6$

6. Length = $(x - y)$.
Width = $(2x + 3y)$.
(a) Area = $(x - y)(2x + 3y) = 2x^2 + 3xy - 2xy - 3y^2$
$= 2x^2 + xy - 3y^2$
(b) Perimeter = $2[(x - y) + (2x + 3y)] = 2[3x + 2y] = 6x + 4y$

7. Length $= x$ cm

Width $= (x - 5)$ cm

Height $= 2x$ cm.

(a) Volume $=$ Length $\times$ Width $\times$ Height

$$= (x)(x-5)(2x) = 2x^3 - 10x^2$$

(b) Surface area $= 2(x)(x-5) + 4(2x)(x-5) + 4(2x)(x)$

$$= 2x^2 - 10x + 8x^2 - 40x + 8x^2$$

$$= 18x^2 - 50x.$$

8. (i) $d(4) =$ the number of diagonals in a 4-sided polygon

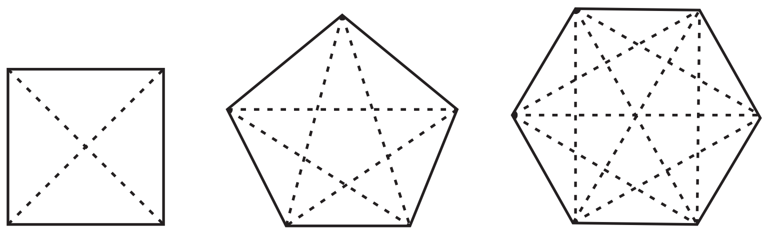
(ii) $d(5) =$ the number of diagonals in a 5-sided polygon.

$$d(4) = \frac{(4)^2}{2} - \frac{3(4)}{2} = 8 - 6 = 2.$$

$$d(5) = \frac{(5)^2}{2} - \frac{3(5)}{2} = \frac{25}{2} - \frac{15}{2} = \frac{10}{2} = 5.$$

$$d(6) = \frac{(6)^2}{2} - \frac{3(6)}{2} = \frac{36}{2} - \frac{18}{2} = \frac{18}{2} = 9.$$

$d(3) = 0$ because a triangle has no diagonal.



9. If $f(x) = x + 5$,

$$f(a^2) - 3f(a) + 2 = a^2 + 5 - 3(a+5) + 2$$

$$= a^2 + 5 - 3a - 15 + 2$$

$$= a^2 - 3a - 8.$$

10. $f(x) = x^2 - 3x + 6.$

(i) $f(-2t) = (-2t)^2 - 3(-2t) + 6 = 4t^2 + 6t + 6.$

(ii) $f(t^2) = (t^2)^2 - 3(t^2) + 6 = t^4 - 3t^2 + 6.$

(iii) $f(t-2) = (t-2)^2 - 3(t-2) + 6 = t^2 - 4t + 4 - 3t + 6 + 6.$

$$= t^2 - 7t + 16.$$

(i) $4t^2 + 6t + 6$ is of degree 2.

(ii) $t^4 - 3t^2 + 6$ is of degree 4.

(iii) $t^2 - 7t + 16$ is of degree 2.

11. $V(r, h) = \frac{1}{3}\pi r^2 h.$

(i) $V(r = 14, h = 21) = \frac{1}{3}\pi \cdot 14^2 \cdot 21 = 1372\pi \text{ cm}^3$

(ii) $V(r, h = r) = \frac{1}{3}\pi \cdot r^2 \cdot r = \frac{1}{3}\pi r^3$

(iii) $V(r = 2h, h) = \frac{1}{3}\pi(2h)^2 \cdot h = \frac{4}{3}\pi h^3$

12. The dimensions of the base of the box are $24 - 2x$ and $18 - 2x$, in cm.

The height of the box is x, in cm.

Thus the volume, V, of the box, in cm^3, is

$V = x(24 - 2x)(18 - 2x)$

$V = x(432 - 48x - 36x + 4x^2)$

$V = x(4x^2 - 84x + 432)$

$V = 4x^3 - 84x^2 + 432x$

The smallest possible value of x is 0. The largest possible value of x is given by

$$18 - 2x = 0$$
$$9 = x$$

Hence $0 < x < 9$, and so $a = 0$ and $b = 9$.

13. $T = 2\pi\sqrt{\frac{l}{g}}$

$\therefore T^2 = 4\pi^2 \frac{l}{g}$

$\therefore \frac{gT^2}{4\pi^2} = l$

$\therefore l = \frac{gT^2}{4\pi^2}$

$\Rightarrow$ when $T = 4$ and $g = 10$, $l = \frac{10.4^2}{4\pi^2} = \frac{40}{\pi^2}$ m.

14. $V = \frac{4}{3}\pi r^3$

$\Rightarrow \frac{4}{3}\pi r^3 = V$

$\Rightarrow \quad r^3 = \frac{3V}{4\pi}$

$\Rightarrow \quad r = \sqrt[3]{\frac{3V}{4\pi}}$

when $V = \frac{792}{7}$ and $\pi = \frac{22}{7}$, $r = \sqrt[3]{\frac{3 \times \overset{198}{\cancel{792}} \times \cancel{7}}{\cancel{4} \times \cancel{7} \times 22}} = \sqrt[3]{27}$

$= 3$ m

15. $H(x) = \frac{x}{2}(x - 1)$, $x =$ number of students.

(i) $x = 5 \Rightarrow H(5) = \frac{5}{2}(5 - 1) = 10$

(ii) $x = 6 \Rightarrow H(6) = \frac{6}{2}(6 - 1) = 15$

(iii) $x = 10 \Rightarrow H(10) = \frac{10}{2}(10 - 1) = 45$

(iv) $H(x) = 136 = \frac{x}{2}(x - 1)$

$272 = x(x - 1)$

$\Rightarrow$ The product of two consecutive numbers $= 272$.

$\Rightarrow$ if $x = 16$, $x - 1 = 15$ $\quad \therefore 16 \times 15 = 240$.

if $x = 17$, $x - 1 = 16$ $\quad \therefore 17 \times 16 = 272$.

$\therefore x = 17$.

or

$272 = x^2 - x$

$\Rightarrow x^2 - x - 272 = 0$

$(x - 17)(x + 16) = 0$

$\therefore x - 17 = 0 \Rightarrow x = 17$

or $x + 16 = 0 \Rightarrow x = -16$ which is invalid

since x stands for the number of students

$\therefore x = 17$.

Exercise 1.3

1. $5x^2 - 10x = 5x(x - 2)$

2. $6ab - 12bc = 6b(a - 2c)$

3. $3x^2 - 6xy = 3x(x - 2y)$

4. $2x^2y - 6x^2z = 2x^2(y - 3z)$

5. $2a^3 - 4a^2 + 8a = 2a(a^2 - 2a + 4)$

6. $5xy^2 - 20x^2y = 5xy(y - 4x)$

7. $2a^2b - 4ab^2 + 12abc = 2ab(a - 2b + 6c)$

8. $3x^2y - 9xy^2 + 15xyz = 3xy(x - 3y + 5z)$

9. $4\pi r^2 + 6\pi rh = 2\pi r(2r + 3h)$

10. $3a(2b - c) - 4(2b - c) = (2b - c)(3a - 4)$

11. $x^2 - ax + 3x - 3a = x(x - a) + 3(x - a)$
$= (x - a)(x + 3)$

12. $2c^2 - 4cd + c - 2d = 2c(c - 2d) + c - 2d$
$= (c - 2d)(2c + 1)$

13. $8ax + 4ay - 6bx - 3by = 4a(2x + y) - 3b(2x + y)$
$= (2x + y)(4a - 3b)$

14. $7y^2 - 21by + 2ay - 6ab = 7y(y - 3b) + 2a(y - 3b)$
$= (y - 3b)(7y + 2a)$

15. $6xy + 12yz - 8xz - 9y^2 = 6xy - 9y^2 + 12yz - 8xz$
$= 3y(2x - 3y) + 4z(3y - 2x)$
$= 3y(2x - 3y) - 4z(2x - 3y)$
$= (2x - 3y)(3y - 4z)$

16. $6x^2 - 3y(3x - 2a) - 4ax = 6x^2 - 4ax - 3y(3x - 2a)$
$= 2x(3x - 2a) - 3y(3x - 2a)$
$= (3x - 2a)(2x - 3y)$

17. $3ax^2 - 3ay^2 - 4bx^2 + 4by^2 = 3a(x^2 - y^2) - 4b(x^2 - y^2)$
$= (x^2 - y^2)(3a - 4b)$
$= (x - y)(x + y)(3a - 4b)$

18. $a^2 - b^2 = (a - b)(a + b)$

19. $x^2 - 4y^2 = (x - 2y)(x + 2y)$

20. $9x^2 - y^2 = (3x - y)(3x + y)$

21. $16x^2 - 25y^2 = (4x)^2 - (5y)^2 = (4x - 5y)(4x + 5y)$

22. $36x^2 - 25 = (6x - 5)(6x + 5)$

23. $1 - 36x^2 = (1 - 6x)(1 + 6x)$

24. $49a^2 - 4b^2 = (7a)^2 - (2b)^2 = (7a - 2b)(7a + 2b)$

25. $x^2y^2 - 1 = (xy - 1)(xy + 1)$

26. $4a^2b^2 - 16c^2 = (2ab)^2 - (4c)^2 = (2ab - 4c)(2ab + 4c)$

27. $3x^2 - 27y^2 = 3(x^2 - 9y^2)$
$= 3(x - 3y)(x + 3y)$

28. $45 - 5x^2 = 5(9 - x^2)$
$= 5(3 - x)(3 + x)$

29. $45a^2 - 20 = 5(9a^2 - 4)$
$= 5(3a - 2)(3a + 2)$

30. $(2x + y)^2 - 4 = (2x + y - 2)(2x + y + 2)$

31. $(3a - 2b)^2 - 9 = (3a - 2b - 3)(3a - 2b + 3)$

32. $a^4 - b^4 = (a^2)^2 - (b^2)^2 = (a^2 - b^2)(a^2 + b^2)$
$= (a - b)(a + b)(a^2 + b^2)$

33. $x^2 + 9x + 14 = (x + 2)(x + 7)$

34. $2x^2 + 7x + 3 = (2x + 1)(x + 3)$

35. $2x^2 + 11x + 14 = (2x + 7)(x + 2)$

36. $x^2 - 9x + 14 = (x - 2)(x - 7)$

37. $x^2 - 11x + 28 = (x - 7)(x - 4)$

38. $2x^2 - 7x + 3 = (2x - 1)(x - 3)$

39. $3x^2 - 17x + 20 = (3x - 5)(x - 4)$

40. $7x^2 - 18x + 8 = (7x - 4)(x - 2)$

41. $2x^2 - 7x - 15 = (2x + 3)(x - 5)$

42. $3x^2 + 11x - 20 = (3x - 4)(x + 5)$

43. $12x^2 - 11x - 5 = (4x - 5)(3x + 1)$

44. $6x^2 + x - 15 = (3x + 5)(2x - 3)$

45. $3x^2 + 13x - 10 = (3x - 2)(x + 5)$

46. $6x^2 - 11x + 3 = (3x - 1)(2x - 3)$

47. $36x^2 - 7x - 4 = (9x - 4)(4x + 1)$

48. $15x^2 - 14x - 8 = (5x + 2)(3x - 4)$

49. $6y^2 + 11y - 35 = (3y - 5)(2y + 7)$

50. $12x^2 + 17xy - 5y^2 = (4x - y)(3x + 5y)$

51. (i) $x^2 + 3\sqrt{3}x + 6.$

$\Rightarrow a = 1, b = 3\sqrt{3}, c = 6.$

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3\sqrt{3} \pm \sqrt{27 - 4(1)(6)}}{2(1)}$$

$$= \frac{-3\sqrt{3} \pm \sqrt{3}}{2}$$

$$\therefore \quad x = \frac{-3\sqrt{3} + \sqrt{3}}{2} \quad \text{or} \quad x = \frac{-3\sqrt{3} - \sqrt{3}}{2}$$

$\therefore \quad x = -\sqrt{3} \quad$ or $\quad -2\sqrt{3}$

$\therefore$ Factors are $(x + \sqrt{3})$ and $(x + 2\sqrt{3})$

(ii) $x^2 + 2\sqrt{5}x - 15.$

$\Rightarrow a = 1, b = 2\sqrt{5}, c = -15.$

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2\sqrt{5} \pm \sqrt{20 - 4(1)(-15)}}{2(1)}$$

$$= \frac{-2\sqrt{5} \pm \sqrt{80}}{2}$$

$$= \frac{-2\sqrt{5} \pm 4\sqrt{5}}{2}$$

$$\therefore \quad x = \frac{-2\sqrt{5} + 4\sqrt{5}}{2} \quad \text{or} \quad x = \frac{-2\sqrt{5} - 4\sqrt{5}}{2}$$

$\therefore \quad x = \sqrt{5} \quad$ or $\quad -3\sqrt{5}$

$\therefore$ Factors are $(x - \sqrt{5})$ and $(x + 3\sqrt{5})$

(iii) $2x^2 - 5\sqrt{2}x - 6.$

$\Rightarrow a = 2, b = -5\sqrt{2}, c = -6.$

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{5\sqrt{2} \pm \sqrt{50 - 4(2)(-6)}}{2(2)}$$

$$= \frac{5\sqrt{2} \pm \sqrt{98}}{4}$$

$$= \frac{5\sqrt{2} \pm 7\sqrt{2}}{4}$$

$$\therefore \quad x = \frac{5\sqrt{2} + 7\sqrt{2}}{4} \quad \text{or} \quad x = \frac{5\sqrt{2} - 7\sqrt{2}}{4}$$

$$\therefore \quad x = 3\sqrt{2} \quad \text{or} \quad \frac{-\sqrt{2}}{2}$$

$\therefore$ Factors are $(x - 3\sqrt{2})$ and $\left(x + \frac{\sqrt{2}}{2}\right)$

But since coefficient of x^2 is 2, one of the factors must contain $2x$. $\quad \therefore x + \frac{\sqrt{2}}{2} = 0$

$\Rightarrow 2x + \sqrt{2} = 0$

$\therefore$ Factors are $(x - 3\sqrt{2})$ and $(2x + \sqrt{2})$

52. (i) $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

(ii) $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

(iii) $8x^3 + y^3 = ((2x)^3 + y^3) = (2x + y)(4x^2 - 2xy + y^2)$

53. (i) $27x^3 - y^3 = (3x)^3 - y^3 = (3x - y)(9x^2 + 3xy + y^2)$

(ii) $x^3 - 64 = x^3 - 4^3 = (x - 4)(x^2 + 4x + 16)$

(iii) $8x^3 - 27y^3 = (2x)^3 - (3y)^3 = (2x - 3y)(4x^2 + 6xy + 9y^2)$

54. (i) $8 + 27k^3 = (2)^3 + (3k)^3 = (2 + 3k)(4 - 6k + 9k^2)$

(ii) $64 - 125a^3 = (4)^3 - (5a)^3 = (4 - 5a)(16 + 20a + 25a^2)$

(iii) $27a^3 + 64b^3 = (3a)^3 + (4b)^3 = (3a + 4b)(9a^2 - 12ab + 16b^2)$

55. (i) $a^3 - 8b^3c^3 = a^3 - (2bc)^3 = (a - 2bc)(a^2 + 2abc + 4b^2c^2)$

(ii) $5x^3 + 40y^3 = 5(x^3 + 8y^3)$

$= 5(x^3 + (2y)^3)$

$= 5(x + 2y)(x^2 - 2xy + 4y^2)$

(iii) $(x + y)^3 - z^3 = (x + y - z)[(x + y)^2 + (x + y)z + z^2]$

Exercise 1.4

1. (i) $\frac{\cancel{8}^{4}\cancel{y}}{\cancel{2}\cancel{y^3}^{2}} = \frac{4}{y^2}$ (ii) $\frac{\cancel{7}\,\cancel{a^6}^{a}\,\cancel{b^3}}{\cancel{14}_{2}\,\cancel{a^5}\,\cancel{b^4}_{b}} = \frac{a}{2b}$

(iii) $\frac{(2x)^2}{4x} = \frac{\cancel{4x^2}^{x}}{\cancel{4x}} = x$

(iv) $\frac{7y + 2y^2}{7y} = \frac{\cancel{y}(7 + 2y)}{7\cancel{y}} = \frac{7 + 2y}{7}$

(v) $\frac{5ax}{15a + 10a^2} = \frac{\cancel{5a}x}{\cancel{5a}(3 + 2a)} = \frac{x}{3 + 2a}$

2. (a) $\frac{2x}{5} + \frac{4x}{3} = \frac{6x}{15} + \frac{20x}{15} = \frac{26x}{15}$

(b) $\frac{3x}{5} - \frac{x}{2} = \frac{6x}{10} - \frac{5x}{10} = \frac{x}{10}$

(c) $\frac{2x + 3}{4} + \frac{x}{3} = \frac{6x + 9}{12} + \frac{4x}{12} = \frac{10x + 9}{12}$

(d) $\dfrac{x+1}{4}+\dfrac{2x-1}{5}=\dfrac{5x+5}{20}+\dfrac{8x-4}{20}=\dfrac{13x+1}{20}$

(e) $\dfrac{3x-4}{6}-\dfrac{2x+1}{3}=\dfrac{3x-4}{6}-\dfrac{4x+2}{6}=\dfrac{-x-6}{6}$

(f) $\dfrac{3x-2}{6}-\dfrac{x-3}{4}=\dfrac{6x-4}{12}-\dfrac{3x-9}{12}=\dfrac{3x+5}{12}$

(g) $\dfrac{5x-1}{4}-\dfrac{2x-4}{5}=\dfrac{25x-5}{20}-\dfrac{8x-16}{20}=\dfrac{17x+11}{20}$

(h) $\dfrac{3x+5}{6}-\dfrac{2x+3}{4}-\dfrac{1}{12}=\dfrac{6x+10}{12}-\dfrac{6x+9}{12}-\dfrac{1}{12}$

$=\dfrac{0}{12}=0$

(i) $\dfrac{3x-2}{4}+\dfrac{3}{5}-\dfrac{2x-1}{10}=\dfrac{15x-10}{20}+\dfrac{12}{20}-\dfrac{4x-2}{20}$

$=\dfrac{11x+4}{20}$

(j) $\dfrac{1}{3x}+\dfrac{1}{5x}=\dfrac{5}{15x}+\dfrac{3}{15x}=\dfrac{8}{15x}$

(k) $\dfrac{3}{4x}-\dfrac{5}{8x}=\dfrac{6}{8x}-\dfrac{5}{8x}=\dfrac{1}{8x}$

(l) $\dfrac{1}{x}+\dfrac{1}{x+3}=\dfrac{x+3+x}{x(x+3)}=\dfrac{2x+3}{x(x+3)}$

(m) $\dfrac{2}{x+2}+\dfrac{3}{x+4}=\dfrac{2(x+4)+3(x+2)}{(x+2)(x+4)}=\dfrac{5x+14}{(x+2)(x+4)}$

(n) $\dfrac{2}{x-2}+\dfrac{3}{2x-1}=\dfrac{2(2x-1)+3(x-2)}{(x-2)(2x-1)}=\dfrac{7x-8}{(x-2)(2x-1)}$

(o) $\dfrac{5}{3x-1}-\dfrac{2}{x+3}=\dfrac{5(x+3)-2(3x-1)}{(3x-1)(x+3)}=\dfrac{-x+17}{(3x-1)(x+3)}$

(p) $\dfrac{3}{2x-7}-\dfrac{1}{5x+2}=\dfrac{3(5x+2)-(2x-7)}{(2x-7)(5x+2)}=\dfrac{13x+13}{(2x-7)(5x+2)}$

(q) $\dfrac{2}{3x-5}-\dfrac{1}{4}=\dfrac{8-(3x-5)}{4(3x-5)}=\dfrac{13-3x}{4(3x-5)}$

(r) $\dfrac{5}{2x-1}-\dfrac{3}{x-2}=\dfrac{5(x-2)-3(2x-1)}{(2x-1)(x-2)}=\dfrac{-x-7}{(2x-1)(x-2)}$

(s) $\dfrac{x}{x-y}-\dfrac{y}{x+y}=\dfrac{x(x+y)-y(x-y)}{(x-y)(x+y)}=\dfrac{x^2+\cancel{xy}+\cancel{xy}+y^2}{(x-y)(x+y)}$

$=\dfrac{x^2+y^2}{x^2-y^2}$

(t) $\dfrac{3}{x}+\dfrac{4}{3y}-\dfrac{2}{3xy}=\dfrac{3(3y)+4(x)-2}{3xy}=\dfrac{9y+4x-2}{3xy}$

(u) $\dfrac{3}{x}-\dfrac{2}{x-1}-\dfrac{4}{x(x-1)}=\dfrac{3(x-1)-2(x)-4}{x(x-1)}=\dfrac{x-7}{x(x-1)}$

3. (i) $\dfrac{2z^2-4z}{2z^2-10z}=\dfrac{\cancel{2z}(z-2)}{\cancel{2z}(z-5)}=\dfrac{z-2}{z-5}$

(ii) $\dfrac{y^2+7y+10}{y^2-25}=\dfrac{\cancel{(y+5)}(y+2)}{\cancel{(y+5)}(y-5)}=\dfrac{y+2}{y-5}$

(iii) $\dfrac{t^2+3t-4}{t^2-3t+2}=\dfrac{(t+4)\cancel{(t-1)}}{(t-2)\cancel{(t-1)}}=\dfrac{t+4}{t-2}$

(iv) $\dfrac{x}{x^2-4}-\dfrac{1}{x+2}=\dfrac{x}{(x+2)(x-2)}-\dfrac{1}{x+2}$

$=\dfrac{x-(x-2)}{(x+2)(x-2)}=\dfrac{2}{(x+2)(x-2)}$

(v) $\dfrac{2}{a+3}-\dfrac{a+2}{a^2-9}=\dfrac{2}{a+3}-\dfrac{a+2}{(a+3)(a-3)}$

$=\dfrac{2(a-3)-(a+2)}{(a+3)(a-3)}=\dfrac{a-8}{(a+3)(a-3)}$

(vi) $\dfrac{x-1}{x^2-4}+\dfrac{1}{x-2}=\dfrac{x-1}{(x-2)(x+2)}+\dfrac{1}{x-2}$

$=\dfrac{x-1+x+2}{(x-2)(x+2)}=\dfrac{2x+1}{(x-2)(x+2)}$

4. (i) $\dfrac{10}{2x^2-3x-2}-\dfrac{2}{x-2}=\dfrac{10}{(2x+1)(x-2)}-\dfrac{2}{(x-2)}$

$=\dfrac{10-2(2x+1)}{(2x+1)(x-2)}$

$=\dfrac{8-4x}{(2x+1)(x-2)}=\dfrac{4(2-x)}{(2x+1)\,(x-2)}$

$=\dfrac{-4\,\cancel{(x-2)}}{(2x+1)\cancel{(x-2)}}$

(ii) $\dfrac{x+2}{2x^2-x-1}-\dfrac{1}{x-1}=\dfrac{x+2}{(2x+1)(x-1)}-\dfrac{1}{x-1}$

$=\dfrac{x+2-(2x+1)}{(2x+1)(x-1)}$

$=\dfrac{-x+1}{(2x+1)(x-1)}=\dfrac{-\cancel{(x-1)}}{(2x+1)\cancel{(x-1)}}=\dfrac{-1}{2x+1}$

5. (i) $\dfrac{1}{x^2-9}-\dfrac{2}{x^2-x-6}=\dfrac{1}{(x-3)(x+3)}-\dfrac{2}{(x-3)(x+2)}$

$=\dfrac{x+2-2(x+3)}{(x-3)(x+3)(x+2)}$

$=\dfrac{-x-4}{(x-3)(x+3)(x+2)}$

(ii) $\dfrac{3}{x^2+x-2}-\dfrac{2}{x^2+3x+2}=\dfrac{3}{(x+2)(x-1)}-\dfrac{2}{(x+2)(x+1)}$

$=\dfrac{3(x+1)-2(x-1)}{(x+2)(x-1)(x+1)}$

$=\dfrac{x+5}{(x+2)(x-1)(x+1)}$

(iii) $\dfrac{2}{6x^2-5x-4}-\dfrac{3}{9x^2-16}=\dfrac{2}{(3x-4)(2x+1)}-\dfrac{3}{(3x-4)(3x+4)}$

$=\dfrac{2(3x+4)-3(2x+1)}{(3x-4)(2x+1)(3x+4)}$

$=\dfrac{\cancel{6x}+8-\cancel{6x}-3}{(3x-4)(2x+1)(3x+4)}=\dfrac{5}{(3x-4)(2x+1)(3x+4)}$

(iv) $\dfrac{1}{xy-x^2}-\dfrac{1}{y^2-xy}=\dfrac{1}{x(y-x)}-\dfrac{1}{y(y-x)}$

$=\dfrac{y-x}{x(y-x)\,y}$

$=\dfrac{\cancel{(y-x)}}{\cancel{(y-x)}\,xy}=\dfrac{1}{xy}$

6. (i) $\dfrac{\frac{1}{2}+\frac{3}{4}}{\frac{1}{4}} = \dfrac{\frac{2}{4}+\frac{3}{4}}{\frac{1}{4}} = \dfrac{\frac{5}{4}}{\frac{1}{4}} = \dfrac{5}{4} \times \dfrac{4}{1} = 5$

(ii) $\dfrac{\frac{2}{3}+\frac{5}{6}}{\frac{3}{8}} = \dfrac{\frac{4}{6}+\frac{5}{6}}{\frac{3}{8}} = \dfrac{\frac{9}{6}}{\frac{3}{8}} = \dfrac{\cancel{9}^{\cancel{3}^{1}}}{\cancel{6}_{\cancel{2}_{1}}} \times \dfrac{\cancel{8}^{4}}{\cancel{3}_{1}} = 4$

(iii) $\dfrac{x-\frac{1}{x}}{1+\frac{1}{x}} = \dfrac{\frac{x^2-1}{x}}{\frac{x+1}{x}} = \dfrac{x^2-1}{x+1} = \dfrac{(x-1)\cancel{(x+1)}}{\cancel{(x+1)}} = x-1$

7. (i) $\dfrac{\frac{1}{x}+1}{\frac{1}{x}-1} = \dfrac{\frac{1+x}{x}}{\frac{1-x}{x}} = \dfrac{(1+x)}{\cancel{x}} \cdot \dfrac{\cancel{x}}{(1-x)} = \dfrac{1+x}{1-x}$

(ii) $\dfrac{\frac{1}{x^2}-4}{\frac{1}{x}-2} = \dfrac{\frac{1-4x^2}{x^2}}{\frac{1-2x}{x}} = \dfrac{(1-4x^2)}{x^{\cancel{2}}} \cdot \dfrac{\cancel{x}^{1}}{(1-2x)}$

$= \dfrac{\cancel{(1-2x)}(1+2x)}{x\cancel{(1-2x)}} = \dfrac{1+2x}{x}$

(iii) $\dfrac{x+y}{\frac{1}{x}+\frac{1}{y}} = \dfrac{x+y}{\frac{y+x}{xy}} = \dfrac{\cancel{(x+y)}}{1} \cdot \dfrac{xy}{\cancel{(x+y)}} = xy$

8. (i) $\dfrac{4y-\frac{3}{2}}{2} = \dfrac{\frac{8y-3}{2}}{2} = \dfrac{8y-3}{2} \times \dfrac{1}{2} = \dfrac{8y-3}{4}$

(ii) $\dfrac{2-\frac{1}{x}}{2} = \dfrac{\frac{2x-1}{x}}{2} = \dfrac{2x-1}{x} \cdot \dfrac{1}{2} = \dfrac{2x-1}{2x}$

(iii) $\dfrac{3x+\frac{1}{x}}{2} = \dfrac{\frac{3x^2+1}{x}}{2} = \dfrac{3x^2+1}{x} \cdot \dfrac{1}{2} = \dfrac{3x^2+1}{2x}$

(iv) $\dfrac{y+\frac{1}{4}}{\frac{1}{2}} = \dfrac{\frac{4y+1}{4}}{\frac{1}{2}} = \dfrac{4y+1}{\cancel{4}_{2}} \cdot \dfrac{\cancel{2}^{1}}{1} = \dfrac{4y+1}{2}$

9. (i) $\dfrac{z-\frac{1}{3}}{z-\frac{1}{2}} = \dfrac{\frac{3z-1}{3}}{\frac{2z-1}{2}} = \dfrac{3z-1}{3} \cdot \dfrac{2}{2z-1} = \dfrac{6z-2}{6z-3}$

(ii) $\dfrac{2x+\frac{1}{2}}{x+\frac{1}{4}} = \dfrac{\frac{4x+1}{2}}{\frac{4x+1}{4}} = \dfrac{\cancel{(4x+1)}}{\cancel{2}_{1}} \cdot \dfrac{\cancel{4}^{2}}{\cancel{(4x+1)}} = 2$

(iii) $\dfrac{z-\frac{1}{2z}}{z-\frac{1}{3z}} = \dfrac{\frac{2z^2-1}{2z}}{\frac{3z^2-1}{3z}} = \dfrac{(2z^2-1)}{2\cancel{z}} \cdot \dfrac{3\cancel{z}}{(3z^2-1)}$

$= \dfrac{6z^2-3}{6z^2-2}$

(iv) $\dfrac{x-\frac{1}{x+1}}{x-1} = \dfrac{\frac{x(x+1)-1}{x+1}}{x-1} = \dfrac{x^2+x-1}{x+1} \cdot \dfrac{1}{x-1}$

$= \dfrac{x^2+x-1}{x^2-1}$

10. (i) $\dfrac{1+\frac{2}{x}}{\frac{x+2}{x-2}} = \dfrac{\frac{x+2}{x}}{\frac{x+2}{x-2}} = \dfrac{\cancel{(x+2)}}{x} \cdot \dfrac{x-2}{\cancel{(x+2)}} = \dfrac{x-2}{x}$

(ii) $\dfrac{2+\frac{1}{x}}{2x^2+x} = \dfrac{\frac{2x+1}{x}}{x(2x+1)} = \dfrac{\cancel{(2x+1)}}{x} \cdot \dfrac{1}{x\cancel{(2x+1)}} = \dfrac{1}{x^2}$

(iii) $\dfrac{x + \frac{2x}{x-2}}{1 + \frac{4}{(x+2)(x-2)}} = \dfrac{\frac{x(x-2)+2x}{(x-2)}}{\frac{(x+2)(x-2)+4}{(x+2)(x-2)}}$

$= \dfrac{x^2 - \cancel{2}x + \cancel{2}x}{(x-2)} \cdot \dfrac{(x+2)(x-2)}{x^2 - \cancel{4} + \cancel{4}}$

$= \dfrac{\cancel{x^2}}{\cancel{(x-2)}} \cdot \dfrac{(x+2)\cancel{(x-2)}}{\cancel{x^2}} = x + 2$

11. (i) $\dfrac{\left(\frac{a+b}{a-b}\right) - \left(\frac{a-b}{a+b}\right)}{1 + \left(\frac{a-b}{a+b}\right)} = \dfrac{\frac{(a+b)(a+b)-(a-b)(a-b)}{(a-b)+(a+b)}}{\frac{\cancel{(a+b)}+\cancel{(a-b)}}{(a+b)}}$

$= \dfrac{a^2 + 2ab + b^2 - (a^2 - 2ab + b^2)}{(a-b)(a+b)} \cdot \dfrac{(a+b)}{2a}$

$= \dfrac{(\cancel{a^2} + 2ab + \cancel{b^2} - \cancel{a^2} + 2ab - \cancel{b^2})}{(a-b)\,(a+b)} \cdot \dfrac{(a+b)}{2a}$

$= \dfrac{{}^{2}\cancel{4}\cancel{a}b\cancel{(a+b)}}{(a-b)\cancel{(a+b)}\cancel{2}\cancel{a}} = \dfrac{2b}{a-b}$

(ii) $\dfrac{x + \frac{3}{x}}{x - \frac{9}{x^3}} = \dfrac{\frac{x^2+3}{x}}{\frac{x^4-9}{x^3}} = \dfrac{(x^2+3)}{\cancel{x}} \cdot \dfrac{x^{\cancel{3}2}}{(x^4-9)}$

$= \dfrac{\cancel{(x^2+3)}(x^2)}{(x^2-3)\cancel{(x^2+3)}} = \dfrac{x^2}{x^2-3}$

(iii) $\dfrac{9 - \frac{1}{y^2}}{9 + \frac{6}{y} + \frac{1}{y^2}} = \dfrac{\frac{9y^2-1}{y^2}}{\frac{9y^2+6y+1}{y^2}}$

$= \dfrac{9y^2 - 1}{\cancel{y^2}} \cdot \dfrac{\cancel{y^2}}{9y^2 + 6y + 1}$

$= \dfrac{(3y-1)\,\cancel{(3y+1)}}{(3y+1)\,\cancel{(3y+1)}} = \dfrac{3y-1}{3y+1}$

12. $\dfrac{3x-5}{x-2} + \dfrac{1}{2-x} = \dfrac{(3x-5)(2-x)+(x-2)}{(x-2)(2-x)}$

$= \dfrac{6x - 3x^2 - 10 + 5x + x - 2}{(x-2)\,(2-x)}$

$= \dfrac{-3x^2 + 12x - 12}{(x-2)(2-x)}$

$= \dfrac{-3(x^2 - 4x + 4)}{(x-2)(2-x)}$

$= \dfrac{-3\,\cancel{(x-2)}\,\cancel{(x-2)}}{-\cancel{(x-2)}\,\cancel{(x-2)}} = 3.$

Exercises 1.5

1. By calculator:

(i) $\binom{7}{4} = 35$ (ii) $\binom{6}{2} = 15$ (iii) $\binom{6}{4} = 15$

(iv) $\binom{15}{4} = 1365$ (v) $\binom{10}{9} = 10$

2. (i) $\binom{9}{0} = 1$ (ii) $\binom{10}{1} = 10$ (iii) $\binom{13}{13} = 1$

(iv) $\binom{30}{0} = 1$ (v) $\binom{18}{17} = \binom{18}{1} = 18$

3. $\binom{12}{3} = \binom{12}{12-3} = \binom{12}{9}$. Hence $k = 9$.

4. $\binom{16}{12} = \binom{16}{16-12} = \binom{16}{4}$. Hence $k = 4$.

5. $$\begin{aligned}(a+2b)^4 &= \binom{4}{0}a^4 + \binom{4}{1}a^3(2b) + \binom{4}{2}a^2(2b)^2 + \binom{4}{3}a(2b)^3 + \binom{4}{4}(2b)^4\\ &= (1)a^4 + (4)a^3(2b) + (6)a^2(4b^2) + (4)a(8b^3) + (1)(16b^4)\\ &= a^4 + 8a^3b + 24a^2b^2 + 32ab^3 + 16b^4\end{aligned}$$

6. The coefficients in the 6th row of Pascal's triangle are 1 6 15 20 15 6 1.
Thus
$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$.

7. The fifth row of Pascal's triangle is 1 5 10 10 ...
The coefficient of the 4th term is thus 10. The 4th term is then
$10(1)^2(2x)^3 = 10(8x^3) = 80x^3$
Thus the coefficient is 80.

8. (i) $$\begin{aligned}(a-2b)^4 &= \binom{4}{0}a^4 + \binom{4}{1}a^3(-2b) + \binom{4}{2}a^2(-2b)^2 + \binom{4}{3}a(-2b)^3 + \binom{4}{4}(-2b)^4\\ &= (1)a^4 + (4)a^3(-2b) + (6)a^2(4b^2) + (4)a(-8b^3) + (1)(16b^4)\\ &= a^4 - 8a^3b + 24a^2b^2 - 32ab^3 + 16b^4\end{aligned}$$

(ii) $$\begin{aligned}(2x-y)^3 &= \binom{3}{0}(2x)^3 + \binom{3}{1}(2x)^2(-y) + \binom{3}{2}(2x)(-y)^2 + \binom{3}{3}(-y)^3\\ &= (1)(8x^3) + (3)(4x^2)(-y) + (3)(2x)(y^2) + (1)(-y^3)\\ &= 8x^3 - 12x^2y + 6xy^2 - y^3\end{aligned}$$

(iii) $$\begin{aligned}(p+3q)^4 &= \binom{4}{0}p^4 + \binom{4}{1}p^3(3q) + \binom{4}{2}p^2(3q)^2 + \binom{4}{3}p(3q)^3 + \binom{4}{4}(3q)^4\\ &= (1)p^4 + 4p^3(3q) + (6)p^2(9q^2) + (4)p(27q^3) + (1)(81q^4)\\ &= p^4 + 12p^3q + 54p^2q^2 + 108pq^3 + 81q^4\end{aligned}$$

(iv) $$\begin{aligned}(1+2y)^5 &= \binom{5}{0}(1)^5 + \binom{5}{1}(1)^4(2y) + \binom{5}{2}(1)^3(2y)^2 + \binom{5}{3}(1)^2(2y)^3\\ &\quad + \binom{5}{4}(1)(2y)^4 + \binom{5}{5}(2y)^5\\ &= (1)(1) + (5)(1)(2y) + (10)(1)(4y^2) + (10)(1)(8y^3) + (5)(1)(16y^4) + (1)(32y^5)\\ &= 1 + 10y + 40y^2 + 80y^3 + 80y^4 + 32y^5\end{aligned}$$

9. (i) $$\begin{aligned}(2+3p)^6 &= \binom{6}{0}2^6 + \binom{6}{1}2^5(3p) + \binom{6}{2}2^4(3p)^2 + \binom{6}{3}2^3(3p)^3 + \binom{6}{4}2^2(3p)^4\\ &\quad + \binom{6}{5}2(3p)^5 + \binom{6}{6}(3p)^6\\ &= (1)(64) + (6)(32)(3p) + (15)(16)(9p^2) + (20)(8)(27p^3) + (15)(4)(81p^4)\\ &\quad + (6)(2)(243p^5) + (1)(729p^6)\\ &= 64 + 576p + 2160p^2 + 4320p^3 + 4860p^4 + 2916p^5 + 729p^6\end{aligned}$$

(ii) $$\begin{aligned}(1-b)^7 &= \binom{7}{0}1^7 + \binom{7}{1}1^6(-b) + \binom{7}{2}1^5(-b)^2 + \binom{7}{3}1^4(-b)^3 + \binom{7}{4}1^3(-b)^4\\ &\quad + \binom{7}{5}1^2(-b)^5 + \binom{7}{6}(1)(-b)^6 + \binom{7}{7}(-b)^7\\ &= (1)(1) + (7)(1)(-b) + (21)(1)(b^2) + (35)(1)(-b^3) + (35)(1)(b^4)\\ &\quad + (21)(1)(-b^5) + (7)(1)(b^6) + (1)(-b^7)\\ &= 1 - 7b + 21b^2 - 35b^3 + 35b^4 - 21b^5 + 7b^6 - b^7\end{aligned}$$

(iii) $(p - 4q)^5 = \binom{5}{0}p^5 + \binom{5}{1}p^4(-4q) + \binom{5}{2}p^3(-4q)^2 + \binom{5}{3}p^2(-4q)^3$
$+ \binom{5}{4}p(-4q)^4 + \binom{5}{5}(-4q)^5$
$= (1)p^5 + (5)p^4(-4q) + (10)p^3(16q^2) + (10)p^2(-64q^3) + (5)p(256q^4)$
$+ (1)(-1024q^5)$
$= p^5 - 20p^4q + 160p^3q^2 - 640p^2q^3 + 1280pq^4 - 1024q^5$

10. The coefficients in the 7th row are then

1 $\quad 1 + 6 \quad 6 + 15 \quad 15 + 20 \quad 20 + 15 \quad 15 + 6 \quad 6 + 1 \quad$ 1

or 1 $\quad$ 7 $\quad$ 21 $\quad$ 35 $\quad$ 35 $\quad$ 21 $\quad$ 7 $\quad$ 1.

11. General term $= \binom{8}{r}x^{8-r}y^r$. For the 5th term, $r = 4$. Thus the 5th term is

$\binom{8}{4}x^{8-4}y^4 = 70x^4y^4$.

12. General term $= \binom{9}{r}x^{9-r}(-y)^r$. For the 4th term, $r = 3$. Thus the 4th term is

$\binom{9}{3}x^{9-3}(-y)^3 = (84)x^6(-y^3) = -84x^6y^3$

13. General term $= \binom{10}{r}(2x)^{10-r}y^r$. For the 6th term, $r = 5$. Thus the 6th term is

$\binom{10}{5}(2x)^5y^5 = (252)(32x^5)y^5 = 8064x^5y^5$

14. The number of terms in $(p + 2q)^6$ is $6 + 1 = 7$.
The middle term is then the 4th term (3 before and 3 after)
The general term is $\binom{6}{r}p^{6-r}(2q)^r$. For the 4th term, $r = 3$. Thus the middle term is
$\binom{6}{3}p^{6-3}(2q)^3 = (20)p^3(8q^3) = 160p^3q^3$.
The coefficient of this term is 160.

15. (i) $(2x - y)^8 = \binom{8}{0}(2x)^8 + \binom{8}{1}(2x)^7(-y) + \binom{8}{2}(2x)^6(-y)^2 + \binom{8}{3}(2x)^5(-y)^3$
$+ \binom{8}{4}(2x)^4(-y)^4 + \binom{8}{5}(2x)^3(-y)^5 + \binom{8}{6}(2x)^2(-y)^6$
$+ \binom{8}{7}(2x)(-y)^7 + \binom{8}{8}(-y)^8$
$= (1)(256x^8) + (8)(128x^7)(-y) + (28)(64x^6)(y^2) + (56)(32x^5)(-y^3)$
$+ (70)(16x^4)(y^4) + (56)(8x^3)(-y^5) + (28)(4x^2)(y^6)$
$+ (8)(2x)(-y^7) + (1)(y^8)$
$= 256x^8 - 1024x^7y + 1792x^6y^2 - 1792x^5y^3 + 1120x^4y^4 - 448x^3y^5$
$+ 112x^2y^6 - 16xy^7 + y^8$

(ii) $(a + 2b)^9 = \binom{9}{0}a^9 + \binom{9}{1}a^8(2b) + \binom{9}{2}a^7(2b)^2 + \binom{9}{3}a^6(2b)^3 + \binom{9}{4}a^5(2b)^4$
$+ \binom{9}{5}a^4(2b)^5 + \binom{9}{6}a^3(2b)^6 + \binom{9}{7}a^2(2b)^7 + \binom{9}{8}a(2b)^8 + \binom{9}{9}(2b)^9$
$= (1)a^9 + (9)a^8(2b) + (36)a^7(4b^2) + (84)a^6(8b^3) + (126)a^5(16b^4)$
$+ (126)a^4(32b^5) + (84)a^3(64b^6) + (36)a^2(128b^7)$
$+ (9)a(256b^8) + (1)(512b^9)$
$= a^9 + 18a^8b + 144a^7b^2 + 672a^6b^3 + 2016a^5b^4 + 4032a^4b^5 + 5376a^3b^6$
$+ 4608a^2b^7 + 2304ab^8 + 512b^9$

16. The general term in the expansion of $(5x+1)^{10}$ is

$\binom{10}{r}(5x)^{10-r}(1)^r$.

For the 3rd term, $r = 2$. Thus the 3rd term is

$\binom{10}{2}(5x)^8(1)^2 = (45)(390625x^8) = 17578125x^8$

17. The general term in the expansion of $\left(x - \frac{3y}{2}\right)^9$ is

$\binom{9}{r}x^{9-r}\left(\frac{-3y}{2}\right)^r$

For the 4th term, $r = 3$. Thus the 4th term is

$\binom{9}{3}x^6\left(\frac{-3y}{2}\right)^3 = (84)x^6\left(\frac{-27y^3}{8}\right) = -\frac{567}{2}x^6y^3$

The coefficient is $-\frac{567}{2} = -283.5$.

Exercise 1.6

1. $ax^2 + bx + c = (2x-3)(3x+4)$ for all x

$= 6x^2 + 8x - 9x - 12$

$= 6x^2 - x - 12$

$\therefore a = 6, b = -1, c = -12$

2. $(3x-2)(x+5) = 3x^2 + px + q$ for all x

$3x^2 + 15x - 2x - 10 = 3x^2 + px + q$

$3x^2 + 13x - 10 = 3x^2 + px + q$

$\therefore p = 13, q = -10$

3. $x^2 + 6x + 16 = (x+a)^2 + b$ for all x

$x^2 + 6x + 16 = x^2 + 2ax + a^2 + b$

$\therefore \quad 2a = 6 \Rightarrow a = 3$

and $a^2 + b = 16$

$\therefore \quad 9 + b = 16 \Rightarrow b = 7$

4. $x^2 + 4x - 6 = (x+a)^2 + b$ for all x

$x^2 + 4x - 6 = x^2 + 2ax + a^2 + b$

$\therefore \; 2a = 4 \Rightarrow a = 2$

and $a^2 + b = -6$

$\therefore \quad 4 + b = -6 \Rightarrow b = -10$

5. $2x^2 + 5x + 6 = p(x+q)^2 + r$ for all x

$= p(x^2 + 2xq + q^2) + r$

$= px^2 + 2pqx + pq^2 + r$

$\therefore p = 2$

and $2pq = 5$

$\therefore 2(2)q = 5 \Rightarrow q = \frac{5}{4}$

and $pq^2 + r = 6$

$\therefore 2\left(\frac{5}{4}\right)^2 + r = 6$

$\frac{25}{8} + r = 6 \Rightarrow r = 6 - \frac{25}{8}$

$= \frac{48-25}{8} = \frac{23}{8}$

6. $(2x + a)^2 = 4x^2 + 12x + b$ for all values of x

$4x^2 + 4ax + a^2 = 4x^2 + 12x + b$

$\therefore\ 4a = 12 \Rightarrow a = 3$

and $a^2 = b \Rightarrow 9 = b$

7. $x^2 - 4x - 5 = (x - n)^2 - m$ for all values of x

$x^2 - 4x - 5 = x^2 - 2nx + n^2 - m$

$\therefore\quad -4 = -2n \Rightarrow n = 2$

and $-5 = n^2 - m$

$\therefore\quad -5 = 4 - m \Rightarrow m = 9$

8. (i) $V(x) = ax^3 + bx^2 + cx + d = (x + 5)(x + 3)(x + 2)$ for all x

$$= (x^2 + 8x + 15)(x + 2)$$
$$= x^3 + 2x^2 + 8x^2 + 16x + 15x + 30$$
$$= x^3 + 10x^2 + 31x + 30$$

$\therefore a = 1, b = 10, c = 31, d = 30.$

(ii) $S(x) = px^2 + qx + r = 2(x + 3)(x + 2) + 2(x + 5)(x + 3) + (x + 5)(x + 2)$

$$= 2(x^2 + 5x + 6) + 2(x^2 + 8x + 15) + (x^2 + 7x + 10)$$
$$= 5x^2 + 33x + 52$$

$\therefore\quad p = 5, q = 33, r = 52$

9. $3(x - p)^2 + q = 3x^2 - 12x + 7$ for all x

$\therefore\ 3(x^2 - 2px + p^2) + q = 3x^2 - 12x + 7$

$\therefore\ 3x^2 - 6px + 3p^2 + q = 3x^2 - 12x + 7$

$\therefore\quad -6p = -12 \Rightarrow p = 2$

and $3p^2 + q = 7$

$\therefore\quad 3(2)^2 + q = 7 \Rightarrow q = -5$

10. $V(x) = x^3 + 12x^2 + bx + 30 = (x^2 + cx + 4)(x + a)$

$$= x^3 + ax^2 + cx^2 + acx + 4x + 4a$$
$$= x^3 + x^2(a + c) + x(ac + 4) + 4a$$

$\therefore\quad a + c = 12$

and $b = ac + 4$

and $4a = 30 \Rightarrow a = \frac{30}{4} = \frac{15}{2}.$

$\therefore\quad a + c = 12 \Rightarrow c = 12 - a$

$$c = 12 - \frac{15}{2} = \frac{9}{2}$$

$\therefore\quad b = ac + 4 \Rightarrow b = \frac{15}{2} \cdot \frac{9}{2} + 4 = \frac{135 + 16}{4} = 37\frac{3}{4}$

11. $(x - 4)^3 = x^3 + px^2 + qx - 64$ for all x

$= (x - 4)(x - 4)(x - 4)$

$= (x^2 - 8x + 16)(x - 4)$

$= x^3 - 4x^2 - 8x^2 + 32x + 16x - 64$

$= x^3 - 12x^2 + 48x - 64$

$\therefore p = -12, q = 48$

12. $(x + a)(x^2 + bx + 2) = x^3 - 2x^2 - x - 6$ for all x

$= x^3 + bx^2 + 2x + ax^2 + abx + 2a$

$= x^3 + x^2(b + a) + x(2 + ab) + 2a$

$\therefore\quad b + a = -2$

and $2 + ab = -1$

and $2a = -6 \Rightarrow a = -3$

$\therefore\quad b + a = -2 \Rightarrow b - 3 = -2 \Rightarrow b = 1$

13. $(x-2)(x^2+bx+c) = x^3+2x^2-5x-6$ for all x

$= x^3+bx^2+cx-2x^2-2bx-2c$

$= x^3+x^2(b-2)+x(c-2b)-2c$

$\therefore \quad b-2 = 2 \Rightarrow b = 4$

and $c-2b = -5 \Rightarrow c-2(4) = -5 \Rightarrow c = 3$

14. $(5a-b)x+b+2c = 0$ for all x

$\therefore (5a-b)x+b+2c = 0.x+0$

$\therefore 5a-b = 0 \Rightarrow b = 5a$

and $\quad b+2c = 0 \Rightarrow b = -2c$

$\therefore 5a = -2c$

$a = \dfrac{-2c}{5}$

15. $(4x+r)(x^2+s) = 4x^3+px^2+qx+2$ for all x

$= 4x^3+4xs+rx^2+rs$

$= 4x^3+rx^2+4xs+rs$

$\therefore \quad r = p$

$\left.\begin{array}{l}\text{and } 4s = q\\ \text{and } rs = 2\end{array}\right\} \Rightarrow pq = r.4s = 4rs$

$\therefore \quad pq = 4rs = 4(2) = 8$

16. $(x+s)(x-s)(ax+t) = ax^3+bx^2+cx+d$ for all x

$= (x^2-s^2)(ax+t)$

$= ax^3+tx^2-as^2x-ts^2$

$\therefore \quad t = b$

$\left.\begin{array}{l}\text{and } -as^2 = c\\ \text{and } -ts^2 = d\end{array}\right\} \therefore \dfrac{-a\cancel{s^2}}{-t\cancel{s^2}} = \dfrac{c}{d}$

$\Rightarrow -ad = -ct$

but $t = b$ $\quad \therefore -ad = -cb$

$\Rightarrow ad = cb$

17. $\dfrac{1}{(x+1)(x-1)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x-1)}$ for all x

$\Rightarrow \dfrac{1}{(x+1)(x-1)} = \dfrac{A(x-1)+B(x+1)}{(x+1)(x-1)}$

$\Rightarrow \quad 1 = A(x-1)+B(x+1)$

$\Rightarrow \quad 1 = Ax-A+Bx+B$

$\Rightarrow \quad 1 = x(A+B)-A+B$

$\therefore \quad 0.x+1 = (A+B)x-A+B$

$\therefore \quad A+B = 0$

and $\underline{-A+B} = 1$

adding: $\quad 2B = 1$

$B = \dfrac{1}{2}$

since $A+B = 0 \Rightarrow A = -B = -\dfrac{1}{2}$.

18. $\dfrac{1}{(x+2)(x-3)} = \dfrac{C}{(x+2)} + \dfrac{D}{(x-3)}$

$\dfrac{1}{(x+2)(x-3)} = \dfrac{C(x-3)+D(x+2)}{(x+2)(x-3)}$

$\Rightarrow \quad 1 = C(x-3)+D(x+2)$

$1 = Cx-3C+Dx+2D$

$0.x+1 = x(C+D)-3C+2D$

$$\begin{aligned} \Rightarrow \quad & C + D = 0 \\ \times 3 \text{ and } & -3C + 2D = 1 \\ & \underline{3C + 3D = 0} \\ \therefore \text{adding: } & 5D = 1 \Rightarrow \\ & D = \frac{1}{5} \end{aligned}$$

since $C + D = 0 \Rightarrow C = -D = -\frac{1}{5}$.

19. $\dfrac{1}{(x+1)(x+4)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x+4)}$

$\Rightarrow \dfrac{1}{(x+1)(x+4)} = \dfrac{A(x+4) + B(x+1)}{(x+1)(x+4)}$

$\Rightarrow 1 = Ax + 4A + Bx + B$

$0.x + 1 = x(A + B) + 4A + B$

$\therefore \quad A + B = 0$

and $\quad \underline{4A + B = 1}$

subtracting: $\quad -3A = -1 \quad \Rightarrow A = \frac{1}{3}$

since $\quad A + B = 0 \quad \Rightarrow B = -A = -\frac{1}{3}$.

20. $(x - 3)^2$ is a factor of $x^3 + ax + b$

$\Rightarrow (x - 3)^2 (x + k) = x^3 + ax + b$

$\therefore (x^2 - 6x + 9)(x + k) = x^3 + ax + b$

$\therefore x^3 + kx^2 - 6x^2 - 6kx + 9x + 9k = x^3 + ax + b$

$\therefore x^3 + x^2(k - 6) + x(-6k + 9) + 9k = x^3 + ax + b = x^3 + 0.x^2 + ax + b$

$\therefore k - 6 = 0 \Rightarrow k = 6$

also $-6k + 9 = a \quad \therefore -6(6) + 9 = a \Rightarrow a = -27$

also $\quad 9k \quad = b \quad \therefore 9(6) = b \Rightarrow b = 54$.

21. $(x - 2)^2$ is a factor of $x^3 + px + q$

$\Rightarrow (x - 2)^2 (x - k) = x^3 + px + q$

$\therefore \ (x^2 - 4x + 4)(x - k) = x^3 + 0.x^2 + px + q$

$\therefore \ x^3 - kx^2 - 4x^2 + 4kx + 4x - 4k = x^3 + 0.x^2 + px + q$

$\therefore \ x^3 + x^2(-k - 4) + x(4k + 4) - 4k =$

$\therefore \ -k - 4 = 0$

$\Rightarrow \ k = -4$

also $p = 4k + 4 = 4(-4) + 4 = -12$

$q = -4k = -4(-4) = +16$.

22. $(x^2 - 4)$ is a factor of $x^3 + cx^2 + dx - 12$

$\therefore \ (x^2 - 4)(x + k) = x^3 + cx^2 + dx - 12$

$\therefore x^3 + kx^2 - 4x - 4k = x^3 + cx^2 + dx - 12$

$\therefore \quad k = c$

also $\quad d = -4$

and $-4k = -12 \Rightarrow k = 3 = c$

$\therefore \ (x^2 - 4)(x + 3) = x^3 + 3x^2 - 4x - 12$

$\Rightarrow (x - 2)(x + 2)(x + 3) = x^3 + 3x^2 - 4x - 12$.

23. $(x^2 + b)$ is a factor of $x^3 - 3x^2 + bx - 15$

$\Rightarrow (x^2 + b)(x + k) = x^3 - 3x^2 + bx - 15$

$\therefore x^3 + kx^2 + bx + bk = x^3 - 3x^2 + bx - 15$

$\Rightarrow \quad k = -3$

also $bk = -15 \Rightarrow b = \dfrac{-15}{k} = \dfrac{-15}{-3} = 5$.

24. $x^2 - px + 9$ is a factor of $x^3 + ax + b$

$\therefore\ (x^2 - px + 9)(x + k) = x^3 + 0.x^2 + ax + b$

$\therefore x^3 + kx^2 - px^2 - pkx + 9x + 9k = x^3 + 0.x^2 + ax + b$

$\therefore x^3 + x^2(k - p) + x(-pk + 9) + 9k = x^3 + 0.x^2 + ax + b$

$\therefore\quad k - p = 0 \Rightarrow k = p$

$-pk + 9 = a \Rightarrow -p(p) + 9 = a$

$\therefore a = 9 - p^2$

also $b = 9k \Rightarrow b = 9p$

$a + b = 17$

$\Rightarrow\quad 9 - p^2 + 9p = 17$

$-p^2 + 9p - 8 = 0$

$p^2 - 9p + 8 = 0$

$(p - 8)(p - 1) = 0$

$\therefore p = 8, 1$

25. $x^2 - kx + 1$ is a factor of $ax^3 + bx + c$

$$\begin{array}{r|l}
 & ax + ak \\ \hline
x^2 - kx + 1 & \cancel{ax^3} + \qquad + bx + c \\
 & \underline{\cancel{ax^3} - akx^2 + ax} \\
 & \qquad akx^2 + bx - ax + c \\
 & \qquad \cancel{akx^2} + (b - a)x + c \\
 & \qquad \underline{\cancel{akx^2} - ak^2x \qquad + ak} \\
 & \qquad\qquad (b - a)x + ak^2x + c - ak \\
 & \qquad\qquad (b - a + ak^2)x + c - ak \quad \text{(remainder)}
\end{array}$$

Since $x^2 - kx + 1$ is a factor, there can be no remainder.

$\therefore\ (b - a + ak^2)x^2 + c - ak = 0 \quad$ for all x

$\Rightarrow b - a + ak^2 = 0$

and $c - ak = 0 \Rightarrow k = \dfrac{c}{a}$

$\therefore\ b - a + a\left(\dfrac{c}{a}\right)^2 = 0$

$\therefore\ b - a + \dfrac{c^2}{a} = 0 \Rightarrow ab - a^2 + c^2 = 0$

$\Rightarrow c^2 = a^2 - ab = a(a - b)$

26. $(x - a)^2$ is a factor of $x^3 + 3px + c$

i.e. $x^2 - 2ax + a^2$ is a factor of $x^3 + 3px + c$.

$$\begin{array}{r|l}
 & x + 2a \\ \hline
\therefore x^2 - 2ax + a^2 & \cancel{x^3} \qquad\quad + 3px + c \\
 & \underline{\cancel{x^3} - 2ax^2 + a^2x} \\
 & \quad 2ax^2 + 3px - a^2x + c \\
 & \quad \cancel{2ax^2} + (3p - a^2)x + c \\
 & \quad \underline{\cancel{2ax^2} - 4a^2x + 2a^3} \\
 & \qquad\quad (3p - a^2)x + 4a^2x + c - 2a^3 \text{ (remainder)}
\end{array}$$

Since $x^2 - 2ax + a^2$ is a factor, there can be no remainder.

$\therefore\ (3p - a^2 + 4a^2)x + c - 2a^3 = 0 \quad$ for all x

$\therefore\ 3p - a^2 + 4a^2 = 0 \Rightarrow 3p = -3a^2 \Rightarrow p = -a^2$

also, $c - 2a^3 = 0 \Rightarrow c = 2a^3$.

27. $x^2 + ax + b$ is a factor of $x^3 - k$.

$$
\begin{array}{r|l}
 & x - a \\
\therefore x^2 + ax + b & \cancel{x^3} \qquad\qquad -k \\
 & \underline{\cancel{x^3} + ax^2 + bx} \\
 & \quad -\cancel{ax^2} - bx - k \\
 & \quad \underline{-\cancel{ax^2} - a^2x - b} \\
 & \qquad\quad - bx + a^2x - k + b \\
 & \qquad\quad x(-b + a^2) - k + ba \ \text{(remainder)}
\end{array}
$$

Since $x^2 + ax + b$ is a factor, there can be no remainder.

$\therefore (-b + a^2) = 0 \Rightarrow b = a^2$

(i) also, $-k + ba = 0 \Rightarrow k = ab = a.a^2 = a^3$

(ii) Since $b = a^2 \Rightarrow b^3 = (a^2)^3 = a^6 = k^2$.

28.

$$
\begin{array}{r|l}
 & 2x - 1 \\
2x - \sqrt{3} & \cancel{4x^2} - 2(1 + \sqrt{3})x + \sqrt{3} \\
 & \underline{\cancel{4x^2} - 2\sqrt{3}x} \\
 & \quad -2(1 + \sqrt{3})x + 2\sqrt{3}x + \sqrt{3} \\
 & \quad -2x - \cancel{2\sqrt{3}x} + \cancel{2\sqrt{3}x} + \sqrt{3} \\
 & \quad -\cancel{2x} + \cancel{\sqrt{3}} \\
 & \quad \underline{-\cancel{2x} + \cancel{\sqrt{3}}} \\
 & \qquad\quad 0
\end{array}
$$

$\therefore 2x - \sqrt{3}$ is a factor

and the second factor is $2x - 1$.

29. $5x + 3 = Ax(x + 3) + Bx(x - 1) + C(x - 1)(x + 3)$ for all x

$$
\begin{aligned}
5x + 3 &= Ax^2 + 3Ax + Bx^2 - Bx + C(x^2 + 3x - x - 3) \\
&= Ax^2 + 3Ax + Bx^2 - Bx + Cx^2 + 3Cx - Cx - 3C \\
&= Ax^2 + 3Ax + Bx^2 - Bx + Cx^2 + 2Cx - 3C \\
&= (A + B + C)x^2 + (3A - B + 2C)x - 3C
\end{aligned}
$$

$\therefore A + B + C = 0$

also $3A - B + 2C = 5$

and $\quad -3C = 3 \Rightarrow C = -1$

$\therefore \quad A + B = 1$

$\quad \underline{3A - B = 7}$

adding: $\quad 4A = 8 \Rightarrow A = 2$

since $\quad A + B = 1 \Rightarrow B = 1 - A = 1 - 2 = -1$.

Exercise 1.7

1. (i) $3x - 2y = 4$

$$
\begin{aligned}
3x &= 4 + 2y \\
x &= \frac{4 + 2y}{3}
\end{aligned}
$$

(ii) $2x - b = 4c$

$$
\begin{aligned}
2x &= 4c + b \\
x &= \frac{4c + b}{2}
\end{aligned}
$$

(iii) $5x - 4 = \dfrac{y}{2}$

$$
\begin{aligned}
5x &= \frac{y}{2} + 4 \\
x &= \frac{\frac{y}{2} + 4}{5} = \frac{y + 8}{10}
\end{aligned}
$$

(iv) $5(x-3) = 2y$

$$x - 3 = \frac{2y}{5}$$

$$x = \frac{2y}{5} + 3 = \frac{2y+15}{5}$$

(v) $3y = \frac{x}{3} - 2$

$$\frac{-x}{3} = -3y - 2$$

$$\frac{x}{3} = 3y + 2$$

$$x = 9y + 6$$

(vi) $xy = xz + yz$

$$xy - xz = yz$$

$$x(y-z) = yz$$

$$x = \frac{yz}{y-z}$$

2. (i) $2x - \frac{y}{3} = \frac{1}{3}$

$$2x = \frac{1}{3} + \frac{y}{3} = \frac{y+1}{3}$$

$$x = \frac{y+1}{6}$$

(ii) $z = \frac{y-2x}{3}$

$$3z = y - 2x$$

$$2x = y - 3z$$

$$x = \frac{y-3z}{2}$$

(iii) $\frac{a}{x} - b = c$

$$a - bx = cx$$

$$-cx - bx = -a$$

$$cx + bx = a$$

$$x(c+b) = a$$

$$x = \frac{a}{b+c}$$

3. (a) $V = \pi r^2 h$

$$\Rightarrow \pi r^2 h = V$$

$$\Rightarrow r^2 = \frac{V}{\pi h}$$

$$r = \sqrt{\frac{V}{\pi h}}$$

(b) $A = 2\pi r h$

$$\Rightarrow 2\pi r h = A$$

$$r = \frac{A}{2\pi h}$$

(c) $r = \sqrt{\frac{V}{\pi h}}$ and $r = \frac{A}{2\pi h}$

$$\Rightarrow \sqrt{\frac{V}{\pi h}} = \frac{A}{2\pi h}$$

$$\therefore \frac{V}{\pi h} = \frac{A^2}{4\pi^2 h^2}$$

$$\Rightarrow A^2 = \frac{4\pi^2 h^2 V}{\cancel{\pi}\cancel{h}} = 4\pi h V$$

4. (a) $A_{circle} = \pi r^2$

(b) The side of the square $= 2r$

$\Rightarrow A_{square} = l^2 = (2r)^2 = 4r^2$

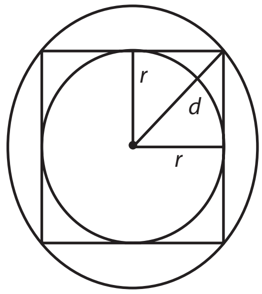
(c) $A_{corners} = 4r^2 - \pi r^2 = (4 - \pi)r^2$

(d) area of new square $= (4r)^2 = 16r^2$

area of new circle $= \pi\left(\frac{r}{2}\right)^2 = \frac{\pi r^2}{4}$

$$\Rightarrow \text{area of new shaded section} = 16r^2 - \frac{\pi r^2}{4}$$
$$= r^2\left(16 - \frac{\pi}{4}\right)$$
$$= \frac{r^2}{2}(64 - \pi)$$

(e) To find the radius of the outer circle we need to find the distance from the centre of the circle to a corner (vertex) of the square.



$d = \sqrt{r^2 + r^2} = \sqrt{2r^2} = \sqrt{2}r.$

$$\Rightarrow \text{Area of outer circle} = \pi d^2$$
$$= \pi(\sqrt{2}r)^2$$
$$= 2\pi r^2$$

= twice the area of the inner circle.

5. (i)

$$f' = \frac{fc}{c-u}$$
$$\Rightarrow f'(c-u) = fc$$
$$c - u = \frac{fc}{f'}$$
$$-u = \frac{fc}{f'} - c$$
$$u = c - \frac{fc}{f'} = \frac{f'c - fc}{f'} = \frac{c(f' - f)}{f'}$$

(ii)

$$f' = \frac{fc}{c-u}$$
$$f'(c-u) = fc$$
$$f'c - f'u = fc$$
$$f'c - fc = f'u$$
$$c(f' - f) = f'u$$
$$c = \frac{f'u}{f' - f}$$

6. (i) $T = 2\pi\sqrt{\frac{l}{g}}$

$T^2 = 4\pi^2\frac{l}{g}$

$l = \frac{gT^2}{4\pi^2}$

(ii) $T = 3, g = 10. \quad \Rightarrow \quad l = \frac{10.3^2}{4\pi^2} = \frac{90}{39.48} \simeq 2.3$ m

7. (i)

$$\frac{x}{y} = \frac{a+b}{a-b}$$
$$x(a-b) = y(a+b)$$
$$ax - bx = ay + by$$
$$ax - ay = bx + by$$
$$a(x-y) = b(x+y)$$
$$a = \frac{b(x+y)}{(x-y)}$$

(ii)

$$bc - ac = ac$$
$$-ac - ac = -bc$$
$$-2ac = -bc$$
$$a = \frac{-bc}{-2c} = \frac{b}{2}$$

8. (i)
$$y = \frac{3(u-v)}{4}$$
$$4y = 3u - 3v$$
$$3v = 3u - 4y$$
$$v = \frac{3u - 4y}{3}$$

(ii)
$$s = \frac{t}{2}(u+v)$$
$$2s = tu + tv$$
$$-tv = tu - 2s$$
$$tv = 2s - tu$$
$$v = \frac{2s - tu}{t}$$

9.
$$A = P\left(1 + \frac{i}{100}\right)^3$$
$$P\left(1 + \frac{i}{100}\right)^3 = A$$
$$\left(1 + \frac{i}{100}\right) = \frac{A}{P}$$
$$1 + \frac{i}{100} = \sqrt[3]{\frac{A}{P}}$$
$$100 + i = 100\sqrt[3]{\frac{A}{P}}$$
$$i = 100\sqrt[3]{\frac{A}{P}} - 100$$
$$P = 2500, \qquad A = 2650$$
$$\therefore i = 100\sqrt[3]{\frac{2650}{2500}} - 100$$
$$= 100(1.0196) - 100$$
$$= 1.961$$
$$\therefore \quad i = 2\%$$

10. (i)
$$d = \sqrt{\frac{a-b}{ac}}$$
$$d^2 = \frac{a-b}{ac}$$
$$acd^2 = a - b$$
$$c = \frac{a-b}{ad^2}$$

(ii)
$$b = \frac{2c-1}{c-1}$$
$$\Rightarrow b(c-1) = 2c - 1$$
$$bc - b = 2c - 1$$
$$bc - 2c = b - 1$$
$$c(b-2) = b - 1$$
$$c = \frac{b-1}{b-2}$$

11. (i) From Pythagoras: $h^2 + r^2 = 15^2$
$$h^2 = 15^2 - r^2$$
$$h = \sqrt{15^2 - r^2}$$

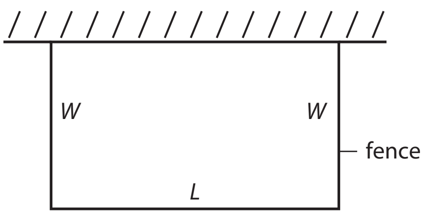
(ii) at $r = 5$ cm: $h = \sqrt{15^2 - 5^2}$
$$= \sqrt{225 - 25} = \sqrt{200}$$
$$= 10\sqrt{2} \text{ cm}$$

(iii) $r = \frac{15}{2}$: $h = \sqrt{15^2 - \left(\frac{15}{2}\right)^2}$
$$= \sqrt{168.75} = 12.99 \text{ cm}$$
$$h = 13 \text{ cm}$$

12. (i) $2W + L = 300$
$$L = 300 - 2W$$

(ii) $A = L \times W$
$$= (300 - 2W) \cdot W = 300W - 2W^2$$

(iii) $10\,000 = 300W - 2W^2$
$$2W^2 - 300W + 10\,000 = 0$$
$$W^2 - 150W + 5000 = 0$$
$$(W - 50)(W - 100) = 0$$



$\Rightarrow \quad W = 50 \quad \text{or} \quad W = 100$

hence, $L = 300 - 2(50) \quad \text{or} \quad L = 300 - 2(100)$

$\qquad = 200 \qquad\qquad\qquad = 100$

answer: $(50, 200)$ or $(100\ 100)$

Exercise 1.8

1. (a) $4, 7, 10, 13, 16 \ldots$ constant 1st difference $\Rightarrow$ linear

(b) $-2, 2, 6, 10, 14 \ldots$ constant 1st difference $\Rightarrow$ linear

(c) $-4, -3, 0, 5, 12$

$+1, +3, +5, +7$

$+2, +2, +2 \ldots.$ constant 2nd difference $\Rightarrow$ quadratic

(d) $2, 1, -2, -7, -14, -23 \ldots$

$-1, -3, -5, -7,$

$-2, -2, -2 \ldots$ constant 2nd difference $\Rightarrow$ quadratic

(e) $2, 7, 22, 47$

$+5, +15, +25$

$+10, +10 \ldots.$ constant 2nd difference $\Rightarrow$ quadratic

(f) $3, 1, -5, -15, -29, \ldots.$

$-2, -6, -10, -14$

$-4, -4, -4, \ldots$ constant 2nd difference $\Rightarrow$ quadratic

(g) $1, -4, -19, -44, -79 \ldots.$

$-5, -15, -25, -35$

$-10, -10, -10 \ldots$ constant 2nd difference $\Rightarrow$ quadratic

(h) $3, -2, -7, -12, -17 \ldots$

$-5, -5, -5, -5 \ldots$ constant 1st difference $\Rightarrow$ linear

(i) $0, 3, 12, 27, 48$

$+3, +9, +15, +21$

$+6, +6, +6 \ldots.$ constant 2nd difference $\Rightarrow$ quadratic

(j) $5, 17, 37, 65, 101$

$+12, +20, +28, +36$

$+8, +8, +8 \ldots$ constant 2nd difference $\Rightarrow$ quadratic.

2. (a) $-1, 3, 15, 35, 63$

first differences $= +4, +12, +20, +28$

second differences $= +8, +8, +8.$

$\Rightarrow$ quadratic pattern of the form $ax^2 + bx + c$

also, $2a = 8 \Rightarrow a = 4.$

$\therefore 4x^2 + bx + c$

let $x = 1 \Rightarrow 4(1)^2 + b(1) + c = -1$

$\Rightarrow \qquad b + c = -1 - 4 = -5$

let $x = 2 \Rightarrow 4(2)^2 + b(2) + c = 3$

$\Rightarrow 2b + c = 3 - 16 = -13$

using simultaneous equations: $2b + c = -13$

$$\underline{b + c = -5}$$

$$b \quad = -8$$

$$\Rightarrow -8 + c = -5$$

$$\Rightarrow \quad c = -5 + 8 = 3$$

$\therefore ax^2 + bx + c = 4x^2 - 8x + 3 \quad$ for $x = 1, 2, 3, \ldots.$

Note also we could let $x = 0 \Rightarrow 4(0)^2 + b(0) + c = -1$

$\Rightarrow \qquad c = -1$

$$\text{let } x = 1 \Rightarrow 4(1)^2 + b(1) + c = 3$$
$$4 + b + c = 3$$
$$4 + b - 1 = 3$$
$$b = 0.$$

$\therefore ax^2 + bx + c = 4x^2 - 1$ for $x = 0, 1, 2, ...$

(b) $4, 3, 0, -5, -12, -21, -32$

first difference $= -1, -3, -5, -7, -9, -11$

second differnce $= -2, -2, -2, -2$

$\Rightarrow$ quadratic pattern of the form $ax^2 + bx + c$

also, $2a = -2 \Rightarrow a = -1$.

$\therefore -x^2 + bx + c$

let $x = 0 \Rightarrow -(0)^2 + b(0) + c = 4$

$c = 4$

let $x = 1 \Rightarrow -(1)^2 + b(1) + c = 3$

$-1 + b + 4 = 3$

$b = 0$

$\therefore ax^2 + bx + c = -x^2 + 4$ is the pattern for $x = 0, 1, 2, ...$

3. (i) $2, 7, 12, 17, 22, \ldots\ldots$

first difference $= 5$, a constant $\Rightarrow$ a linear pattern.

$\therefore f(x) = ax + b = 5x + b.$

Let $x = 0$ be the 1st term of the pattern.

$\Rightarrow f(0) = 5(0) + b = 2$

$\therefore \quad b = 2$

$\therefore f(x) = 5x + 2$ for $x = 0, 1, 2, \ldots$

(ii) $-6, -2, 2, 6, 10\ldots$

first difference $= 4$, a constant $\Rightarrow$ a linear pattern.

$\therefore f(x) = ax + b = 4x + b$

Let $x = 0$ be the 1st term of the pattern.

$\Rightarrow f(0) = 4(0) + b = -6$

$\therefore \quad b = -6.$

$\therefore f(x) = 4x - 6$ for $x = 0, 1, 2, \ldots$

(iii) $3, 2, 1, 0, -1, -2, \ldots$

first difference $= -1$, a constant $\Rightarrow$ a linear relationship

$\therefore f(x) = ax + b = -x + b$

Let $x = 0$ be the 1st term of the pattern.

$\Rightarrow \quad f(0) = -(0) + b = 3$

$\therefore \quad b = 3$

$\therefore f(x) = -x + 3$

$= 3 - x.$

(iv) $-2, 7, -12, -17, -22, \ldots$

first difference $= -5$, a constant $\Rightarrow$ a linear relationship

$\therefore f(x) = ax + b = -5x + b$

Let $x = 0$ be the first term of the pattern.

$\Rightarrow f(0) = -5(0) + b = -2$

$\therefore \quad b = -2$

$\therefore f(x) = -5x - 2$

(v) $3, 3.5, 4, 4.5, 5, \ldots$

first difference $= 0.5$, a constant $\Rightarrow$ a linear relationship.

$\therefore f(x) = ax + b = 0.5x + b$

Let $x = 0$ be the first term of the pattern.

$\therefore f(0) = 0.5(0) + b = 3$

$\Rightarrow \qquad b = 3$

$\therefore f(x) = 0.5(x) + 3$

$= \frac{x}{2} + 3$ for $x = 0, 1, 2, \ldots$

(vi) $-1, -0.8, -0.6, -0.4, -0.2, \ldots$

first difference $= 0.2$, a constant $\Rightarrow$ a linear relationship.

$\therefore f(x) = ax + b = 0.2x + b$

Let $x = 0$ be the first term of the pattern.

$\therefore\ f(0) = 0.2(0) + b = -1$

$\therefore \qquad b = -1$

$\therefore\ f(x) = 0.2x - 1$ for $x = 0, 1, 2, \ldots$

4. $11, 13, 15, 17, 19, \ldots$

first difference $= 2$, a constant $\Rightarrow$ a linear relationship.

$\therefore f(x) = ax + b = 2x + b$

Let $x = 3$ be the first term of the pattern.

$\Rightarrow\ f(3) = 2(3) + b = 11$

$\Rightarrow \qquad b = 5$

$\therefore f(x) = 2x + 5$ for $x = 3, 4, 5, \ldots$

5. $1, 3, 5, 7, 9, \ldots$

first difference $= 2$, a constant $\Rightarrow$ a linear relationship.

$\therefore f(x) = ax + b = 2x + b$

Let $x = -2$ be the first term of the pattern.

$\therefore f(-2) = 2(-2) + b = 1$

$b = 1 + 4 = 5.$

$\therefore f(x) = 2x + 5$ for $x = -2, -1, 0, \ldots$

6. (a) $3, 6, 9, \ldots$

a first difference $= 3$ (a constant) $\Rightarrow$ a linear pattern.

$\Rightarrow f(x) = ax + b = 3x + b$

Let $x = 1$ be the first element of the pattern.

$\therefore f(1) = 3(1) + b = 3$

$\Rightarrow \qquad b = 0.$

$\therefore f(x) = 3x$ for $x = 1, 2, 3, \ldots$

$\Rightarrow$ for the 15^{th} element, $x = 15$.

$\therefore f(15) = 3(15) = 45$ matchsticks are needed.

(b) $4, 8, 12, \ldots$

a first difference $= 4$ (a constant) $\Rightarrow$ a linear pattern.

$\Rightarrow f(x) = ax + b = 4x + b$

Let $x = 1$ be the first element of the pattern.

$\therefore f(1) = 4(1) + b = 4$

$\Rightarrow \qquad b = 0.$

$\Rightarrow\ \therefore f(x) = 4x$ for $x = 1, 2, 3, \ldots$

$\Rightarrow$ for the 15^{th} element, $x = 15$.

$\therefore f(15) = 4(15) = 60$ matchsticks are needed.

(c) $3, 5, 7, \ldots$

a first difference $= 2$ (constant) $\Rightarrow$ a linear pattern.

$\Rightarrow f(x) = ax + b = 2x + b$

Let $x = 1$ be the first element of the pattern.
$\therefore f(1) = 2(1) + b = 3$
$\Rightarrow \quad b = 1$
$\therefore f(x) = 2x + 1$ for $x = 1, 2, 3, \dots$
$\Rightarrow$ For the 15th element, $x = 15$.
$\therefore f(15) = 2(15) + 1 = 31$ matchsticks are needed.

7. Plan $A = 35x + 70$
Plan $B = 24x + 125$
Both plans repay the same amount if
$35x + 70 = 24x + 125$
$\Rightarrow \quad 11x = 55$
$x = 5$ months.

8. 4, 7, 14, 25, 40
first difference: 3, 7, 11, 15
second difference : 4, 4, 4 $\Rightarrow$ a quadratic pattern, $f(t) = at^2 + bt + c$
$\therefore 2a = 4 \Rightarrow a = 2.$
$\therefore f(t) = 2t^2 + bt + c$
At $t = 0$, $f(0) = 2(0)^2 + b(0) + c = 4$ (i.e. at the start there were 4 bacteria)
$\Rightarrow \quad c = 4$
At $t = 1$, $f(1) = 2(1)^2 + b(1) + c = 7$
$\Rightarrow 2 + b + 4 = 7$
$\Rightarrow b = 1$
$\therefore f(t) = 2t^2 + t + 4$ for $t = 0, 1, 2, 3, \dots$.
when is $f(t) = 529$, assuming $f(t) = 2t^2 + t + 4$?
if $t = 10 \Rightarrow 2(10)^2 + 10 + 4 = 214$ too small
$t = 15 \Rightarrow 2(15)^2 + 15 + 4 = 469$ too small
$t = 16 \Rightarrow 2(16)^2 + 16 + 4 = 532$ too large
$\therefore$ In the 16th hour, the number of bacteria was 529.

Exercise 1.9

1. (i) $y = 2x^2 + 2x - 1$ is not linear because the highest power is not 1.
(ii) $y = 2(x - 1)^{-1}$ is not linear because the highest power of x is not 1.
(iii) $y^2 = 3x + 4$
$\Rightarrow \quad y = \sqrt{3x + 4}$ is not linear because the highest power of x is not 1.
$= (3x + 4)^{\frac{1}{2}}$

2. (i) Solve $5x - 3 = 32$
$\Rightarrow 5x = 35$
$x = \frac{35}{5} = 7$
(ii) Solve $3x + 2 = x + 8$
$\Rightarrow \quad 3x - x = 8 - 2$
$2x = 6$
$x = 3$
(iii) Solve $2 - 5x = 8 - 3x$
$\Rightarrow \quad 3x - 5x = 8 - 2$
$-2x = 6$
$x = \frac{6}{-2} = -3$

3. (i) Solve $2(x-3)+5(x-1)=3$

$$\Rightarrow 2x-6+5x-5=3$$
$$\Rightarrow 7x=3+11=14$$
$$x=2$$

(ii) Solve $2(4x-1)-3(x-2)=14$

$$\Rightarrow 8x-2-3x+6=14$$
$$5x=14-4=10$$
$$x=2$$

(iii) Solve $3(x-1)-4(x-2)=6(2x+3)$

$$\Rightarrow 3x-3-4x+8=12x+18$$
$$-x-12x=18-5$$
$$-13x=13$$
$$x=-1$$

(iv) Solve $3(x+5)+2(x+1)-3x=22$

$$\cancel{3x}+15+2x+2-\cancel{3x}=22$$
$$2x=22-17$$
$$2x=5$$
$$x=\frac{5}{2}=2.5$$

4. (i) $\dfrac{2x+1}{5}=1$

$$\Rightarrow 2x+1=5$$
$$2x=4$$
$$x=2$$

(ii) $\dfrac{3x-1}{4}=8$

$$\Rightarrow 3x-1=32$$
$$3x=33$$
$$x=11$$

(iii) $\dfrac{x-3}{4}=\dfrac{x-2}{5}$

$$\Rightarrow 5(x-3)=4(x-2)$$
$$5x-15=4x-8$$
$$5x-4x=15-8$$
$$x=7$$

5. (i) $\dfrac{2a}{3}-\dfrac{a}{4}=\dfrac{5}{6}$

$\Rightarrow$ multiplying each term by 12 : $4(2a)-3(a)=2(5)$

$$\Rightarrow 8a-3a=10$$
$$5a=10$$
$$a=2$$

(ii) $\dfrac{b+2}{4}-\dfrac{b-3}{3}=\dfrac{1}{2}$

$\Rightarrow$ multiplying each term by 12 : $3(b+2)-4(b-3)=6$

$$\Rightarrow 3b+6-4b+12=6$$
$$-b=6-18$$
$$-b=-12$$
$$b=12$$

(iii) $\frac{3c-1}{6} - \frac{c-3}{4} = \frac{4}{3}$

$\Rightarrow$ multiplying each term by 12 : $2(3c-1) - 3(c-3) = 4(4)$

$$\Rightarrow 6c - 2 - 3c + 9 = 16$$
$$3c = 16 - 7$$
$$3c = 9$$
$$c = 3$$

6. (i) $\frac{x-2}{5} + \frac{2x-3}{10} = \frac{1}{2}$

multiplying each term by 10 we get : $2(x-2) + (2x-3) = 5(1)$

$$\Rightarrow 2x - 4 + 2x - 3 = 5$$
$$4x = 5 + 7$$
$$4x = 12$$
$$x = 3$$

(ii) $\frac{3y-12}{5} + 3 = \frac{3(y-5)}{2}$

multiplying each term by 10 we get : $2(3y-12) + 10(3) = 5.3(y-5)$

$$\Rightarrow 6y - 24 + 30 = 15y - 75$$
$$\therefore \quad 6y - 15y = 24 - 30 - 75$$
$$-9y = -81$$
$$y = \frac{-81}{-9} = +9$$

(iii) $\frac{3p-2}{6} - \frac{3p+1}{4} = \frac{2}{3}$

multiplying each term by 12 we get : $2(3p-2) - 3(3p+1) = 4(2)$

$$\Rightarrow 6p - 4 - 9p - 3 = 8$$
$$-3p = 8 + 7$$
$$-3p = 15$$
$$p = -5$$

(iv) $\frac{3r-2}{5} - \frac{2r-3}{4} = \frac{1}{2}$

multiplying each term by 20 we get : $4(3r-2) - 5(2r-3) = 10(1)$

$$\Rightarrow 12r - 8 - 10r + 15 = 10$$
$$2r = 10 - 7$$
$$2r = 3$$
$$r = \frac{3}{2} = 1.5$$

7. (i) $\frac{3}{4}(2x-1) - \frac{2}{3}(4-x) = 2$

multiplying each term by 12 we get : $3.3(2x-1) - 4.2(4-x) = 12.2$

$$\Rightarrow 18x - 9 - 32 + 8x = 24$$
$$26x = 24 + 41$$
$$26x = 65$$
$$x = \frac{65}{26} = 2.5$$

(ii) $\frac{2}{3}(x-1) - \frac{1}{5}(x-3) = x + 1$

multiplying each term by 15 we get : $5.2(x-1) - 3.1(x-3) = 15x + 15$

$$\Rightarrow 10x - 10 - 3x + 9 = 15x + 15$$
$$7x - 15x = 15 + 1$$
$$-8x = 16$$
$$x = -2.$$

Exercise 1.10

1. (i) $3x - 2y = 8$, $x + y = 6$ $\Rightarrow$

$$\begin{aligned} 3x - 2y &= 8 \\ 2x + 2y &= 12 \\ \hline \text{(adding)} \quad 5x &= 20 \\ x &= 4 \end{aligned}$$

since $x + y = 6 \Rightarrow 4 + y = 6$

$\therefore y = 2.$

$\therefore$ solution $(x, y) = (4, 2)$

(ii) $3x - y = 1$, $x - 2y = -8$ $\Rightarrow$

$$\begin{aligned} 6x - 2y &= 2 \\ x - 2y &= -8 \\ \hline \text{(subtracting)} \quad 5x &= 10 \\ x &= 2 \end{aligned}$$

since $x - 2y = -8 \Rightarrow 2 - 2y = -8$

$-2y = -10$

$y = 5$

$\therefore$ solution $(x, y) = (2, 5)$

(iii) $2x - 5y = 1$, $4x - 3y - 9 = 0$ $\Rightarrow$

$$\begin{aligned} 4x - 10y &= 2 \\ 4x - 3y &= 9 \\ \hline \text{(subtracting)} \quad -7y &= -7 \\ y &= 1 \end{aligned}$$

since $2x - 5y = 1 \Rightarrow 2x - 5(1) = 1$

$\Rightarrow \quad 2x = 6$

$x = 3$

$\therefore$ solution $(x, y) = (3, 1)$

2. (i) $4x - 5y = 22$, $7x + 3y - 15 = 0$ $\Rightarrow$

$$\begin{aligned} 12x - 15y &= 66 \\ 35x + 15y &= 75 \\ \hline \text{(adding)} \quad 47x &= 141 \\ x &= \frac{141}{47} = 3 \end{aligned}$$

since $4x - 5y = 22 \Rightarrow 4(3) - 5y = 22$

$-5y = 22 - 12$

$-5y = 10$

$y = -2$

$\therefore$ solution $(x, y) = (3, -2)$

(ii) $\frac{x}{2} - \frac{y}{6} = \frac{1}{6}$, $x - 2y = -8$ $\Rightarrow$ $3x - y = 1$, $x - 2y = -8$

$\Rightarrow$

$$\begin{aligned} 6x - 2y &= 2 \\ x - 2y &= -8 \\ \hline \text{(subtracting)} \quad 5x &= 10 \\ x &= 2 \end{aligned}$$

since $3x - y = 1 \Rightarrow 3(2) - y = 1$

$6 - y = 1$

$-y = -5$

$y = 5$

$\therefore$ solution $(x, y) = (2, 5)$

(iii) $\dfrac{4x-2}{5} = \dfrac{8y}{10}$ $\Rightarrow$ $8x - 4 = 8y$
$18x - 20y = 4$ $\quad$ $18x - 20y = 4$

$\Rightarrow$ $8x - 8y = 4$
$18x - 20y = 4$

$\Rightarrow$ $40x - 40y = 20$
$\underline{36x - 40y = 8}$

(subtracting) $4x = 12$
$x = 3$

since $18x - 20y = 4$

$\Rightarrow 18(3) - 20y = 4$

$-20y = 4 - 54 = -50$

$y = 2\frac{1}{2}$

$\therefore$ solution $(x, y) = (3, 2\frac{1}{2})$

3. $\dfrac{2x-5}{3} + \dfrac{y}{5} = 6 \Rightarrow 5(2x-5) + 3y = 15.6$

$\Rightarrow 10x - 25 + 3y = 90$

$10x + 3y = 115$

$\dfrac{3x}{10} + 2 = \dfrac{3y-5}{2} \Rightarrow 3x + 10.2 = 5(3y - 5)$

$\Rightarrow 3x + 20 = 15y - 25$

$\Rightarrow 3x - 15y = -45$

$\therefore$ $10x + 3y = 115$ $\Rightarrow$ $50x + 15y = 575$
$3x - 15y = -45$ $\quad$ $\underline{3x - 15y = -45}$

(adding) $53x = 530$
$x = 10$

since $10x + 3y = 115$
$10(10) + 3y = 115$

$3y = 15$
$y = 5$

$\therefore$ solution $(x, y) = (10, 5)$

4. $y = 3x - 23$ $\Rightarrow$ $y = 3x - 23$

$y = \frac{x}{2} + 2$ $\Rightarrow$ $2y = x + 4$

$\Rightarrow$ $2y = 6x - 46$
$\underline{2y = x + 4}$

(subtracting) $0 = 5x - 50$
$\Rightarrow 5x = 50$
$x = 10$

since $y = 3x - 23$
$\Rightarrow$ $y = 3(10) - 23 = 7$
$\therefore$ solution $(x, y) = (10, 7)$

5. (i)

$$\begin{array}{lll}
A: 2x + y + z = 8 & & 3A: \; 6x + \cancel{3y} + 3z = 24 \\
B: 5x - 3y + 2z = 3 & & B: \; \underline{5x - \cancel{3y} + 2z = 3} \\
C: 7x + y + 3z = 20 & D: \text{(adding)} & 11x \qquad + 5z = 27
\end{array}$$

$$\begin{array}{rl}
\text{also} & B: 5x - \cancel{3y} + 2z = 3 \\
& 3C: \underline{21x + \cancel{3y} + 9z = 60} \\
E: \text{(adding)} & 26x \qquad + 11z = 63
\end{array}$$

$\Rightarrow 11D: 121x + \cancel{55z} = 297$

$5E: \underline{130x + \cancel{55z} = 315}$

(subtracting) $-9x = -18$

$x = 2$

since $11x + 5z = 27$

$11(2) + 5z = 27$

$5z = 27 - 22 = 5$

$z = 1$

since $2x + y + z = 8$

$2(2) + y + 1 = 8$

$y = 3$

$\therefore$ solution $(x, y, z) = (2, 3, 1)$

(ii)

$$\begin{array}{lll}
A: 2x - y - z = 6 & & 2A: 4x - \cancel{2y} - 2z = 12 \\
B: 3x + 2y + 3z = 3 & \Rightarrow & B: \underline{3x + \cancel{2y} + 3z = 3} \\
C: 4x + y - 2z = 3 & & D: \text{(adding) } 7x \quad + z = 15
\end{array}$$

$$\begin{array}{rl}
\text{also} & B: 3x + \cancel{2y} + 3z = 3 \\
& 2C: \underline{8x + \cancel{2y} - 4z = 6} \\
E: \text{(subtracting)} & -5x \qquad + 7z = -3
\end{array}$$

since D: $7x + z = 15 \Rightarrow 7D = 49x + \cancel{7z} = 105$

also $E = \underline{-5x + \cancel{7z} = -3}$

subtracting: $54x = 108$

$x = 2.$

since D: $7x + z = 15$

$\Rightarrow \quad 7(2) + z = 15$

$z = 1$

also, since A: $2x - y - z = 6$

$\Rightarrow \quad 2(2) - y - 1 = 6$

$-y = 3$

$y = -3.$

$\therefore$ solution $(x, y, z) = (2, -3, 1)$

(iii)

$$\begin{array}{lll}
A: 2x + y - z = 9 & & A: 2x + y - \cancel{z} = 9 \\
B: x + 2y + z = 6 & \Rightarrow & B: \underline{x + 2y + \cancel{z} = 6} \\
C: 3x - y + 2z = 17 & & D \text{ (adding)}: 3x + 3y = 15
\end{array}$$

$$\begin{array}{rl}
& \text{also } 2B: 2x + 4y + \cancel{2z} = 12 \\
& C: \underline{3x - y + \cancel{2z} = 17} \\
E \text{ (subtracting)}: & -x + 5y = -5
\end{array}$$

$$\begin{aligned}
\text{since } D:&\quad 3x + 3y = 15\\
3E:&\quad \underline{-3x + 15y = -15}\\
\text{adding:}&\quad 18y = 0\\
&\quad y = 0.\\
\text{since } E:&\quad -x + 5y = -5\\
&\quad -x + 5(0) = -5\\
&\quad x = 5.
\end{aligned}$$

$$\begin{aligned}
\text{also, } A:\ 2x + y - z &= 9\\
2(5) + 0 - z &= 9\\
-z &= 9 - 10 = -1\\
+z &= 1
\end{aligned}$$

$\therefore$ solution $(x, y, z) = (5, 0, 1)$

6. (i) $A: 2a + b + c = 8 \quad \Rightarrow \quad 3A: 6a + \cancel{3b} + 3c = 24$

$B: 5a - 3b + 2c = 3 \qquad B: \underline{5a - \cancel{3b} + 2c = -3}$

$C: 7a - 3b + 3c = 1 \qquad D\text{: adding: } 11a + 5c = 21$

also $\quad B: 5a - \cancel{3b} + 2c = -3$

$C: \underline{7a - \cancel{3b} + 3c = 1}$

E: subtracting: $-2a - c = -4.$

$$\begin{aligned}
\text{since } D:&\quad 11a + \cancel{5c} = 21\\
\text{and } 5E:&\quad \underline{-10a - \cancel{5c} = -20}\\
\text{adding:}&\quad a = 1\\
\text{since}&\quad D: 11a + 5c = 21\\
\Rightarrow&\quad 11(1) + 5c = 21\\
&\quad 5c = 10\\
&\quad c = 2.
\end{aligned}$$

$$\begin{aligned}
\text{also, } A:\ 2a + b + c &= 8\\
2(1) + b + 2 &= 8\\
b &= 4
\end{aligned}$$

$\therefore$ solution $(a, b, c) = (1, 4, 2)$

(ii) $A: x + y + 2z = 3 \quad \Rightarrow \quad 2A: 2x + \cancel{2y} + 4z = 6$

$B: 4x + 2y + z = 13 \qquad B: \underline{4x + \cancel{2y} + z = 13}$

$C: 2x + y - 2z = 9 \qquad D$: (subtracting): $-2x + 3z = -7$

also $\quad B: \cancel{4x} + \cancel{2y} + z = 13$

$2C: \underline{\cancel{4x} + \cancel{2y} - 4z = 18}$

E: (subtracting): $5z = -5$

$\Rightarrow \quad z = -1$

$$\begin{aligned}
\text{since}\quad D:\ -2x + 3z &= -7\\
\Rightarrow\quad -2x + 3(-1) &= -7\\
-2x - 3 &= -7\\
-2x &= -7 + 3\\
-2x &= -4\\
x &= 2.
\end{aligned}$$

$$\text{also, } A: \quad x + y + 2z = 3$$
$$2 + y + 2(-1) = 3$$
$$y = 3$$
$\therefore$ solution $(x, y, z) = (2, 3, -1)$.

(iii) $A: x + y + z = 2$ $\qquad$ $A: x + y + z = 2$

$B: 2x + 3y + z = 7$

$C: \frac{x}{2} + \frac{y}{2} + \frac{z}{3} = \frac{2}{3} \quad \Rightarrow \quad C: \underline{3x - y + 2z = 4}$

D: (adding) $\quad 4x \quad + 3z = 6$

also $B: 2x + 3y + z = 7$

$\underline{3C: 9x - 3y + 6z = 12}$

E: (adding) $11x \quad + 7z = 19$

$\therefore$ $11D: 44x + 33z = 66$

$4E: \underline{44x + 28z = 76}$

subtracting: $\quad 5z = -10$

$$z = -2.$$

since $D: 4x + 3z = 6$
$$4x + 3(-2) = 6$$
$$4x = 12$$
$$x = 3$$
also, $A: x + y + z = 2$
$$\Rightarrow \quad 3 + y - 2 = 2$$
$$y = 2 - 1$$
$$y = 1$$
$\therefore$ solution $(x, y, z) = (3, 1, -2)$

7. $A: 6x + 4y - 2z - 5 = 0 \quad \Rightarrow \quad A: 6x + \cancel{4y} - 2z = 5$

$B: 3x - 2y + 4z + 10 = 0 \qquad \underline{2B: 6x - \cancel{4y} + 8z = -20}$

$C: 5x - 2y + 6z + 13 = 0 \qquad D$: (adding): $12x \quad + 6z = -15$

also $B: 3x - \cancel{2y} + 4z = -10$

$C: \underline{5x - \cancel{2y} + 6z = -13}$

E (subtracting): $-2x \quad -2z = 3$

also, $D: \quad 12x + \cancel{6z} = -15$

$3E: \quad \underline{-6x - \cancel{6z} = 9}$

adding: $\quad 6x \quad = -6$
$$x = -1$$
since $D: \quad 12x + 6z = -15$
$$12(-1) + 6z = -15$$
$$6z = -3$$
$$z = -\frac{1}{2}$$
also, $A: \quad 6x + 4y - 2z = 5$
$$\Rightarrow 6(-1) + 4y - 2\left(-\frac{1}{2}\right) = 5$$
$$-6 + 4y + 1 = 5$$

$$4y = 10 \qquad\qquad y = 2\tfrac{1}{2}$$

$\therefore$ solution $(x, y, z) = (-1, 2\frac{1}{2}, -\frac{1}{2})$.

8. Curve $f(x) = ax^2 + bx + c$

$(1, 2)$ on curve $\Rightarrow$ when $x = 1, f(x) = 2$

$$\Rightarrow 2 = a(1)^2 + b(1) + c$$
$$\Rightarrow 2 = a + b + c \qquad : A$$

$(2, 4)$ on curve $\Rightarrow$ when $x = 2, f(x) = 4$

$$\Rightarrow 4 = a(2)^2 + b(2) + c$$
$$\Rightarrow 4 = 4a + 2b + c \qquad : B$$

$(3, 8)$ on curve $\Rightarrow$ when $x = 3, f(x) = 8$

$$\Rightarrow 8 = a(3)^2 + b(3) + c$$
$$\Rightarrow 8 = 9a + 3b + c \qquad : C$$

$$\begin{aligned} \text{since } A:&\quad a + b + \cancel{c} = 2 \\ \text{and } B:&\quad \underline{4a + 2b + \cancel{c} = 4} \\ D\text{(subtracting)}:&\quad -3a - b \quad = -2 \end{aligned}$$

$$\begin{aligned} \text{since } B:&\ 4a + 2b + \cancel{c} = 4 \\ \text{and } C:&\ \underline{\ a + 3b + \cancel{c} = 8} \\ E\text{(subtracting)}:&\ -5a - b \quad = -4 \end{aligned}$$

$$\begin{aligned} \text{also} \quad D:&\quad -3a - \cancel{b} = -2 \\ E:&\quad \underline{-5a - \cancel{b} = -4} \\ \text{subtracting:}&\quad 2a \quad = 2 \\ \Rightarrow&\quad a = 1 \\ \text{since} \quad D:&\quad -3a - b = -2 \\ \Rightarrow&\quad -3(1) - b = -2 \\ &\quad -b = +1 \\ &\quad b = -1 \\ \text{also} \quad A:&\ a + b + c = 2 \\ \Rightarrow&\quad 1 - 1 + c = 2 \\ &\quad c = 2 \end{aligned}$$

$\therefore$ solution $(a, b, c) = (1, -1, 2)$

9. Point 1, $(1, 1)$
point 2, $(0, -6)$
point 3, $(-2, -8)$
Curve $= f(x) = ax^2 + bx + c$
$(1,1)$ on curve $\Rightarrow$ when $x = 1, f(x) = 1$

$$\Rightarrow 1 = a(1)^2 + b(1) + c$$
$$\Rightarrow 1 = a + b + c \qquad : A$$

$(0, -6)$ on curve $\Rightarrow$ when $x = 0, f(x) = -6$

$$\Rightarrow -6 = a(0) + b(0) + c$$
$$\Rightarrow -6 = c \qquad : B$$

$(-2, -8)$ on curve $\Rightarrow$ when $x = -2, f(x) = -8$

$$\Rightarrow -8 = a(-2)^2 + b(-2) + c$$

$$-8 = 4a - 2b + c \qquad : C$$

also A: $\quad a + b - 6 = 1$

C: $\quad 4a - \cancel{2b} - 6 = -8$

$\Rightarrow 2A$: $\quad 2a + \cancel{2b} - 12 = 2$

adding: $\quad 6a - 18 = -6$

$$6a = 12$$

$$a = 2$$

since A: $\quad a + b - 6 = 1$

$\Rightarrow \quad 2 + b - 6 = 1$

$$b = 1 + 4 = 5$$

$(a, b, c) = (2, 5, -6)$

solution: $f(x) = 2x^2 + 5x - 6$

10. Let x = number of people paying €20

Let y = number of people paying €30

$\therefore \quad x + y = 44\,000 : A$

also, $20x + 30y = 1\,200\,000 : B$

$\therefore \quad 20A : \cancel{20x} + 20y = 880\,000$

$B : \cancel{20x} + 30y = 1\,200\,000$

subtracting: $\quad -10y = -320\,000$

$$y = 32\,000$$

$\therefore$ 32 000 paid the higher price.

11. Let x be Lydia's age now.

$\therefore$ five years ago, Lydia was $(x - 5)$ years old.

Let Callum be y years old now.

$\therefore$ three years from now, Callum will be $(y + 3)$ years old

$\Rightarrow (y + 3) = 2(x - 5) \qquad : A$

also, $\dfrac{x + y}{2} = 16$

$\Rightarrow x + y = 32 \qquad : B$

From A: $\quad y + 3 = 2x - 10$

$$13 = 2x - y$$

$\therefore \quad A$: $\quad 2x - y = 13$

B: $\quad x + y = 32$

adding: $\quad 3x = 45$

$$x = 15$$

since $x + y = 32$

$\Rightarrow 15 + y = 32$

$$y = 17$$

Lydia is 15 years old, Callum is 17 years old.

12. Equation of line: $y = ax + b$.

(6,7) on line $\Rightarrow$ when $x = 6, y = 7$

$\Rightarrow 7 = a(6) + b$

$\Rightarrow 6a + b = 7 \qquad : A$

also, $(-2, 3)$ on line $\Rightarrow$ when $x = -2, y = 3$

$\Rightarrow 3 = a(-2) + b$

$$\Rightarrow -2a + b = 3 \qquad : B$$

$$\begin{array}{lll} \text{since} & A: & 6a + \cancel{b} = 7 \\ \text{and} & B: & \underline{-2a + \cancel{b} = 3} \\ \text{subtracting:} & & 8a \quad = 4 \\ \Rightarrow & & a \quad = \frac{4}{8} = \frac{1}{2}. \end{array}$$

since A: $6a + b = 7$

$$\Rightarrow 6\left(\frac{1}{2}\right) + b = 7$$

$$b = 4$$

$\therefore$ line $y = \frac{1}{2}x + 4$.

Verify (4, 6) is on line $\Rightarrow 6 = \frac{1}{2}(4) + 4$

$= 6$, which is true.

13. $\frac{N_1}{4} - N_2 = 0 \quad \Rightarrow \quad N_1 - 4N_2 = 0 \qquad : A$

$N_1 + \frac{1}{2}N_2 - 99 = 0 \quad \Rightarrow \quad 2N_1 + N_2 = 198 \qquad : B$

$$\begin{array}{lll} \text{since} & A: & N_1 - 4N_2 = 0 \\ \text{and} & 4B: & \underline{8N_1 + 4N_2 = 792} \\ \text{adding:} & & 9N_1 \quad = 792 \\ & & N_1 \quad = 88 \end{array}$$

$$\begin{array}{ll} \text{also, } A: & N_1 - 4N_2 = 0 \\ \Rightarrow & 88 - 4N_2 = 0 \\ & -4N_2 = -88 \\ & N_2 = 22 \end{array}$$

$(N_1, N_2) = (88, 22)$

14. $\frac{a}{x-2} + \frac{b}{x+2} = \frac{4}{(x-2)(x+2)}$

$\Rightarrow \frac{a(x+2) + b(x-2)}{(x-2)(x+2)} = \frac{4}{(x-2)(x+2)}$

$\Rightarrow ax + 2a + bx - 2b = 4$

$$(a+b)x + 2a - 2b = 4 + 0.x$$

$$\begin{array}{lll} \Rightarrow & a + b = 0 & : A \\ \text{and} & 2a - 2b = 4 & : B \\ \text{also} & \underline{2a + 2b = 0} & : 2A \\ \text{adding:} & 4a \quad = 4 & \\ & a \quad = 1 & \\ \text{since} & a + b = 0 & \\ \Rightarrow & 1 + b = 0 & \\ & b = -1 & \end{array}$$

$\therefore (a, b) = (1, -1)$

$\therefore \frac{1}{x-2} + \frac{-1}{x+2} = \frac{4}{(x-2)(x+2)}$

$\frac{x + 2 - (x-2)}{(x-2)(x+2)} = \frac{4}{(x-2)(x+2)}$

$= \frac{\cancel{x} + 2 - \cancel{x} + 2}{(x-2)(x+2)}$

$\frac{4}{(x-2)(x+2)} = \frac{4}{(x-2)(x+2)} \qquad$ qed

15. $\dfrac{c}{z-3}+\dfrac{d}{z+2}=\dfrac{4}{(z-3)\,(z+2)}$

$\Rightarrow \dfrac{c(z+2)+d(z-3)}{(z-3)(z+2)}=\dfrac{4}{(z-3)(z+2)}$

$\Rightarrow cz+2c+dz-3d=4$

$(c+d)z+2c-3d=0.z+4$

$\therefore \quad c+d=0 \quad :A$

and $\quad 2c-3d=4 \quad :B$

$\Rightarrow \quad \underline{2c+2d=0} \quad :2A$

subtracting: $\quad -5d=4$

$d=\dfrac{-4}{5}$

since $\quad c+d=0$

$\Rightarrow \quad c-\dfrac{4}{5}=0$

$\Rightarrow \quad c=\dfrac{4}{5}$

$\therefore (c,d)=\left(\dfrac{4}{5},-\dfrac{4}{5}\right)$

$\therefore \dfrac{4}{5(z-3)}+\dfrac{-4}{5(z+2)}=\dfrac{4}{(z-3)(z+2)}$

$\Rightarrow \dfrac{4(z+2)-4(z-3)}{5(z-3)(z+2)}$

$=\dfrac{\cancel{4z}+8-\cancel{4z}+12}{5(z-3)(z+2)}$

$=\dfrac{20}{5(z-3)(z+2)}$

$=\dfrac{4}{(z-3)(z+2)}=\dfrac{4}{(z-3)(z+2)}$ qed

16. Let x = number of litres of 70% alcohol

$\Rightarrow x(70\%)+50(40\%)=(x+50)(50\%)$

$\Rightarrow \quad 0.7x+0.4(50)=0.5x+0.5(50)$

$0.7x+20=0.5x+25$

$0.7x-0.5x=25-20$

$0.2x=5$

$x=25$ litres.

17. x = bigger number

y = smaller number

$A: \quad x+\ \ y=26$

$B: \quad 4x-5y=5$

$\Rightarrow 5A: \quad 5x+5y=130$

$B: \quad \underline{4x-5y=\ \ \ 5}$

adding: $\quad 9x \quad =135$

$x \quad =15$

since $\quad x+y=26$

$\Rightarrow \quad 15+y=26$

$y=11$

$\therefore (x, y) = (15, 11)$

18. $v = u + at$ $\quad v = \text{speed}$

$t = \text{time.}$

at $t = 7, v = 2 \quad \Rightarrow \quad 2 = u + 7a \quad :A$

at $t = 13, v = 5 \quad \Rightarrow \quad 5 = u + 13a \quad :B$

since $\quad A: u + 7a = 2$

$\quad B: \underline{u + 13a = 5}$

subtracting: $\quad -6a = -3$

$6a = 3$

$a = \frac{3}{6} = \frac{1}{2}$

since $u + 7a = 2$

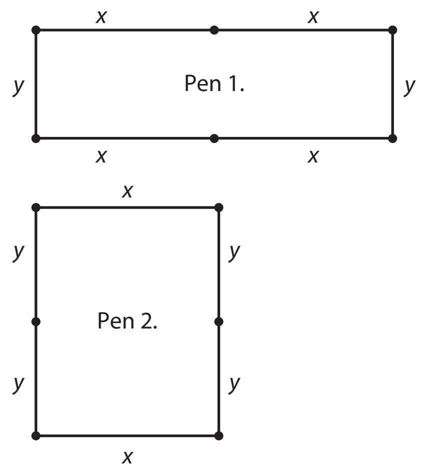
$\Rightarrow \; u + 7\left(\frac{1}{2}\right) = 2$

$u = 2 - 3\frac{1}{2}$

$u = -1\frac{1}{2}$

19. $\therefore \; 4x + 2y = 60 \quad :A$

also, $2x + 4y = 42 \quad :B$



$\therefore \quad 4x + 2y = 60 \quad :A$

also $\quad \underline{4x + 8y = 84} \quad :2B$

subtracting: $\quad -6y = -24$

$y = 4$

since $4x + 2y = 60$

$\Rightarrow 4x + 2(4) = 60$

$4x = 52$

$x = 13$

$\therefore$ Original pen dimensions: 26×4

new pen dimensions: 13×8

Pen 1 requires $4x$ and $2y$ lengths

Pen 2 requires $2x$ and $4y$ lengths

$\Rightarrow$ they both have $2x$ and $2y$ length in common

$\therefore$ in pen 2, $2x$ lengths are swapped for $2y$ lengths

if $y < x$, less fencing is needed in pen 2.

Note: Area of pen 1 $= (2x) \times (y) = 2xy$

Area of pen 2 $= (x) \times (2y) = 2xy$

$\Rightarrow$ the areas are the same.

20. $y = ax^2 + bx + c$

the point $(0, 1) \in$ curve $\Rightarrow$ when $x = 0, y = 1$

$\Rightarrow 1 = a(0)^2 + b(0) + c$

$\Rightarrow 1 = c$

the point $(2, 9) \in$ curve $\Rightarrow$ when $x = 2, y = 9$

$\Rightarrow 9 = a(2)^2 + b(2) + c$

$\Rightarrow 9 = 4a + 2b + c$

$\Rightarrow 9 = 4a + 2b + 1 \quad$ since $c = 1$

$$\Rightarrow 8 = 4a + 2b$$
$$A: \quad \Rightarrow 4 = 2a + b \quad \text{dividing each term by 2.}$$

the point $(4, 41) \in$ curve $\Rightarrow$ when $x = 4, y = 41$

$$\Rightarrow 41 = a(4)^2 + b(4) + c$$
$$\Rightarrow 41 = 16a + 4b + c$$
$$\Rightarrow 41 = 16a + 4b + 1 \quad \text{since } c = 1$$
$$\Rightarrow 40 = 16a + 4b$$
$$B: \quad \Rightarrow 10 = 4a + b \quad \text{dividing each term by 4.}$$

since $\quad A: \quad 2a + \cancel{b} = 4$

and $\quad B: \quad \underline{4a + \cancel{b} = 10}$

subtracting $\quad -2a = -6$

$$a = 3$$

since $\quad A: \quad 2a + b = 4$

$$\Rightarrow \underline{2(3) + b = 4}$$
$$b = -2$$

$\therefore (a, b, c) = (3, -2, 1)$

21. (i) $A: \quad y - z = 3$

$B: \quad x - 2y + z = -4$

$C: \quad x + 2y = 11$

$\therefore \quad A: \quad y - \cancel{z} = 3$

$B: \quad \underline{x - 2y + \cancel{z} = -4}$

D(adding): $\quad \cancel{x} - y = -1$

$C: \quad \underline{\cancel{x} + 2y = 11}$

subtracting: $\quad -3y = -12$

$$y = 4$$

since $x - y = -1$

$\Rightarrow \quad x - 4 = -1$

$$x = 3$$

also $\quad y - z = 3$

$\Rightarrow \quad 4 - z = 3$

$$-z = -1$$
$$z = 1$$

solution $(x, y, z) = (3, 4, 1)$

(ii) $A: \quad \frac{x}{3} + \frac{y}{2} - z = 7 \Rightarrow 2x + 3y - 6z = 42$

$B: \quad \frac{x}{4} - \frac{3y}{2} + \frac{z}{2} = -6 \Rightarrow x - 6y + 2z = -24$

$C: \quad \frac{x}{6} - \frac{y}{4} - \frac{z}{3} = 1 \Rightarrow 2x - 3y - 4z = 12$

$\therefore \quad A: \quad \cancel{2x} + 3y - 6z = 42$

$2B: \quad \underline{\cancel{2x} - 12y + 4z = -48}$

D(subtracting): $\quad 15y - 10z = 90$

also $\quad 2B: \quad \cancel{2x} - 12y + 4z = -48$

$C: \quad \underline{\cancel{2x} - 3y - 4z = 12}$

E(subtracting): $\quad -9y + 8z = -60$

$\therefore \quad 4D: \quad 60y - \cancel{40z} = 360$

$5E: \quad \underline{-45y + \cancel{40z} = -300}$

$$\begin{aligned} \text{adding:} \quad 15y &= 60 \\ y &= 4 \end{aligned}$$

$$\begin{aligned} \text{since} \quad D: 15 - 10z &= 90 \\ 15(4) - 10z &= 90 \\ -10z &= 30 \\ z &= -3 \end{aligned}$$

$$\begin{aligned} \text{also } A: \quad 2x + 3y - 6z &= 42 \\ 2x + 3(4) - 6(-3) &= 42 \\ 2x + 12 + 18 &= 42 \\ 2x &= 12 \\ x &= 6 \end{aligned}$$

$\therefore \quad (x, y, z) = (6, 4, -3)$

22. Curve: $x^2 + y^2 + ax + by + c = 0.$

$(1, 0) \in$ curve $\Rightarrow$ when $x = 1, y = 0$

$\Rightarrow 1^2 + 0^2 + a(1) + b(0) + c = 0$

$\Rightarrow 1 + a + c = 0$

$\Rightarrow a + c = -1 \qquad : A$

$(1, 2) \in$ curve $\Rightarrow$ when $x = 1, y = 2$

$\Rightarrow 1^2 + 2^2 + a(1) + b(2) + c = 0$

$\Rightarrow 1 + 4 + a + 2b + c = 0$

$\Rightarrow a + 2b + c = -5 \qquad : B$

$(2,1) \in$ curve $\Rightarrow$ when $x = 2, y = 1$

$\Rightarrow 2^2 + 1^2 + a(2) + b(1) + c = 0$

$\Rightarrow 4 + 1 + 2a + b + c = 0$

$\Rightarrow 2a + b + c = -5 \qquad : C$

$$\begin{aligned} \text{since} \quad B: \quad a + \cancel{2b} + c &= -5 \\ \text{and} \quad 2C: \quad 4a + \cancel{2b} + 2c &= -10 \end{aligned}$$

$$\begin{aligned} D\text{(subtracting)}: \quad -3a - \cancel{c} &= 5 \\ A: \quad a + \cancel{c} &= -1 \end{aligned}$$

$$\begin{aligned} \text{adding:} \quad -2a &= 4 \\ a &= -2 \end{aligned}$$

$$\begin{aligned} \text{since } A: \quad a + c &= -1 \\ \Rightarrow \quad -2 + c &= -1 \\ c &= 1 \end{aligned}$$

$$\begin{aligned} \text{also } B: \quad a + 2b + c &= -5 \\ -2 + 2b + 1 &= -5 \\ 2b &= -4 \\ b &= -2 \end{aligned}$$

$\therefore (a, b, c) = (-2, -2, 1)$

23. Let x be the number of small bags, y be the number of medium bags and z be the number of large bags. Then

$x + y + z = 15 \qquad$ **... 1**

$3x + 5y + 7z = 77 \qquad$ **... 2**

$y = x + 2$

$-x + y = 2 \qquad$ **... 3**

Then

1: $\quad x + y + z = 15$

$$\begin{array}{lrl} \textbf{3}: & -x+y & =6 \\ \hline & 2y+z & =17 \quad \textbf{... 4} \end{array}$$

Also

$$\begin{array}{lrl} \textbf{2}: & 3x+5y+7z & =77 \\ \textbf{3}\times 3: & -3x+3y & =6 \\ \hline & 8y+7z & =83 \quad \textbf{... 5} \end{array}$$

Then

$$\begin{array}{lrl} \textbf{4}\times -4: & -8y-4z & =-68 \\ \textbf{5}: & 8y+7z & =83 \\ \hline & 3z & =15 \\ & z & =5 \\ \textbf{4}: & 2y+5 & =17 \\ & 2y & =12 \\ & y & =6 \\ \textbf{3}: & 6 & =x+2 \\ & x & =4 \end{array}$$

The shop sells 4 small, 6 medium and 5 large bags of popcorn.

24. Let €x be the amount in the mutual fund, €y be the amount in government bonds and €z be the amount in her local bank. Then

$$x+y+z=10500 \quad \textbf{... 1}$$
$$x=2z$$
$$x-2z=0 \quad \textbf{... 2}$$
$$0.11x+0.07y+0.05z=825$$
$$11x+7y+5z=82500 \quad \textbf{... 3}$$

Then

$$\begin{array}{lrl} \textbf{1}\times 2: & 2x+2y+2z & =21\,000 \\ \textbf{2}: & x-2z & =0 \\ \hline & 3x+2y & =21\,000 \quad \textbf{... 4} \\ \textbf{1}\times -5: & -5x-5y-5z & =-52\,500 \\ \textbf{3}: & 11x+7y+5z & =82\,500 \\ \hline & 6x+2y & =30\,000 \quad \textbf{... 5} \end{array}$$

Then

$$\begin{array}{lrl} \textbf{4}\times -1: & -3x-2y & =-21\,000 \\ \textbf{5}: & 6x+2y & =30\,000 \\ \hline & 3x & =9000 \\ & x & =3000 \\ \textbf{2}: & x & =2z \\ & 3000 & =2z \\ & z & =1500 \\ \textbf{1}: & 3000+y+1500 & =10\,500 \\ & y & =6000 \end{array}$$

She invests €3000 in the mutual fund, €6000 in a government bond and €1500 in her local bank.

25. Let x be the number of white beads, y be the number of blue beads and z be the number of green beads. Then

$$\begin{aligned} y &= (x + z) - 4 \\ x - y + z &= 4 && \textbf{... 1} \\ z &= x + y \\ x + y - z &= 0 && \textbf{... 2} \\ y &= 2x \\ 2x - y &= 0 && \textbf{... 3} \end{aligned}$$

Then

1: $x - y + z = 4$

2: $\underline{x + y - z = 0}$

$2x = 4$

$x = 2$

3: $y = 2x$

$y = 4$

2: $z = x + y$

$z = 2 + 4$

$z = 6$

He uses 2 white, 4 blue and 6 green beads.

Revision Exercise 1 (Core)

1. (i) $\dfrac{12m^2n^3}{(6m^4n^5)^2} = \dfrac{\cancelto{1}{12}\, m^2n^3}{\cancelto{3}{36}\, m^8n^{10}} = \dfrac{1}{3m^6\, n^7}$

(ii) $\dfrac{3 + \frac{1}{x}}{\frac{5}{x} + 4} = \dfrac{\left(\frac{3x + 1}{x}\right)}{\left(\frac{5 + 4x}{x}\right)} = \left(\dfrac{3x + 1}{x}\right) \cdot \left(\dfrac{x}{5 + 4x}\right) = \dfrac{3x + 1}{5 + 4x}$

(iii) $\dfrac{2 + \frac{x}{2}}{x^2 - 16} = \dfrac{\frac{4 + x}{2}}{(x - 4)(x + 4)} = \dfrac{\cancel{(x + 4)}}{2(x - 4)\cancel{(x + 4)}} = \dfrac{1}{2x - 8}$

2. (i) $y = x + 4$
$5y + 2x = 6$ $\Rightarrow$

$$\begin{aligned} y - x &= 4 &&: \ A \\ 5y + \cancel{2x} &= 6 &&: \ B \\ \therefore \quad \underline{2y - \cancel{2x}} &\underline{= 8} &&: \ 2A \\ \text{adding} \quad 7y &= 14 \\ y &= 2 \end{aligned}$$

since $y = x + 4$

$\Rightarrow \quad 2 = x + 4$

$\Rightarrow \quad x = -2$

$\therefore (x, y) = (-2, 2)$

(ii) $3x + y = 7 \quad \Rightarrow \quad y = 7 - 3x$

$$\begin{aligned} x^2 + y^2 = 13 \quad &\Rightarrow \quad x^2 + (7 - 3x)^2 = 13 \\ &\Rightarrow \quad x^2 + [49 - 42x + 9x^2] = 13 \\ &\qquad x^2 + 49 - 42x + 9x^2 = 13 \\ &\qquad 10x^2 - 42x + 36 = 0 \\ &\qquad 5x^2 - 21x + 18 = 0 \\ &\qquad (5x - 6)(x - 3) = 0 \end{aligned}$$

$\therefore \quad x = 3 \quad \text{or} \quad x = \dfrac{6}{5}$

$\Rightarrow \quad y = 7 - 3(3) \quad \text{or} \quad y = 7 - 3\left(\dfrac{6}{5}\right)$

$y = 7 - 9 \quad \text{or} \quad y = 7 - \dfrac{18}{5}$

$y = -2 \quad \text{or} \quad y = \dfrac{17}{5}$

$\therefore (x, y) = (3, -2)$ or $\left(\dfrac{6}{5}, \dfrac{17}{5}\right)$

3.

$$
\begin{array}{r|l}
 & x^2 + 2x - 1 \\ \hline
x - 3 & x^3 - x^2 - 7x + 3 \\
 & \underline{x^3 - 3x^2} \quad \text{(subtracting)} \\
 & \quad 2x^2 - 7x + 3 \\
 & \quad \underline{2x^2 - 6x} \quad \text{(subtracting)} \\
 & \qquad -x + 3 \\
 & \qquad \underline{-x + 3} \quad \text{(subtracting)} \\
 & \qquad\quad 0
\end{array}
$$

answer: $x^2 + 2x - 1$

4.

$$
\begin{array}{r|l}
 & 3x^3 + 6x^2 + 3x + 33 \\ \hline
x - 2 & 3x^4 \qquad - 9x^2 + 27x - 66 \\
 & \underline{3x^4 - 6x^3} \quad \text{(subtracting)} \\
 & \quad 6x^3 - 9x^2 + 27x - 66 \\
 & \quad \underline{6x^3 - 12x^2} \quad \text{(subtracting)} \\
 & \qquad 3x^2 + 27x - 66 \\
 & \qquad \underline{3x^2 - 6x} \quad \text{(subtracting)} \\
 & \qquad\quad 33x - 66 \\
 & \qquad\quad \underline{33x - 66} \quad \text{(subtracting)} \\
 & \qquad\qquad 0
\end{array}
$$

answer: $3x^3 + 6x^2 + 3x + 33$

5. (i) $x^4 - 9x^2 = 0$

$\Rightarrow x^2 (x^2 - 9) = 0$

$\Rightarrow x^2 (x - 3)(x + 3) = 0$

$\therefore x = 0, 3, -3$

(ii) $(2x - 1)^3 (2 - x) = 0$

$\Rightarrow (2x - 1)^3 = 0$

$\Rightarrow (2x - 1) = 0$

$x = \dfrac{1}{2}$

or $2 - x = 0$

$x = 2$

$\therefore x = 2, \dfrac{1}{2}.$

6. $4x^2 + 20x + k$ is a perfect square

$\Rightarrow (2x + a)(2x + a) = 4x^2 + 20x + k$

$\Rightarrow 4x^2 + 4ax + a^2 = 4x^2 + 20x + k$

$\Rightarrow 4a = 20$

$a = 5$

$\therefore a^2 = 5^2 = 25 = k$

$\therefore k = 25(5^2)$

7. (i) $(2x + 3)^5 = \binom{5}{0}(2x)^5 + \binom{5}{1}(2x)^4(3) + \binom{5}{2}(2x)^3(3)^2 + \binom{5}{3}(2x)^2(3)^3 + \binom{5}{4}(2x)(3)^4 + \binom{5}{5}(3)^5$

$= (1)(32x^5) + (5)(16x^4)(3) + (10)(8x^3)(9) + (10)(4x^2)(27) + (5)(2x)(81) + (1)(243)$

$= 32x^5 + 240x^4 + 720x^3 + 1080x^2 + 810x + 243$

(ii) $(x - 2)^6 = \binom{6}{0}x^6 + \binom{6}{1}x^5(-2) + \binom{6}{2}x^4(-2)^2 + \binom{6}{3}x^3(-2)^3 + \binom{6}{4}x^2(-2)^4 + \binom{6}{5}x(-2)^5 + \binom{6}{6}(-2)^6$

$= (1)x^6 + (6)x^5(-2) + (15)x^4(4) + (20)x^3(-8) + (15)x^2(16) + (6)(x(-32) + (1)(64)$

$= x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64$

8. $x^3 - 27 = x^3 - 3^3 = (x - 3)(x^2 + 3x + 9)$

9. $p(x-q)^2 + r = 2x^2 - 12x + 5$ for all values of x

$\Rightarrow p(x^2 - 2xq + q^2) + r =$

$\Rightarrow px^2 - 2pqx + pq^2 + r = 2x^2 - 12x + 5.$

$\therefore p = 2, -2pq = -12$ and $pq^2 + r = 5.$

$\Rightarrow \quad -2(2)q = -12$

$\Rightarrow \quad q = 3$ and $2(3)^2 + r = 5$

$r = 5 - 18$

$r = -13$

$(p, q, r) = (2, 3, -13)$

10. $A: \quad 3x + 5y - z = -3$

$B: \quad 2x + y - 3z = -9$

$C: \quad x + 3y + 2z = 7$

$$\begin{array}{rl} \Rightarrow 2A: & 6x + 10y - \cancel{2z} = -6 \\ C: & \underline{x + 3y + \cancel{2z} = 7} \\ D(\text{adding}): & 7x + 13y = 1 \end{array}$$

$$\begin{array}{rl} \text{also} \quad 3A: & 9x + 15y - \cancel{3z} = -9 \\ B: & \underline{2x + y - \cancel{3z} = -9} \\ E(\text{subtracting}): & 7x + 14y = 0 \end{array}$$

$$\begin{array}{rl} \therefore \quad D: & 7x + 13y = 1 \\ E: & \underline{7x + 14y = 0} \\ (\text{subtracting}): & -y = 1 \\ & y = -1 \end{array}$$

since $7x + 14y = 0$

$7x + 14(-1) = 0$

$7x = 14$

$x = 2.$

also, $C: \quad x + 3y + 2z = 7$

$2 + 3(-1) + 2z = 7$

$2z = 8$

$z = 4.$

$\therefore \quad (x, y, z) = (2, -1, 4)$

11. $(b+1)^3 - (b-1)^3$

$= b^3 + 3b^2 + 3b + 1 - (b^3 - 3b^2 + 3b - 1)$

$= \cancel{b^3} + 3b^2 + \cancel{3b} + 1 - \cancel{b^3} + 3b^2 - \cancel{3b} + 1$

$= 6b^2 + 2.$

12. (i) $3, 12, 27, 48, 75, \ldots$

first difference: $9, 15, 21, 27, \ldots$

second difference: $6, 6, 6\ldots \Rightarrow$ quadratic pattern of the form $an^2 + bn + c$

$\Rightarrow \quad 2a = 6$

$a = 3$

$\therefore \quad 3n^2 + bn + c$ represents the pattern

let $\quad n = 1 \Rightarrow 3(1)^2 + b(1) + c = 3$

$b + c = 0 \quad : \quad A$

let $\quad n = 2 \Rightarrow 3(2)^2 + b(2) + c = 12$

$$2b + c = 0 \quad : \quad B$$

$$\Rightarrow \quad A: \quad b + c = 0$$

$$B: \quad \underline{2b + c = 0}$$

$$\text{(subtracting)}: \quad -b \quad = 0$$

$$\Rightarrow \quad b \quad = 0$$

since $b + c = 0$

$$\Rightarrow 0 + c = 0$$

$$\Rightarrow \quad c = 0$$

$\therefore \quad an^2 + bn + c = 3n^2$

(ii) 5, 20, 45, 80, 125, …

first difference: 15, 25, 35, 45, …

second difference: 10, 10, 10, …

$\therefore$ quadratic pattern of the form $an^2 + bn + c$

$\Rightarrow 2a = 10$

$a = 5 \quad \therefore \quad 5n^2 + bn + c$ represents the pattern

We note that for $n = 1$: $5n^2 = 5$

$n = 2$: $5n^2 = 20$

$n = 3$: $5n^2 = 45$, etc.

$\Rightarrow$ b and c must equal zero

[alternatively, set up simultaneous equations in b and c and solve]

$\therefore$ the quadratic pattern is $5n^2$

(iii) 0.5, 2, 4.5, 8, 12.5, …

first difference $=$ 1.5, 2.5, 3.5, 4.5, …

second difference $=$ 1, 1, 1

$\therefore$ quadratic pattern of the form $an^2 + bn + c$

$\Rightarrow 2a = 1$

$a = \frac{1}{2} \quad \therefore 0.5n^2 + bn + c$ represents this pattern

By comparison to part (ii), we can deduce that the quadratic pattern is $0.5n^2$.

13. 6, 12, 20, 30, 42, …

first difference: 6, 8, 10, 12, …

second difference: 2, 2, 2

$\therefore$ quadratic pattern of the form $an^2 + bn + c$

$\Rightarrow 2a = 2$

$a = 1 \quad \Rightarrow \quad n^2 + bn + c$ represents this pattern

let $n = 1 \quad \Rightarrow \quad 1^2 + b(1) + c = 6$

$$b + c = 5 \quad : A$$

let $n = 2 \quad \Rightarrow \quad 2^2 + b(2) + c = 12$

$$2b + c = 8 \quad : B$$

since $\quad A: \; b + c = 5$

and $\quad B: \; \underline{2b + c = 8}$

subtracting: $-b \quad = -3$

$$b = 3$$

also $\quad A: \; b + c = 5$

$$3 + c = 5$$
$$c = 2$$

$\therefore$ quadratic pattern is $n^2 + 3n + 2$

When $n = 100 \Rightarrow 100^2 + 3(100) + 2 = 10,302$

14. Let the width $= x$ cm.

Let the length $= y$ cm.

$\Rightarrow$ A: $3x = 2y + 3$

B: $4y = 2(x + y) + 12$

$\Rightarrow$ B: $4y = 2x + 2y + 12$

$\Rightarrow$ B: $2y = 2x + 12.$

since A: $3x - \cancel{2y} = 3$

and B: $2x - \cancel{2y} = -12$

(subtracting): $x = 15 \Rightarrow$ width $= 15$ cm.

also B: $2x - 2y = -12$

$$2(15) - 2y = -12$$
$$-2y = -42$$
$$y = 21 \Rightarrow \text{length} = 21 \text{ cm.}$$

15. A: $\dfrac{1}{u} + \dfrac{1}{v} = \dfrac{2}{r}$

B: $m = \dfrac{v - r}{r - u}$

since A: $\dfrac{1}{u} + \dfrac{1}{v} = \dfrac{2}{r}$

$$\Rightarrow \frac{v + u}{uv} = \frac{2}{r}$$
$$\Rightarrow \frac{2}{r} = \frac{v + u}{uv}$$
$$\Rightarrow \frac{r}{2} = \frac{u.v}{v + u}$$
$$\Rightarrow r = \frac{2uv}{v + u}$$

also, $m = \dfrac{v - r}{r - u} = \dfrac{v - \dfrac{2uv}{v + u}}{\dfrac{2uv}{v + u} - u}$

$$= \frac{\left(\dfrac{v(v + u) - 2uv}{v + u}\right)}{\left(\dfrac{2uv - u(v + u)}{v + u}\right)}$$

$$= \frac{v^2 + vu - 2uv}{\cancel{(v + u)}} \cdot \frac{\cancel{(v + u)}}{2uv - uv - u^2}$$

$$= \frac{v^2 - uv}{uv - u^2} = \frac{v\cancel{(v - u)}}{u\cancel{(v - u)}} = \frac{v}{u}$$

Revision Exercise 1 (Advanced)

1. 1, 3, 6, 10, …
first difference: 2, 3, 4, …
second difference: 1, 1, …

$\Rightarrow$ quadratic pattern of the form $an^2 + bn + c$

$\Rightarrow 2a = 1$

$a = \frac{1}{2} \Rightarrow \frac{1}{2}n^2 + bn + c$ represents this pattern

let $n = 1$: $\frac{1}{2}(1)^2 + b(1) + c = 1$

$$\frac{1}{2} + b + c = 1$$

$$b + c = \frac{1}{2} \quad : A$$

let $n = 2$: $\frac{1}{2}(2)^2 + b(2) + c = 3$

$$2 + 2b + c = 3$$

$$2b + c = 1 \quad : B$$

since A: $b + \cancel{c} = \frac{1}{2}$

and B: $\underline{2b + \cancel{c} = 1}$

(subtracting): $-b = -\frac{1}{2}$

$$b = \frac{1}{2}$$

also A: $b + c = \frac{1}{2}$

$\Rightarrow$ $\frac{1}{2} + c = \frac{1}{2}$

$\Rightarrow$ $c = 0$

$\therefore$ the quadratic pattern is $\frac{1}{2}n^2 + \frac{1}{2}n$

2. If x m^3 of soil is needed,

$\Rightarrow x(55\%) + 1(25\%) = (x + 1)(35\%)$

$\Rightarrow \quad 0.55x + 0.25 = 0.35x + 0.35$

$\Rightarrow \quad 0.55x - 0.35x = 0.35 - 0.25$

$$0.2x = 0.1$$

$$x = 0.5 \text{ m}^3.$$

3. (i) Let x kg of alloy 1 be added to
y kg of alloy 2.

$\Rightarrow x + y = 8.4 \quad : A$

alloy 1 contains 60% gold

$\Rightarrow$ the amount of gold in the new alloy $= x(60\%) = 0.6x$

alloy 2 contains 40% gold

$\Rightarrow$ the amount of gold in the new alloy $= y(40\%) = 0.4y$

also, the total amount of gold in the new alloy $= (x + y)(50\%)$
$= 0.5(x + y)$

$\Rightarrow 0.6x + 0.4y = 0.5(x + y) \quad : B$

since A: $x + y = 8.4$

and B: $6x + 4y = 5(x + y)$

$\Rightarrow$ B: $x - \cancel{y} = 0$ (simplifying B)

$$\begin{aligned}
\text{and} \quad A: \quad x + y &= 8.4 \\
\Rightarrow \quad 2x &= 8.4 \\
x &= 4.2 \text{ kg} \\
\text{since} \quad B: \quad x - y &= 0 \\
\Rightarrow \quad 4.2 - y &= 0 \\
\Rightarrow \quad y &= 4.2 \text{ kg also.}
\end{aligned}$$

4. $(3p - 2t)x + r - 4t^2 = 0$ for all x.

$\Rightarrow (3p - 2t)x + r - 4t^2 = 0.x + 0$

$\Rightarrow \quad 3p - 2t = 0 \qquad \Rightarrow t = \dfrac{3p}{2}$

and $\quad r - 4t^2 = 0.$

$\Rightarrow r - 4\left(\dfrac{3p}{2}\right)^2 = 0$

$\Rightarrow \quad r - \dfrac{\cancel{4}.9p^2}{\cancel{4}} = 0$

$\Rightarrow \qquad r = 9p^2.$

5. $\dfrac{x + y^2}{x^2} + \dfrac{x - 1}{x} = -1$

$\Rightarrow x + y^2 + x(x - 1) = -x^2$ [multiplying each term by x^2]

$\Rightarrow x + y^2 + x^2 - x = -x^2$

$\Rightarrow \cancel{x} + y^2 + x^2 - \cancel{x} + x^2 = 0$

$2x^2 + y^2 = 0.$

also, $\dfrac{2x^2}{y^2} + 1 = 0.$

$\dfrac{2x^2}{y^2} = -1$

$\dfrac{x^2}{y^2} = -\dfrac{1}{2}$

6. Let the students take x litres of 10% solution and y litres of 30% solution.

$\Rightarrow \quad x + y = 10$ litres

also, $x(10\%) + y(30\%) = (x + y)15\%$

$\Rightarrow \quad 10x + 30y = 15(x + y)$ [multiplying each term by 100]

$\therefore \quad x + y = 10 \quad : A$

and $10x + 30y = 15x + 15y$

$\Rightarrow \quad -5x + 15y = 0 \quad : B$

since $A: \quad x + y = 10$

and $\quad B: -\cancel{5x} + 15y = 0$

$\Rightarrow \quad 5A: \quad \cancel{5x} + 5y = 50$

adding : $\quad 20y = 50$

$y = \dfrac{50}{20} = 2\dfrac{1}{2}$ litres.

also, since $\quad x + y = 10$

$\Rightarrow \quad x + 2\dfrac{1}{2} = 10$

$x = 7\dfrac{1}{2}$ litres.

(i) $7\dfrac{1}{2}$ litres of 10% mixed with (ii) $2\dfrac{1}{2}$ litres of 30%.

7. $3x - y = 0$ **... (1)**

$x + y + z = 23\,000$ ... **(2)**

$75x + 55y + 30z = 870\,000$

$15x + 11y + 6z = 174\,000$ ... **(3)**

(2) $\times$ 6: $6x + 6y + 6z = 138\,000$

$$\begin{aligned}15x + 11y + 6z &= 174\,000\\ 6x + 6y + 6z &= 138\,000\\ \hline 9x + 5y &= 36\,000\end{aligned}$$

(1) $\times$ 3: $9x - 3y = 0$

$$\begin{aligned}8y &= 36\,000\\ y &= 4500\\ \therefore\quad 3x &= 4500\\ x &= 1500\end{aligned}$$

$\therefore\quad x + y + z = 1500 + 4500 + z = 23\,000$

$\therefore\quad z = 17\,000$

$\therefore\quad x = 1500, y = 4500, z = 17\,000$

8. Brian's time for 50 m = $50a$ seconds.
Luke's time for 50 m = $50b$ seconds.
Luke faster than Brian $\Rightarrow 50a = 50b + 1$
Brian's time for 47 m = $47a$ seconds.
Luke's time for 50 m = $50b$ seconds.
Luke faster than Brian $\Rightarrow 47a = 50b + 0.1$
Subtracting : $3a = 0.9$
$a = 0.3$

$\Rightarrow 50(0.3) = 50b + 1$
$14 = 50b$
$0.28 = b$

Luke's speed $= \dfrac{1}{b} = \dfrac{1}{0.28} = 3.57$ m/sec.

9. (i) $\left(2 - \dfrac{x}{3}\right)^9$

General term: $T_{r+1} = \dbinom{9}{r}(2)^{9-r}\left(\dfrac{-x}{3}\right)^r$

$$= \binom{9}{r}2^{9-r}\frac{(-1)^r x^r}{3^r}$$

For the 6th term, $r = 5$. Thus the 6th term is

$$\binom{9}{5}2^{9-5}\frac{(-1)^5x^5}{3^5}$$
$$= (126)(16)\frac{(-1)x^5}{243}$$
$$= -\frac{224}{27}x^5$$

(ii) $(3x + 1)^{12}$

General term: $T_{r+1} = \dbinom{12}{r}(3x)^{12-r}(1)^r$

$$= \binom{12}{r}3^{12-r}x^{12-r}$$

For the 8th term, $r = 7$. Thus the 8th term is

$$\binom{12}{7}3^5x^5$$
$$= (792)(243)x^5$$
$$= 192\,456x^5$$

10. $(3x - 1)^{11}$

General term: $T_{r+1} = \binom{11}{r}(3x)^{11-r}(-1)^r$

$= (-1)^r\binom{11}{r}3x^{11-r}x^{11-r}$

For the 6th term, $r = 5$. Thus the 6th term is

$$(-1)^5\binom{11}{5}3^6x^6$$
$$= (-1)(462)(729)x^6$$
$$= -336\,798x^6$$

11. $(2x - 3)^{14}$

General term: $T_{r+1} = \binom{14}{r}(2x)^{14-r}(-3)^r$

$= \binom{14}{r}2^{14-r}x^{14-r}(-1)^r3^r$

For the term containing x^{10},

$$14 - r = 10$$
$$r = 4$$

The coefficient of x^{10} is then

$$\binom{14}{4}2^{10}(-1)^4 3^4$$
$$= (1001)(1024)(1)(81)$$
$$= 83\,026\,944$$

Revision Exercise 1 (Extended-Response Questions)

1. (a) Let x be the number of adults

y be the number of children.

$\therefore \quad x + y = 548 \quad : A$

also, $5x + 2.5y = 2460 : B$

since $A: \quad x + y = 548$

and $B: \quad \cancel{5x} + 2.5y = 2460$

$\Rightarrow \quad 5A: \quad \cancel{5x} + 5y = 2740$

(subtracting): $\quad -2.5y = -280$

$$2.5y = 280$$

$$y = \frac{280}{2.5} = 112.$$

since $A: \quad x + y = 548$

$\Rightarrow \quad x + 112 = 548$

$\Rightarrow \quad x = 548 - 112$

$x = 436$

$\Rightarrow$ (i) Number of adult tickets $(x) = 436$

(ii) Number of children tickets $(y) = 112$.

(iii) Proportion of adult tickets sold $= \frac{436}{548} = 0.7956$.

(b) attendance (predicted) $= 13\,000$

$\Rightarrow$ adults $= 0.7956 \times 13\,000 = 10\,343$

$\Rightarrow$ children $= 13\,000 - 10\,343 = 2657$

Revenue $= 10\,343 \times (€5) + 2657 \times (€2.5) = €58\,357.50$

2. (i) x standard sofas

y deluxe sofas

Standard sofas require 2 hours of work $\Rightarrow 2x =$ time for x sofas

Deluxe sofas require 2.5 hours of work $\Rightarrow 2.5y =$ time for y sofas

$\Rightarrow$ with 48 hours of manufacturing time: $2x + 2.5y = 48 : A$

(ii) also, standard sofas need 1 hour finishing $\Rightarrow x =$ time for x sofas

deluxe sofas need 1.5 hours finishing $\Rightarrow 1.5y =$ time for y sofas

$\Rightarrow$ with 26 hours of finishing time: $x + 1.5y = 26 : B$

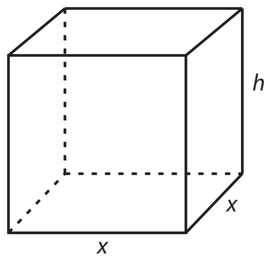
$$\begin{aligned}
\text{since } A:&\quad 2x + 2.5y = 48\\
\text{and } B:&\quad x + 1.5y = 26\\
\Rightarrow 2B:&\quad \cancel{2x} + 3y = 52\\
\text{and } A:&\quad \underline{\cancel{2x} + 2.5y = 48}\\
\text{(subtracting)}:&\quad 0.5y = 4\\
&\quad y = 8
\end{aligned}$$

also, since B: $x + 1.5y = 26$

$$\begin{aligned}
x + 1.5(8) &= 26\\
x + 12 &= 26\\
x &= 14
\end{aligned}$$

(iii) 14 standard sofas and 8 deluxe sofas.

3. Rectangular box with square base of length x cm.



Height of box $= h$ cm.

Volume of box $= l \times b \times h$

$= x \times x \times h = x^2h.$

(i) If $V = 40 \Rightarrow \quad 40 = x^2h$

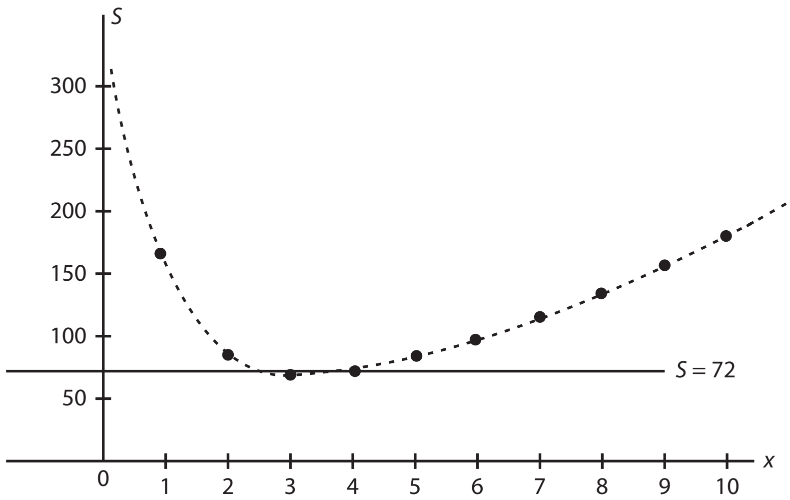
$$\Rightarrow \quad h = \frac{40}{x^2}.$$

(ii) Surface Area $= 4 \times (x \times h) + 2x^2$

$$= 4\cancel{x} \cdot \frac{40}{x^{\cancel{2}}} + 2x^2$$

$$S = \frac{160}{x} + 2x^2.$$

(iii)



(iv) $72 = \dfrac{160}{x} + 2x^2$

$72x = 160 + 2x^3$

$\Rightarrow 2x^3 - 72x + 160 = 0.$

Using trail + error at $x = 2$: $2(2)^3 - 72(2) + 160 = 32 > 0$

$$\text{at } x = 2.8 \quad : \quad 2(2.8)^3 - 72(2.8) + 160 = 2.3 > 0$$
$$\text{at } x = 2.9 \quad : \quad 2(2.9)^3 - 72(2.9) + 160 = -0.02 < 0$$

$\therefore$ at $x = 2.9$ cm (approximately), $S = 72$ cm^2

also, at $x = 4$: $\quad 2(4)^3 - 72(4) + 160 = 0$

$\therefore$ at $x = 4$ cm, $\quad S = 72$ cm^2

when $x = 4$, $\quad h = \dfrac{40}{(4)^2} = 2.5$ cm.

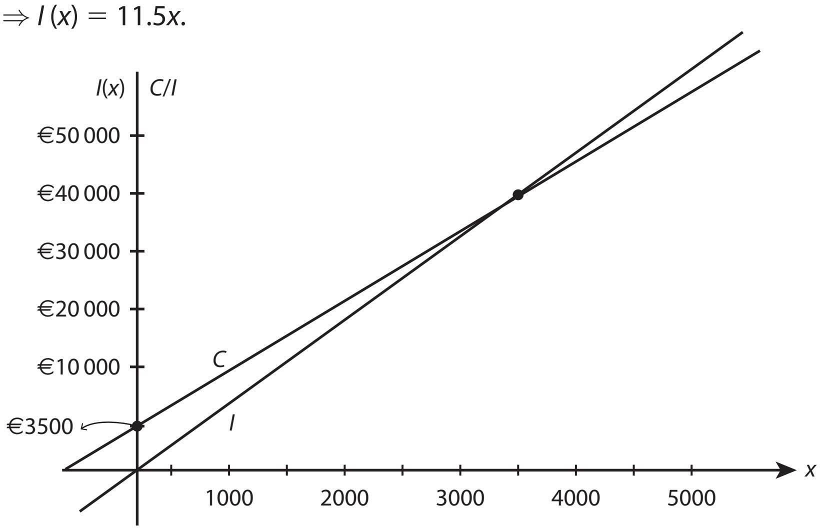
when $x = 2.9$, $h = \dfrac{40}{(2.9)^2} = 4.76$ cm.

4. Selling price of game $=$€11.50.
Production cost for each game $=$ €10.50.
Initial production costs $=$ €3500.

(i) Cost of producing x games $= C(x)$
$\Rightarrow C(x) = 10.5x + 3500$

(ii) Income $= I(x)$
$\Rightarrow I(x) = 11.5x$.

(iii)



(iv) To recoup costs : $\quad 11.5x = 10.5x + 3500$
$\Rightarrow \quad x = 3500$ need to be sold.

(v) $P = I - C \quad \Rightarrow \quad P =$ profit.

(vi) To make a profit of a 2000 :
$$\Rightarrow \quad P = I - C$$
$$\therefore \quad 2000 = 11.5x - [10.5x + 3500]$$
$$2000 = 11.5x - 10.5x - 3500$$
$$5500 = x$$
$\therefore$ 5500 need to be sold.

5. 15 days to complete quilt.
x blue squares at a rate of 4 squares a day.
y white squares at a rate of 7 squares a day.
96 squares in quilt $\Rightarrow x + y = 96 \qquad : A$

15 days to finish $\quad \Rightarrow \quad \dfrac{x}{4} + \dfrac{y}{7} = 15 : B$

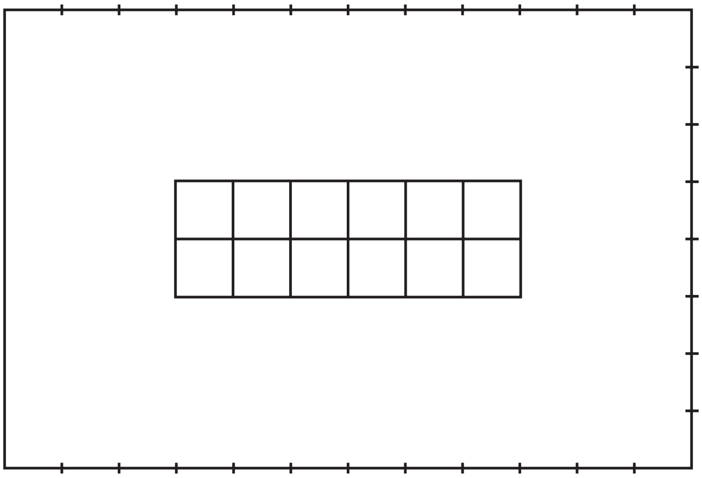
$$\begin{array}{lrl}
\therefore & A: & x + y = 96 \\
 & 28B: & 7x + \cancel{4y} = 420 \\
\text{also} & 4A: & 4x + \cancel{4y} = 384 \\
\text{(subtracting):} & & 3x \qquad = 36 \\
 & & \qquad x = 12
\end{array}$$

since A : $\quad x + y = 96$

$$\Rightarrow \quad 12 + y = 96$$
$$y = 84$$

(a) Cost $= x(0.8) + y(1.20)$
$= 12(0.8) + 84(1.20)$
$= €110.40$

(b) $l : w = 3x : 2x$
$\Rightarrow 3x.2x = 96$
$6x^2 = 96$
$x^2 = 16$
$x = 4$
$\therefore \quad l = 3 \times 4 = 12$
$w = 2 \times 4 = 8$



The 12 blue squares could form 2 rows of 6 in the centre. (There are many different possibilities.)

6. Overheads $=$ €30 000 per year
Cost of manufacture $=$ €40 per wheelbarrow

(i) $C(x) = 40x + 30\,000$

(ii) 6000 wheelbarrows per year $\Rightarrow \dfrac{30\,000}{6000} =$ €5 overhead per wheelbarrow
$\Rightarrow$ Total cost per wheelbarrow $=$ €40 $+$ €5 $=$ €45

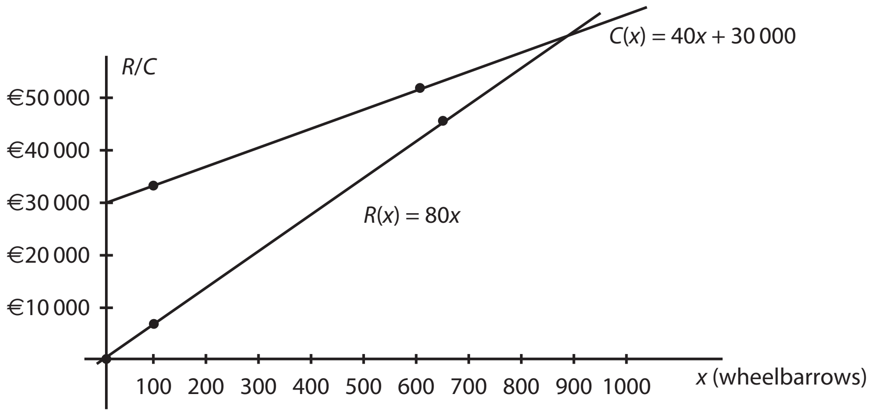
(iii) To get a cost of €46 per wheelbarrow:

$$\frac{30\,000}{x} + 40 = 46.$$

$$\Rightarrow \quad 30\,000 + 40x = 46x$$
$$\Rightarrow \quad 30\,000 = 6x$$
$$x = 5000 \text{ wheelbarrows}$$

(iv) Selling price $=$ €80 per wheelbarrow
$\Rightarrow$ Revenue $=$ €$80x$

(v)



(vi) To make a profit $80x > 40x + 30\,000$

$$\Rightarrow 40x > 30\,000$$

$$x > \frac{30\,00\cancel{0}}{4\cancel{0}} = 750$$

The minimum number of wheelbarrows $= 751$.

(vii) Profit €$P = 80x - [40x + 30\,000]$

€$P = 40x - 30\,000$

Chapter 2

Exercise 2.1

1. (a) (i) $x = 4, x = -5$

(ii) $(x - 3)(x - 4) = 0$

$\Rightarrow x = 3, x = 4$

(iii) $(x + 1)(x - 5) = 0$

$\Rightarrow x = -1, x = 5$

(b) (i) $(x + 3)(x - 5) = 0$

$\Rightarrow x = -3, x = 5$

(ii) $(2x - 3)(x + 5) = 0$

$\Rightarrow 2x = 3, x = -5$

$\Rightarrow x = \frac{3}{2}, x = -5$

(iii) $(3x + 2)(x - 5) = 0$

$\Rightarrow 3x = -2, x = 5$

$\Rightarrow x = \frac{-2}{3}, x = 5$

(c) (i) $(5x + 2)(x - 3) = 0$

$\Rightarrow 5x = -2, x = 3$

$\Rightarrow x = \frac{-2}{5}, x = 3$

(ii) $(3x + 5)(3x - 4) = 0$

$\Rightarrow 3x = -5, 3x = 4$

$\Rightarrow x = \frac{-5}{3}, x = \frac{4}{3}$

(iii) $(4x + 5)(2x - 3) = 0$

$\Rightarrow 4x = -5, 2x = 3$

$\Rightarrow x = \frac{-5}{4}, x = \frac{3}{2}$

(d) (i) $(x + 3)(x - 3) = 0$

$\Rightarrow x = -3, x = 3$

(ii) $x(3x - 10) = 0$

$\Rightarrow x = 0, 3x = 10$

$\Rightarrow x = 0, x = \frac{10}{3}$

(iii) $x(5x - 8) = 0$

$\Rightarrow x = 0, 5x = 8$

$\Rightarrow x = 0, x = \frac{8}{5}$

(e) (i) $(5 + x)(3 - 2x) = 0$

$\Rightarrow x = -5, 2x = 3$

$\Rightarrow x = -5, x = \frac{3}{2}$

(ii) $(5 + 3x)(2 - x) = 0$

$\Rightarrow 3x = -5, x = 2$

$\Rightarrow x = \frac{-5}{3}, x = 2$

(iii) $(2 + x)(1 - x) = 0$

$\Rightarrow x = -2, x = 1$

(f) (i) $(x + 5)(x + 4)(x - 4) = 0$

$\Rightarrow x = -5, x = -4, x = 4$

(ii) $(x - 3)(x + 1)(x - 1) = 0$

$\Rightarrow x = 3, x = -1, x = 1$

(g) (i) $(x + 2)(x - 2)(3x + 4) = 0$

$\Rightarrow x = -2, x = 2, 3x = -4$

$\Rightarrow x = -2, x = 2, x = \frac{-4}{3}$

(ii) $(x + 4)(x + 3)(x - 5) = 0$

$\Rightarrow x = -4, x = -3, x = 5$

2. (a) (i) $x = \frac{2 \pm \sqrt{4 - 4(1)(-2)}}{2(1)}$

$= \frac{2 \pm \sqrt{12}}{2}$

$= \frac{2 \pm \sqrt{4}\sqrt{3}}{2}$

$= 1 \pm \sqrt{3}$

$= 2.7, 20.7$

(ii) $x = \frac{-3 \pm \sqrt{9 - 4(+1)(-2)}}{2(1)}$

$= \frac{-3 \pm \sqrt{17}}{2}$

$= 0.56, -3.56$

$= 0.6, -3.6$

(iii) $x = \frac{6 \pm \sqrt{36 - 4(2)(3)}}{2(2)}$

$= \frac{6 \pm \sqrt{12}}{(4)}$

$= 2.36, 0.635$

$= 2.4, 0.6$

(b) (i) $x = \dfrac{6 \pm \sqrt{36 - 4(1)(3)}}{2(1)}$

$= \dfrac{6 \pm \sqrt{24}}{2}$

$= 5.44, 0.55$

$= 5.4, 0.6$

(ii) $x = \dfrac{8 \pm \sqrt{64 - 4(3)(1)}}{2(3)}$

$= \dfrac{8 \pm \sqrt{52}}{6}$

$= 2.535, 0.13$

$= 2.5, 0.1$

(iii) $x = \dfrac{-4 \pm \sqrt{16 - 4(2)(-5)}}{2(2)}$

$= \dfrac{-4 \pm \sqrt{56}}{4}$

$= 0.87, -2.87$

$= 0.9, -2.9$

3. (a) (i) $x = \dfrac{-4 \pm \sqrt{16 - 4(3)(-5)}}{2(3)}$

$= \dfrac{-4 \pm \sqrt{76}}{6}$

$= \dfrac{-2 \pm \sqrt{19}}{3}$

(ii) $x = \dfrac{12 \pm \sqrt{144 - 4(2)(-5)}}{2(2)}$

$= \dfrac{12 \pm \sqrt{184}}{4}$

$= \dfrac{6 \pm \sqrt{46}}{2}$

(iii) $4x^2 - 12x + 9 = 8$

$\Rightarrow 4x^2 - 12x + 1 = 0$

$\Rightarrow x = \dfrac{12 \pm \sqrt{144 - 4(4)(1)}}{2(4)}$

$= \dfrac{12 \pm \sqrt{128}}{8}$

$= \dfrac{3 \pm 2\sqrt{2}}{2}$

(b) (i) $x = \dfrac{-4 \pm \sqrt{16 - 4(1)(-8)}}{2(1)}$

$= \dfrac{-4 \pm \sqrt{48}}{2}$

$= \dfrac{-4 \pm 4\sqrt{3}}{2}$

$= -2 \pm 2\sqrt{3}$

(ii) $x = \dfrac{-4 \pm \sqrt{16 - 4(5)(-2)}}{2(5)}$

$= \dfrac{-4 \pm \sqrt{56}}{10}$

$= \dfrac{-2 \pm \sqrt{14}}{5}$

(iii) $x = \dfrac{1 \pm \sqrt{1 - 4(1)(-1)}}{2(1)}$

$= \dfrac{1 \pm \sqrt{5}}{2}$

4. (a) (i) $x(x + 7) + 2(3) = 4(3)(x)$

$\Rightarrow x^2 + 7x + 6 = 12x$

$\Rightarrow x^2 - 5x + 6 = 0$

$\Rightarrow (x - 2)(x - 3) = 0$

$\Rightarrow x = 2, x = 3$

(ii) $1(x) + 4(x - 1) = 3(x)(x - 1)$

$\Rightarrow x + 4x - 4 = 3x^2 - 3x$

$\Rightarrow 3x^2 - 8x + 4 = 0$

$\Rightarrow (3x - 2)(x - 2) = 0$

$\Rightarrow 3x = 2, x = 2$

$\Rightarrow x = \dfrac{2}{3}, x = 2$

(iii) $3(x + 1) - 2(x - 1) = 1(x - 1)(x + 1)$

$\Rightarrow 3x + 3 - 2x + 2 = x^2 - 1$

$\Rightarrow x^2 - x - 6 = 0$

$\Rightarrow (x + 2)(x - 3) = 0$

$\Rightarrow x = -2, x = 3$

(b) (i) $1(x - 2) + 2(x) = 3(x)(x - 2)$

$\Rightarrow x - 2 + 2x = 3x^2 - 6x$

$\Rightarrow 3x^2 - 9x + 2 = 0$

$\Rightarrow x = \dfrac{9 \pm \sqrt{81 - 4(3)(2)}}{2(3)} = \dfrac{9 \pm \sqrt{57}}{6}$

(ii) $(x + 2)(x - 2) = (2x + 1)(x - 4)$

$\Rightarrow x^2 - 4 = 2x^2 - 7x - 4$

$\Rightarrow x^2 - 7x = 0$

$\Rightarrow x(x - 7) = 0$

$\Rightarrow x = 0, x = 7$

(iii) $2(x)(x - 4) + 3(x - 2)(x - 4)$

$= 5(x)(x - 2)$

$\Rightarrow 2x^2 - 8x + 3x^2 - 12x - 6x + 24$

$= 5x^2 - 10x$

$\Rightarrow 5x^2 - 26x + 24 = 5x^2 - 10x$

$\Rightarrow \quad -16x = -24$

$\Rightarrow \quad x = \dfrac{-24}{-16} = \dfrac{3}{2}$

5. (a) (i) $y = x^2 \Rightarrow y^2 - 7y + 10 = 0$
$\Rightarrow (y-2)(y-5) = 0$
$\Rightarrow y = 2, y = 5$
Hence: $x^2 = 2, x^2 = 5$
$\Rightarrow x = \pm\sqrt{2}, x = \pm\sqrt{5}$

(ii) $y = x + 1 \Rightarrow y^2 + 3y - 2 = 0$

$$\Rightarrow y = \frac{-3 \pm \sqrt{9 - 4(1)(-2)}}{2}$$
$$\Rightarrow y = \frac{-3 \pm \sqrt{17}}{2}$$
$$\Rightarrow x + 1 = \frac{-3 \pm \sqrt{17}}{2}$$
$$\Rightarrow x = \frac{-5 \pm \sqrt{17}}{2}$$

(iii) $y = x^2 \Rightarrow y^2 - 2y - 2 = 0$

$$\Rightarrow y = \frac{2 \pm \sqrt{4 - 4(1)(-2)}}{2}$$
$$\Rightarrow y = \frac{2 \pm \sqrt{12}}{2} = 1 \pm \sqrt{3}$$

Hence: $x^2 = 1 \pm \sqrt{3}$
$\Rightarrow x = \pm\sqrt{1 \pm \sqrt{3}}$

(iv) $y = k - 2 \Rightarrow 2y^2 - 3y - 4 = 0$

$$\Rightarrow y = \frac{3 \pm \sqrt{9 - 4(2)(-4)}}{2(2)}$$
$$= \frac{3 \pm \sqrt{41}}{4}$$

Hence: $k - 2 = \dfrac{3 \pm \sqrt{41}}{4}$

$$\Rightarrow k = \frac{11 \pm \sqrt{41}}{4}$$

(b) (i) $k = 2y - 1 \quad \Rightarrow k^2 - 3k - 28 = 0$
$\Rightarrow (k+4)(k-7) = 0$
$\Rightarrow k = -4, k = 7$
Hence: $2y - 1 = -4, 2y - 1 = 7$
$\Rightarrow 2y = -3, 2y = 8$
$\Rightarrow y = \dfrac{-3}{2}, y = 4$

(ii) $k = 2y - 3 \quad \Rightarrow k^2 - 1 = 0$
$\Rightarrow (k+1)(k-1) = 0$
$\Rightarrow k = -1, k = 1$
Hence: $2y - 3 = -1, 2y - 3 = 1$
$\Rightarrow 2y = 2, 2y = 4$
$\Rightarrow y = 1, y = 2$

(c) $k = y + \dfrac{4}{y} \Rightarrow k^2 - 9k + 20 = 0$
$\Rightarrow (k-4)(k-5) = 0$
$\Rightarrow k = 4, k = 5$

Hence: $y + \dfrac{4}{y} = 4, y + \dfrac{4}{y} = 5$
$\Rightarrow y^2 + 4 = 4y, y^2 + 4 = 5y$
$\Rightarrow y^2 - 4y + 4 = 0,$
$y^2 - 5y + 4 = 0$
$\Rightarrow (y-2)(y-2) = 0,$
$(y-1)(y-4) = 0$
$\Rightarrow y = 2, y = 1, y = 4$

(d) $x = 2t - \dfrac{5}{t} = 3, \Rightarrow x^2 - 12x + 27 = 0$
$\Rightarrow (x-3)(x-9) = 0$
$\Rightarrow x = 3, x = 9$

Hence: $2t - \dfrac{5}{t} = 3, 2t - \dfrac{5}{t} = 9$
$\Rightarrow 2t^2 - 5 = 3t, 2t^2 - 5 = 9t$
$\Rightarrow 2t^2 - 3t - 5 = 0, 2t^2 - 9t - 5 = 0$
$\Rightarrow (t+1)(2t-5) = 0, (2t+1)(t-5) = 0,$
$\Rightarrow t = -1, 2t = 5, 2t = -1, t = 5$
$\Rightarrow t = -1, t = \dfrac{5}{2}, t = -\dfrac{1}{2}, t = 5$

6. $$x = \frac{\sqrt{3} \pm \sqrt{3 - 4(2)(-3)}}{2(2)}$$
$$= \frac{\sqrt{3} \pm \sqrt{27}}{4}$$
$$= \frac{\sqrt{3} \pm 3\sqrt{3}}{4}$$
$$= \frac{4\sqrt{3}}{4}, -\frac{2\sqrt{3}}{4}$$
$$= \sqrt{3}, -\frac{\sqrt{3}}{2}$$

7. (a) $x = -4.2, 1.2$
(b) $x = -2.3, 1.3$
(c) $x = -9.5, -0.5$
(d) $-1.6 < x < 0.6$
(e) $x = -1.6, 0.6$
(f) $x = -3, 1$
(g) $x = -3.6, 0.6$
(h) $-4.1 < x < -0.9$

8. Graph does not intersect the x-axis

9. (a) $x_2 - x_1 = -22.5 + 2.5 = -20$
(b) $x_2 + x_1 = -22.5 - 2.5 = -25$

10. (a) $x = -0.5, 2$
(b) $x = -0.8$
(c) $x = -0.5, 2.4$

11. (i) $u = \sqrt{x}$ Equation is:

$$2u^2 + 3u - 5 = 0$$
$$(2u + 5)(u - 1) = 0$$
$$2u + 5 = 0 \quad \text{or} \quad u - 1 = 0$$
$$u = -\frac{5}{2} \quad \text{or} \quad u = 1$$

As $u = \sqrt{x}$ must be positive, $u = -\frac{5}{2}$ is not possible. Thus
$u = \sqrt{x} = 1$
$u^2 = x = 1$
is the only solution for x.

(ii) $u = \sqrt{x}$ Equation is:

$$u^2 - 3u - 4 = 0$$
$$(u + 1)(u - 4) = 0$$
$$u + 1 = 0 \quad \text{or} \quad u - 4 = 0$$
$$u = -1 \quad \text{or} \quad u = 4$$

As $u = \sqrt{x}$ must be positive, $u = -1$ is not possible. Thus
$u = \sqrt{x} = 4$
$u^2 = x = 16$
is the only solution for x.

12. (i) $$x = \frac{\sqrt{7} \pm \sqrt{7 - 4(1)(-14)}}{2(1)} = \frac{\sqrt{7} \pm \sqrt{63}}{2} = \frac{\sqrt{7} + 3\sqrt{7}}{2}, \frac{\sqrt{7} - 3\sqrt{7}}{2} = 2\sqrt{7}, -\sqrt{7}$$

(ii) $$x = \frac{-7\sqrt{5} \pm \sqrt{245 - 4(2)(15)}}{2(2)} = \frac{-7\sqrt{5} \pm \sqrt{125}}{4} = \frac{-7\sqrt{5} + 5\sqrt{5}}{4}, \frac{-7\sqrt{5} - 5\sqrt{5}}{4} = -\frac{\sqrt{5}}{2}, -3\sqrt{5}$$

Exercise 2.2

1. (i) f
(ii) h
(iii) g
(iv) Curve f has roots $= 1.6, 4.4$
Curve h has root $= 3$

2. $y = 0 \Rightarrow x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\Rightarrow A = \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}, 0\right)$$

$$\Rightarrow B = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}, 0\right)$$

3. (i) $b^2 - 4ac = (1)^2 - 4(2)(5) = -39 < 0 \Rightarrow$ Imaginary roots
(ii) $b^2 - 4ac = (3)^2 - 4(-2)(1) = 17 > 0$
$\Rightarrow$ Roots are real and different
(iii) $b^2 - 4ac = (2)^2 - 4(3)(-1) = 16 > 0$
$\Rightarrow$ Roots are real and different

(iv) $b^2 - 4ac = (2)^2 - 4(-1)(-3) = -8 < 0$

$\Rightarrow$ Imaginary roots

(v) $b^2 - 4ac = (8)^2 - 4(1)(16) = 0$

$\Rightarrow$ Roots are real and equal

(vi) $b^2 - 4ac = (-10)^2 - 4(1)(25) = 0$

$\Rightarrow$ Roots are real and equal

4. $y = x^2 - 2x + 5$ is positive for all values of x

$3x^2 - kx + 12$ is positive

$\Rightarrow$ Roots are imaginary

$\Rightarrow b^2 - 4ac < 0$

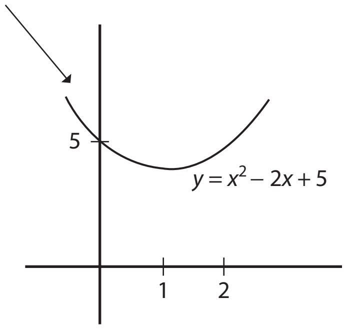
$\Rightarrow (-k^2) - 4(3)(12) < 0$

$\Rightarrow k^2 - 144 < 0$

$\Rightarrow$ Factors: $(k + 12)(k - 12)$

$\Rightarrow$ Roots: $k = -12, 12$

$\Rightarrow$ Solution for $k^2 - 144 < 0 \Rightarrow -12 < k < 12$



5. Equal Roots $\Rightarrow b^2 - 4ac = 0$

(i) $(-10)^2 - 4(1)(k) = 0$

$\Rightarrow 100 - 4k = 0$

$\Rightarrow -4k = -100$

$\Rightarrow k = 25$

(ii) $(k)^2 - 4(4)(9) = 0$

$\Rightarrow k^2 - 144 = 0$

$\Rightarrow (k + 12)(k - 12) = 0$

$\Rightarrow k = -12, 12$

(iii) $(-2k - 2)^2 - 4(1)(5k + 1) = 0$

$\Rightarrow 4k + 8k + 4 - 20k - 4 = 0$

$\Rightarrow 4k^2 - 12k = 0$

$\Rightarrow k^2 - 3k = 0$

$\Rightarrow k(k - 3) = 0$

$\Rightarrow k = 0, k = 3$

6. Equal Roots $\Rightarrow (2k + 2)^2 - 4(k^2)(4) = 0$

$\Rightarrow 4k^2 + 8k + 4 - 16k^2 = 0$

$\Rightarrow 12k^2 - 8k - 4 = 0$

$\Rightarrow 3k^2 - 2k - 1 = 0$

$\Rightarrow (3k + 1)(k - 1) = 0$

$\Rightarrow k = -\frac{1}{3}, k = 1$

7. (i) Real Roots $\Rightarrow (-3k)^2 - 4(1)(-k^2) \geq 0$

$\Rightarrow 9k^2 + 4k^2 \geq 0$

$\Rightarrow 13k^2 \geq 0$ True

(ii) Real Roots $\Rightarrow (2)^2 - 4(k)(2 - k) \geq 0$

$\Rightarrow 4 - 8k + 4k^2 \geq 0$

$\Rightarrow k^2 - 2k + 1 \geq 0$

$\Rightarrow (k - 1)^2 \geq 0$ True

8. Real Roots $\Rightarrow (-3)^2 - 4(1)(2 - c^2) \geq 0$

$\Rightarrow 9 - 8 + 4c^2 \geq 0$

$\Rightarrow 1 + 4c^2 \geq 0$ True

9. Real Roots $\Rightarrow (2)^2 - 4(k - 2)(-k) \geq 0$

$\Rightarrow 4 + 4k^2 - 8k \geq 0$

$\Rightarrow k^2 - 2k + 1 \geq 0$

$\Rightarrow (k - 1)^2 \geq 0$ True

10. Equal Roots $\Rightarrow (2k+1)^2 - 4(k-2)(k) = 0$
$\Rightarrow 4k^2 + 4k + 1 - 4k^2 + 8k = 0$
$\Rightarrow 12k = -1$
$\Rightarrow k = -1$

11. Equal Roots $\Rightarrow (6-2m)^2 - 4(m+3)(m-1) = 0$
$\Rightarrow 36 - 24m + 4m^2 - 4m^2 - 12m + 4m + 12 = 0$
$\Rightarrow -32m + 48 = 0$
$\Rightarrow -2m + 3 = 0$
$\Rightarrow -2m = -3$
$\Rightarrow m = 3$

12. Equal Roots $\Rightarrow b^2 - 4(a)(1) = 0$
$\Rightarrow b^2 - 4a = 0$
$\Rightarrow -4a = -b^2$
$\Rightarrow a = \dfrac{b^2}{4}$

Hence, $x = \dfrac{-b + \sqrt{0}}{2\left(\dfrac{b^2}{4}\right)}$
$= -b \cdot \dfrac{2}{b^2} = -\dfrac{2}{b}$

13. No Real Roots $\Rightarrow (-2p)^2 - 4(1)(3p^2 + q^2) < 0$
$\Rightarrow 4p^2 - 12p^2 - 4q^2 < 0$
$\Rightarrow -8p^2 - 4q^2 < 0$
$\Rightarrow -2p^2 - q^2 < 0$ True

Exercises 2.3

1. $y = -2x + 3 \cap y = x^2$
$\Rightarrow x^2 = -2x + 3$
$\Rightarrow x^2 + 2x - 3 = 0$
$\Rightarrow (x+3)(x-1) = 0$
$\Rightarrow x = -3, \quad x = 1$
$\Rightarrow y = (-3)^2, y = (1)^2$
$\Rightarrow y = 9, \quad y = 1$
ANS $= (-3, 9), (1, 1)$

2. $x = y - 1 \cap x^2 + y^2 = 5$
$\Rightarrow (y-1)^2 + y^2 - 5 = 0$
$\Rightarrow y^2 - 2y + 1 + y^2 - 5 = 0$
$\Rightarrow 2y^2 - 2y - 4 = 0$
$\Rightarrow y^2 - y - 2 = 0$
$\Rightarrow (y+1)(y-2) = 0$
$\Rightarrow y = -1, \quad y = 2$
$\Rightarrow x = -1 - 1, x = 2 - 1$
$\Rightarrow x = -2, x = 1$
ANS $= (-2, -1), (1, 2)$

3. $y = -2x + 2 \cap 4x^2 = y$
$\Rightarrow 4x^2 = -2x + 2$
$\Rightarrow 4x^2 + 2x - 2 = 0$
$\Rightarrow 2x^2 + x - 1 = 0$
$\Rightarrow (x+1)(2x-1) = 0$
$\Rightarrow x = -1, x = \dfrac{1}{2}$
$\Rightarrow y = -2(-1) + 2, y = -2\left(\dfrac{1}{2}\right) + 2$
$\Rightarrow y = 4, \quad y = 1$
ANS $= (-1, 4), \left(\dfrac{1}{2}, 1\right)$

4. $y = -x + 1 \cap y = x^2 - 6x + 5$
$\Rightarrow x^2 - 6x + 5 = -x + 1$
$\Rightarrow x^2 - 5x + 4 = 0$
$\Rightarrow (x-1)(x-4) = 0$
$\Rightarrow x = 1, \quad x = 4$
$\Rightarrow y = -1 + 1, \quad y = -4 + 1$
$\Rightarrow y = 0, \quad y = -3$
ANS $= (1, 0), (4, -3)$

5. $y = -x + 7 \cap x^2 + y^2 = 25$
$\Rightarrow x^2 + (x + 7)^2 - 25 = 0$
$\Rightarrow x^2 + x^2 - 14x + 49 - 25 = 0$
$\Rightarrow 2x^2 - 14x + 24 = 0$
$\Rightarrow x^2 - 7x + 12 = 0$
$\Rightarrow (x - 3)(x - 4) = 0$
$\Rightarrow x = 3, x = 4$
$\Rightarrow y = -3 + 7, y = -4 + 7$
$\Rightarrow y = 4, y = 3$
ANS = (3, 4), (4, 3)

6. $2x + 1 = y \cap 3x^2 - y^2 = 3$
$\Rightarrow 3x^2 - (2x + 1)^2 - 3 = 0$
$\Rightarrow 3x^2 - 4x^2 - 4x - 1 - 3 = 0$
$\Rightarrow -x^2 - 4x - 4 = 0$
$\Rightarrow x^2 + 4x + 4 = 0$
$\Rightarrow (x + 2)(x + 2) = 0$
$\Rightarrow x = -2$
$\Rightarrow y = 2(-2) + 1 = -3$
ANS = (−2, −3)

7. $y = 3x - 4 \cap y = x^2 - 4x + 6$
$\Rightarrow x^2 - 4x + 6 = 3x - 4$
$\Rightarrow x^2 - 7x + 10 = 0$
$\Rightarrow (x - 2)(x - 5) = 0$
$\Rightarrow x = 2, x = 5$
$\Rightarrow y = 3(2) - 4, y = 3(5) - 4$
$\Rightarrow y = 2, y = 11$
ANS = (2, 2), (5, 11)

8. $y = -x + 4 \cap x^2 + y^2 - 4x + 2 = 0$
$\Rightarrow x^2 + (-x + 4)^2 - 4x + 2 = 0$
$\Rightarrow x^2 + x^2 - 8x + 16 - 4x + 2 = 0$
$\Rightarrow 2x^2 - 12x + 18 = 0$
$\Rightarrow x^2 - 6x + 9 = 0$
$\Rightarrow (x - 3)(x - 3) = 0$
$\Rightarrow x = 3$
$\Rightarrow y = -3 + 4 = 1$
ANS = (3, 1)

9. $x = -2y + 2 \cap x^2 + 4y^2 = 4$
$\Rightarrow (-2y + 2)^2 + 4y^2 - 4 = 0$
$\Rightarrow 4y^2 - 8y + 4 + 4y^2 - 4 = 0$
$\Rightarrow 8y^2 - 8y = 0$
$\Rightarrow y^2 - y = 0$
$\Rightarrow y(y - 1) = 0$
$\Rightarrow y = 0, y = 1$
$\Rightarrow x = -2(0) + 2, x = -2(1) + 2$
$\Rightarrow x = 2, x = 0$
ANS = (2, 0), (0, 1)

10. $2x + 2 = y \cap xy = 4$
$\Rightarrow x(2x + 2) = 4$
$\Rightarrow 2x^2 + 2x - 4 = 0$
$\Rightarrow x^2 + x - 2 = 0$
$\Rightarrow (x + 2)(x - 1) = 0$
$\Rightarrow x = -2, x = 1$
$\Rightarrow y = 2(-2) + 2, y = 2(1) + 2$
$\Rightarrow y = -2, y = 4$
ANS = (−2, −2), (1, 4)

11. $y = -2x + 3 \cap y^2 + xy = 2$
$\Rightarrow (-2x + 3)^2 + x(-2x + 3) - 2 = 0$
$\Rightarrow 4x^2 - 12x + 9 - 2x^2 + 3x - 2 = 0$
$\Rightarrow 2x^2 - 9x + 7 = 0$
$\Rightarrow (2x - 7)(x - 1) = 0$
$\Rightarrow x = \frac{7}{2}, x = 1$
$\Rightarrow y = -2\left(\frac{7}{2}\right) + 3, y = -2(1) + 3$
$\Rightarrow y = -4, y = 1$
ANS $\left(\frac{7}{2}, -4\right)$, (1, 1)

12. $x = y - 3 \cap x^2 + y^2 + 2x - 4y + 3 = 0$
$\Rightarrow (y - 3)^2 + y^2 + 2(y - 3) - 4y + 3 = 0$
$\Rightarrow y^2 - 6y + 9 + y^2 + 2y - 6 - 4y + 3 = 0$
$\Rightarrow 2y^2 - 8y + 6 = 0$
$\Rightarrow y^2 - 4y + 3 = 0$
$\Rightarrow (y - 1)(y - 3) = 0$
$\Rightarrow y = 1, y = 3$
$\Rightarrow x = 1 - 3, x = 3 - 3$
$\Rightarrow x = -2, x = 0$
ANS = (−2, 1), (0, 3)

13. $s = 2t - 1 \cap 3t^2 - 2ts + s^2 = 9$
$\Rightarrow 3t^2 - 2t(2t - 1) + (2t - 1)^2 = 9$
$\Rightarrow 3t^2 - 4t^2 + 2t + 4t^2 - 4t + 1 - 9 = 0$
$\Rightarrow 3t^2 - 2t - 8 = 0$
$\Rightarrow (3t + 4)(t - 2) = 0$
$\Rightarrow t = -\frac{4}{3}, t = 2$
$\Rightarrow s = 2\left(-\frac{4}{3}\right) - 1, s = 2(2) - 1$
$\Rightarrow s = -\frac{11}{3}, s = 3$
ANS = $\left(-\frac{11}{3}, -\frac{4}{3}\right)$, (3, 2)

14. $2s + 3 = t \cap 2s^2 = t^2 + 1$
$\Rightarrow 2s^2 = (2s + 3)^2 + 1$
$\Rightarrow 2s^2 = 4s^2 + 12s + 9 + 1$
$\Rightarrow 2s^2 + 12s + 10 = 0$
$\Rightarrow s^2 + 6s + 5 = 0$
$\Rightarrow (s + 1)(s + 5) = 0$
$\Rightarrow s = -1, s = -5$
$\Rightarrow t = 2(-1) + 3, t = 2(-5) + 3$
$\Rightarrow t = 1, t = -7$
ANS $= (-1, 1), (-5, -7)$

15. $2t = 3s + 1 \cap t^2 + ts - 4s^2 = 2$
$\Rightarrow t = \dfrac{3s + 1}{2} \Rightarrow \left(\dfrac{3s + 1}{2}\right)^2 + s\left(\dfrac{3s + 1}{2}\right) - 4s^2 - 2 = 0$
$\Rightarrow \dfrac{9s^2 + 6s + 1}{2} + \dfrac{3s^2 + s}{2} - 4s^2 - 2 = 0$
$\Rightarrow 9s^2 + 6s + 1 + 6s^2 + 2s - 16s^2 - 8 = 0$
$\Rightarrow -s^2 + 8s - 7 = 0$
$\Rightarrow s^2 - 8s + 7 = 0$
$\Rightarrow (s - 1)(s - 7) = 0$
$\Rightarrow s = 1, s = 7$
$\Rightarrow t = \dfrac{3(1) + 1}{2}, t = \dfrac{3(7) + 1}{2}$
$\Rightarrow t = 2, t = 11$
ANS $= (1, 2), (7, 11)$

Exercise 2.4

1. $x, x + 1 \Rightarrow (x)^2 + (x + 1)^2 = 61$
$\Rightarrow x^2 + x^2 + 2x + 1 - 61 = 0$
$\Rightarrow 2x^2 + 2x - 60 = 0$
$\Rightarrow x^2 + x - 30 = 0$
$\Rightarrow (x + 6)(x - 5) = 0$
$\Rightarrow x = -6, x = 5$
$\Rightarrow x + 1 = -5, x + 1 = 6$
Numbers are $-6, -5$ or $5, 6$

2. $x, x + 2 \Rightarrow (x)^2 + (x + 2)^2 = 52$
$\Rightarrow x^2 + x^2 + 4x + 4 - 52 = 0$
$\Rightarrow 2x^2 + 4x - 48 = 0$
$\Rightarrow x^2 + 2x - 24 = 0$
$\Rightarrow (x + 6)(x - 4) = 0$
$\Rightarrow x = -6, x = 4$
$\Rightarrow x + 2 = -4, x + 2 = 6$
Numbers are $-6, -4$ or $4, 6$

3. (i) Perimeter $= 2x + 2y = 62$
$\Rightarrow \quad x + y = 31$
$\Rightarrow \quad y = -x + 31$
Area $= (x)(y) = 198$
$\Rightarrow xy = 198$
(ii) $\Rightarrow x(-x + 31) - 198 = 0$
$\Rightarrow -x^2 + 31x - 198 = 0$
$\Rightarrow x^2 - 31x + 198 = 0$
$\Rightarrow (x - 9)(x - 22) = 0$
$\Rightarrow x = 9$ m, $x = 22$ m
$\Rightarrow y = -9 + 31, y = -22 + 31$
$= 22$ m, $\quad = 9$ m
Dimensions $= 9$ m, 22 m

4. Sides $= x, x+1, x+2 \Rightarrow$ Pythagoras $\Rightarrow (x+2)^2 = (x)^2 + (x+1)^2$

$$\Rightarrow x^2 + 4x + 4 = x^2 + x^2 + 2x + 1$$
$$\Rightarrow x^2 - 2x - 3 = 0$$
$$\Rightarrow (x-3)(x+1) = 0$$
$$\Rightarrow x = 3, x = -1 \text{ (Not valid)}$$
$$\Rightarrow x + 1 = 4$$
$$\Rightarrow x + 2 = 5$$

Perimeter $= 3 + 4 + 5 = 12$ units

5. $s = 25 \Rightarrow 12t - t^2 = 25$

$$\Rightarrow t^2 - 12t + 25 = 0$$
$$\Rightarrow t = \frac{12 \pm \sqrt{(-12)^2 - 4(1)(25)}}{2(1)}$$
$$= \frac{12 \pm \sqrt{144 - 100}}{2}$$
$$= \frac{12 \pm \sqrt{44}}{2}$$
$$= \frac{12 + 6.633}{2}, \frac{12 - 6.633}{2}$$
$$= 9.316, 2.683$$
$$= 9.32, 2.68$$

6. $x^2 - 15 = 2x$

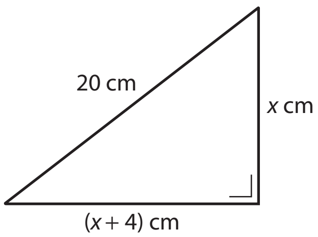
$$\Rightarrow x^2 - 2x - 15 = 0$$
$$\Rightarrow (x+3)(x-5) = 0$$
$$\Rightarrow x = -3, x = 5$$

7. $h = 6 \Rightarrow -16t^2 + 24t + 1 = 6$

$$\Rightarrow 16t^2 - 24t + 5 = 0$$
$$\Rightarrow (4t-1)(4t-5) = 0$$
$$\Rightarrow 4t = 1, 4t = 5$$
$$\Rightarrow t = \frac{1}{4} \text{ sec}, t = \frac{5}{4} \text{ sec}$$

8. Pythagoras $\Rightarrow (x)^2 + (x+4)^2 = (20)^2$

$$\Rightarrow x^2 + x^2 + 8x + 16 = 400$$
$$\Rightarrow 2x^2 + 8x - 384 = 0$$
$$\Rightarrow x^2 + 4x - 192 = 0$$
$$\Rightarrow (x-12)(x+16) = 0$$
$$\Rightarrow x = 12, x = -16 \text{ (Not valid)}$$



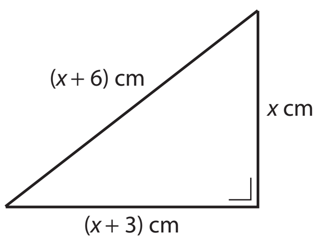
9. $x, x+2 \Rightarrow x(x+2) = 4(2x+2) - 1$

$$\Rightarrow x^2 + 2x = 8x + 8 - 1$$
$$\Rightarrow x^2 - 6x - 7 = 0$$
$$\Rightarrow (x+1)(x-7) = 0$$
$$\Rightarrow x = -1, x = 7$$
$$\Rightarrow x + 2 = 1, x + 2 = 9$$

ANS $= -1, 1$ or $7, 9$

10. Pythagoras $\Rightarrow (x)^2 + (x+3)^2 = (x+6)^2$

$$\Rightarrow x^2 + x^2 + 6x + 9 = x^2 + 12x + 36$$
$$\Rightarrow x^2 - 6x - 27 = 0$$
$$\Rightarrow (x-9)(x+3) = 0$$
$$\Rightarrow x = 9, x = -3 \text{ (Not valid)}$$



11. $x, x+4 \Rightarrow \text{Area} = x(x+4) = 60$

$\Rightarrow x^2 + 4x - 60 = 0$

$\Rightarrow (x-6)(x+10) = 0$

$\Rightarrow x = 6, x = -10$ (Not valid)

Dimensions are 6 m, 10 m

12. $x, x+1, x+2 \Rightarrow 3(x + x + 1 + x + 2) = (x+1)(x+2)$

$\Rightarrow 9x + 9 = x^2 + 3x + 2$

$\Rightarrow x^2 - 6x - 7 = 0$

$\Rightarrow (x+1)(x-7) = 0$

$\Rightarrow x = -1, x = 7$

ANS $= -1, 0, 1$ or $7, 8, 9$

13. $\text{Area} = \pi(x+14)^2 - \pi(14)^2 = 60\pi$

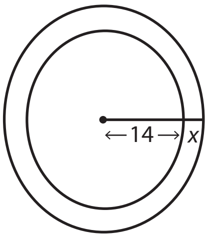
$\Rightarrow x^2 + 28x + 196 - 196 = 60$

$\Rightarrow x^2 + 28x - 60 = 0$

$\Rightarrow (x-2)(x+30) = 0$

$\Rightarrow x = 2, x = -30$ (Not valid)

width of desk = 2 m



14. $\text{Area} = x^2 + 96 = (2x)(x-2)$

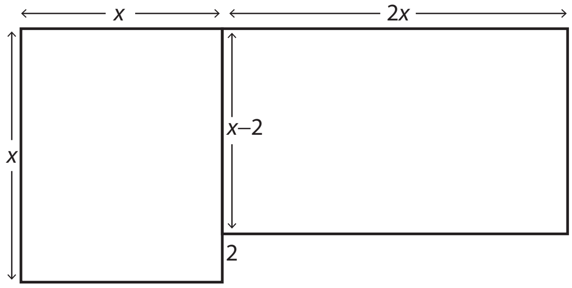
$\Rightarrow x^2 + 96 = 2x^2 - 4x$

$\Rightarrow x^2 - 4x - 96 = 0$

$\Rightarrow (x-12)(x+8) = 0$

$\Rightarrow x = 12, x = -8$ (Not valid)

Dimensions = 10 m, 24 m



15. Point G $\Rightarrow 0.1x^2 - x + 2.5 = 1.5$

$\Rightarrow 0.1x^2 - x + 1 = 0$

$\Rightarrow x^2 - 10x + 10 = 0$

$$\Rightarrow x = \frac{10 \pm \sqrt{100 - 4(1)(10)}}{2(1)}$$

$$= \frac{10 \pm \sqrt{60}}{2}$$

$$= \frac{10 - \sqrt{60}}{2}, \frac{10 + \sqrt{60}}{2}$$

$= 1.127, 8.873$ (Not valid)

Point C $\Rightarrow 0.1x^2 - x + 2.5 = 3$

$\Rightarrow 0.1x^2 - x - 0.5 = 0$

$\Rightarrow x^2 - 10x - 5 = 0$

$$\Rightarrow x = \frac{10 \pm \sqrt{100 - 4(1)(-5)}}{2(1)}$$

$$= \frac{10 \pm \sqrt{120}}{2}$$

$$= \frac{10 + \sqrt{120}}{2}, \frac{10 - \sqrt{120}}{2}$$

$= 10.477, -0.477$ (Not valid)

Distance $= 10.477 - 1.127 = 9.35$ m

16. $3t - 4 = s \cap 2t^2 + s^2 = 43$

$\Rightarrow 2t^2 + (3t - 4)^2 - 43 = 0$

$\Rightarrow 2t^2 + 9t^2 - 24t + 16 - 43 = 0$

$\Rightarrow 11t^2 - 24t - 27 = 0$

$\Rightarrow (t - 3)(11t + 9) = 0$

$\Rightarrow t = 3, t = -\frac{9}{11}$ (Not valid)

$\Rightarrow s = 3(3) - 4$

$= 5$

Point $= (3, 5)$

17. $x + 3y = 5 \cap x^2 + 6y^2 = 40$

$\Rightarrow x = -3y + 5 \Rightarrow (-3y + 5)^2 + 6y^2 - 40 = 0$

$\Rightarrow 9y^2 - 30y + 25 + 6y^2 - 40 = 0$

$\Rightarrow 15y^2 - 30y - 15 = 0$

$\Rightarrow y^2 - 2y - 1 = 0$

$\Rightarrow y = \dfrac{2 \pm \sqrt{4 - 4(1)(-1)}}{2(1)}$

$\Rightarrow y = \dfrac{2 \pm \sqrt{8}}{2}$

$\Rightarrow y = 1 + \sqrt{2}, 1 - \sqrt{2}$

$\Rightarrow y = 2.414, -0.414$

$\Rightarrow x = -3(2.414) + 5, x = -3(-0.414) + 5$

$= -2.242, \quad = 6.242$

Plane crosses front at 2 points; $(-2.2, 2.4)$ and $(6.2, -0.4)$

$k = 10 \Rightarrow x + 3y = 10 \cap x^2 + 6y^2 = 40$

$\Rightarrow x = -3y + 10 \Rightarrow (-3y + 10)^2 + 6y^2 - 40 = 0$

$\Rightarrow 15y^2 - 60y + 60 = 0$

$\Rightarrow y^2 - 4y + 4 = 0$

$\Rightarrow (y - 2)(y - 2) = 0$

$\Rightarrow y = 2$ and $x = -3(2) + 10 = 4$

Plane crosses front at one point; $(4, 2)$

$\Rightarrow$ Plane will avoid weather front when $k > 10$

$\Rightarrow$ Minimum value of $k = 11$

Exercise 2.5

1. (a) (i) sum $= -9$ (ii) product $= 4$

(b) (i) sum $= 2$ (ii) product $= -5$

(c) (i) sum $= 7$ (ii) product $= 2$

(d) (i) sum $= 9$ (ii) product $= -3$

(e) (i) sum $= \frac{7}{2}$ (ii) product $= -\frac{1}{2}$

(f) (i) sum $= -\frac{1}{7}$ (iii) product $= -\frac{1}{7}$

(g) (i) sum $= -\frac{10}{3}$ (ii) product $= -\frac{2}{3}$

(h) (i) sum $= -\frac{10}{5} = -2$ (ii) product $= \frac{1}{5}$

(i) (i) sum $= -2$ (ii) product $= -3$

(j) (i) sum $= \frac{3}{4}$ (ii) product $= \frac{5}{4}$

2. (a) $x^2 + 3x - 1 = 0$

(b) $x^2 - 6x - 4 = 0$

(c) $x^2 - 7x - 5 = 0$

(d) $x^2 + \frac{2}{3}x - \frac{7}{3} = 0$

$\Rightarrow 3x^2 + 2x - 7 = 0$

(e) $x^2 + \frac{5}{2}x - 2 = 0$

$\Rightarrow 2x^2 + 5x - 4 = 0$

(f) $x^2 + \frac{3}{2}x - 5 = 0$

$\Rightarrow 2x^2 + 3x - 10 = 0$

(g) $x^2 + \frac{1}{4}x - \frac{1}{3} = 0$

$\Rightarrow 12x^2 + 3x - 4 = 0$

(h) $x^2 + 1\frac{2}{3}x + \frac{1}{2} = 0$

$\Rightarrow 6x^2 + 10x + 3 = 0$

3. (i) $x^2 - (4 + 6)x + (4)(6) = 0$

$\Rightarrow x^2 - 10x + 24 = 0$

(ii) $x^2 - (2 - 3)x + (2)(-3) = 0$

$\Rightarrow x^2 + x - 6 = 0$

(iii) $x^2 - (-5 - 1)x + (-5)(-1) = 0$

$\Rightarrow x^2 + 6x + 5 = 0$

(iv) $x^2 - (4 + \sqrt{5})x + (4)(\sqrt{5}) = 0$

$\Rightarrow x^2 - (4 + \sqrt{5})x + 4\sqrt{5} = 0$

(v) $x^2 - (a + 3a)x + (a)(3a) = 0$

$\Rightarrow x^2 - 4ax + 3a^2 = 0$

(vi) $x^2 - \left(\frac{2}{5} + \frac{3}{5}\right)x + \left(\frac{2}{5}\right)\left(\frac{3}{5}\right) = 0$

$\Rightarrow x^2 - x + \frac{6}{25} = 0$

$\Rightarrow 25x^2 - 25x + 6 = 0$

(vii) $x^2 - \left(\frac{2}{b} + \frac{3}{b}\right)x + \left(\frac{2}{b}\right)\left(\frac{3}{b}\right) = 0$

$\Rightarrow x^2 - \frac{5}{b}x + \frac{6}{b^2} = 0$

$\Rightarrow b^2x^2 - 5bx + 6 = 0$

(viii) $x^2 - \left(\frac{5}{2} + \frac{3}{5}\right)x + \left(\frac{5}{2}\right)\cdot\left(\frac{3}{5}\right) = 0$

$\Rightarrow x^2 - \frac{31}{10}x + \frac{3}{2} = 0$

$\Rightarrow 10x^2 - 31x + 15 = 0$

Exercise 2.6

1. (i) $c = \left(\frac{28}{2}\right)^2 = (14)^2 = 196$ (ii) $c = \left(\frac{-6}{2}\right)^2 = (-3)^2 = 9$ (iii) $c = \left(\frac{-5}{2}\right)^2 = \frac{25}{4}$

2. (i) $x^2 - 8x - 3 = 0$

$\Rightarrow x^2 - 8x + 16 - 16 - 3 = 0$

$\Rightarrow (x - 4)^2 - 19 = 0$

(ii) $x^2 - 2x - 5 = 0$

$\Rightarrow x^2 - 2x + 1 - 1 - 5 = 0$

$\Rightarrow (x - 1)^2 - 6 = 0$

(iii) $x^2 - 2x + 1 = 0$

$\Rightarrow (x - 1)^2 = 0$

3. (i) $x^2 + 4x - 6 = 0$

$\Rightarrow x^2 + 4x + 4 - 4 - 6 = 0$

$\Rightarrow (x + 2)^2 - 10 = 0$

(ii) $x^2 + 9x + 4 = 0$

$\Rightarrow x^2 + 9x + 20\frac{1}{4} - 20\frac{1}{4} + 4 = 0$

$\Rightarrow \left(x + 4\frac{1}{2}\right)^2 - 16\frac{1}{4} = 0$

(iii) $x^2 - 7x - 3 = 0$

$\Rightarrow x^2 - 7x + 12\frac{1}{4} - 12\frac{1}{4} - 3 = 0$

$\Rightarrow \left(x - 3\frac{1}{2}\right)^2 - 15\frac{1}{4} = 0$

4. (i) $2x^2 + 4x - 5 = 0$

$= 2\left(x^2 + 2x - \frac{5}{2}\right) = 0$

$= 2\left(x^2 + 2x + 1 - 1 - \frac{5}{2}\right) = 0$

$= 2\left[(x + 1)^2 - \frac{7}{2}\right] = 0$

$= 2(x + 1)^2 - 7 = 0$

$\Rightarrow$ Minimum Point $= (-1, -7)$

(ii) $3x^2 - 6x - 1 = 0$

$\Rightarrow 3\left(x^2 - 2x - \frac{1}{3}\right) = 0$

$\Rightarrow 3\left(x^2 - 2x + 1 - 1 - \frac{1}{3}\right) = 0$

$\Rightarrow 3\left[(x - 1)^2 - 1\frac{1}{3}\right] = 0$

$\Rightarrow 3(x - 1)^2 - 4 = 0$

$\Rightarrow$ Minimum Point $= (1, -4)$

(iii) $4x^2 + x + 3 = 0$

$\Rightarrow 4\left(x^2 + \frac{1}{4}x + \frac{3}{4}\right) = 0$

$\Rightarrow 4\left(x^2 + \frac{1}{4}x + \frac{1}{64} - \frac{1}{64} + \frac{3}{4}\right) = 0$

$\Rightarrow 4\left[\left(x + \frac{1}{8}\right)^2 + \frac{47}{64}\right] = 0$

$\Rightarrow 4\left(x + \frac{1}{8}\right)^2 + \frac{47}{16} = 0$

$\Rightarrow$ Minimum Point $= \left(-\frac{1}{8}, \frac{47}{16}\right)$

5. $x^2 - 6x + k = 0$

$= x^2 - 6x + 9 - 9 + k$

$= (x - 3)^2 + k - 9$

hence, $k - 9 > 0$

$\Rightarrow k > 9$

Minimum value of $k = 10$

6. $2x^2 - 12x + 7$

$= 2\left(x^2 - 6x + \frac{7}{2}\right)$

$= 2\left(x^2 - 6x + 9 - 9 + \frac{7}{2}\right)$

$= 2\left[(x - 3)^2 - \frac{11}{2}\right]$

$= 2(x - 3)^2 - 11$

7. $g(x) = x^2 + 8x + 20$

$= x^2 + 8x + 16 - 16 + 20$

$= (x + 4)^2 + 4$

Since $(x + 4)^2 \geqslant 0$

$\Rightarrow g(x) = (x + 4)^2 + 4 \geqslant 4$

8. (i) Minimum for $f(x) = (-1, -5)$

$\Rightarrow f(x) = (x + 1)^2 - 5$

$= x^2 + 2x + 1 - 5$

$= x^2 + 2x - 4$

Choose $(-2, -4) \Rightarrow f(-2) = (-2)^2 + 2(-2) - 4$

$= 4 - 4 - 4$

$= -4$ True

Minimum for $g(x) = (2, -1)$

$\Rightarrow g(x) = (x - 2)^2 - 1$

$= x^2 - 4x + 4 - 1$

$= x^2 - 4x + 3$

Choose (3, 0) $\Rightarrow g(3) = (3)^2 - 4(3) + 3$
$= 9 - 12 + 3$
$= 0$ True

Minimum for $h(x) = (4, 1)$
$\Rightarrow h(x) = (x - 4)^2 + 1$
$= x^2 - 8x + 16 + 1$
$= x^2 - 8x + 17$

Choose (5, 2) $\Rightarrow h(5) = (5)^2 - 8(5) + 17$
$= 25 - 40 + 17$
$= 42 - 40 = 2$ True

9. $f(x) = x^2 + 4x + 7$
$= x^2 + 4x + 4 + 3$
$= (x + 2)^2 + 3$

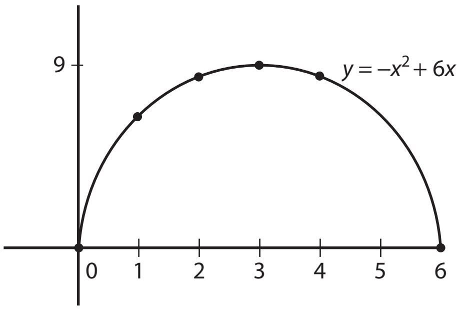
(i) smallest value of $f(x) = 3$

(ii) $x = -2$

(iii) $\dfrac{1}{(x^2 + 4x + 7)} = \dfrac{1}{3}$ (greatest value)

10. $y = -x^2 + 6x$
$= -(x^2 - 6x)$
$= -(x^2 - 6x + 9 - 9)$
$= -[(x - 3)^2 - 9]$
$= -(x - 3)^2 + 9$

maximum point $= (3, 9)$
greatest height $= 9$



11. (i) C : $x^2 - 6x + 8$
$= x^2 - 6x + 9 - 9 + 8$
$= (x - 3)^2 - 1$

(ii) B: $x^2 - 6x + 9$
$= (x - 3)^2$

(iii) A: $x^2 - 6x + 10$
$= x^2 - 6x + 9 - 9 + 10$
$= (x - 3)^2 + 1$

12. Curve C: Maximum point $= (2, 4) = (p, q)$
$\Rightarrow y = 4 - a(x - 2)^2$
point (0, 3) $\Rightarrow 3 = 4 - a(0 - 2)^2$
$\Rightarrow 3 = 4 - a(4)$
$\Rightarrow 4a = 1$
$\Rightarrow a = \dfrac{1}{4}$

Curve D: Maximum point $= (2, 4) = (p, q)$
$\Rightarrow y = 4 - a(x - 2)^2$
point (1, 3) $\Rightarrow 3 = 4 - a(1 - 2)^2$
$\Rightarrow 3 = 4 - a(1)$
$\Rightarrow a = 1$

13. Roots are $x = 6, x = -3$
Hence, $y = a(x - 6)(x + 3)$
$\Rightarrow y = a(x^2 - 3x - 18)$
point (1,10) $\Rightarrow 10 = a[(1)^2 - 3(1) - 18]$
$\Rightarrow 10 = a(1 - 3 - 18)$
$\Rightarrow 10 = -20a$
$\Rightarrow a = -\dfrac{1}{2}$

$\Rightarrow y = -\frac{1}{2}(x^2 - 3x - 18) = -\frac{1}{2}x^2 + \frac{3}{2}x + 9$

14. Minimum point $= (-1, 3)$

$\Rightarrow y = a(x + 1)^2 + 3$

point $(0, 4) \Rightarrow 4 = a(0 + 1)^2 + 3$

$\Rightarrow 4 = a + 3$

$\Rightarrow a = 1$

$\Rightarrow y = 1(x + 1)^2 + 3 \quad = x^2 + 2x + 4$

15. (i) Maximum point $= (6, 4)$

$\Rightarrow f(x) = 4 - (0.1)(x - 6)^2$

(ii) $f(x) = 4 - (0.1)(x^2 - 12x + 36) = 0$

$\Rightarrow 40 - (x^2 - 12x + 36) = 0$

$\Rightarrow -x^2 + 12x + 4 = 0$

$$\Rightarrow x = \frac{-12 \pm \sqrt{144 - 4(-1)(4)}}{2(-1)}$$

$$= \frac{-12 \pm \sqrt{160}}{-2}$$

$$= \frac{-12 \pm 4\sqrt{10}}{-2}$$

$= 6 - 2\sqrt{10}, 6 + 2\sqrt{10}$

point where ball started is $(6 - 2\sqrt{10}, 0)$

and point where ball finished is $(6 + 2\sqrt{10}, 0)$

(iii) Horizontal Distance $= 6 + 2\sqrt{10} - (6 - 2\sqrt{10})$

$= 6 + 2\sqrt{10} - 6 + 2\sqrt{10}$

$= 4\sqrt{10}$

16. (i) Area of trapezium $= \frac{1}{2}(a + b)h$

$= \frac{1}{2}(x + 20)(2x)$

$= x(x + 20)$

$= x^2 + 20x$

(ii) Given: $x^2 + 20x = 400$

$x^2 + 20x + 100 = 400 + 100$

$(x + 10)^2 = 500$

$x + 10 = \pm\sqrt{500}$

$x + 10 = \pm 10\sqrt{5}$

$x = -10 + 10\sqrt{5}$ (as $x = -10 - 10\sqrt{5}$ is not possible)

$x = 10(\sqrt{5} - 1)$

17. $T = 36 + 4t - t^2$

$t^2 - 4t = -T + 36$

$t^2 - 4t + 4 = -T + 36 + 4$

$(t - 2)^2 = -T + 40$

$T = -(t - 2)^2 + 40$

The maximum temperature is then 40°.

Then

$T = 36$

$-(t - 2)^2 + 40 = 36$

$-(t - 2)^2 = -4$

$(t - 2)^2 = 4$

$(t - 2) = \pm\sqrt{4}$

$t = 2 \pm 2$

$t = 0$ or $t = 4$

Thus it will take 4 hours for temperature returns to 36°

Exercise 2.7

1. (i) $\sqrt{8} = \sqrt{4}.\sqrt{2} = 2\sqrt{2}$

(ii) $\sqrt{27} = \sqrt{9}.\sqrt{3} = 3\sqrt{3}$

(iii) $\sqrt{45} = \sqrt{9}.\sqrt{5} = 3\sqrt{5}$

(iv) $\sqrt{200} = \sqrt{100}.\sqrt{2} = 10\sqrt{2}$

(v) $3\sqrt{18} = 3.\sqrt{9}.\sqrt{2} = 3.3\sqrt{2} = 9\sqrt{2}$

2. (i) $2\sqrt{2} + 6\sqrt{2} - 3\sqrt{2} = 5\sqrt{2}$

(ii) $2\sqrt{2} + \sqrt{18}$
$= 2\sqrt{2} + \sqrt{9}.\sqrt{2}$
$= 2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2}$

(iii) $\sqrt{32} + \sqrt{18}$
$= \sqrt{16}.\sqrt{2} + \sqrt{9}.\sqrt{2}$
$= 4\sqrt{2} + 3\sqrt{2}$
$= 7\sqrt{2}$

(iv) $\sqrt{27} + \sqrt{48} - 2\sqrt{3}$
$= \sqrt{9}\sqrt{3} + \sqrt{16}\sqrt{3} - 2\sqrt{3}$
$= 3\sqrt{3} + 4\sqrt{3} - 2\sqrt{3} = 5\sqrt{3}$

(v) $\sqrt{8} + \sqrt{200} - \sqrt{18}$
$= \sqrt{4}\sqrt{2} + \sqrt{100}\sqrt{2} - \sqrt{9}\sqrt{2}$
$= 2\sqrt{2} + 10\sqrt{2} - 3\sqrt{2}$
$= 9\sqrt{2}$

(vi) $7\sqrt{5} + 2\sqrt{20} - \sqrt{80}$
$= 7\sqrt{5} + 2\sqrt{4}.\sqrt{5} - \sqrt{16}.\sqrt{5}$
$= 7\sqrt{5} + 4\sqrt{5} - 4\sqrt{5}$
$= 7\sqrt{5}$

3. (i) $\dfrac{1}{\sqrt{3}} = \dfrac{1}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

(ii) $\dfrac{2}{\sqrt{8}} = \dfrac{2}{\sqrt{4}\sqrt{2}} = \dfrac{2}{2\sqrt{2}} = \dfrac{1}{\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

(iii) $\dfrac{2}{5\sqrt{2}} = \dfrac{2}{5\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{2\sqrt{2}}{10} = \dfrac{\sqrt{2}}{5}$

(iv) $\dfrac{20}{\sqrt{50}} = \dfrac{20}{\sqrt{25}\sqrt{2}} = \dfrac{20}{5\sqrt{2}} = \dfrac{4}{\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{4\sqrt{2}}{2} = 2\sqrt{2}$

(v) $\dfrac{8}{\sqrt{128}} = \dfrac{8}{\sqrt{64}\sqrt{2}} = \dfrac{8}{8\sqrt{2}} = \dfrac{1}{\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

4. (i) $\sqrt{8} \times \sqrt{12} = \sqrt{96} = \sqrt{16}\sqrt{6} = 4\sqrt{6}$

(ii) $3\sqrt{2} \times 5\sqrt{2} = 15.\sqrt{4} = 15.2 = 30$

(iii) $\sqrt{2}(\sqrt{6} + 3\sqrt{2}) = \sqrt{12} + 3\sqrt{4} = \sqrt{4}\sqrt{3} + 3.2 = 2\sqrt{3} + 6$

(iv) $(5 - \sqrt{3})(5 + \sqrt{3})$
$= 5(5 + \sqrt{3}) - \sqrt{3}(5 + \sqrt{3})$
$= 25 + 5\sqrt{3} - 5\sqrt{3} - \sqrt{9}$
$= 25 - 3 = 22$

(v) $(\sqrt{7} + \sqrt{5})(\sqrt{7} - \sqrt{5})$
$= \sqrt{7}(\sqrt{7} - \sqrt{5}) + \sqrt{5}(\sqrt{7} - \sqrt{5})$
$= \sqrt{49} - \sqrt{35} + \sqrt{35} - \sqrt{25} = 7 - 5 = 2$

(vi) $(a + 2\sqrt{b})(a - 2\sqrt{b})$
$= a(a - 2\sqrt{b}) + 2\sqrt{b}(a - 2\sqrt{b})$
$= a^2 - 2a\sqrt{b} + 2a\sqrt{b} - 4\sqrt{b^2} = a^2 - 4b$

5. (i) $\dfrac{4}{\sqrt{5}+1} . \dfrac{\sqrt{5}-1}{\sqrt{5}-1} = \dfrac{4(\sqrt{5}-1)}{\sqrt{25}-1} = \dfrac{4(\sqrt{5}-1)}{5-1} = \dfrac{4(\sqrt{5}-1)}{4} = \sqrt{5} - 1$

(ii) $\dfrac{12}{3-\sqrt{2}} . \dfrac{3+\sqrt{2}}{3+\sqrt{2}} = \dfrac{36 + 12\sqrt{2}}{9 - \sqrt{4}} = \dfrac{36 + 12\sqrt{2}}{7}$

(iii) $\dfrac{2-\sqrt{5}}{2+\sqrt{5}} . \dfrac{2-\sqrt{5}}{2-\sqrt{5}} = \dfrac{4 - 2\sqrt{5} - 2\sqrt{5} + \sqrt{25}}{4 - \sqrt{25}}$
$= \dfrac{9 - 4\sqrt{5}}{-1} = -9 + 4\sqrt{5}$

(iv) $\dfrac{1}{\sqrt{8} - \sqrt{2}} = \dfrac{1}{\sqrt{4}\sqrt{2} - \sqrt{2}} = \dfrac{1}{2\sqrt{2} - \sqrt{2}} = \dfrac{1}{\sqrt{2}} . \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{\sqrt{2}}{\sqrt{4}} = \dfrac{\sqrt{2}}{2}$

6. (i) $\frac{1}{\sqrt{2}-1} - \frac{1}{\sqrt{2}+1} = \frac{1(\sqrt{2}+1) - 1(\sqrt{2}-1)}{(\sqrt{2}-1)(\sqrt{2}+1)}$

$= \frac{\sqrt{2}+1-\sqrt{2}+1}{\sqrt{4}+\sqrt{2}-\sqrt{2}-1} = \frac{2}{2-1} = 2$

(ii) $\frac{1}{2+\sqrt{3}} + \frac{1}{2-\sqrt{3}} = \frac{1(2-\sqrt{3}) + 1(2+\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})}$

$= \frac{2-\sqrt{3}+2+\sqrt{3}}{4-2\sqrt{3}+2\sqrt{3}-\sqrt{9}}$

$= \frac{4}{4-3} = 4$

7. (i) $(2\sqrt{3}-\sqrt{5})(2\sqrt{3}+\sqrt{5})$

$= 2\sqrt{3}(2\sqrt{3}+\sqrt{5}) - \sqrt{5}(2\sqrt{3}+\sqrt{5})$

$= 4.3 + 2\sqrt{15} - 2\sqrt{15} - \sqrt{25} = 12 - 5 = 7$

(ii) $\frac{4}{2-\sqrt{5}} + \frac{2}{2+\sqrt{5}} = \frac{4(2+\sqrt{5}) + 2(2-\sqrt{5})}{(2-\sqrt{5})(2+\sqrt{5})}$

$= \frac{8+4\sqrt{5}+4-2\sqrt{5}}{4+2\sqrt{5}-2\sqrt{5}-\sqrt{25}}$

$= \frac{12+2\sqrt{5}}{-1}$

$= -12-2\sqrt{5}$

8. (i) $X+Y = \frac{4+\sqrt{3}}{\sqrt{2}} + \frac{4-\sqrt{3}}{\sqrt{2}} = \frac{8}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{8\sqrt{2}}{2} = 4\sqrt{2}$

(ii) $X-Y = 4\frac{+\sqrt{3}}{\sqrt{2}} - \frac{4-\sqrt{3}}{\sqrt{2}} = \frac{4+\sqrt{3}-4+\sqrt{3}}{\sqrt{2}}$

$= \frac{2\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{6}}{2} = \sqrt{6}$

(iii) $XY = \frac{4+\sqrt{3}}{\sqrt{2}} \cdot \frac{4-\sqrt{3}}{\sqrt{2}} = \frac{16-4\sqrt{3}+4\sqrt{3}-\sqrt{9}}{2} = \frac{16-3}{2} = \frac{13}{2}$

(iv) $\frac{X}{Y} = \frac{\frac{4+\sqrt{3}}{\sqrt{2}}}{\frac{4-\sqrt{3}}{\sqrt{2}}} = \frac{4+\sqrt{3}}{4-\sqrt{3}} \cdot \frac{4+\sqrt{3}}{4+\sqrt{3}} = \frac{16+4\sqrt{3}+4\sqrt{3}+\sqrt{9}}{16+4\sqrt{3}-4\sqrt{3}-\sqrt{9}}$

$= \frac{19+8\sqrt{3}}{13}$

9. $(2\sqrt{5}-3\sqrt{2})(2\sqrt{5}+3\sqrt{2})$

$= 2\sqrt{5}(2\sqrt{5}+3\sqrt{2}) - 3\sqrt{2}(2\sqrt{5}+3\sqrt{2})$

$= 4\sqrt{25} + 6\sqrt{10} - 6\sqrt{10} - 9\sqrt{4}$

$= 20 - 18 = 2$

10. $\frac{5}{2+\sqrt{3}} = \frac{5}{2+\sqrt{3}} \cdot \frac{2-\sqrt{3}}{2-\sqrt{3}}$

$= \frac{5(2-\sqrt{3})}{4-3} = 5(2-\sqrt{3})$

11. $\frac{2+\sqrt{2}}{1+\sqrt{2}} \cdot \frac{1-\sqrt{2}}{1-\sqrt{2}} = \frac{2-2\sqrt{2}+\sqrt{2}-2}{1+\sqrt{2}-\sqrt{2}-2} = \frac{-\sqrt{2}}{-1} = \sqrt{2}$

$\frac{\sqrt{5}-2}{\sqrt{5}-1} \cdot \frac{\sqrt{5}+1}{\sqrt{5}+1} = \frac{5+\sqrt{5}-2\sqrt{5}-2}{5+\sqrt{5}-\sqrt{5}-1} = \frac{3-\sqrt{5}}{4}$

Hence, $\sqrt{2} \cdot \frac{(3-\sqrt{5})}{4} \cdot (3+\sqrt{5}) = \frac{\sqrt{2}}{4}[3(3+\sqrt{5}) - \sqrt{5}(3+\sqrt{5})]$

$= \frac{\sqrt{2}}{4}[9+3\sqrt{5}-3\sqrt{5}-5]$

$= \frac{\sqrt{2}}{4}[4] = \sqrt{2}$

12. $\dfrac{-1+\sqrt{3}}{1+\sqrt{3}} = \dfrac{-1+\sqrt{3}}{1+\sqrt{3}} \times \dfrac{1-\sqrt{3}}{1-\sqrt{3}}$

$= \dfrac{-1+\sqrt{3}+\sqrt{3}-3}{(1)^2-(\sqrt{3})^2}$

$= \dfrac{-4+2\sqrt{3}}{1-3}$

$= \dfrac{-4+2\sqrt{3}}{-2}$

$= 2-\sqrt{3}$

13. $\dfrac{\sqrt{3}}{1-\sqrt{3}} - \dfrac{1}{\sqrt{3}} = \dfrac{(\sqrt{3})^2-(1-\sqrt{3})}{\sqrt{3}(1-\sqrt{3})}$

$= \dfrac{3-1+\sqrt{3}}{\sqrt{3}-3}$

$= \dfrac{2+\sqrt{3}}{\sqrt{3}-3} \times \dfrac{\sqrt{3}+3}{\sqrt{3}+3}$

$= \dfrac{2\sqrt{3}+6+3+3\sqrt{3}}{(\sqrt{3})^2-9}$

$= \dfrac{9+5\sqrt{3}}{3-9}$

$= \dfrac{9+5\sqrt{3}}{-6}$

$= \dfrac{-9-5\sqrt{3}}{6}$

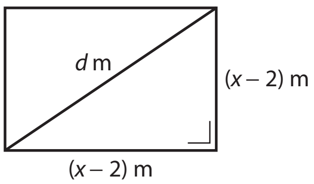
Exercise 2.8

1. $d^2 = (x+2)^2 + (x-2)^2$

$= x^2 + 4x + 4 + x^2 - 4x + 4$

$= 2x^2 + 8$

$\Rightarrow d = \sqrt{2x^2+8}$



2. (a) $|AC|^2 = (2+\sqrt{3})^2 + (2-\sqrt{3})^2$

$= 4 + 4\sqrt{3} + 3 + 4 - 4\sqrt{3} - 3$

$= 14$

$\Rightarrow |AC| = \sqrt{14}$

(b) (i) Distance, 1st Runner $= 2-\sqrt{3}+2+\sqrt{3}+2-\sqrt{3}+2+\sqrt{3} = 8$

Distance, 2nd Runner $= 2\sqrt{14}$

$\Rightarrow$ Difference $= 8 - 2\sqrt{14} = 2(4-\sqrt{14})$km

(ii) Time, 1st Runner $= \dfrac{8000}{1.5} = 5333.33$

Time, 2nd Runner $= \dfrac{2\sqrt{14} \times 1000}{1.4} = 5345.22$

$\Rightarrow$ Difference $= 5345.22 - 5333.33 = 11.89 = 12$ secs

3. $|EG|^2 = (4)^2 + (2)^2 = 16 + 4 = 20$

$\Rightarrow |EG| = \sqrt{20}$

$|KG|^2 = (\sqrt{20})^2 + (2)^2 = 20 + 4 = 24$

$\Rightarrow |KG| = \sqrt{24} = \sqrt{4}\sqrt{6} = 2\sqrt{6}$

Hence: Distance $= 4 + 2 + 2 + 2\sqrt{6} = 8 + 2\sqrt{6}$

4. (i) $x + y = \sqrt{a} + \frac{1}{\sqrt{a}} + \sqrt{a} - \frac{1}{\sqrt{a}} = 2\sqrt{a}$

(ii) $x - y = \sqrt{a} + \frac{1}{\sqrt{a}} - \left(\sqrt{a} - \frac{1}{\sqrt{a}}\right)$

$= \sqrt{a} + \frac{1}{\sqrt{a}} - \sqrt{a} + \frac{1}{\sqrt{a}} = \frac{2}{\sqrt{a}}$

Hence, $\sqrt{x^2 - y^2} = \sqrt{(x + y)(x - y)}$

$= \sqrt{2\sqrt{a} \cdot \frac{2}{\sqrt{a}}} = \sqrt{4} = 2$

5. (i) $\sqrt{2x + 1} = 3$

$\Rightarrow 2x + 1 = 9$

$\Rightarrow 2x = 8$

$\Rightarrow x = 4$

Check $x = 4 \Rightarrow \sqrt{2(4) + 1} = \sqrt{9} = 3$ True

(ii) $\sqrt{3x + 10} = x$

$\Rightarrow 3x + 10 = x^2$

$\Rightarrow x^2 - 3x - 10 = 0$

$\Rightarrow (x + 2)(x - 5) = 0$

$\Rightarrow x = -2, x = 5$

Check $x = -2 \Rightarrow \sqrt{3(-2) + 10} = \sqrt{4} = 2 \neq -2$ False

Check $x = 5 \Rightarrow \sqrt{3(5) + 10} = \sqrt{25} = 5$ True

(iii) $\sqrt{2x - 1} = \sqrt{x + 8}$

$\Rightarrow 2x - 1 = x + 8$

$\Rightarrow \quad x = 9$

Check $x = 9 \Rightarrow \sqrt{2(9) - 1} = \sqrt{9 + 8}$

$\Rightarrow \quad \sqrt{17} = \sqrt{17}$ True

(iv) $\sqrt{3x - 5} = x - 1$

$\Rightarrow (\sqrt{3x - 5})^2 = (x - 1)^2$

$\Rightarrow 3x - 5 = x^2 - 2x + 1$

$\Rightarrow x^2 - 5x + 6 = 0$

$\Rightarrow (x - 2)(x - 3) = 0$

$\Rightarrow x = 2, x = 3$

Check $x = 2 \Rightarrow \sqrt{3(2) - 5} = 2 - 1$

$\Rightarrow \quad 1 = 1$ True

Check $x = 3 \Rightarrow \sqrt{3(3) - 5} = 3 - 1$

$\Rightarrow \quad 2 = 2$ True

(v) $\sqrt{2x + 5} = x + 1$

$\Rightarrow (\sqrt{2x - 5})^2 = (x + 1)^2$

$\Rightarrow 2x + 5 = x^2 + 2x + 1$

$\Rightarrow x^2 = 4$

$\Rightarrow x = -2, x = 2$

Check $x = -2 \Rightarrow \sqrt{2(-2) + 5} = -2 + 1$

$\Rightarrow \quad 1 = -1$ False

Check $x = 2 \Rightarrow \sqrt{2(2) + 5} = 2 + 1$

$\Rightarrow \quad 3 = 3$ True

(vi) $\sqrt{2x^2 - 7} = x + 3$

$\Rightarrow (\sqrt{2x^2 - 7})^2 = (x + 3)^2$

$\Rightarrow 2x^2 - 7 = x^2 + 6x + 9$

$\Rightarrow x^2 - 6x - 16 = 0$

$\Rightarrow (x + 2)(x - 8) = 0$

$\Rightarrow x = -2, x = 8$

Check $x = -2 \Rightarrow \sqrt{2(-2)^2 - 7} = -2 + 3$

$\Rightarrow \quad 1 = 1$ True

Check $x = 8 \Rightarrow \sqrt{2(8)^2 - 7} = 8 + 3$

$\Rightarrow \quad 11 = 11$ True

6. (i) $\sqrt{x + 5} = 5 - \sqrt{x}$

$\Rightarrow (\sqrt{x + 5})^2 = (5 - \sqrt{x})^2$

$\Rightarrow x + 5 = 25 - 10\sqrt{x} + x$

$\Rightarrow 10\sqrt{x} = 20$

$\Rightarrow \sqrt{x} = 2$

$\Rightarrow x = 4$

Check $x = 4 \Rightarrow \sqrt{4 + 5} = 5 - \sqrt{4}$

$\Rightarrow \quad 3 = 3$ True

(ii) $\sqrt{5x+6} = \sqrt{2x} + 2$

$\Rightarrow (\sqrt{5x+6})^2 = (\sqrt{2x} + 2)^2$

$\Rightarrow 5x + 6 = 2x + 4\sqrt{2x} + 4$

$\Rightarrow 3x + 2 = 4\sqrt{2x}$

$\Rightarrow (3x + 2)^2 = (4\sqrt{2x})^2$

$\Rightarrow 9x^2 + 12x + 4 = 16(2x)$

$\Rightarrow 9x^2 - 20x + 4 = 0$

$\Rightarrow (9x - 2)(x - 2) = 0$

$\Rightarrow x = \frac{2}{9}, x = 2$

Check $x = \frac{2}{9} \Rightarrow \sqrt{5\left(\frac{2}{9}\right) + 6} = \sqrt{\frac{4}{9}} + 2$

$\Rightarrow \sqrt{\frac{64}{9}} = \frac{2}{3} + 2$

$\Rightarrow \frac{8}{3} = 2\frac{2}{3}$ True

Check $x = 2 \Rightarrow \sqrt{5(2) + 6} = \sqrt{2(2)} + 2$

$\Rightarrow \sqrt{16} = 2 + 2$

$\Rightarrow 4 = 4$ True

(iii) $\sqrt{x+7} = 7 - \sqrt{x}$

$\Rightarrow (\sqrt{x+7})^2 = (7 - \sqrt{x})^2$

$\Rightarrow x + 7 = 49 - 14\sqrt{x} + x$

$\Rightarrow 14\sqrt{x} = 42$

$\Rightarrow \sqrt{x} = 3$

$\Rightarrow x = 9$

Check $x = 9 \Rightarrow \sqrt{9+7} + \sqrt{9} = 7$

$\Rightarrow 4 + 3 = 7$ True

(iv) $\sqrt{3x-2} = \sqrt{x-2} + 2$

$\Rightarrow (\sqrt{3x-2})^2 = (\sqrt{x-2} + 2)^2$

$\Rightarrow 3x - 2 = x - 2 + 4\sqrt{x-2} + 4$

$\Rightarrow 2x - 4 = 4\sqrt{x-2}$

$\Rightarrow x - 2 = 2\sqrt{x-2}$

$\Rightarrow (x-2)^2 = (2\sqrt{x-2})^2$

$\Rightarrow x^2 - 4x + 4 = 4(x-2) = 4x - 8$

$\Rightarrow x^2 - 8x + 12 = 0$

$\Rightarrow (x-2)(x-6) = 0$

$\Rightarrow x = 2, x = 6$

Check $x = 2 \Rightarrow \sqrt{3(2) - 2} = \sqrt{2-2} + 2$

$\Rightarrow \sqrt{4} = 0 + 2$

$\Rightarrow 2 = 2$ True

Check $x = 6 \Rightarrow \sqrt{3(6) - 2} = \sqrt{6-2} + 2$

$\Rightarrow \sqrt{16} = \sqrt{4} + 2$

$\Rightarrow 4 = 2 + 2$ True

7. $$\frac{1}{\sqrt{x+2}} - \frac{1}{\sqrt{4x+8}} = \frac{1}{\sqrt{x+2}} - \frac{1}{\sqrt{4(x+2)}}$$

$$= \frac{1}{\sqrt{x+2}} - \frac{1}{2\sqrt{x+2}}$$

$$= \frac{2-1}{2\sqrt{x+2}}$$

$$= \frac{1}{2\sqrt{x+2}}$$

Thus

$$\frac{1}{\sqrt{x+2}} - \frac{1}{\sqrt{4x+8}} = 2$$

$$\frac{1}{2\sqrt{x+2}} = 2$$

$$1 = 4\sqrt{x+2}$$

$$1 = 16(x+2)$$

$$1 = 16x + 32$$

$$-31 = 16x$$

$$x = \frac{-31}{16}$$

8. $x = \sqrt{4x + 5}$

$x^2 = 4x + 5$

$x^2 - 4x - 5 = 0$

$(x - 5)(x + 1) = 0$

$x = 5 \quad \text{or} \quad x = -1$

check: $\quad x = 5: \quad 5 = \sqrt{25}$

$5 = 5$ True.

$x = -1: -1 = \sqrt{1}$

$-1 = 1$ False.

Thus the only solution is $x = 5$.

9. $x = \sqrt{a} + \frac{1}{\sqrt{a}} + 1$

$\Rightarrow x - 2 = \sqrt{a} + \frac{1}{\sqrt{a}} + 1 - 2 = \sqrt{a} + \frac{1}{\sqrt{a}} - 1$

Hence, $x^2 - 2x = x(x - 2)$

$$= \left(\sqrt{a} + \frac{1}{\sqrt{a}} + 1\right)\left(\sqrt{a} + \frac{1}{\sqrt{a}} - 1\right)$$

$$= \sqrt{a}\left(\sqrt{a} + \frac{1}{\sqrt{a}} - 1\right) + \frac{1}{\sqrt{a}}\left(\sqrt{a} + \frac{1}{\sqrt{a}} - 1\right) + 1\left(\sqrt{a} + \frac{1}{\sqrt{a}} - 1\right)$$

$$= a + 1 - \sqrt{a} + 1 + \frac{1}{a} - \frac{1}{\sqrt{a}} + \sqrt{a} + \frac{1}{\sqrt{a}} - 1$$

$$= a + \frac{1}{a} + 1$$

10. $(a + \sqrt{3})(b - \sqrt{3}) = 7 + 3\sqrt{3}$

$\Rightarrow ab - \sqrt{3}a + \sqrt{3}b - 3 = 7 + 3\sqrt{3}$

$\Rightarrow (ab - 3) + (-a + b)\sqrt{3} = 7 + 3\sqrt{3}$

$\Rightarrow ab - 3 = 7 \text{ and } -a + b = 3$

$\Rightarrow \qquad b = a + 3$

$\Rightarrow a(a + 3) - 3 - 7 = 0$

$\Rightarrow a^2 + 3a - 10 = 0$

$\Rightarrow (a + 5)(a - 2) = 0$

$\Rightarrow a = -5$ (Not valid), $a = 2$ (Valid)

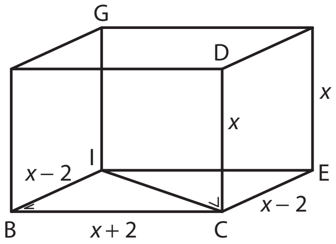
$\Rightarrow b = 2 + 3 = 5$

11. (i) $|IC|^2 = (x + 2)^2 + (x - 2)^2$

$= x^2 + 4x + 4 + x^2 - 4x + 4$

$= 2x^2 + 8$

$\Rightarrow |IC| = \sqrt{2x^2 + 8}$



(ii) $|ID|^2 = (\sqrt{2x^2 + 8})^2 + x^2$

$= 2x^2 + 8 + x^2$

$= 3x^2 + 8$

$\Rightarrow |ID| = \sqrt{3x^2 + 8}$

Hence, $\sqrt{3x^2 + 8} = \sqrt{56}$

$\Rightarrow \quad 3x^2 + 8 = 56$

$\Rightarrow \quad 3x^2 = 48$

$\Rightarrow \quad x^2 = 16$

$\Rightarrow x = -4$ (Not valid), $x = 4$ (Valid)

Exercises 2.9

1. $f(x) = x^2 - 8x + 15$ If $(x - 3)$ is a factor, then $x = 3$ is a root.

$\Rightarrow f(3) = (3)^2 - 8(3) + 15$
$= 9 - 24 + 15$
$= 24 - 24 = 0 \quad \Rightarrow (x - 3)$ is a factor

2. If $(x - 1)$ is a factor, then $x = 1$ is a root.

$f(x) = x^3 - x^2 - 9x + 9$
$\Rightarrow f(1) = (1)^3 - (1)^2 - 9(1) + 9$
$= 1 - 1 - 9 + 9 = 10 - 10 = 0$
$\Rightarrow (x - 1)$ is a factor

3. If $(x + 2)$ is a factor, then $x = -2$ is a root.

$f(x) = x^3 + 6x^2 + 11x + 6$
$\Rightarrow f(-2) = (-2)^3 + 6(-2)^2 + 11(-2) + 6$
$= -8 + 24 - 22 + 6$
$= 30 - 30 = 0$
$\Rightarrow (x + 2)$ is a factor

4. If $(x - 2)$ is a factor, then $x = 2$ is a root.

$f(x) = 2x^3 - 3x^2 - 12x + 20$
$\Rightarrow f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 20$
$= 16 - 12 - 24 + 20$
$= 36 - 36 = 0 \quad \Rightarrow x - 2$ is a factor

5. If $(x - 2)$ is a factor, then $x = 2$ is a root.

$f(x) = x^3 - 5x^2 + 8x - 4$
$\Rightarrow f(2) = (2)^3 - 5(2)^2 + 8(2) - 4$
$= 8 - 20 + 16 - 4$
$= 24 - 24 = 0 \quad \Rightarrow (x - 2)$ is a factor

6. If $(2x - 1)$ is a factor $\Rightarrow x = \frac{1}{2}$ is a root.

$f(x) = 2x^3 + 7x^2 + 2x - 3$

$$\Rightarrow f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 + 7\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) - 3$$

$$= \frac{1}{4} + \frac{7}{4} + 1 - 3$$

$= 3 - 3 = 0 \quad \Rightarrow (2x - 1)$ is a factor

7. If $(2x + 1)$ is a factor $\Rightarrow x = -\frac{1}{2}$ is a root.

$f(x) = 2x^3 - x^2 - 5x - 2$

$$\Rightarrow f\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^3 - \left(-\frac{1}{2}\right)^2 - 5\left(-\frac{1}{2}\right) - 2$$

$$= -\frac{1}{4} - \frac{1}{4} + \frac{5}{2} - 2$$

$= \frac{5}{2} - \frac{5}{2} = 0 \quad \Rightarrow (2x + 1)$ is a factor

8. If $(x - 1)$ is a factor $\Rightarrow x = 1$ is a root.

$f(x) = x^3 + kx^2 - x - 8$
$\Rightarrow f(1) = (1)^3 + k(1)^2 - (1) - 8 = 0$
$\Rightarrow 1 + k - 1 - 8 = 0$
$\Rightarrow k = 8$

9. If $(x + 2)$ is a factor $\Rightarrow x = -2$ is a root.

$f(x) = x^3 + 6x^2 + px + 6$

$\Rightarrow f(-2) = (-2)^3 + 6(-2)^2 + p(-2) + 6 = 0$

$\Rightarrow -8 + 24 - 2p + 6 = 0$

$-2p = -22$

$\Rightarrow p = 11$

10. If $(x - 3)$ is a factor $\Rightarrow x = 3$ is a root.

$f(x) = x^3 - 2x^2 - 5x + 6$

$\Rightarrow f(3) = (3)^3 - 2(3)^2 - 5(3) + 6$

$= 27 - 18 - 15 + 6 = 33 - 33 = 0 \Rightarrow x = 3$ is a root.

$$
\begin{array}{r|l}
 & x^2 + x - 2 \\
\hline
x - 3 & x^3 - 2x^2 - 5x + 6 \\
 & \underline{x^3 - 3x^2} \\
 & \quad x^2 - 5x \\
 & \quad \underline{x^2 - 3x} \\
 & \qquad -2x + 6 \\
 & \qquad \underline{-2x + 6} \\
 & \qquad\qquad 0
\end{array}
$$

$\Rightarrow x^2 + x - 2$

$= (x + 2)(x - 1)$

$\therefore$ the other two factors are $(x + 2)$ and $(x - 1)$

11. If $(x + 3)$ is a factor $\Rightarrow x = -3$ is a root.

$f(x) = x^3 - 2x^2 - 9x + 18$

$\Rightarrow f(-3) = (-3)^3 - 2(-3)^2 - 9(-3) + 18$

$= -27 - 18 + 27 + 18 = 45 - 45 = 0 \Rightarrow x = -3$ is a root.

$$
\begin{array}{r|l}
 & x^2 - 5x + 6 \\
\hline
x + 3 & x^3 - 2x^2 - 9x + 18 \\
 & \underline{x^3 + 3x^2} \\
 & \quad -5x^2 - 9x \\
 & \quad \underline{-5x^2 - 15x} \\
 & \qquad\quad 6x + 18 \\
 & \qquad\quad \underline{6x + 18} \\
 & \qquad\qquad\quad 0
\end{array}
$$

$\Rightarrow x^2 - 5x + 6$

$= (x - 2)(x - 3)$

$\therefore$ the other two factors are $(x - 2)$ and $(x - 3)$

12. (i) $f(x) = x^3 - 4x^2 - x + 4$

$\Rightarrow f(1) = (1)^3 - 4(1)^2 - (1) + 4$

$= 1 - 4 - 1 + 4 = 5 - 5 = 0$ $\Rightarrow x = 1$ is a root

$\Rightarrow (x - 1)$ is a factor.

$$
\begin{array}{r|l}
 & x^2 - 3x - 4 \\
\hline
x - 1 & x^3 - 4x^2 - x + 4 \\
 & \underline{x^3 - x^2} \\
 & \quad -3x^2 - x \\
 & \quad \underline{-3x^2 + 3x} \\
 & \qquad\quad -4x + 4 \\
 & \qquad\quad \underline{-4x + 4} \\
 & \qquad\qquad\quad 0
\end{array}
$$

$\Rightarrow x^2 - 3x - 4 = (x + 1)(x - 4)$

(ii) $f(x) = x^3 - 8x^2 + 19x - 12$

$\Rightarrow f(1) = (1)^3 - 8(1)^2 + 19(1) - 12$

$= 1 - 8 + 19 - 12 = 20 - 20 = 0$ $\Rightarrow x = 1$ is a root

$\Rightarrow (x - 1)$ is a factor.

$$\begin{array}{r|l} & x^2 - 7x + 12 \\ \hline x - 1 & x^3 - 8x^2 + 19x - 12 \\ & \underline{x^3 - x^2} \\ & \quad -7x^2 + 19x \\ & \quad \underline{-7x^2 + 7x} \\ & \qquad\quad 12x - 12 \\ & \qquad\quad \underline{12x - 12} \\ & \qquad\qquad\quad 0 \end{array}$$

$\Rightarrow x^2 - 7x + 12$

$= (x - 3)(x - 4)$

(iii) $f(x) = x^3 + 6x^2 - x - 30$

$\Rightarrow f(2) = (2)^3 + 6(2)^2 - (2) - 30$

$= 8 + 24 - 2 - 32 = 32 - 32 = 0$ $\Rightarrow x = 2$ is a root

$\Rightarrow (x - 2)$ is a factor.

$$\begin{array}{r|l} & x^2 + 8x + 15 \\ \hline x - 2 & x^3 + 6x^2 - x - 30 \\ & \underline{x^3 - 2x^2} \\ & \quad 8x^2 - x \\ & \quad \underline{8x^2 - 16x} \\ & \qquad\quad 15x - 30 \\ & \qquad\quad \underline{15x - 30} \\ & \qquad\qquad\quad 0 \end{array}$$

$\Rightarrow x^2 + 8x + 15$

$= (x + 3)(x + 5)$

(iv) $f(x) = 3x^3 - 4x^2 - 3x + 4$

$\Rightarrow f(1) = 3(1)^3 - 4(1)^2 - 3(1) + 4$

$= 3 - 4 - 3 + 4 = 7 - 7 = 0$ $\Rightarrow x = 1$ is a root

$\Rightarrow (x - 1)$ is a factor.

$$\begin{array}{r|l} & 3x^2 - x - 4 \\ \hline x - 1 & 3x^3 - 4x^2 - 3x + 4 \\ & \underline{3x^3 - 3x^2} \\ & \quad -x^2 - 3x \\ & \quad \underline{-x^2 + x} \\ & \qquad\quad -4x + 4 \\ & \qquad\quad \underline{-4x + 4} \\ & \qquad\qquad\quad 0 \end{array}$$

$\Rightarrow 3x^2 - x - 4$

$= (3x - 4)(x + 1)$

(v) $f(x) = 2x^3 - 3x^2 - 8x - 3$

$f(-1) = 2(-1)^3 - 3(-1)^2 - 8(-1) - 3$

$= -2 - 3 + 8 - 3 = 8 - 8 = 0$ $\Rightarrow x = -1$ is a root

$\Rightarrow (x + 1)$ is a factor.

$$\begin{array}{r|l} & 2x^2 - 5x - 3 \\ \hline x + 1 & 2x^3 - 3x^2 - 8x - 3 \\ & \underline{2x^3 + 2x^2} \\ & \quad -5x^2 - 8x \\ & \quad \underline{-5x^2 - 5x} \\ & \qquad\quad -3x - 3 \\ & \qquad\quad \underline{-3x - 3} \\ & \qquad\qquad\quad 0 \end{array}$$

$\Rightarrow 2x^2 - 5x - 3$

$= (2x + 1)(x - 3)$

(vi) $f(x) = 2x^3 - 3x^2 - 12x + 20$

$f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 20$

$= 16 - 12 - 24 + 20 = 36 - 36 = 0$ $\Rightarrow x = 2$ is a root

$\Rightarrow (x - 2)$ is a factor.

$$
\begin{array}{r|l}
 & 2x^2 + x - 10 \\
\hline
x - 2 & 2x^3 - 3x^2 - 12x + 20 \\
 & \underline{2x^3 - 4x^2} \\
 & \quad x^2 - 12x \\
 & \quad \underline{x^2 - 2x} \\
 & \qquad -10x + 20 \\
 & \qquad \underline{-10x + 20} \\
 & \qquad\qquad 0
\end{array}
$$

$\Rightarrow 2x^2 + x - 10$

$\Rightarrow (2x + 5)(x - 2)$

13. $f(x) = 2x^3 + 13x^2 + 13x - 10$

$f(-2) = 2(-2)^3 + 13(-2)^2 + 13(-2) - 10$

$= -16 + 52 - 26 - 10 = 52 - 52 = 0$ $\Rightarrow x = -2$ is a root

$\Rightarrow (x + 2)$ is a factor.

$$
\begin{array}{r|l}
 & 2x^2 + 9x - 5 \\
\hline
x + 2 & 2x^3 + 13x^2 + 13x - 10 \\
 & \underline{2x^3 + 4x^2} \\
 & \quad 9x^2 + 13x \\
 & \quad \underline{9x^2 + 18x} \\
 & \qquad -5x - 10 \\
 & \qquad \underline{-5x - 10} \\
 & \qquad\qquad 0
\end{array}
$$

$\Rightarrow 2x^2 + 9x - 5$

$\Rightarrow (2x - 1)(x + 5)$

14. $(x + 2)$ is a factor $\Rightarrow x = -2$ is a root

$f(x) = x^3 + ax^2 - x - 2$

$\Rightarrow f(-2) = (-2)^3 + a(-2)^2 - (-2) - 2 = 0$

$\Rightarrow -8 + 4a + 2 - 2 = 0$

$\Rightarrow 4a = 8$

$\Rightarrow a = 2$

$$
\begin{array}{r|l}
 & x^2 - 1 \\
\hline
x + 2 & x^3 + 2x^2 - x - 2 \\
 & \underline{x^3 + 2x^2} \\
 & \qquad 0 - x - 2 \\
 & \qquad \underline{-x - 2} \\
 & \qquad\qquad 0
\end{array}
$$

$\Rightarrow x^2 - 1 = (x + 1)(x - 1)$

15. $f(x) = x^3 - x^2 - 14x + 24$

$f(2) = (2)^3 - (2)^2 - 14\,(2) + 24$

$= 8 - 4 - 28 + 24 = 32 - 32 = 0$ $\Rightarrow x = 2$ is a root

$\Rightarrow (x - 2)$ is a factor.

$$
\begin{array}{r|l}
 & x^2 + x - 12 \\
\hline
x - 2 & x^3 - x^2 - 14x + 24 \\
 & \underline{x^3 - 2x^2} \\
 & \quad x^2 - 14x \\
 & \quad \underline{x^2 - 2x} \\
 & \qquad -12x + 24 \\
 & \qquad \underline{-12x + 24} \\
 & \qquad\qquad 0
\end{array}
$$

$\Rightarrow x^2 + x - 12$

$= (x - 3)(x + 4)$

Hence, roots $x = 2, 3, -4$

16. $f(x) = x^3 + 5x^2 + 2x - 8$

$\Rightarrow f(1) = (1)^3 + 5(1)^2 + 2(1) - 8$

$= 1 + 5 + 2 - 8 = 8 - 8 = 0$ $\qquad \Rightarrow x = 1$ is a root

$\Rightarrow (x - 1)$ is a factor.

$$\begin{array}{r l}
 & x^2 + 6x + 8 \\
x - 1 & \overline{)\, x^3 + 5x^2 + 2x - 8} \\
 & \underline{x^3 - x^2} \\
 & \quad 6x^2 + 2x \\
 & \quad \underline{6x^2 - 6x} \\
 & \qquad 8x - 8 \\
 & \qquad \underline{8x - 8} \\
 & \qquad\quad 0
\end{array}$$

$\Rightarrow x^2 + 6x + 8 = 0$

$\Rightarrow (x + 2)(x + 4) = 0$

$\Rightarrow$ roots $x = -2, -4$

17. (i) $f(x) = x^3 - 4x^2 - x + 4$

$f(1) = (1)^3 - 4(1)^2 - (1) + 4$

$= 1 - 4 - 1 + 4 = 5 - 5 = 0$ $\qquad \Rightarrow x = 1$ is a root

$\Rightarrow (x - 1)$ is a factor.

$$\begin{array}{r l}
 & x^2 - 3x - 4 \\
x - 1 & \overline{)\, x^3 - 4x^2 - x + 4} \\
 & \underline{x^3 - x^2} \\
 & \quad -3x^2 - x \\
 & \quad \underline{-3x^2 + 3x} \\
 & \qquad -4x + 4 \\
 & \qquad \underline{-4x + 4} \\
 & \qquad\quad 0
\end{array}$$

$\Rightarrow x^2 - 3x - 4 = 0$

$\Rightarrow (x + 1)(x - 4) = 0$

$\Rightarrow$ roots $x = -1, 4$

(ii) $f(x) = x^3 + 2x^2 - 11x - 12$

$f(-1) = (-1)^3 + 2(-1)^2 - 11(-1) - 12$

$= -1 + 2 + 11 - 12 = 13 - 13 = 0$ $\qquad \Rightarrow x = -1$ is a root

$\Rightarrow (x + 1)$ is a factor.

$$\begin{array}{r l}
 & x^2 + x - 12 \\
x + 1 & \overline{)\, x^3 + 2x^2 - 11x - 12} \\
 & \underline{x^3 + x^2} \\
 & \quad x^2 - 11x \\
 & \quad \underline{x^2 + x} \\
 & \qquad -12x - 12 \\
 & \qquad \underline{-12x - 12} \\
 & \qquad\quad 0
\end{array}$$

$\Rightarrow x^2 + x - 12 = 0$

$\Rightarrow (x + 4)(x - 3) = 0$

$\Rightarrow$ roots $x = -4, 3$

(iii) $f(x) = 3x^3 - 4x^2 - 3x + 4$

$f(1) = 3(1)^3 - 4(1)^2 - 3(1) + 4$

$= 3 - 4 - 3 + 4 = 7 - 7 = 0$ $\qquad \Rightarrow x = 1$ is a root

$\Rightarrow (x - 1)$ is a factor.

$$\begin{array}{r l}
 & 3x^2 - x - 4 \\
x - 1 & \overline{)\, 3x^3 - 4x^2 - 3x + 4} \\
 & \underline{3x^3 - 3x^2} \\
 & \quad -x^2 - 3x \\
 & \quad \underline{-x^2 + x} \\
 & \qquad -4x + 4 \\
 & \qquad \underline{-4x + 4} \\
 & \qquad\quad 0
\end{array}$$

$\Rightarrow 3x^2 - x - 4 = 0$

$\Rightarrow (3x - 4)(x + 1) = 0$

$3x = 4$ or $x = -1$

$\Rightarrow$ roots $x = \frac{4}{3}, -1$

(iv) $f(x) = x^3 - 7x - 6$

$f(-1) = (-1)^3 - 7(-1) - 6$

$= -1 + 7 - 6 = 7 - 7 = 0$ $\quad \Rightarrow x = -1$ is a root

$\Rightarrow (x + 1)$ is a factor.

$$\begin{array}{r l}
 & x^2 - x - 6 \\
x + 1 & \overline{)\, x^3 - 7x - 6} \\
 & \underline{x^3 + x^2} \\
 & -x^2 - 7x \\
 & \underline{-x^2 - x} \\
 & -6x - 6 \\
 & \underline{-6x - 6} \\
 & 0
\end{array}$$

$\Rightarrow x^2 - x - 6 = 0$

$\Rightarrow (x + 2)(x - 3) = 0$

$\Rightarrow$ roots $x = -2, 3$

18. If $(x + 1)$ is a factor $\Rightarrow x = -1$ is a root

$f(x) = 2x^3 + ax^2 + bx - 3$

$f(-1) = 2(-1)^3 + a(-1)^2 + b(-1) - 3 = 0$

$\Rightarrow -2 + a - b - 3 = 0 \Rightarrow a - b = 5$

If $(x + 3)$ is a factor $\Rightarrow x = -3$ is a root

$\Rightarrow f(-3) = 2(-3)^3 + a(-3)^2 + b(-3) - 3 = 0$

$\Rightarrow -54 + 9a - 3b - 3 = 0$

$\Rightarrow \quad 9a - 3b = 57$

$\Rightarrow \quad 3a - b = 19$

and $\quad \underline{a - b = 5}$

$\Rightarrow \quad 2a = 14$

$\Rightarrow \quad a = 7$

$\Rightarrow \quad 7 - b = 5$

$\Rightarrow \quad b = 2$

Hence, $(x + 1)(x + 3) = x^2 + 4x + 3$

$$\begin{array}{r l}
 & 2x - 1 \\
\Rightarrow x^2 + 4x + 3 & \overline{)\, 2x^3 + 7x^2 + 2x - 3} \\
 & \underline{2x^3 + 8x^2 + 6x} \\
 & -x^2 - 4x - 3 \\
 & \underline{-x^2 - 4x - 3} \\
 & 0
\end{array}$$

$2x - 1 = 0$

$\Rightarrow 3^{rd}$ root $x = \dfrac{1}{2}$

19. If $(x + 1)$ is a factor $\Rightarrow x = -1$ is a root

$f(x) = x^3 + 5x^2 + kx - 12$

$f(-1) = (-1)^3 + 5(-1)^2 + k(-1) - 12 = 0$

$\Rightarrow -1 + 5 - k - 12 = 0$

$-k - 8 = 0$

$\Rightarrow k = -8$

$$\begin{array}{r l}
 & x^2 + 4x - 12 \\
\Rightarrow x + 1 & \overline{)\, x^3 + 5x^2 - 8x - 12} \\
 & \underline{x^3 + x^2} \\
 & 4x^2 - 8x \\
 & \underline{4x^2 + 4x} \\
 & -12x - 12 \\
 & \underline{-12x - 12} \\
 & 0
\end{array}$$

$\Rightarrow x^2 + 4x - 12$

$\Rightarrow (x + 6)(x - 2)$ other 2 factors

20. If $(x+2)$ is a factor $\Rightarrow x=-2$ is a root

$f(x)=2x^3+ax^2-17x+b$

$f(-2)=2(-2)^3+a(-2)^2-17(-2)+b=0$

$\Rightarrow -16+4a+34+b=0 \Rightarrow 4a+b=-18$

If $(x-3)$ is a factor $\Rightarrow x=3$ is a root

$\Rightarrow f(3)=2(3)^3+a(3)^2-17(3)+b=0$

$\Rightarrow 54+9a-51+b=0$

$\Rightarrow 9a+b=-3$

and $\underline{4a+b=-18}$

$\Rightarrow 5a=15$

$\Rightarrow a=3$

$\Rightarrow 4(3)+b=-18$

$b=-30$

Hence, $(x+2)(x-3)=x^2-x-6$

$$\begin{array}{r|l} & 2x+5 \\ \Rightarrow x^2-x-6 & 2x^3+3x^2-17x-30 \\ & \underline{2x^3-2x^2-12x} \\ & \quad 5x^2-5x-30 \\ & \quad \underline{5x^2-5x-30} \\ & 0 \end{array}$$

$\Rightarrow$ 3rd factor $(2x+5)$

21. $x^2-2x-3=(x+1)(x-3)=0$

$=$ roots $x=-1, x=3$

$f(x)=ax^3+8x^2+bx+6$

$\Rightarrow f(-1)=a(-1)^3+8(-1)^2+b(-1)+6=0$

$\Rightarrow -a+8-b+6=0$

$\Rightarrow a+b=14$

$f(3)=a(3)^3+8(3)^2+b(3)+6=0$

$\Rightarrow 27a+72+3b+6=0$

$\Rightarrow 27a+3b=-78$

$\Rightarrow 9a+b=-26$

and $\underline{a+b=14}$

$8a=-40$

$\Rightarrow a=-5$

$\Rightarrow -5+b=14$

$\Rightarrow b=19$

$$\begin{array}{r|l} & -5x-2 \\ \text{Hence,}\quad x^2-2x-3 & -5x^3+8x^2+19x+6 \\ & \underline{-5x^3+10x^2+15x} \\ & \quad -2x^2+4x+6 \\ & \quad \underline{-2x^2+4x+6} \\ & 0 \end{array}$$

$\Rightarrow$ 3rd factor $=-5x-2$

$\Rightarrow$ 3rd root: $-5x=2$

$\Rightarrow x=-\frac{2}{5}$

22. (i) $ax^3-b=c$

$\Rightarrow ax^3=b+c$

$\Rightarrow x^3=\frac{b+c}{a}$

$\Rightarrow x=\sqrt[3]{\frac{b+c}{a}}$

(ii) $a(x+b)^3=c$

$\Rightarrow (x+b)^3=\frac{c}{a}$

$\Rightarrow x+b=\left(\frac{c}{a}\right)^{\frac{1}{3}}$

$\Rightarrow x=\left(\frac{c}{a}\right)^{\frac{1}{3}}-b$

Exercise 2.10

1. (i) Graph crosses the x-axis at $x = -1, 1, 3$

$\Rightarrow f(x) = a(x+1)(x-1)(x-3)$

Point $(0, 3) \Rightarrow f(0) = a(0+1)(0-1)(0-3) = 3$

$\Rightarrow 3a = 3$

$\Rightarrow \quad a = 1$

$\therefore f(x) = 1(x+1)(x-1)(x-3) = x^3 - 3x^2 - x + 3$

(ii) Graph crosses the x-axis at $x = -4, 1, 2$

$\Rightarrow f(x) = a(x+4)(x-1)(x-2)$

Point $(0, 8) \Rightarrow f(0) = a(0+4)(0-1)(0-2) = 8$

$\Rightarrow 8a = 8$

$\Rightarrow \quad a = 1$

$\therefore f(x) = 1(x+4)(x-1)(x-2) = x^3 + x^2 - 10x + 8$

2. (i) Graph crosses the x-axis at $x = -3, 0, 2$

$\Rightarrow f(x) = a(x+3)(x)(x-2)$

Point $(1, -4) \Rightarrow f(1) = a(1+3)(1)(1-2) = -4$

$\Rightarrow -4a = -4$

$\Rightarrow \quad a = 1$

$\therefore f(x) = 1(x+3)(x)(x-2) = x^3 + x^2 - 6x$ (Green Graph)

Point $(1, -12) \Rightarrow f(1) = a(1+3)(1)(1-2) = -12$

$\Rightarrow -4a = -12$

$\Rightarrow \quad a = 3$

$\therefore f(x) = 3(x+3)(x)(x-2) = 3x^3 + 3x^2 - 18x$ (Blue Graph)

(ii) Graph crosses the x-axis at $x = 1, 2, 3$

$\Rightarrow f(x) = a(x-1)(x-2)(x-3)$

Point $(0, 6) \Rightarrow f(0) = a(0-1)(0-2)(0-3) = 6$

$\Rightarrow -6a = 6 \Rightarrow a = -1$

$\therefore f(x) = -1(x-1)(x-2)(x-3) = -x^3 + 6x^2 - 11x + 6$ (Red Graph)

Point $(0, 12) \Rightarrow f(0) = a(0-1)(0-2)(0-3) = 12$

$\Rightarrow -6a = 12$

$\Rightarrow \quad a = -2$

$\therefore f(x) = -2(x-1)(x-2)(x-3) = -2x^3 + 12x^2 - 22x + 12$ (Blue Graph)

3. Graph crosses the x-axis at $x = 1, -2, \frac{1}{2}$

$\Rightarrow f(x) = a(x-1)(x+2)(2x-1)$

Point $(0, 6) \Rightarrow f(0) = a(0-1)(0+2)(0-1) = 6$

$\Rightarrow 2a = 6 \Rightarrow a = 3$

$\therefore f(x) = 3(x-1)(x+2)(2x-1)$

$= 6x^3 + 3x^2 - 15x + 6$

4. $f(x) = (x-3)(x+1)(x+2)$

$= (x-3)(x^2 + 3x + 2)$

$= x^3 - 7x - 6 \Rightarrow a = 0, b = -7, c = -6$

5. (i) $f(x) = x^3 + 2$ Point $(0, 2) \Rightarrow f(0) = 0^3 + 2 = 2$ True

(ii) $g(x) = x^3$ Point $(1,1) \Rightarrow f(1) = (1)^3 = 1$ True

(iii) $h(x) = 2x^3$ Point $(1, 2) \Rightarrow f(1) = 2(1)^3 = 2$ True

$A(2\frac{1}{3}, 4)$

6. $f(x) = x(x-4)(x-6)$

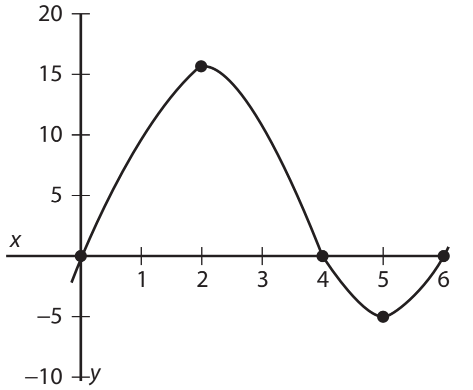
$f(2) = 2(2-4)(2-6)$

$= 2(-2)(-4) = 16$

$f(5) = 5(5-4)(5-6)$

$= 5(1)(-1) = -5$

Points: (0, 0) (4, 0) (6, 0) (2, 16) (5, −5)



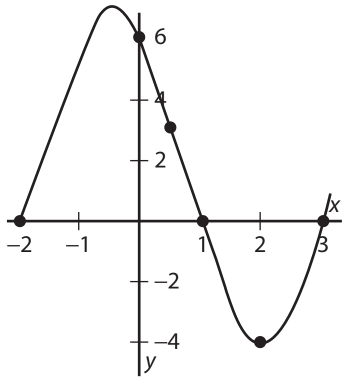
7. $f(x) = (x+2)(x-1)(x-3)$

$f(0) = (0+2)(0-1)(0-3) = 6$

$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}+2\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-3\right) = 3\frac{1}{8}$

$f(2) = (2+2)(2-1)(2-3) = -4$

Points: (−2, 0) (1, 0) (3, 0) (0, 6) $\left(\frac{1}{2}, 3\frac{1}{8}\right)$ (2, −4)



8. (i) Roots are $x = -1, 1, 2$

$\Rightarrow f(x) = a(x+1)(x-1)(x-2)^2$

(ii) Point (0, 4) $\Rightarrow f(0) = a(0+1)(0-1)(0-2)^2 = 4$

$\Rightarrow -4a = 4$

$a = -1$

Hence, $f(x) = -1(x^2-1)(x^2-4x+4)$

$= -1x^4 + 4x^3 - 3x^2 - 4x + 4 = ax^4 + bx^3 + cx^2 + dx + e$

$\Rightarrow a = -1, b = 4, c = -3, d = -4, e = 4$

9. (i) $(0, 4) \in f(x) \Rightarrow f(0) = 4$

$(0, -2) \in g(x) \Rightarrow g(0) = -2$

Hence, $4 = a(-2)$

$\Rightarrow a = -2$

(ii) $f(x)$ has roots $x = -2, 1$ (double roots)

$\Rightarrow f(x) = a(x+2)^2(x-1)^2$

Point (0, 4) $\Rightarrow f(0) = a(0+2)^2(0-1)^2 = 4$

$\Rightarrow 4a = 4$

$\Rightarrow a = 1$

Hence, $f(x) = (x+2)^2(x-1)^2$

$g(x)$ has roots $x = -2, 1$ (double roots)

$\Rightarrow g(x) = a(x+2)^2(x-1)^2$

Point (0, −2) $\Rightarrow g(0) = a(0+2)^2(0-1)^2 = -2$

$\Rightarrow 4a = -2$

$\Rightarrow a = -\frac{1}{2}$

Hence, $g(x) = -\frac{1}{2}(x+2)^2(x-1)^2$

10. (i) Roots are $x = -1, 2, 5$

$\Rightarrow f(x) = (x + 1)(x - 2)(x - 5) = 0$

$\Rightarrow x^3 - 6x^2 + 3x + 10 = 0$

(ii) Roots are $x = -3, -1, 0$

$\Rightarrow f(x) = (x + 3)(x + 1)(x - 0) = 0$

$\Rightarrow x^3 - 4x^2 + 3x = 0$

(iii) Roots are $x = -2, \frac{1}{4}, 3$

$\Rightarrow f(x) = (x + 2)(x - \frac{1}{4})(x - 3) = 0$

$\Rightarrow (x + 2)(4x - 1)(x - 3) = 0$

$\Rightarrow 4x^3 - 5x^2 - 23x + 6 = 0$

(iv) Roots are $x = \frac{1}{2}, 2, 4$

$\Rightarrow f(x) = \left(x - \frac{1}{2}\right)(x - 2)(x - 4) = 0$

$\Rightarrow (2x - 1)(x - 2)(x - 4) = 0$

$\Rightarrow 2x^3 - 13x^2 + 22x - 8 = 0$

11. (i) Roots are $x = -\frac{1}{2}, 3, 6$

$\Rightarrow f(x) = a\left(x + \frac{1}{2}\right)(x - 3)(x - 6)$

$\Rightarrow f(x) = a(2x + 1)(x - 3)(x - 6)$

Point $(1, 30) \Rightarrow f(1) = a(2(1) + 1)(1 - 3)(1 - 6) = 30$

$\Rightarrow 30a = 30$

$\Rightarrow a = 1$

$\Rightarrow f(x) = 1(2x + 1)(x - 3)(x - 6)$

$\Rightarrow f(x) = 2x^3 - 17x^2 + 27x + 18$

(ii) Roots are $x = -4, -\frac{1}{2}, 2\frac{1}{2}$

$\Rightarrow f(x) = a(x + 4)(x + \frac{1}{2})(x - 2\frac{1}{2})$

$\Rightarrow f(x) = a(x + 4)(2x + 1)(2x - 5)$

Point $(0, 20) \Rightarrow f(0) = a(0 + 4)(0 + 1)(0 - 5) = 20$

$\Rightarrow -20a = 20 \Rightarrow a = -1$

$\Rightarrow f(x) = -1(x + 4)(2x + 1)(2x - 5)$

$\Rightarrow f(x) = -4x^3 - 8x^2 + 37x + 20$

12. 2 Roots are $x = 2, 4$

$\Rightarrow (x - 2)(x - 4) = (x^2 - 6x + 8)$

$$
\begin{array}{r|l}
 & -3x - 1 \\
\hline
\text{Hence, } x^2 - 6x + 8 & -3x^3 + 17x^2 + bx - 8 \\
 & \underline{-3x^3 + 18x^2 - 24x} \\
 & \quad -x^2 + (b + 24)x - 8 \\
 & \quad \underline{-x^2 + 6x \qquad\quad - 8} \\
 & \qquad\qquad\qquad\qquad 0
\end{array}
$$

Hence, $b + 24 - 6 = 0$ and $-3x - 1 = 0$

$\Rightarrow b = -18$ $\qquad \Rightarrow x = -\frac{1}{3}$

$\Rightarrow a = -\frac{1}{3}$

13. (i) Graph $f(x) = 0$ meets x-axis at $x = -1, 0, 2$

(ii) $f(x) = g(x) \Rightarrow$ Curve meets line at $x = -1.3, 0, 2.3$

(iii) $f(x) = g(x) \Rightarrow x^3 - x^2 - 2x = x$

$\Rightarrow x^3 - x^2 - 3x = 0$

$\Rightarrow x(x^2 - x - 3) = 0$

$\Rightarrow x = 0 \quad \text{OR} \quad x = \dfrac{1 \pm \sqrt{(-1)^2 - 4(1)(-3)}}{2(1)}$

$x = \dfrac{1 \pm \sqrt{13}}{2}$

$x = \dfrac{1 + 3.60555}{2}, \dfrac{1 - 3.60555}{2}$

$= 2.303, -1.303$

$= 2.3, \quad -1.3$

$\Rightarrow x = 0, 2.3$ or -1.3

14. (i) Volume $= x(x - 1)(x + 1) = (x^3 - x)$ cm^3

(ii) Volume $= 24 = x^3 - x$

$\Rightarrow f(x) = x^3 - x - 24$

$f(3) = (3)^3 - (3) - 24 = 27 - 27 = 0$

$\Rightarrow x = 3$ is a root

15. $h = 2r \Rightarrow r = \dfrac{h}{2}$

$V = \pi\left(\dfrac{h}{2}\right)^2 \cdot h = \dfrac{\pi}{4} h^3$

(i) Hence, $ah^3 = \dfrac{\pi}{4}h^3 \Rightarrow a = \dfrac{3.14}{4} = 0.785 = 0.79$

(ii) Volume cylinder $= ah^3 = (0.79)(11)^3 = 1051.49$ cm^3

(iii) Volume $= 215.58 = ah^3$

$\Rightarrow (0.79)h^3 = 215.58$

$\Rightarrow h^3 = \dfrac{215.58}{0.79} = 272.886$

$\Rightarrow h = \sqrt[3]{272.886} = 6.486 = 6.5$ cm

16. Volume of cube $= (3)^3 = 27$

Volume of sphere $= (4.19)(27) = 113.13$

Volume of cube $= x^3 = 113.13$

$\Rightarrow x = \sqrt[3]{113.13} = 4.836 = 4.8$ cm

Hence, $4.19x^3 - 150 = x^3$

$\Rightarrow 3.19x^3 = 150$

$\Rightarrow x^3 = \dfrac{150}{3.19} = 47.0219$

$\Rightarrow x = \sqrt[3]{47.0219} = 3.609 = 3.6$ cm

Revision Exercise 2 (Core)

1. $x^2 - 6x + 5 = 0$

$\Rightarrow (x - 1)(x - 5) = 0$

$\Rightarrow x = 1, \quad x = 5$

Hence, $t - \frac{6}{t} = 1$ OR $t - \frac{6}{t} = 5$

$\Rightarrow t^2 - 6 = t$ OR $t^2 - 6 = 5t$

$\Rightarrow t^2 - t - 6 = 0$ OR $t^2 - 5t - 6 = 0$

$\Rightarrow (t + 2)(t - 3) = 0$ OR $(t + 1)(t - 6) = 0$

$\Rightarrow t = -2, 3, -1, 6$

2. $2(x + 1)(x - 4) - (x - 2)^2 = 0$

$\Rightarrow 2x^2 - 6x - 8 - (x^2 - 4x + 4) = 0$

$\Rightarrow x^2 - 2x - 12 = 0$

$$\Rightarrow x = \frac{2 \pm \sqrt{(2)^2 - 4(1)(-12)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{52}}{2}$$

$$= \frac{2 \pm 2\sqrt{13}}{2} = 1 \pm \sqrt{13}$$

3. $px^2 + 2x + 1 = 0$

$$\Rightarrow x = \frac{-2 \pm \sqrt{(2)^2 - 4(p)(1)}}{2(p)}$$

$$= \frac{-2 \pm \sqrt{4 - 4p}}{2p}$$

No solution $\Rightarrow 4 - 4p < 0$

$\Rightarrow -4p < -4$

$\Rightarrow 4p > 4$

$\Rightarrow p > 1$

4. Real roots $\Rightarrow b^2 - 4ac \geq 0$

$\Rightarrow (-a - d)^2 - 4(1)(ad - b^2) \geq 0$

$\Rightarrow a^2 + 2ad + d^2 - 4ad + 4b^2 \geq 0$

$\Rightarrow a^2 - 2ad + d^2 + 4b^2 \geq 0$

$\Rightarrow (a - d)^2 + (2b)^2 \geq 0$ True

5. $(x + 1)$ and $(x - 2)$ are factors $\Rightarrow x = -1$ and $x = 2$ are roots

$f(x) = 6x^4 - x^3 + ax^2 - 6x + b$

$\Rightarrow f(-1) = 6(-1)^4 - (-1)^3 + a(-1)^2 - 6(-1) + b = 0$

$\Rightarrow 6 + 1 + a + 6 + b = 0$

$\Rightarrow a + b = -13$

$f(2) = 6(2)^4 - (2)^3 + a(2)^2 - 6(2) + b = 0$

$\Rightarrow 96 - 8 + 4a - 12 + b = 0$

$\Rightarrow 4a + b = -76$

and $\underline{a + b = -13}$

$\Rightarrow 3a = -63$

$\Rightarrow a = -21$

$\Rightarrow -21 + b = -13 \Rightarrow b = 8$

6. (i) $f(x) = x^3 - 4x^2 - 11x + 30$

$f(2) = (2)^3 - 4(2)^2 - 11(2) + 30$

$= 8 - 16 - 22 + 30 = 38 - 38 = 0$

$\Rightarrow x = 2$ is a root $\Rightarrow (x - 2)$ is a factor

(ii)

$$\begin{array}{r|l} & x^2 - 2x - 15 \\ \hline x - 2 & x^3 - 4x^2 - 11x + 30 \\ & \underline{x^3 - 2x^2} \\ & \quad -2x^2 - 11x \\ & \quad \underline{-2x^2 + 4x} \\ & \qquad\quad -15x + 30 \\ & \qquad\quad \underline{-15x + 30} \\ & \qquad\qquad\qquad 0 \end{array}$$

$\Rightarrow x^2 - 2x - 15$
$= (x + 3)(x - 5)$

(iii) $f(x) = (x - 2)(x + 3)(x - 5) = 0$
$\Rightarrow$ Roots : $x = 2, -3, 5$

7. (i) $b^2 - 4ac = (-2)^2 - (4)(1)(-5)$
$= 24 > 0 \quad \Rightarrow$ 2 distinct real roots

(ii) $b^2 - 4ac = (-4)^2 - 4(1)(6)$
$= 16 - 24 = -8 < 0 \quad \Rightarrow$ 2 imaginary roots

(iii) $b^2 - 4ac = (4)^2 - 4(-1)(-6)$
$= 16 - 24 = -8 < 0 \quad \Rightarrow$ 2 imaginary roots

8. $3^{2x} - 12(3^x) + 27 = 0$
$\Rightarrow (3^x)^2 - 12(3^x) + 27 = 0$
If $y = 3^x \Rightarrow y^2 - 12y + 27 = 0$
$\Rightarrow (y - 3)(y - 9) = 0$
$\Rightarrow y = 3 \quad$ OR $\quad y = 9$
$\Rightarrow 3^x = 3^1 \quad$ OR $\quad 3^x = 9 = 3^2$
$\Rightarrow x = 1 \quad$ OR $\quad x = 2$

Revision Exercise 2 (Advanced)

1. $2x^2 - 4x - 5$
$= 2(x^2 - 2x) - 5$
$= 2(x^2 - 2x + 1) - 2 - 5$
$= 2(x - 1)^2 - 7$

(i) $2x^2 - 4x - 5 = 0$
$\Rightarrow 2(x - 1)^2 = 7$
$\Rightarrow (x - 1)^2 = \frac{7}{2}$
$\Rightarrow x - 1 = \pm\sqrt{\frac{7}{2}}$
$\Rightarrow x = 1 \pm \sqrt{\frac{7}{2}}$

(ii) Minimum Point $= (1, -7)$

2. $(2\sqrt{2} - \sqrt{3})^2 = (2\sqrt{2})^2 - 2(2\sqrt{2})(\sqrt{3}) + (-\sqrt{3})^2$
$= 8 - 4\sqrt{6} + 3$
$= 11 - 4\sqrt{6}$

3. $\frac{\sqrt{7} + \sqrt{5}}{\sqrt{80} + \sqrt{5}} = \frac{\sqrt{7} + \sqrt{5}}{4\sqrt{5} + \sqrt{5}} = \frac{\sqrt{7} + \sqrt{5}}{5\sqrt{5}}$
$= \frac{\sqrt{7} + \sqrt{5}}{5\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{35} + 5}{25}$

4. $\sqrt{x + 2} = x - 4$
$\Rightarrow x + 2 = (x - 4)^2 = x^2 - 8x + 16$
$\Rightarrow x^2 - 9x + 14 = 0$
$\Rightarrow (x - 2)(x - 7) = 0$

$\Rightarrow x = 2, 7$

Test $x = 2$ $\quad \Rightarrow \sqrt{2 + 2} = 2 - 4$

$\Rightarrow \sqrt{4} = -2$ False

Test $x = 7$ $\quad \Rightarrow \sqrt{7 + 2} = 7 - 4$

$\Rightarrow \sqrt{9} = 3$ True

5. (i) Test $t = 1 \Rightarrow s = 8(1)^2 + 4(1) = 8 + 4 = 12$ m

$\Rightarrow t < 1 \Rightarrow$ Test $t = 0.9 \Rightarrow s = 8(0.9)^2 + 4(0.9) = 10.08$ m

(ii) Solve $8t^2 + 4t = 10$

$$\Rightarrow 4t^2 + 2t - 5 = 0$$

$$\Rightarrow t = \frac{-2 \pm \sqrt{(2)^2 - 4(4)(-5)}}{2(4)}$$

$$= \frac{-2 \pm \sqrt{84}}{8}$$

$$= \frac{-2 + 9.165}{8}, \frac{-2 - 9.165}{8}$$

$= 0.8956$ (Valid), -1.3956 (Not valid)

$= 0.90$

(iii) Answer $= 0.8956$

Corrected answer $= 0.90$ $\quad \Rightarrow$ error $= 0.0044$

$$\Rightarrow \text{Percentage error} = \frac{0.0044}{0.8956} \times 100$$

$$= 0.491\%$$

$$= 0.49\%$$

6. $\sigma = \dfrac{\sqrt{p(1 + p)}}{n}$

$$\Rightarrow \sqrt{p + p^2} = \sigma n$$

$$\Rightarrow p + p^2 = \sigma^2 n^2$$

$$\Rightarrow p^2 + p - \sigma^2 n^2 = 0$$

$$\Rightarrow p = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-\sigma^2 n^2)}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{(1)^2 + 4\sigma^2 n^2}}{2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{1 + 4\sigma^2 n^2}}{2}$$

7.

	$k < 0$	$0 < k < \frac{1}{4}$	$k > \frac{1}{4}$
k	Negative	Positive	Positive
$4k$	Negative	Positive	Positive
$4k - 1$	Negative	Negative	Positive
$k(4k - 1)$	Positive	Negative	Positive

Solve $x^2 + 4kx + k = 0$.

$$\Rightarrow x = \frac{-4k \pm \sqrt{(4k)^2 - 4(1)(k)}}{2(1)}$$

$$\Rightarrow x = \frac{-4k \pm \sqrt{16k^2 - 4k}}{2}$$

$$\Rightarrow x = \frac{-4k \pm \sqrt{4} \cdot \sqrt{4k^2 - k}}{2}$$

$\Rightarrow$ Roots: $x = -2k \pm \sqrt{k(4k - 1)}$

Hence, $x^2 + 4kx + k > 0$ for $0 < k < \frac{1}{4}$

8. $ax^2 + 2bx + c = 0$

$$\Rightarrow \text{Roots: } x = \frac{-2b \pm \sqrt{(2b)^2 - 4(a)(c)}}{2a}$$

$$\Rightarrow x = \frac{-2b \pm \sqrt{4b^2 - 4ac}}{2a}$$

$$\Rightarrow x = \frac{-2b \pm \sqrt{4} \cdot \sqrt{b^2 - ac}}{2a}$$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - ac}}{a} \qquad \text{(Real)}$$

Similarly, $bx^2 + 2cx + a = 0 \Rightarrow$ Roots: $x = \dfrac{-c \pm \sqrt{c^2 - ab}}{b}$ (Real)

True if $c > a$ $\Rightarrow bc > a^2$

and $b > a$ $\Rightarrow a^2 - bc < 0$

Hence, $cx^2 + 2ax + b = 0 \Rightarrow$ Roots: $x = \dfrac{-a \pm \sqrt{a^2 - bc}}{c}$

Since $a^2 - bc < 0 \Rightarrow$ Roots are not real

9. Points of intersection at A and B occur when $g(x) = f(x)$.

Hence, $x^2 + 5x - 1 = -x^2 + 5x + 3$

$\Rightarrow 2x^2 - 4 = 0$

$\Rightarrow x^2 - 2 = 0$

$\Rightarrow x^2 = 2$

$\Rightarrow x = -\sqrt{2}, x = \sqrt{2}$

$\Rightarrow g(-\sqrt{2}) = (-\sqrt{2})^2 + 5(-\sqrt{2}) - 1$

$= 2 - 5\sqrt{2} - 1 = 1 - 5\sqrt{2}$

and $g(\sqrt{2}) = (\sqrt{2})^2 + 5(\sqrt{2}) - 1$

$= 2 + 5\sqrt{2} - 1 = 1 + 5\sqrt{2}$

Hence, A $= (\sqrt{2}, 1 - 5\sqrt{2})$, B $= (-\sqrt{2}, 1 + 5\sqrt{2})$

10. $kx^2 - 2kx - 3k - 12 = 0$

$[a = k, b = -2k, c = -3k - 12]$

For equal roots:

$b^2 - 4ac = 0$

$(-2k)^2 - 4(k)(-3k - 12) = 0$

$4k^2 + 12k^2 + 48k = 0$

$16k^2 + 48k = 0$

$k^2 + 3k = 0$

$k(k + 3) = 0$

$k = 0$ or $k = -3$

When $k = 0$, the equation is untrue.

Thus the only solution is $k = -3$.

11. $x^2 - \sqrt{3}x - 6 = 0$

$$\Rightarrow x = \frac{\sqrt{3} \pm \sqrt{(\sqrt{3})^2 - 4(1)(-6)}}{2(1)}$$

$$= \frac{\sqrt{3} \pm \sqrt{27}}{2} = \frac{\sqrt{3} + 3\sqrt{3}}{2}, \frac{\sqrt{3} - 3\sqrt{3}}{2}$$

$$= 2\sqrt{3}, -\sqrt{3}$$

Hence, $r_1 r_2 = (2\sqrt{3})(-\sqrt{3}) = -6$

12. $3x + y = -1 \cap x^2 + y^2 = 53$

$\Rightarrow y = -3x - 1 \Rightarrow x^2 + (-3x - 1)^2 = 53$

$\Rightarrow x^2 + 9x^2 + 6x + 1 - 53 = 0$

$\Rightarrow 10x^2 + 6x - 52 = 0$

$\Rightarrow 5x^2 + 3x - 26 = 0$

$\Rightarrow (5x + 13)(x - 2) = 0$

$\Rightarrow 5x = -13, x = 2$

$\Rightarrow x = -\dfrac{13}{5}$ $\quad\quad \Rightarrow y = -3(2) - 1$

$\quad\quad\quad\quad\quad\quad\quad = -7$

$\Rightarrow y = -3\left(\dfrac{-13}{5}\right) - 1$

$= \dfrac{34}{5}$

$\therefore$ the solutions are: $\left(\dfrac{-13}{5}, \dfrac{34}{5}\right)$ and $(2, -7)$

13. length $= x$ and width $= y$.

$\Rightarrow 2x + 2y = 48$ and $x = \dfrac{1}{2}y^2$

$\Rightarrow x + y = 24$

$\Rightarrow x = -y + 24 \quad\quad \Rightarrow -y + 24 = \dfrac{1}{2}y^2$

$\Rightarrow -2y + 48 = y^2$

$\Rightarrow y^2 + 2y - 48 = 0$

$\Rightarrow (y + 8)(y - 6) = 0$

$\Rightarrow y = -8$(Not valid), $y = 6$ m (Valid)

$\Rightarrow x = -6 + 24$

$= 18$ m

$\Rightarrow$ Dimensions are 18 m and 6 m

14. $\dfrac{y}{1} = \dfrac{x^2}{x+1} \quad \Rightarrow x^2 = xy + y$

$\Rightarrow x^2 - xy - y = 0$

$\Rightarrow x = \dfrac{y \pm \sqrt{(-y)^2 - 4(1)(-y)}}{2(1)}$

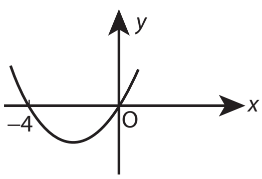
$= \dfrac{y \pm \sqrt{y^2 + 4y}}{2}$

Real roots $\Rightarrow b^2 - 4ac \geqslant 0 \quad\quad \Rightarrow y^2 + 4y \geqslant 0$

Factors: $y(y + 4)$

$\Rightarrow$ Roots: $y = 0, -4$

For $y^2 + 4y \geqslant 0 \quad \Rightarrow y \leqslant -4 \quad$ or $\quad y \geqslant 0$



15. $y = ax^2 + bx + c$

Point $(-2, -1) \Rightarrow -1 = a(-2)^2 + b(-2) + c$

$\Rightarrow -1 = 4a - 2b + c$

Point $(1, 2) \quad \Rightarrow 2 = a(1)^2 + b(1) + c$

$\Rightarrow 2 = a + b + c$

Point $(3, -16) \quad \Rightarrow -16 = a(3)^2 + b(3) + c$

$\Rightarrow -16 = 9a + 3b + c$

$\Rightarrow 4a - 2b + c = -1 \quad\quad 9a + 3b + c = -16$

and $\underline{a + b + c = 2} \quad\quad \underline{a + b + c = 2}$

$\Rightarrow 3a - 3b = -3 \quad\quad 8a + 2b = -18$

$\Rightarrow a - b = -1 \quad\quad \Rightarrow 4a + b = -9$

and $\underline{a - b = -1}$

$5a = -10$

$\Rightarrow a = -2$

$\Rightarrow -2 - b = -1 \quad\quad$ and $\quad a + b + c = 2$

$\Rightarrow \quad b = -1 \quad\quad \Rightarrow -2 - 1 + c = 2$

$\Rightarrow \quad c = 5$

Hence, $\quad y = -2x^2 - x + 5$

16. (ii) $f(x) = x^3 - 6x^2 + 11x - 6$

$\Rightarrow f(0) = (0)^3 - 6(0)^2 + 11(0) - 6 = -6$

$\Rightarrow f(1) = (1)^3 - 6(1)^2 + 11(1) - 6 = 12 - 12 = 0$

$\Rightarrow f(2) = (2)^3 - 6(2)^2 + 11(2) - 6 = 30 - 30 = 0$

$\Rightarrow f(3) = (3)^3 - 6(3)^2 + 11(3) - 6 = 60 - 60 = 0$

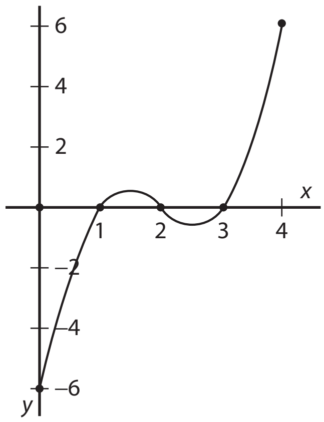
$\Rightarrow f(4) = (4)^3 - 6(4)^2 + 11(4) - 6 = 108 - 102 = 6$

$\Rightarrow$ Roots are $x = 1, 2, 3$

(iii) Points are $(0, -6)(1, 0)$

$(2, 0)(3, 0)$

$(4, 6)$



17. (i) Roots $x = 2, 5$ (double root)

(ii) $f(x) = a(x - 2)(x - 5)^2$

(iii) $f(x) = a(x^3 - 12x^2 + 45x - 50)$

Point $(3, 8) \Rightarrow f(3) = a[(3)^3 - 12(3)^2 + 45(3) - 50] = 8$

$\Rightarrow a[27 - 108 + 135 - 50] = 8$

$\Rightarrow 4a = 8 \Rightarrow a = 2$

Hence, $f(x) = 2(x^3 - 12x^2 + 45x - 50)$

$= 2x^3 - 24x^2 + 90x - 100 \qquad \Rightarrow a = 2, b = -24$

$c = 90, d = -100$

(iv) Point $(3, -8) \Rightarrow f(3) = a[(3)^3 - 12(3)^2 + 45(3) - 50] = -8$

$\Rightarrow 4a = -8 \Rightarrow a = -2$

$\Rightarrow f(x) = -2(x^3 - 12x^2 + 45x - 50)$

$= -2x^3 + 24x^2 - 90x + 100$

(v) Reflection in y-axis $\Rightarrow x$ is replaced by $(-x)$

$f(x) = 2x^3 - 24x^2 + 90x - 100$

$\Rightarrow f(-x) = 2(-x)^3 - 24(-x)^2 + 90(-x) - 100$

$= -2x^3 - 24x^2 - 90x - 100$

Revision Exercise 2 (Extended–Response)

1. (i) $C(t) = 0.02t - at^3$

$t = 5 \Rightarrow C(5) = (0.02)(5) - a(5)^3 = 0.075$

$\Rightarrow 0.1 - 125a = 0.075$

$\Rightarrow -125a = -0.025$

$\Rightarrow a = \dfrac{-0.025}{-125} = 0.0002$

(ii) $C(t) = 0 \Rightarrow 0.02t - 0.0002t^3 = 0$

$\Rightarrow 200t - 2t^3 = 0$

$\Rightarrow 100t - t^3 = 0$

$\Rightarrow t(100 - t^2) = 0$

$\Rightarrow t = 0$ (not valid), $t = 10$ (valid)

(iii) Beyond $t = 10 \Rightarrow C(t)$ is negative

2. (i) Area $= (2x + y)(3x + 2y)$

$= 6x^2 + 7xy + 2y^2$

(ii) Area dividing strips $=$ Area full poster $-$ Area 6 purple squares

$= 6x^2 + 7xy + 2y^2 - 6x^2$

$= 7xy + 2y^2 \quad \Rightarrow k = 7$

(iii) Area 6 purple squares $= 6x^2 = 1.5$

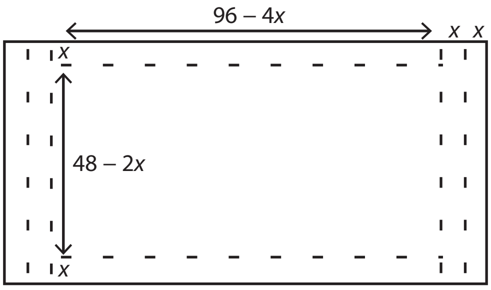
$$\Rightarrow x^2 = 0.25 \quad \Rightarrow x = 0.5\text{ m} = \frac{1}{2}\text{ m}$$

Hence, $7xy + 4y^2 = 3.5y + 2y^2 = 1$

$$\Rightarrow 4y^2 + 7y - 2 = 0$$
$$\Rightarrow (4y - 1)(y + 2) = 0$$
$$\Rightarrow y = \frac{1}{4}\text{ (valid)}, y = -2\text{ (Not valid)}$$

3. (a) Volume $= (96 - 4x)(48 - 2x)(x)$

$V = 4608x - 384x^2 + 8x^3$



(b) (i) $0 < x < 24$

(ii) At C, $x = 0 \Rightarrow$ Volume $= 0$

At A, $x = 8 \Rightarrow$ Maximum volume occurs ($= 16\,384$ cm^3)

At B, $x = 24 \Rightarrow$ Minimum volume occurs ($= 0$ cm^3)

(iii) Maximum volume $= 16\,300$ cm^3 (using graph)

(iv) $x = 10 \Rightarrow V = [96 - 4(10)][48 - 2(10)][10]$

$\Rightarrow (56)(28)(10) = 15\,680$ cm^3

(v) $x = 5 \Rightarrow$ Volume $= 14\,440$ cm^3

$x < 5 \Rightarrow$ Volume $< 14\,440$

$\Rightarrow$ Volume $= 14\,439.9$

(vi) If $5 \leq x \leq 15$, minimum volume occurs at $x = 15$ (using graph)

$\Rightarrow x = 15 \Rightarrow$ Volume $= [96 - 4(15)][48 - 2(15)][15]$

$\Rightarrow (36)(18)(15) = 9720$ cm^3

(c) Area $= 2(x)(96 - 4x) + 2(x)(48 - 2x) + 2(96 - 4x)(48 - 2x)$

$= 8x(24 - x) + 4x(24 - x) + 8(24 - x)(24 - x)$

$= 4(24 - x)(2x + x + 48 - 2x)$

$= 4(24 - x)(48 + x)$

$\Rightarrow a = 4, b = 24, c = 48$

4. (i) $4l + 3w = 120$

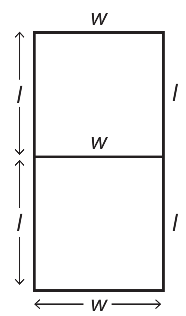
$\Rightarrow 4l = 120 - 3w$

$\Rightarrow l = 30 - \frac{3}{4}w$

Hence, area $= 2lw$

$= 2w\left(30 - \frac{3}{4}w\right)$

$= 60w - \frac{3}{4}w^2$



(ii) $60w - \frac{3}{2}w^2 = 0$

$\Rightarrow w\left(60 - \frac{3}{2}w\right) = 0$

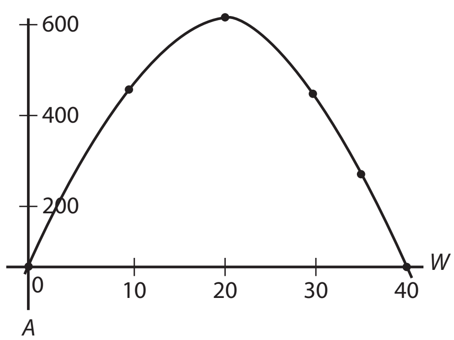
$\Rightarrow w = 0,\ 60 - \frac{3}{2}w = 0$

$120 - 3w = 0$

$\Rightarrow 3w = 120$

$\Rightarrow w = 40$

w =	0	10	20	30	40
A =	0	450	600	450	0



(iii) $A = -\frac{3}{2}w^2 + 60w$

$= -\frac{3}{2}(w^2 - 40w)$

$= -\frac{3}{2}(w^2 - 40w + 400) + 600$

$= 600 - \frac{3}{2}(w - 20)^2$

Hence, maximum area $= 600$ m^2

(iv) $w = 20$ m

(v) Dimensions are: $w = 20$ m, $l = 30 - \frac{3}{4}(20) = 15$ m.

5. (a) (i) $h = 5 \Rightarrow t = 1, 3$

(ii) $t = 4.5$ secs

(b) $2 + 4t - t^2 = 0$

$\Rightarrow t = \dfrac{-4 \pm \sqrt{(4)^2 - 4(-1)(2)}}{2(-1)} = \dfrac{-4 \pm \sqrt{16 + 8}}{-2}$

$\Rightarrow t = \dfrac{-4 + \sqrt{24}}{-2} = 2 + \sqrt{6}$ (valid), $t = 2 - \sqrt{6} = -0.449$ (Not valid)

$\Rightarrow t = 4.449$

$\Rightarrow t = 4.45$

(c) $h = 2 + 4t - t^2$

$= 2 - (t^2 - 4t)$

$= 2 - (t^2 - 4t + 4) + 4$

$= 6 - (t - 2)^2 \Rightarrow (p, q) = (2, 6)$

6.

No. of Price hikes	Price per Rental	Number of Rentals	Total Income (I)
	€12	36	€432
1 Price hike	€12.50	34	€425
2 Price hikes	€13	32	€416
3 Price hikes	€13.50	30	€405
x price hikes	€$(12 + 0.5x)$	$36 - 2x$	€$(12 + 0.5x)(36 - 2x)$

(i) $I = (12 + 0.5x)(36 - 2x) = 432 - 6x - x^2$

(ii) $I = 432 - 6x - x^2$

$= 432 - (x^2 + 6x)$

$= 432 - (x^2 + 6x + 9) + 9$

$= 441 - (x + 3)^2$

(iii) Maximum Income = €441

(iv) Price per rental

7. (a) Area of garden = 2 Area BAC + Area ACED

$= 2\left(\frac{\pi}{4}x^2\right) + (x)(y)$

$= \frac{\pi}{4}x^2 + xy$

(b) (i) Length $= 2\left(\frac{2\pi x}{4}\right) + y = 100$

$\Rightarrow \pi x + y = 100$

$\Rightarrow y = 100 - \pi x$

(ii) Area $= \frac{\pi}{2}x^2 + x(100 - \pi x)$

$= \frac{\pi}{2}x^2 + 100x - \pi x^2$

$A = 100x - \frac{\pi}{2}x^2$

(iii) $100 - \pi x = 0$

$\Rightarrow \quad \pi x = 100$

$\Rightarrow \quad x = \dfrac{100}{\pi}$

Domain: $0 < x < \dfrac{100}{\pi}$

(c) Area $= 100x - \dfrac{\pi}{2}x^2 = 1000$

$\Rightarrow 200x - \pi x^2 = 2000$

$\Rightarrow \pi x^2 - 200x + 2000 = 0$

$$\Rightarrow x = \frac{200 \pm \sqrt{(-200)^2 - 4(\pi)(2000)}}{2(\pi)}$$

$$= \frac{200 \pm \sqrt{40\,000 - 8000\pi}}{2\pi}$$

$$= \frac{200 \pm \sqrt{14\,867.25877}}{2\pi}$$

$$= \frac{200 + 121.9314}{2\pi}, \frac{200 - 121.9314}{2\pi}$$

$= 51.236, 12.425$

$= 51.2, 12.4$

(d) (i) Volume $=$ (Area)(height)

$$= \left(100x - \frac{\pi}{2}x^2\right)\left(\frac{x}{50}\right)$$

$$= 2x^2 - \frac{\pi}{100}x^3$$

(ii) $x = 12.4 \Rightarrow$ Volume $= 2(12.4)^2 - \dfrac{\pi}{100}(12.4)^3$

$= 307.52 - 59.898$

$= 247.62 = 247.6\text{ m}^3$

$x = 51.2 \Rightarrow$ Volume $= 2(51.2)^2 - \dfrac{\pi}{100}(51.2)^3$

$= 5242.88 - 4216.574$

$= 1026.306 = 1026.3\text{ m}^3$

(iii) Volume $= 2x^2 - \dfrac{\pi}{100}x^3 = 500$

$\Rightarrow 200x^2 - \pi x^3 = 50\,000$

$\Rightarrow \quad f(x) = \pi x^3 - 200x^2 + 50\,000$

$\Rightarrow \quad f(18) = 18321.77 - 64\,800 + 50\,000 = 3521.77$

$\Rightarrow f(18.8) = \pi(18.8)^3 - 200(18.8)^2 + 50\,000$

$= 20\,874.85 - 70\,688 + 50\,000 = 186$

$\Rightarrow f(18.9) = \pi(18.9)^3 - 200(18.9)^2 + 50\,000$

$= 21\,209.74 - 71\,442 + 50\,000 = -232.26$

Hence, $x = 18.8$

8. (a) $y = 2x \cap y = 6 + 5x - x^2$

$\Rightarrow 2x = 6 + 5x - x^2$

$\Rightarrow x^2 - 3x - 6 = 0$

$$\Rightarrow x = \frac{3 \pm \sqrt{(3)^2 - 4(1)(-6)}}{2(1)}$$

$$= \frac{3 \pm \sqrt{33}}{2}$$

$$\Rightarrow x_1 = \frac{3 + \sqrt{33}}{2} \Rightarrow y_1 = 2\left(\frac{3 + \sqrt{33}}{2}\right) = 3 + \sqrt{33}$$

Hence, A $= \left(\frac{3+\sqrt{33}}{2}, 3+\sqrt{33}\right)$

$\Rightarrow x_2 = \frac{3-\sqrt{33}}{2} \Rightarrow y_2 = 2\left(\frac{3-\sqrt{33}}{2}\right) = 3-\sqrt{33}$

Hence, B $= \left(\frac{3-\sqrt{33}}{2}, 3-\sqrt{33}\right)$

(b) $(x, 6+5x-x^2)$ and $(x, 2x)$

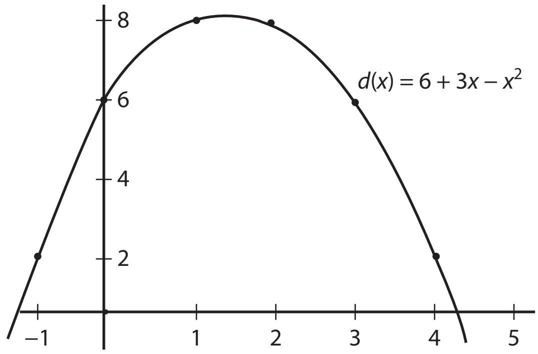
$d = \sqrt{(x-x^2)+(6+5x-x^2-2x)^2}$

$= \sqrt{(6+3x-x^2)^2}$

$\Rightarrow d(x) = 6+3x-x^2$

(c)

$x =$	-1	0	1	2	3	4
$d(x) =$	2	6	8	8	6	2



(d) $d(x) = 6+3x-x^2$

$= 6 - \left(x^2 - 3x - \left(1\frac{1}{2}\right)^2\right) + 2\frac{1}{4}$

$= 8\frac{1}{4} - \left(x - 1\frac{1}{2}\right)^2$

(e) Maximum point $= \left(1\frac{1}{2}, 8\frac{1}{4}\right)$

$\Rightarrow$ Maximum vertical distance $= 8\frac{1}{4}$ occurs at $x = 1\frac{1}{2}$

(f) $0 \leq d(x) \leq 8\frac{1}{4}$

9. (i) $y = x \cap y = \sqrt{x-b} + c$

$\Rightarrow x = \sqrt{x-b} + c$

$\Rightarrow x - c = \sqrt{x-b}$

$\Rightarrow (x-c)^2 = x - b$

$\Rightarrow x^2 - 2cx + c^2 - x + b = 0$

$\Rightarrow x^2 - x(2c+1) + c^2 + b = 0$

At $(a, a) \Rightarrow x = a \Rightarrow a^2 - a(2c+1) + c^2 + b = 0$

(ii) Tangent $\Rightarrow b^2 - 4ac = 0$

$\Rightarrow [-(2c+1)]^2 - 4(1)(c^2+b) = 0$

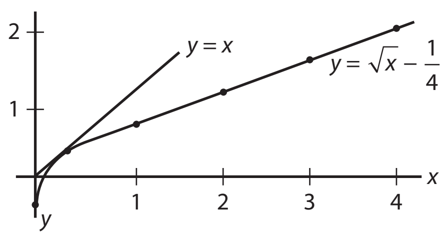
$\Rightarrow 4c^2 + 4c + 1 - 4c^2 - 4b = 0$

$\Rightarrow \quad 4c = 4b - 1$

$\Rightarrow \quad c = \frac{4b-1}{4}$

(iii)

$x =$	0	1	2	3	4
$y =$	-0.25	0.75	1.16	1.48	1.75



(iv) $y = x \cap y = \sqrt{x} - \frac{1}{4}$

$$\Rightarrow \sqrt{x} - \frac{1}{4} = x$$

$$\Rightarrow \sqrt{x} = x + \frac{1}{4}$$

$$\Rightarrow x = \left(x + \frac{1}{4}\right)^2 = x^2 + \frac{1}{2}x + \frac{1}{16}$$

$$\Rightarrow x^2 - \frac{1}{2}x + \frac{1}{16} = 0$$

$$\Rightarrow \left(x + \frac{1}{4}\right)^2 = 0 \quad \Rightarrow x = \frac{1}{4} \Rightarrow y = \frac{1}{4} \Rightarrow \left(\frac{1}{4}, \frac{1}{4}\right)$$

(v) $y = x + k \cap y = \sqrt{x} - \frac{1}{4}$

$$\Rightarrow x + k = \sqrt{x} - \frac{1}{4}$$

$$\Rightarrow x + \left(k + \frac{1}{4}\right) = \sqrt{x}$$

$$\Rightarrow x^2 + 2\left(k + \frac{1}{4}\right) + k^2 + \frac{1}{2}k + \frac{1}{16} = x$$

$$\Rightarrow x^2 + \left(2k - \frac{1}{2}\right) + x + k^2 + \frac{1}{2}k + \frac{1}{16} = 0$$

(a) Twice $\Rightarrow b^2 - 4ac > 0$

$$\Rightarrow \left(2k - \frac{1}{2}\right)^2 - 4(1)\left(k^2 + \frac{1}{2}k + \frac{1}{16}\right) > 0$$

$$\Rightarrow 4k^2 - 2k + \frac{1}{4} - 4k^2 - 2k - \frac{1}{4} > 0$$

$$\Rightarrow -4k^2 > 0$$

$$\Rightarrow k > 0$$

(b) Once $\Rightarrow b^2 - 4ac = 0$

$$\Rightarrow k = 0$$

(c) At no point $\Rightarrow b^2 - 4ac < 0$

$$\Rightarrow k > 0$$

Chapter 12

Exercise 12.1

1. (i) $3x - 5 > x + 3$

$\Rightarrow \quad 2x > 8$

$\Rightarrow \quad x > 4$

(ii) $6x - 5 \leq 2x - 1$

$\Rightarrow \quad 4x \leq 4$

$\Rightarrow \quad x \leq 1$

(iii) $1 - 3x > 10$

$\Rightarrow \quad -3x > 9$

$\Rightarrow \quad 3x < -9$

$\Rightarrow \quad x < -3$

2. (i) $\frac{x}{2} + 2 < 7, x \in N$

$\Rightarrow \quad \frac{x}{2} < 5$

$\Rightarrow \quad x < 10$

(ii) $\frac{1}{6}(x - 1) \geq \frac{1}{3}(x - 4), x \in Z$

$\Rightarrow \quad 1(x - 1) \geq 2(x - 4)$

$\Rightarrow \quad x - 1 \geq 2x - 8$

$\Rightarrow \quad -x \geq -7$

$\Rightarrow \quad x \leq 7$

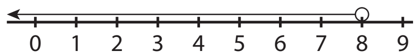
(iii) $\frac{4 - x}{2} > \frac{2 - x}{3}, x \in R$

$\Rightarrow \quad 3(4 - x) > 2(2 - x)$

$\Rightarrow \quad 12 - 3x > 4 - 2x$

$\Rightarrow \quad -x > -8$

$\Rightarrow \quad x < 8$

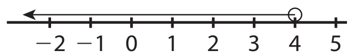


3. (i) $12x - 3(x - 3) < 45, x \in R$

$\Rightarrow \quad 12x - 3x + 9 < 45$

$\Rightarrow \quad 9x < 36$

$\Rightarrow \quad x < 4$



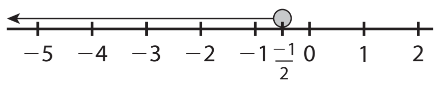
(ii) $x(x - 4) \geq x^2 + 2, x \in R$

$\Rightarrow x^2 - 4x \geq x^2 + 2$

$\Rightarrow -4x \geq 2$

$\Rightarrow \quad 4x \leq -2$

$\Rightarrow \quad x \leq -\frac{1}{2}$



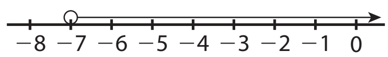
(iii) $x - 2(5 + 2x) < 11$

$\Rightarrow \quad x - 10 - 4x < 11$

$\Rightarrow \quad -3x < 21$

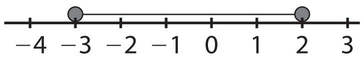
$\Rightarrow \quad 3x > -21$

$\Rightarrow \quad x > -7$



4. (i) $-2 \leqslant x + 1 \leqslant 3, x \in R$

$\Rightarrow \quad -3 \leqslant x \leqslant 2$

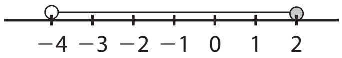


(ii) $13 > 1 - 3x \geqslant 7, x \in R$

$\Rightarrow \quad 12 > -3x \geqslant 6$

$\Rightarrow \quad -12 < 3x \leqslant 6$

$\Rightarrow \quad -4 < x \leqslant 2$

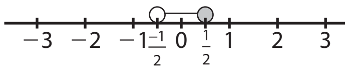


(iii) $3 \geq 4x + 1 > -1$

$\Rightarrow \quad 2 \geq 4x > -2$

$\Rightarrow \quad \frac{1}{2} \geq x > -\frac{1}{2}$

$\Rightarrow \quad -\frac{1}{2} < x \leqslant \frac{1}{2}$



5. (i) $3 > \frac{3}{5}(x - 2) > 0$

$\Rightarrow \quad 15 > 3x - 6 > 0$

$\Rightarrow \quad 21 > 3x > 6$

$\Rightarrow \quad 7 > x > 2$

$\Rightarrow \quad 2 < x < 7$

(ii) $-4 \leqslant \frac{2}{5}(1 - 3x) \leqslant 1$

$\Rightarrow -20 \leqslant 2 - 6x \leqslant 5$

$\Rightarrow -22 \leqslant -6x \leqslant 3$

$\Rightarrow \quad 22 \geqslant 6x \geqslant -3$

$\Rightarrow \quad 3\frac{2}{3} \geqslant x \geqslant -\frac{1}{2}$

$\Rightarrow -\frac{1}{2} \leqslant x \leqslant 3\frac{2}{3}$

(iii) $3 \leqslant 2 - \frac{x}{7} < 4$

$\Rightarrow 21 \leqslant 14 - x \leqslant 28$

$\Rightarrow 7 \leqslant -x \leqslant 14$

$\Rightarrow -7 \geqslant x \geqslant -14$

$\Rightarrow -14 \leqslant x \leqslant -7$

6. $3(x - 2) > x - 4$ and $4x + 12 > 2x + 17, x \in R$

$\Rightarrow \quad 3x - 6 > x - 4 \qquad \Rightarrow \quad 2x > 5$

$\Rightarrow \quad 2x > 2 \qquad \Rightarrow \quad x > 2\frac{1}{2}$

$\Rightarrow \quad x > 1$

$\Rightarrow \quad$ ANS : $x > 2\frac{1}{2}$

7. (i) $2x - 5 < x - 1, x \in R$

$\Rightarrow \quad x < 4$

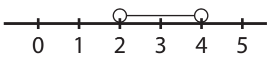
(ii) $7(x + 1) > 23 - x, x \in R$

$\Rightarrow \quad 7x + 7 > 23 - x$

$\Rightarrow \quad 8x > 16$

$\Rightarrow \quad x > 2$

(iii) $2 < x < 4, x \in R$



8. (i) $2x - 3 > 2, x \in R$

$\Rightarrow \quad 2x > 5$

$\Rightarrow \quad x > 2\frac{1}{2}$

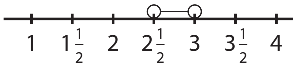
(ii) $3(x + 2) < 12 + x, x \in R$

$\Rightarrow \quad 3x + 6 < 12 + x$

$\Rightarrow \quad 2x < 6$

$\Rightarrow \quad x < 3$

(iii) $2\frac{1}{2} < x < 3, x \in R$



9. (i) $15 - x < 2(11 - x), x \in Z$

$\Rightarrow \quad 15 - x < 22 - 2x$

$\Rightarrow \quad x < 7$

(ii) $5(3x - 1) > 12x + 19, x \in Z$

$\Rightarrow \quad 15x - 5 > 12x + 19$

$\Rightarrow \quad 3x > 24$

$\Rightarrow \quad x > 8$

(iii) $\therefore$ Null set

10. (i) $3x + 8 \leqslant 20, x \in N$

$\Rightarrow 3x \leqslant 12$

$\Rightarrow x \leqslant 4$

(ii) $2(3x - 7) \geq x + 6, x \in N$

$\Rightarrow \quad 6x - 14 \geqslant x + 6$

$\Rightarrow \quad 5x \geqslant 20$

$\Rightarrow \quad x \geqslant 4$

(iii) $x = 4$

11. Width $= x$, Length $= x + 1$

Perimeter $= 2(x) + 2(x + 1) \leqslant 38$

$\Rightarrow 2x + 2x + 2 \leqslant 38$

$\Rightarrow \quad 4x \leqslant 36$

$\Rightarrow \quad x \leqslant 9$

Width $= 9$ m, length $= 10$ m

12. $100 < 2^n < 200$

$\Rightarrow \quad 2^{6.645} < 2^n < 2^{7.645}$

$\Rightarrow \quad 6.645 < n < 7.645$

Hence $a = 6.645, b = 7.645, n = 7$

13. Example 1: $4 > 3 > 0$ Thus $4^2 > 3^2$

Example 2: $4 > 3 > 0$ Thus $4^{-2} < 3^{-2}$, as $\frac{1}{16} < \frac{1}{9}$.

(i) if $a < b < 0$ and $n(odd) > 0$, then $a^n < b^n$.

(ii) if $a < b < 0$ and $n(even) > 0$, then $a^n > b^n$.

(iii) if $a < b < 0$ and $n(odd) < 0$, then $a^n > b^n$.

(iv) if $a < b < 0$ and $n(even) < 0$, then $a^n < b^n$.

14. $5 - 3x < -10 \quad \cap \quad 4x + 6 < 32, \quad x \in Z$

$\Rightarrow \quad -3x < -15 \quad \cap \quad 4x < 26$

$\Rightarrow \quad 3x > 15 \quad \cap \quad x < 6\frac{1}{2}$

$\Rightarrow \quad x > 5 \quad \cap \quad x < 6\frac{1}{2}, \quad x \in Z$

$\Rightarrow \quad x = 6$

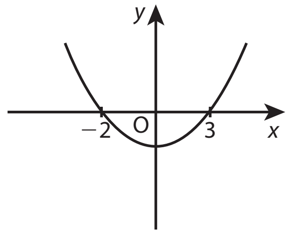
Exercise 12.2

1. (i) Let $x^2 - x - 6 = 0$

$\Rightarrow \quad (x + 2)(x - 3) = 0$

$\Rightarrow \quad$ Roots: $x = -2, 3$

Hence $\quad x^2 - x - 6 \geqslant 0 \quad \Rightarrow \quad x \leqslant -2 \quad$ or $\quad x \geqslant 3$

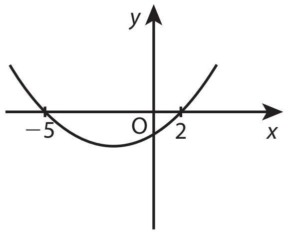


(ii) Let $x^2 + 3x - 10 = 0$

$\Rightarrow \quad (x + 5)(x - 2) = 0$

$\Rightarrow \quad$ Roots: $x = -5, 2$

Hence $\quad x^2 + 3x - 10 \leqslant 0 \quad \Rightarrow \quad -5 \leqslant x \leqslant 2$

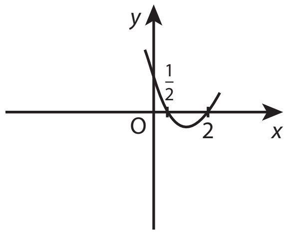


(iii) Let $2x^2 - 5x + 2 = 0$

$\Rightarrow \quad (2x - 1)(x - 2)=0$

$\Rightarrow \quad$ Roots: $x = \frac{1}{2}, 2$

Hence $\quad 2x^2 - 5x + 2 < 0 \quad \Rightarrow \quad \frac{1}{2} < x < 2$

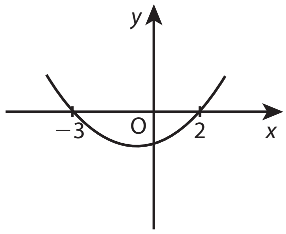


2. (i) Let $6 - x - x^2 = 0$

$\Rightarrow \quad (3 + x)(2 - x) = 0$

$\Rightarrow \quad$ Roots: $x = -3, 2$

Hence $\quad 6 - x - x^2 \geq 0 \quad \Rightarrow \quad -3 \leqslant x \leqslant 2$

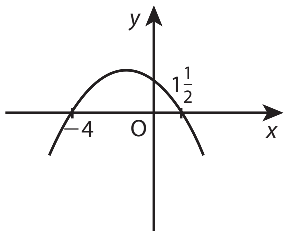


(ii) Let $12 - 5x - 2x^2 = 0$

$\Rightarrow \quad (4 + x)(3 - 2x) = 0$

$\Rightarrow \quad$ Roots: $x = -4, 1\frac{1}{2}$

Hence $\quad 12 - 5x - 2x^2 > 0 \quad \Rightarrow \quad -4 < x < 1\frac{1}{2}$



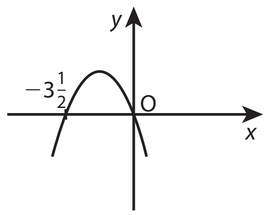
(iii) Let $-2x^2 - 7x = 0$

$\Rightarrow \quad 2x^2 + 7x = 0$

$\Rightarrow \quad x(2x + 7) = 0$

$\Rightarrow \quad$ Roots: $x = 0, -3\frac{1}{2}$

Hence $\quad -2x^2 - 7x \geqslant 0 \quad \Rightarrow \quad -3\frac{1}{2} \leqslant x \leqslant 0$

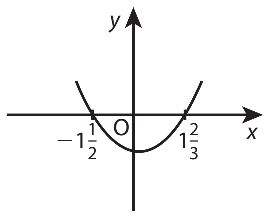


3. (i) Let $6x^2 - x - 15 = 0$

$\Rightarrow \quad (2x + 3)(3x - 5) = 0$

$\Rightarrow \quad$ Roots: $x = -1\frac{1}{2}, 1\frac{2}{3}$

Hence $\quad 6x^2 - x - 15 > 0 \quad \Rightarrow \quad x < -1\frac{1}{2} \quad$ or $\quad x > 1\frac{2}{3}$

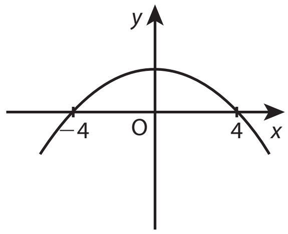


(ii) Let $16 - x^2 = 0$

$\Rightarrow \quad (4 + x)(4 - x) = 0$

$\Rightarrow \quad$ Roots: $x = -4, 4$

Hence $\quad 16 - x^2 \leqslant 0 \quad \Rightarrow \quad x \leqslant -4 \quad$ or $\quad x \geq 4$



(iii) Let $2(x^2 - 6) = 5x$

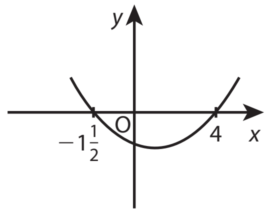
$\Rightarrow \quad 2x^2 - 12 = 5x$

$\Rightarrow \quad 2x^2 - 5x - 12 = 0$

$\Rightarrow \quad (2x + 3)(x - 4) = 0$

$\Rightarrow \quad$ Roots: $x = -1\frac{1}{2}, 4$

Hence $\quad 2(x^2 - 6) \geq 5x \quad \Rightarrow \quad x \leqslant -1\frac{1}{2} \quad$ or $\quad x \geq 4$



4. Let $(4 - x)(1 - x) = x + 11$

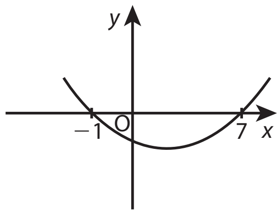
$\Rightarrow \quad 4 - 5x + x^2 - x - 11 = 0$

$\Rightarrow \quad x^2 - 6x - 7 = 0$

$\Rightarrow \quad (x + 1)(x - 7) = 0$

$\Rightarrow \quad$ Roots: $x = -1, 7$

Hence $\quad (4 - x)(1 - x) < x + 11 \quad \Rightarrow \quad -1 < x < 7$



5. Let $x^2 - 6x + 2 = 0$

$$\Rightarrow \quad \text{Roots:} \quad x = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(2)}}{2(1)}$$

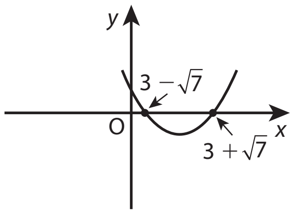
$$= \frac{6 \pm \sqrt{36 - 8}}{2}$$

$$= \frac{6 \pm \sqrt{28}}{2}$$

$$= \frac{6 \pm 2\sqrt{7}}{2}$$

$$= 3 - \sqrt{7}, 3 + \sqrt{7}$$

Hence $\quad x^2 - 6x + 2 \leqslant 0 \quad \Rightarrow \quad 3 - \sqrt{7} \leqslant x \leqslant 3 + \sqrt{7}$



6. $x^2 + (k + 1)x + 1 = 0$

Real Roots $\quad \Rightarrow \quad b^2 - 4ac \geq 0$

$\Rightarrow \quad (k + 1)^2 - 4(1)(1) \geqslant 0$

$\Rightarrow \quad k^2 + 2k + 1 - 4 \geqslant 0$

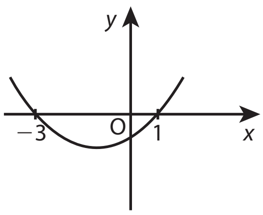
$\Rightarrow \quad k^2 + 2k + 1 - 3 \geqslant 0$

Solve $k^2 + 2k - 3 = 0$

$\Rightarrow \quad (k + 3)(k - 1) = 0$

$\Rightarrow \quad$ Roots $= -3, 1$

Hence $\quad k^2 + 2k - 3 \geqslant 0 \quad \Rightarrow \quad k \leqslant -3$ or $k \geqslant 1$



7. $kx^2 + 4x + 3 + k = 0$

Real Roots $\quad \Rightarrow \quad b^2 - 4ac \geqslant 0$

$\Rightarrow \quad (4)^2 - 4(k)(3 + k) \geqslant 0$

$\Rightarrow \quad 16 - 12k - 4k^2 \geqslant 0$

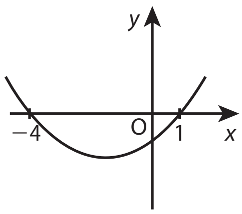
$\Rightarrow \quad k^2 + 3k - 4 \leqslant 0$

solve $k^2 + 3k - 4 = 0$

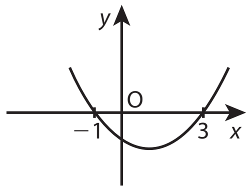
$\Rightarrow \quad (k + 4)(k - 1) = 0$

$\Rightarrow \quad$ Roots: $\quad k = -4, 1$

Hence $\quad k^2 + 3k - 4 \leqslant 0 \quad \Rightarrow \quad -4 \leqslant k \leqslant 1$



8. $px^2 + (p+3)x + p = 0$



Real Roots $\Rightarrow \quad b^2 - 4ac \geqslant 0$

$\Rightarrow \quad (p+3)^2 - 4(p)(p) \geqslant 0$

$\Rightarrow \quad p^2 + 6p + 9 - 4p^2 \geqslant 0$

$\Rightarrow \quad -3p^2 + 6p + 9 \geqslant 0$

$\Rightarrow \quad p^2 - 2p - 3 \leqslant 0$

Solve $p^2 - 2p - 3 = 0$

$\Rightarrow \quad (p+1)(p-3) = 0$

$\Rightarrow$ Roots: $p = -1, 3$

Hence $p^2 - 2p - 3 \leqslant 0 \quad \Rightarrow \quad -1 \leqslant p \leqslant 3$

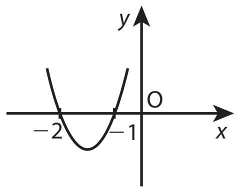
$x = -2 \quad \Rightarrow \quad p(-2)^2 + (p+3)(-2) + p = 0$

$\Rightarrow \quad 4p - 2p - 6 + p = 0$

$\Rightarrow \quad 3p = 6$

$\Rightarrow \quad p = 2$

9. (i) $\dfrac{x+3}{x+2} < 2, x \neq -2$



$\Rightarrow \quad \dfrac{x+3}{x+2}(x+2)^2 < 2(x+2)^2$

$\Rightarrow \quad (x+3)(x+2) < 2(x^2 + 4x + 4)$

$\Rightarrow \quad x^2 + 5x + 6 < 2x^2 + 8x + 8$

$\Rightarrow \quad x^2 + 3x + 2 > 0$

Solve $x^2 + 3x + 2 = 0$

$\Rightarrow \quad (x+2)(x+1) = 0$

$\Rightarrow$ Roots $x = -2, -1$

Hence $x^2 + 3x + 2 > 0 \quad \Rightarrow \quad x < -2 \quad$ or $\quad x > -1$

(ii) $\dfrac{x+5}{x-3} > 1, x \neq 3$

$\Rightarrow \quad \dfrac{x+5}{x-3}(x-3)^2 > 1(x-3)^2$

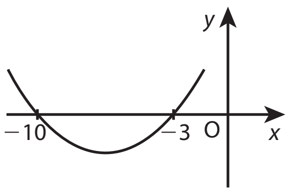
$\Rightarrow \quad (x+5)(x-3) > 1(x^2 - 6x + 9)$

$\Rightarrow \quad x^2 + 2x - 15 > x^2 - 6x + 9$

$\Rightarrow \quad 8x > 24$

$\Rightarrow \quad x > 3$

(iii) $\dfrac{2x-1}{x+3} > 3, x \neq 3$



$\Rightarrow \quad \dfrac{2x-1}{x+3}(x+3)^2 > 3(x+3)^2$

$\Rightarrow \quad (2x-1)(x+3) > 3(x^2 + 6x + 9)$

$\Rightarrow \quad 2x^2 + 5x - 3 > 3x^2 + 18x + 27$

$\Rightarrow \quad x^2 + 13x + 30 < 0$

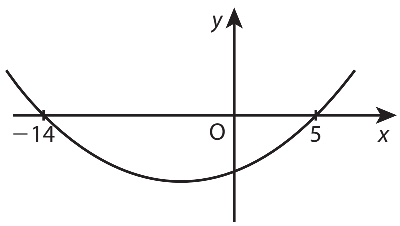
Solve $x^2 + 13x + 30 = 0$

$\Rightarrow \quad (x+10)(x+3) = 0$

$\Rightarrow \quad x = -10, -3$

Hence $x^2 + 13x + 30 < 0 \quad \Rightarrow \quad -10 < x < -3$

10. (i) $\dfrac{3x+4}{x-5} > 2, x \neq 5$



$\Rightarrow \quad \dfrac{3x+4}{x-5}(x-5)^2 > 2(x-5)^2$

$\Rightarrow \quad (3x+4)(x-5) > 2(x^2 - 10x + 25)$

$\Rightarrow \quad 3x^2 - 11x - 20 > 2x^2 - 20x + 50$

$\Rightarrow \quad x^2 + 9x - 70 > 0$

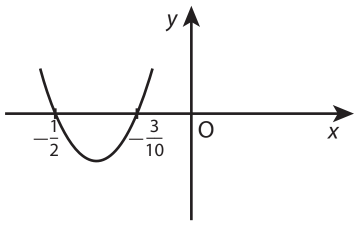
Solve $x^2 + 9x - 70 = 0$

$\Rightarrow (x + 14)(x - 5) = 0$

$\Rightarrow x = -14, 5$

Hence $x^2 + 9x - 70 > 0 \Rightarrow x < -14$ or $x > 5$

(ii) $\dfrac{1 - 2x}{4x + 2} > 2, x \neq -\dfrac{1}{2}$



$\Rightarrow \dfrac{1 - 2x}{4x + 2}(4x + 2)^2 > 2(4x + 2)^2$

$\Rightarrow (1 - 2x)(4x + 2) > 2(16x^2 + 16x + 4)$

$\Rightarrow 4x + 2 - 8x^2 - 4x > 32x^2 + 32x + 8$

$\Rightarrow 40x^2 + 32x + 6 < 0$

$\Rightarrow 20x^2 + 16x + 3 < 0$

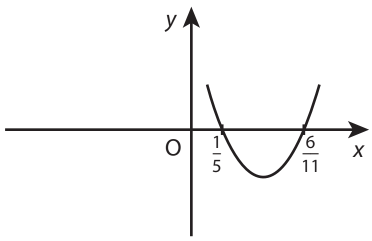
Solve $20x^2 + 16x + 3 = 0$

$\Rightarrow (2x + 1)(10x + 3) = 0$

$\Rightarrow$ Roots: $x = -\dfrac{1}{2}, -\dfrac{3}{10}$

Hence $20x^2 + 16x + 3 < 0 \Rightarrow -\dfrac{1}{2} < x < -\dfrac{3}{10}$

(iii) $\dfrac{3 + 4x}{5x - 1} > 3, x \neq \dfrac{1}{5}$



$\Rightarrow \dfrac{3 + 4x}{5x - 1}(5x - 1)^2 > 3(5x - 1)^2$

$\Rightarrow (3 + 4x)(5x - 1) > 3(25x^2 - 10x + 1)$

$\Rightarrow 15x - 3 + 20x^2 - 4x > 75x^2 - 30x + 3$

$\Rightarrow 55x^2 - 41x + 6 > 0$

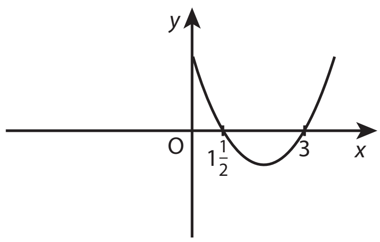
Solve $55x^2 - 41x + 6 = 0$

$\Rightarrow (5x - 1)(11x - 6) = 0$

$\Rightarrow$ Roots: $x = \dfrac{1}{5}, \dfrac{6}{11}$

Hence $55x^2 - 41x + 6 < 0 \Rightarrow \dfrac{1}{5} < x < \dfrac{6}{11}$

11. (i) $\dfrac{x}{2x - 3} \leqslant 1, \quad x \neq \dfrac{3}{2}$



$\Rightarrow \dfrac{x}{2x - 3}(2x - 3)^2 \leqslant 1(2x - 3)^2$

$\Rightarrow x(2x - 3) \leqslant 1(4x^2 - 12x + 9)$

$\Rightarrow 2x^2 - 3x \leqslant 4x^2 - 12x + 9$

$\Rightarrow 2x^2 - 9x + 9 \geqslant 0$

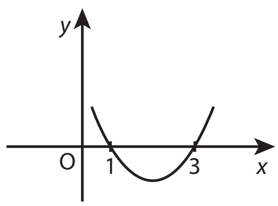
Solve $2x^2 - 9x + 9 = 0$

$\Rightarrow (2x - 3)(x - 3) = 0$

$\Rightarrow$ Roots: $x = 1\dfrac{1}{2}, 3$

Hence $2x^2 - 9x + 9 \geqslant 0 \Rightarrow x \leqslant 1\dfrac{1}{2}$ or $x \geqslant 3$

(ii) $\dfrac{2x - 4}{x - 1} < 1, \quad x \neq 1$



$\Rightarrow \dfrac{2x - 4}{x - 1}(x - 1)^2 < 1(x - 1)^2$

$\Rightarrow (2x - 4)(x - 1) < 1(x^2 - 2x + 1)$

$\Rightarrow 2x^2 - 6x + 4 < x^2 - 2x + 1$

$\Rightarrow x^2 - 4x + 3 < 0$

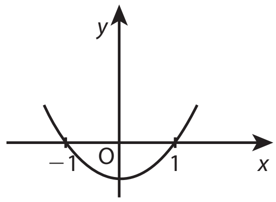
Solve $x^2 - 4x + 3 = 0$

$\Rightarrow (x - 1)(x - 3) = 0$

$\Rightarrow x = 1, 3$

Hence $x^2 - 4x + 3 < 0 \Rightarrow 1 < x < 3$

(iii) $\dfrac{x-5}{x-1} \leqslant 3, \quad x \neq 1$



$\Rightarrow \quad \dfrac{x-5}{x-1}(x-1)^2 \leqslant 3(x-1)^2$

$\Rightarrow \quad (x-5)(x-1) \leqslant 3(x^2-2x+1)$

$\Rightarrow \quad x^2-6x+5 \leqslant 3x^2-6x+3$

$\Rightarrow \quad 2x^2-2 \geq 0$

$\Rightarrow \quad x^2-1 \geq 0$

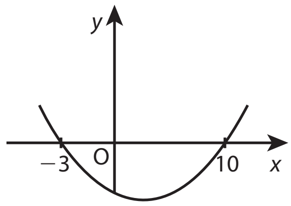
Solve $\quad x^2-1=0$

$\Rightarrow \quad (x+1)(x-1)=0$

$\Rightarrow \quad$ Roots: $x=-1, 1$

Hence $\quad x^2-1 \geqslant 0 \Rightarrow x \leqslant -1 \quad$ or $\quad x \geqslant 1$

12. (i) $\dfrac{2x-7}{x+3} < 1, \quad x \neq -3$



$\Rightarrow \quad \dfrac{2x-7}{x+3}(x+3)^2 < 1(x+3)^2$

$\Rightarrow \quad (2x-7)(x+3) < 1(x^2+6x+9)$

$\Rightarrow \quad 2x^2-x-21 < x^2+6x+9$

$\Rightarrow \quad x^2-7x-30 < 0$

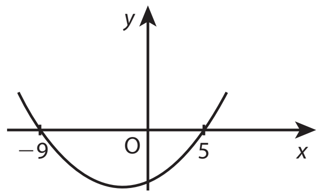
Solve $\quad x^2-7x-30=0$

$\Rightarrow \quad (x+3)(x-10)=0$

$\Rightarrow \quad$ Roots: $x=-3, 10$

Hence $\quad x^2-7x-30<0 \quad \Rightarrow \quad -3<x<10$

(ii) $\dfrac{2x-3}{x-5} < \dfrac{3}{2}, \quad x \neq 5$



$\Rightarrow \quad \dfrac{2x-3}{x-5}(x-5)^2 < \dfrac{3}{2}(x-5)^2$

$\Rightarrow \quad (2x-3)(x-5) < \dfrac{3}{2}(x^2-10x+25)$

$\Rightarrow \quad 2x^2-13x+15 < \dfrac{3x^2-30x+75}{2}$

$\Rightarrow \quad 4x^2-26x+30 < 3x^2-30x+75$

$\Rightarrow \quad x^2+4x-45<0$

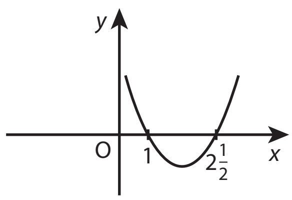
Solve $\quad x^2+4x-45=0$

$\Rightarrow \quad (x+9)(x-5)=0$

$\Rightarrow \quad x=-9, 5$

Hence $\quad x^2+4x-45<0 \quad \Rightarrow \quad -9<x<5$

(iii) $\dfrac{x+2}{x-1} \leqslant 3, \quad x \neq 1$



$\Rightarrow \quad \dfrac{x+2}{x-1}(x-1)^2 \leqslant 3(x-1)^2$

$\Rightarrow \quad (x+2)(x-1) \leqslant 3(x^2-2x+1)$

$\Rightarrow \quad x^2+x-2 \leqslant 3x^2-6x+3$

$\Rightarrow \quad 2x^2-7x+5 \geqslant 0$

Solve $\quad 2x^2-7x+5=0$

$\Rightarrow \quad (x-1)(2x-5)=0$

$\Rightarrow \quad$ Roots: $x=1, 2\frac{1}{2}$

Hence $\quad 2x^2-7x+5 \geqslant 0 \quad \Rightarrow \quad x \leqslant 1 \quad$ or $\quad x \geqslant 2\frac{1}{2}$

13. From graph: $-3 > x > -2$

Using inequality $\Rightarrow 2x^2 + 4x > x^2 - x - 6, \quad x \in R$

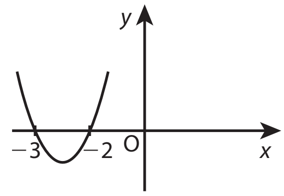
$\Rightarrow x^2 + 5x + 6 > 0$

Solve $x^2 + 5x + 6 = 0$

$\Rightarrow (x + 3)(x + 2) = 0$

$\Rightarrow$ Roots: $x = -3, -2$

Hence $x^2 + 5x + 6 > 0 \Rightarrow x < -3$ or $x > -2$



14. No real roots if $b^2 - 4ac < 0$

$x^2 + x + 1 = 0 \quad \Rightarrow \quad b^2 - 4ac = (1)^2 - 4(1)(1) = -3 < 0$

Hence $x^2 + x + 1 > 0$

$\Rightarrow \quad x^2 + x + \frac{1}{4} + \frac{3}{4} > 0$

$\Rightarrow \quad \left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 > 0$ True

15. (i) $f(t) \leqslant 4 \Rightarrow -11 + 13t - 2t^2 \leqslant 4$

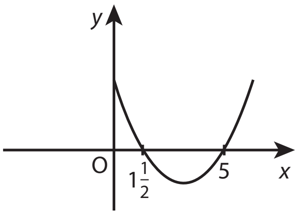
$\Rightarrow \quad 2t^2 - 13t + 15 \geq 0$

Solve $2t^2 - 13t + 15 = 0$

$\Rightarrow (2t - 3)(t - 5) = 0$

$\Rightarrow t = 1\frac{1}{2}, 5$

Hence $2t^2 - 13t + 15 \geq 0 \quad \Rightarrow \quad t \leqslant 1\frac{1}{2}$ or $t \geqslant 5$



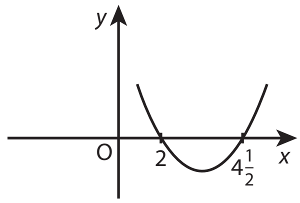
(ii) $f(t) \geq 7 \quad \Rightarrow \quad -11 + 13t - 2t^2 \geqslant 7$

$\Rightarrow \quad 2t^2 - 13t + 18 \leqslant 0$

$\Rightarrow \quad (t - 2)(2t - 9) = 0$

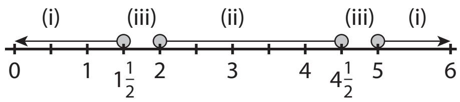
$\Rightarrow \quad$ Roots: $t = 2, 4\frac{1}{2}$

Hence $2t^2 - 13t + 18 \leqslant 0 \quad \Rightarrow \quad 2 \leqslant t \leqslant 4\frac{1}{2}$



(iii) $4 < f(t) < 7 \Rightarrow$

$1\frac{1}{2} < t < 2$ and $4\frac{1}{2} < t < 5$



16. (i) $x < -3$ or $x > -\frac{1}{2}$

(ii) $x \leqslant 1$ or $x \geq 3$

(iii) $-1\frac{1}{2} \leqslant x \leqslant \frac{1}{2}$

(iv) $-1 < x < 5$

17. Length $= x$ and Width $= x - 3$

Ratio $< 5 \quad \Rightarrow \quad \frac{x}{x-3} < 5$

$\Rightarrow \quad \frac{x}{x-3}(x - 3)^2 < 5(x - 3)^2$

$\Rightarrow \quad x(x - 3) < 5(x^2 - 6x + 9)$

$\Rightarrow \quad x^2 - 3x < 5x^2 - 30x + 45$

$\Rightarrow \quad 4x^2 - 27x + 45 > 0$

Solve $4x^2 - 27x + 45 = 0$

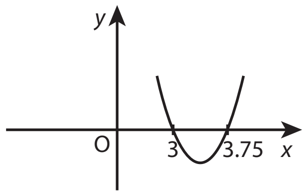
$\Rightarrow \quad (x - 3)(4x - 15) = 0$

$\Rightarrow \quad$ Roots: $x = 3, 3.375$

Hence $4x^2 - 27x + 45 > 0 \quad \Rightarrow \quad x < 3$ or $x > 3.75$

Since $x < 3$ is not valid, hence (i) Length > 3.75

(ii) Width > 0.75



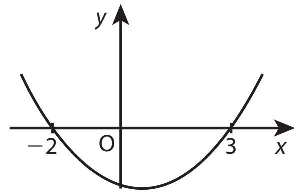
18. Positive graphs $\Rightarrow$ No real roots

$\Rightarrow b^2 - 4ac < 0$

$x^2 - 2px + p + 6 = 0 \Rightarrow (-2p)^2 - 4(1)(p + 6) < 0$

$\Rightarrow 4p^2 - 4p - 24 < 0$

$\Rightarrow p^2 - p - 6 < 0$



Solve $p^2 - p - 6 = 0$

$\Rightarrow (p + 2)(p - 3) = 0$

$\Rightarrow$ Roots: $p = -2, 3$

Hence $p^2 - p - 6 < 0 \Rightarrow -2 < p < 3$

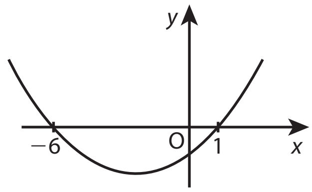
19. (i) Perimeter $< 50 \Rightarrow 2(x + 3) + 2(x + 2) < 50$

$\Rightarrow 2x + 6 + 2x + 4 < 50$

$\Rightarrow 4x < 50 - 10$

$\Rightarrow 4x < 40$

$\Rightarrow x < 10$



(ii) Area $> 12 \Rightarrow (x + 3)(x + 2) > 12$

$\Rightarrow x^2 + 5x + 6 > 12$

$\Rightarrow x^2 + 5x - 6 > 0$

Solve $x^2 + 5x - 6 = 0$

$\Rightarrow (x + 6)(x - 1) = 0$

$\Rightarrow x = -6$ (Not valid), $x = 1$ (valid)

Hence $x^2 + 5x - 6 > 0 \Rightarrow x > 1$

(iii) $1 < x < 10$, by combining (i) and (ii) above

20. Use Pythagoras $\Rightarrow h^2 = x^2 + 3^2 = x^2 + 9$

$\Rightarrow h = \sqrt{x^2 + 9}$

Perimeter $> 8 \Rightarrow \sqrt{x^2 + 9} + x + 3 > 8$

$\Rightarrow \sqrt{x^2 + 9} > -x + 5$

$\Rightarrow x^2 + 9 > (-x + 5)^2$

$\Rightarrow x^2 + 9 > x^2 - 10x + 25$

$\Rightarrow 10x > 16$

$\Rightarrow x > 1.6$

Perimeter $< 12 \Rightarrow \sqrt{x^2 + 9} + x + 3 < 12$

$\Rightarrow \sqrt{x^2 + 9} < -x + 9$

$\Rightarrow x^2 + 9 < (-x + 9)^2$

$\Rightarrow x^2 + 9 < x^2 - 18x + 81$

$\Rightarrow 18x < 72$

$\Rightarrow x < 4$

Hence $1.6 < x < 4$

Since $x \in Z \Rightarrow x = 2$ m or 3 m

Exercise 12.3

1. (i) $x + 3 = -1$ OR $x + 3 = 1$

$\Rightarrow x = -4$ OR $x = -2$

(ii) $x - 2 = -4$ OR $x - 2 = 4$

$\Rightarrow x = -2$ OR $x = 6$

(iii) $2x - 1 = -5$ OR $2x - 1 = 5$

$\Rightarrow 2x = -4$ OR $2x = 6$

$\Rightarrow x = -2$ OR $x = 3$

(iv) $3x - 2 = -x$ OR $3x - 2 = x$

$\Rightarrow$ $4x = 2$ OR $2x = 2$

$\Rightarrow$ $x = \frac{1}{2}$ OR $x = 1$

(v) $2(x - 3) = -2$ OR $2(x - 3) = 2$

$\Rightarrow$ $x - 3 = -1$ OR $x - 3 = 1$

$\Rightarrow$ $x = 2$ OR $x = 4$

(vi) $x - 5 = -(x + 1)$ OR $x - 5 = +(x + 1)$

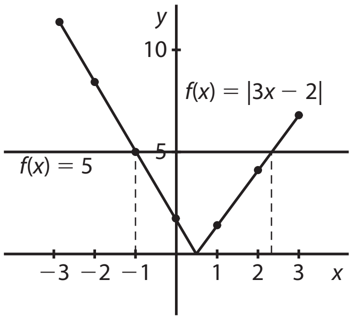
$\Rightarrow$ $x - 5 = -x - 1$ OR $x - 5 = x + 1$

$\Rightarrow$ $2x = 4$ OR $-5 = +1$ (Not valid)

$\Rightarrow$ $x = 2$

2. Copy and complete the following table and hence sketch a graph of $f(x) = |3x - 2|$

x	-3	-2	-1	0	1	2	3
$f(x) = \lvert 3x - 2 \rvert$	11	8	5	2	1	4	7



Solve $|3x - 2| = 5$.

$\Rightarrow x = -1, 2\frac{1}{3}$

3. $f(x) = |x|$, $g(x) = |x - 4|$, $h(x) = |x + 3|$

$f(-2) = |-2| = 2 \Rightarrow$ point $(-2, 2) \in f(x)$

$g(2) = |2 - 4| = |-2| = 2 \Rightarrow$ point $(2, 2) \in g(x)$

$h(-5) = |-5 + 3| = |-2| = 2 \Rightarrow$ point $(-5, 2) \in h(x)$

4. 2 points on $f(x)$ are $(-1, 0)$ and $(0, 1)$

$f(-1) = |a(-1) + b| = 0$ and $f(0) = |a(0) + b| = 1$

$\Rightarrow$ $-a + b = 0$ $\Rightarrow$ $b = 1$

Hence, $-a + 1 = 0$

$\Rightarrow$ $a = 1$

Hence, $f(x) = |x + 1|$.

2 points on $g(x)$ are $(-1, 0)$ and $(0, 2)$

$g(-1) = |a(-1) + b| = 0$ and $g(0) = |a(0) + b| = 2$

$\Rightarrow$ $-a + b = 0$ $\Rightarrow$ $b = 2$

Hence $-a + 2 = 0$

$\Rightarrow$ $a = 2$

Hence, $g(x) = |2x + 2|$

2 points on $h(x)$ are $(-1, 0)$ and $(0, 3)$

$h(-1) = |a(-1) + b| = 0$ and $h(0) = |a(0) + b| = 3$

$\Rightarrow$ $-a + b = 0$ $\Rightarrow$ $b = 3$

Hence $-a + 3 = 0$

$\Rightarrow$ $a = 3$

Hence, $h(x) = |3x + 3|$

$x = -2 \Rightarrow f(-2) = |-2 + 1| = 1$ $\Rightarrow (-2, 1) \in f(x)$

$x = -2 \Rightarrow g(-2) = |2(-2) + 2| = 2$ $\Rightarrow (-2, 2) \in g(x)$

$x = -2 \Rightarrow h(-2) = |3(-2) + 3| = 3$ $\Rightarrow (-2, 3) \in h(x)$

5. $f(x) = |x - 2|$

$\Rightarrow f(0) = |0 - 2| = 2 \quad (0, 2) \in f(x)$

$\Rightarrow f(2) = |2 - 2| = 0 \quad (2, 0) \in f(x)$

$\Rightarrow f(4) = |4 - 2| = 2 \quad (4, 2) \in f(x)$

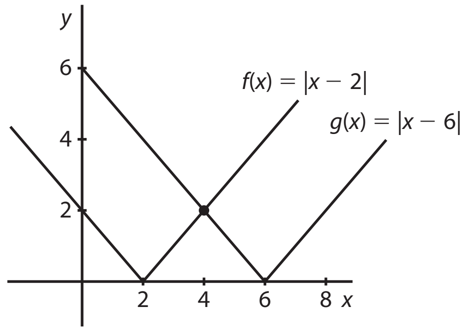
$g(x) = |x - 6|$

$\Rightarrow g(0) = |0 - 6| = 6 \quad (0, 6) \in g(x)$

$\Rightarrow g(6) = |6 - 6| = 0 \quad (6, 0) \in g(x)$

$\Rightarrow g(8) = |8 - 6| = 2 \quad (8, 2) \in g(x)$

Hence, $|x - 2| = |x - 6|$ at $x = 4$.



Solve $|x - 2| = |x - 6|$

$\Rightarrow \quad x - 2 = -(x - 6) \quad$ OR $\quad x - 2 = +(x - 6)$

$\Rightarrow \quad x - 2 = -x + 6 \quad$ OR $\quad x - 2 = x - 6$

$\Rightarrow \quad 2x = 8 \quad$ OR $\quad -2 = -6$ (Not valid)

$\Rightarrow \quad x = 4$

6. (i) $|x - 6| < 2$

$\Rightarrow -2 < x - 6 < 2$

$\Rightarrow 4 < x < 8$

(ii) $|x + 2| \leqslant 4$

$\Rightarrow -4 \leqslant x + 2 \leqslant 4$

$\Rightarrow -6 \leqslant x \leqslant 2$

(iii) $|2x - 1| \geqslant 5$

$\Rightarrow -5 \geqslant 2x - 1 \geqslant 5$

$\Rightarrow -4 \geqslant 2x \geqslant 6 \qquad \Rightarrow -4 \geqslant 2x$ or $2x \geqslant 6$

$\Rightarrow 2x \leqslant -4$ or $x \geqslant 3$

$\Rightarrow x \leqslant -2$ or $x \geqslant 3$

(iv) $|2x - 1| \geqslant 11$

$\Rightarrow -11 \geqslant 2x - 1 \geqslant 11$

$\Rightarrow -10 \geqslant 2x \geqslant 12 \qquad \Rightarrow -10 \geqslant 2x$ or $2x \geqslant 12$

$\Rightarrow \quad 2x \leqslant -10$ or $x \geqslant 6$

$\Rightarrow \quad x \leqslant -5$ or $x \geqslant 6$

(v) $|3x + 5| < 4$

$\Rightarrow -4 < 3x + 5 < 4$

$\Rightarrow -9 < 3x < -1$

$\Rightarrow -3 < x < -\frac{1}{3}$

(vi) $|x - 4| < 3$

$\Rightarrow -3 < x - 4 < 3$

$\Rightarrow 1 < x < 7$

7. (i) $|2x - 1| \geqslant 7$

$\Rightarrow -7 \geqslant 2x - 1 \geqslant 7$

$\Rightarrow -6 \geqslant 2x \geqslant 8$

$\Rightarrow -3 \geqslant x \geqslant 4$

(ii) $|3x + 4| \leqslant |x + 2|$

$\Rightarrow (3x + 4)^2 \leqslant (x + 2)^2$

$\Rightarrow 9x^2 + 24x + 16 \leqslant x^2 + 4x + 4$

$\Rightarrow 8x^2 + 20x + 12 \leqslant 0$

$\Rightarrow 2x^2 + 5x + 3 \leqslant 0$

Solve $2x^2 + 5x + 3 = 0$

$\Rightarrow (2x + 3)(x + 1) = 0$

$\Rightarrow$ Roots: $x = -1\frac{1}{2}, -1$

Hence, $\quad 2x^2 + 5x + 3 \leqslant 0 \Rightarrow -1\frac{1}{2} \leqslant x \leqslant -1$

(iii) $2|x - 1| \leqslant |x + 3|$

$\Rightarrow [2(x - 1)]^2 \leqslant (x + 3)^2$

$\Rightarrow 4(x^2 - 2x + 1) \leqslant x^2 + 6x + 9$

$\Rightarrow 4x^2 - 8x + 4 \leqslant x^2 + 6x + 9$

$\Rightarrow 3x^2 - 14x - 5 \leqslant 0$

Solve $3x^2 - 14x - 5 = 0$

$\Rightarrow \quad (3x + 1)(x - 5) = 0$

$\Rightarrow \quad$ Roots: $x = -\frac{1}{3}, 5$

Hence, $\quad 3x^2 - 14x - 5 \leqslant 0 \Rightarrow -\frac{1}{3} \leqslant x \leqslant 5$

8. $f(x) = |x| - 4$

$f(-4) = |-4| - 4 = 4 - 4 = 0 \qquad (-4, 0) \in f(x)$

$f(0) = |0| - 4 = -4 \qquad (0, -4) \in f(x)$

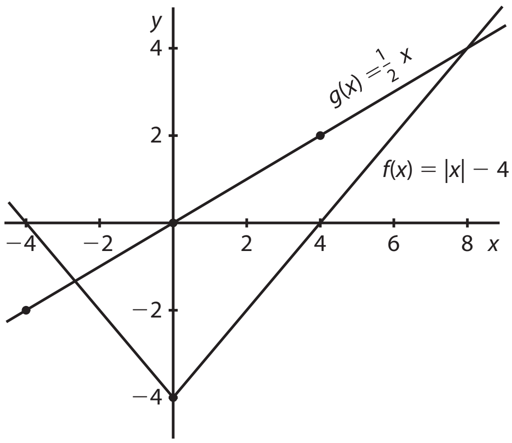
$f(4) = |4| - 4 = 4 - 4 = 0 \qquad (4, 0) \in f(x)$

$g(x) = \frac{1}{2}x$

$g(-4) = \frac{1}{2}(-4) = -2 \qquad (-4, -2) \in g(x)$

$g(0) = \frac{1}{2}(0) = 0 \qquad (0, 0) \in g(x)$

$g(4) = \frac{1}{2}(4) = 2 \qquad (4, 2) \in g(x)$



$f(x) \leqslant g(x) \Rightarrow |x| - 4 \leqslant \frac{1}{2}x$

Solve $|x| = \frac{1}{2}x + 4$

$\Rightarrow x^2 = \frac{1}{4}x^2 + 4x + 16$

$\Rightarrow 4x^2 = x^2 + 16x + 64$

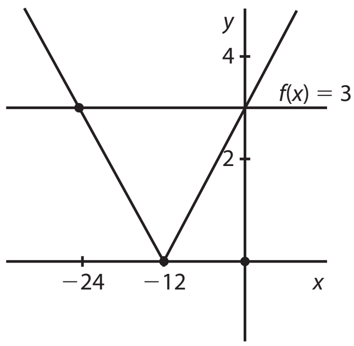
$\Rightarrow 3x^2 - 16x - 64 = 0$

$\Rightarrow (3x + 8)(x - 8) = 0$

$\Rightarrow$ Roots $x = -2\frac{2}{3}, 8$

Hence, $\quad 3x^2 - 16x - 64 \leqslant 0 \Rightarrow -2\frac{2}{3} \leqslant x \leqslant 8$

9. $f(x) = \left|\frac{1}{4}x + 3\right|$



$f(-24) = \left|\frac{1}{4}(-24) + 3\right| = |-6 + 3| = |-3| = 3 \Rightarrow (-24, 3) \in f(x)$

$f(-12) = \left|\frac{1}{4}(-12) + 3\right| = |-3 + 3| = 0 \Rightarrow (-12, 0) \in f(x)$

$f(0) = \left|\frac{1}{4}(0) + 3\right| = |0 + 3| = 3 \Rightarrow (0, 3) \in f(x)$

Hence, $\quad \left|\frac{1}{4}x + 3\right| \geqslant 3 \Rightarrow x \leqslant -24$ or $x \geqslant 0$

10. $|1 + 2x| < |x + 2|$
$\Rightarrow (1 + 2x)^2 < (x + 2)^2$
$\Rightarrow 1 + 4x + 4x^2 < x^2 + 4x + 4$
$\Rightarrow 3x^2 - 3 < 0$
$\Rightarrow x^2 - 1 < 0$
Solve $x^2 - 1 = 0$
$\Rightarrow (x + 1)(x - 1) = 0$
$\Rightarrow$ Roots: $x = -1, 1$
Hence, $x^2 - 1 < 0 \Rightarrow -1 < x < 1$

11. $\left|\frac{1}{1 + 2x}\right| = 1 \Rightarrow \left(\frac{1}{1 + 2x}\right)^2 = (1)^2$
$\Rightarrow \frac{1}{1 + 4x + 4x^2} = \frac{1}{1}$
$\Rightarrow 1 + 4x + 4x^2 = 1$
$\Rightarrow 4x^2 + 4x = 0$
$\Rightarrow x^2 + x = 0$
$\Rightarrow x(x + 1) = 0$
$\Rightarrow$ Roots: $x = 0, -1$
Solve $\left|\frac{1}{1 + 2x}\right| < 1$
Hence, $\frac{1}{1 + 4x + 4x^2} < 1$
$\Rightarrow 1 + 4x + 4x^2 > 1$
$\Rightarrow 4x^2 + 4x > 0$
$\Rightarrow x^2 + x > 0$
$\Rightarrow -1 > x > 0$

12. (i) $-4 < x < 2$
(ii) $x < -4$ or $x > 2$
(iii) $1\frac{1}{4} < x < 3\frac{1}{2}$
(iv) $2 < x < 3$
(v) $1\frac{1}{4} < x < 2$
(vi) $2 < x < 3$
(vii) $3 < x < 3\frac{1}{2}$

13. (i) $\frac{x}{2x - 1} < -2$
$\Rightarrow + \left(\frac{x}{2x - 1}\right) < -2$
$\Rightarrow \frac{x}{2x - 1}(2x - 1)^2 < -2(2x - 1)^2$
$\Rightarrow x(2x - 1) < -2(4x^2 - 4x + 1)$
$\Rightarrow 2x^2 - x < -8x^2 + 8x - 2$
$\Rightarrow 10x^2 - 9x + 2 < 0$
Solve $10x^2 - 9x + 2 = 0$
$\Rightarrow (5x - 2)(2x - 1) = 0$
$\Rightarrow$ Roots: $x = \frac{2}{5}, \frac{1}{2}$
Hence, $10x^2 - 9x + 2 < 0 \Rightarrow \frac{2}{5} < x\frac{1}{2}$

(ii) $|x-3| = 2|x-1|$

$\Rightarrow (x-3)^2 = [2(x-1)]^2$

$\Rightarrow x^2 - 6x + 9 = 4(x^2 - 2x + 1)$

$\Rightarrow x^2 - 6x + 9 = 4x^2 - 8x + 4$

$\Rightarrow 3x^2 - 2x - 5 = 0$

$\Rightarrow (x+1)(3x-5) = 0$

$\Rightarrow$ Roots: $x = -1, 1\frac{2}{3}$

(iii) $|x-1| - |2x+1| > 0$

$\Rightarrow |x-1| > |2x+1|$

$\Rightarrow (x-1)^2 > (2x+1)^2$

$\Rightarrow x^2 - 2x + 1 > 4x^2 + 4x + 1$

$\Rightarrow 3x^2 + 6x < 0$

$\Rightarrow x^2 + 2x < 0$

Solve $x^2 + 2x = 0$

$\Rightarrow x(x+2) = 0$

$\Rightarrow$ Roots: $x = -2, 0$

Hence, $x^2 + 2x < 0 \Rightarrow -2 < x < 0$

Exercise 12.4

1. Proof: Assume x and y are both positive integers and $x^2 - y^2 = (x+y)(x-y)$.

If $x > y$, then $(x+y)$ and $(x-y)$ are positive integers; hence, $(x+y)(x-y)$ is a positive integer $\neq 1$.

If $x < y$, then $(x+y)$ is a positive integer and $(x-y)$ is a negative integer; hence, $(x+y)(x-y)$ is a negative integer $\neq 1$.

Hence, there is a contradiction in both cases.

$\therefore$ There are no positive integer solutions to $x^2 - y^2 = 1$.

2. Proof: Assume $(a+b)$ is a rational number.

Hence, $(a+b)$ can be written as $\frac{p}{q}$ where $p, q \in Z, q \neq 0$.

Since "a" is a rational number $= \frac{m}{n}$ where $m, n \in Z, n \neq 0$

Then, $a + b = \frac{p}{q}$

$\Rightarrow \frac{m}{n} + b = \frac{p}{q}$

$\Rightarrow b = \frac{p}{q} - \frac{m}{n}$

$= \frac{pn - mq}{qn}$ which is rational.

This is a contradiction as b is a irrational number; hence, $a + b$ is an irrational number.

3. Proof: Assume x and y are both positive integers
and $x^2 - y^2 = (x + y)(x - y)$.
If $x > y$, then $(x + y)$ and $(x - y)$ they are positive integers;
hence, $(x + y)(x - y)$ is a positive integer $\neq 10$.
Note: $x + y = 5$ and $x + y = 10$

$$\begin{array}{rl} x - y = 2 \\ \hline 2x = 7 \\ \Rightarrow x = 3\frac{1}{2} \notin Z \end{array} \qquad \begin{array}{rl} x - y = 1 \\ \hline 2x = 11 \\ \Rightarrow x = 5\frac{1}{2} \in Z \end{array}$$

If $x < y$, then $(x + y)$ is a positive integer
and $(x - y)$ is a negative integer;
hence, $(x + y)(x - y)$ is a negative integer $\neq 10$.
Hence, there is a contradiction in both cases.
$\therefore$ There are no positive integer solutions to $x^2 - y^2 = 10$.

4. Proof: a divides b $\Rightarrow$ $b = k(a), k \in N$
b divides c $\Rightarrow$ $c = m(b), m \in N$
$\Rightarrow$ $c = m[k(a)]$
$\Rightarrow$ $c = mk(a), m, k \in N$
Then a divides c.

5. Proof: a divides b $\Rightarrow$ $b = k(a), k \in N$
a divides c $\Rightarrow$ $c = m(a), m \in N$
hence, $(b + c) = k(a) + m(a)$
$= (k + m)(a), k, m \in N$
Then a divides $(b + c)$.

6. Proof: $a^2 + b^2 \geqslant 2ab$
$\Rightarrow$ $a^2 - 2ab + b^2 \geqslant 0$
$\Rightarrow$ $(a - b)(a - b) \geqslant 0$
$\Rightarrow$ $(a - b)^2 \geqslant 0$, true for $a, b \in R$
Hence, $a^2 + b^2 \geqslant 2ab$.

7. Proof: a is a rational number
Hence, a can be written as $\frac{p}{q}$ where $p, q \in Z, q \neq 0$.
b is a rational number.
Hence, b can be written as $= \frac{m}{n}$ where $m, n \in Z, n \neq 0$.
Hence, $a + b = \frac{p}{q} + \frac{m}{n} = \frac{pn + mq}{qn}$
= a rational number, as $p, q, m, n \in Z$
$qn \neq 0$.

8. Proof: x is an odd number $= 2a + 1$, where $a \in N$.
y is an odd number $= 2b + 1$, where $b \in N$.
Hence, $x + y = 2a + 1 + 2b + 1$
$= 2a + 2b + 2$
$= 2(a + b + 1)$
Hence, $x + y$ has a factor of 2.
$\Rightarrow x + y$ must be even.
Hence, the sum of 2 odd numbers is always even.

Exercise 12.5

1. (i) $a^2 + 2ab + b^2 \geq 0$

$\Rightarrow$ $(a + b)(a + b) \geq 0$

$\Rightarrow$ $(a + b)^2 \geq 0$, true for $a, b \in R$

(ii) $a^2 + 2ab + b^2 \geq 0$

$\Rightarrow$ $a^2 + 2ab + b^2 + b^2 \geq 0$

$\Rightarrow$ $(a + b)^2 + (b)^2 \geq 0$, true for $a, b \in R$

2. $(a + b)^2 \geq 4ab$

$\Rightarrow$ $a^2 + 2ab + b^2 - 4ab \geq 0$

$\Rightarrow$ $a^2 - 2ab + b^2 \geq 0$

$\Rightarrow$ $(a - b)^2 \geq 0$, true for $a, b \in R$

3. $-(a^2 + 2ab + b^2) \leq 0$

$\Rightarrow$ $a^2 + 2ab + b^2 \geq 0$

$\Rightarrow$ $(a + b)^2 \geq 0$, true for $a, b \in R$

4. (i) $a + \dfrac{1}{a} \geq 2$

$\Rightarrow$ $a^2 + 1 \geq 2a$

$\Rightarrow$ $a^2 - 2a + 1 \geq 0$

$\Rightarrow$ $(a - 1)^2 \geq 0$, true for $a > 0$ and $a \in R$

(ii) $\dfrac{1}{a} + \dfrac{1}{b} \geq \dfrac{2}{a + b}$

$$(a)(b)(a + b) \cdot \frac{1}{a} + (a)(b)(a + b) \cdot \frac{1}{b} \geq (a)(b)(a + b) \cdot \frac{2}{a + b}$$

$\Rightarrow$ $b(a + b) + a(a + b) \geq 2ab$

$\Rightarrow$ $ab + b^2 + a^2 + ab \geq 2ab$

$\Rightarrow$ $a^2 + b^2 \geq 0$, true for $a > 0$, $b > 0$ and $a, b \in R$

5. $a^2 - 6a + 9 + b^2 \geq 0$

$\Rightarrow$ $(a - 3)(a - 3) + b^2 \geq 0$

$\Rightarrow$ $(a - 3)^2 + b^2 \geq 0$, true for $a, b \in R$

6. (i) $x^2 + 6x + 9 \geq 0$

$\Rightarrow$ $(x + 3)^2 \geq 0$, true for $x \in R$

(ii) $x^2 - 10x + 25 \geq 0$

$\Rightarrow$ $(x - 5)^2 \geq 0$, true for $x \in R$

(iii) $x^2 + 4x + 6 \geq 0$

$\Rightarrow$ $x^2 + 4x + 4 + 2 \geq 0$

$\Rightarrow$ $(x + 2)^2 + 2 \geq 0$, true for $x \in R$

(iv) $x^2 - 6x + 10 \geq 0$

$\Rightarrow$ $x^2 - 6x + 9 + 1 \geq 0$

$\Rightarrow$ $(x - 3)^2 + 1 \geq 0$, true for $x \in R$

(v) $4x^2 + 12x + 11 \geq 0$

$\Rightarrow$ $x^2 + 3x + \dfrac{11}{4} \geq 0$

$\Rightarrow$ $x^2 + 3x + \dfrac{9}{4} + \dfrac{2}{4} \geq 0$

$\Rightarrow$ $\left(x + 1\dfrac{1}{2}\right)^2 + \dfrac{1}{2} \geq 0$, true for $x \in R$

(vi) $4x^2 - 4x + 2 \geq 0$

$\Rightarrow \quad x^2 - x + \frac{1}{2} \geq 0$

$\Rightarrow \quad x^2 - x + \frac{1}{4} + \frac{1}{4} \geq 0$

$\Rightarrow \quad \left(x - \frac{1}{2}\right)^2 + \frac{1}{4} \geq 0,$ true for $x \in R$

7. (i) $-x^2 + 10x - 25 \geq 0$

$\Rightarrow \quad x^2 - 10x + 25 \geq 0$

$\Rightarrow \quad (x - 5)^2 \geq 0,$ true for $x \in R$

(ii) $-x^2 - 4x - 7 \leq 0$

$\Rightarrow \quad x^2 + 4x + 7 \geq 0$

$\Rightarrow \quad x^2 + 4x + 4 + 3 \geq 0$

$\Rightarrow \quad (x + 2)^2 + 3 \geq 0,$ true for $x \in R$

8. (i) $p^2 + 4q^2 \geq 4pq$

$\Rightarrow \quad p^2 - 4pq + 4q^2 \geq 0$

$\Rightarrow \quad (p - 2q)(p - 2q) \geq 0$

$\Rightarrow \quad (p - 2q)^2 \geq 0,$ true for $p, q \in R$

(ii) $(p + q)^2 \leq 2(p^2 + q^2)$

$\Rightarrow \quad p^2 + 2pq + q^2 \leq 2p^2 + 2q^2$

$\Rightarrow \quad -p^2 + 2pq - q^2 \leq 0$

$\Rightarrow \quad p^2 - 2pq + q^2 \geq 0$

$\Rightarrow \quad (p - q)^2 \geq 0,$ true for $p, q \in R$

9. $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Proof: $a^3 + b^3 > a^2b + ab^2$

$\Rightarrow \quad a^3 + b^3 - a^2b - ab^2 > 0$

$\Rightarrow \quad (a + b)(a^2 - ab + b^2) - ab(a + b) > 0$

$\Rightarrow \quad (a + b)(a^2 - ab + b^2 - ab) > 0$

$\Rightarrow \quad (a + b)(a - b)^2 > 0,$ true for $a > 0, b > 0$

10. Given $a^2 + b^2 \geq 2ab$.

(i) $a^2 + c^2 \geq 2ac$

(ii) $b^2 + c^2 \geq 2bc$

$a^2 + b^2 \geq 2ab$

$a^2 + c^2 \geq 2ac$

$b^2 + c^2 \geq 2bc$

Add: $\Rightarrow \quad 2a^2 + 2b^2 + 2c^2 \geq 2ab + 2bc + 2ac$

$\Rightarrow \quad a^2 + b^2 + c^2 \geq ab + bc + ac$

11. $\frac{p + q}{2} > \sqrt{pq}$

$\Rightarrow \quad p + q > 2\sqrt{pq}$

$\Rightarrow \quad (p + q)^2 > (2\sqrt{pq})^2$

$\Rightarrow \quad p^2 + 2pq + q^2 > 4pq$

$\Rightarrow \quad p^2 - 2pq + q^2 > 0$

$\Rightarrow \quad (p - q)^2 > 0$ … true

12. $(ax + by)^2 \leq (a^2 + b^2)(x^2 + y^2)$

$\Rightarrow \quad a^2x^2 + 2axby + b^2y^2 \leq a^2x^2 + a^2y^2 + b^2x^2 + b^2y^2$

$\Rightarrow \quad -a^2y^2 + 2axby - b^2x^2 \leq 0$

$\Rightarrow \quad a^2y^2 - 2axby + b^2x^2 \geq 0$

$\Rightarrow \quad (ay - bx)^2 \geq 0,$ true for $a, b, x, y \in R$

13. $a^4 + b^4 \geq 2a^2b^2$

$\Rightarrow \quad a^4 - 2a^2b^2 + b^4 \geq 0$

$\Rightarrow \quad (a^2 - b^2)(a^2 - b^2) \geq 0$

$\Rightarrow \quad (a^2 - b^2)^2 \geq 0,$ true for $a, b \in R$

14. $(a + 2b)\left(\frac{1}{a} + \frac{1}{2b}\right) \geq 4$

$\Rightarrow \quad a \cdot \frac{1}{a} + \frac{a}{2a} + \frac{2b}{a} + 2b \cdot \frac{1}{2b} \geq 4$

$\Rightarrow \quad 1 + \frac{a}{2b} + \frac{2b}{a} + 1 \geq 4$

$\Rightarrow \quad \frac{a}{2b} + \frac{2b}{a} \geq 2$

$\Rightarrow \quad 2ab \cdot \frac{a}{2b} + 2ab \cdot \frac{2b}{a} \geq 2ab \,.\, 2$ ... since $a > 0, b > 0$

$\Rightarrow \quad a^2 + 4b^2 \geq 4ab$

$\Rightarrow \quad a^2 - 4ab + 4b^2 \geq 0$

$\Rightarrow \quad (a - 2b)^2 \geq 0,$ true for a, b positive

15. $\frac{a}{(a + 1)^2} \leq \frac{1}{4}$

$\Rightarrow \quad \frac{a}{(a + 1)^2} \cdot (a + 1)^2 \leq \frac{1}{4}(a + 1)^2$

$\Rightarrow \quad a \leq \frac{a^2 + 2a + 1}{4}$

$\Rightarrow \quad 4a \leq a^2 + 2a + 1$

$\Rightarrow \quad -a^2 + 2a - 1 \leq 0$

$\Rightarrow \quad a^2 - 2a + 1 \geq 0$

$\Rightarrow \quad (a - 1)^2 \geq 0,$ true for $a \in R$

16. (i) $a^4 - b^4 = (a^2 + b^2)(a^2 - b^2)$

$= (a^2 + b^2)(a + b)(a - b)$

(ii) $a^5 - a^4b - ab^4 + b^5$

$= a^4(a - b) - b^4(a - b)$

$= (a - b)(a^4 - b^4)$

$= (a - b)(a^2 + b^2)(a + b)(a - b)$

$= (a^2 + b^2)(a + b)(a - b)^2$

(iii) $a^5 + b^5 > a^4b + ab^4$

$\Rightarrow \quad a^5 - a^4b - ab^4 + b^5 > 0$

$\Rightarrow \quad (a^2 + b^2)(a + b)(a - b)^2 > 0,$ true if a and b are positive unequal real numbers.

17. Given: $a^2 + b^2 = 1$ and $c^2 + d^2 = 1.$

$a^2 + b^2 \geq 2ab \Rightarrow a^2 - 2ab + b^2 \geq 0$

$\Rightarrow \quad (a - b)^2 \geq 0$

Hence, $(a - b)^2 + (c - d)^2 \geq 0$

$\Rightarrow \quad a^2 - 2ab + b^2 + c^2 - 2cd + d^2 \geq 0$

$\Rightarrow \quad 1 - 2ab + 1 - 2cd \geq 0$

$\Rightarrow \quad -2ab - 2cd \geq -2$

$\Rightarrow \quad 2ab + 2cd \leq 2$

$\Rightarrow \quad ab + cd \leq 1$

18. $\sqrt{ab} > \dfrac{2ab}{a+b}$

$\Rightarrow \quad (a+b).\sqrt{ab} > (a+b)\dfrac{2ab}{a+b}$ if a and b are positive and unequal

$\Rightarrow \quad (a+b)\sqrt{ab} > 2ab$

$\Rightarrow \quad (a+b) > \dfrac{2ab}{\sqrt{ab}}$

$\Rightarrow \quad a+b > 2\sqrt{ab}$

$\Rightarrow \quad (a+b)^2 > (2\sqrt{ab})^2$

$\Rightarrow \quad a^2 + 2ab + b^2 > 4ab$

$\Rightarrow \quad a^2 - 2ab + b^2 > 0$

$\Rightarrow \quad (a-b)^2 > 0,$ true if a and b are positive and unequal.

19. $a + \dfrac{9}{a+2} \geq 4$

$\Rightarrow \quad a(a+2) + \dfrac{9}{a+2}(a+2) \geq 4(a+2)$, where $(a+2) > 0$

$\Rightarrow \quad a^2 + 2a + 9 \geq 4a + 8$

$\Rightarrow \quad a^2 - 2a + 1 \geq 0$

$\Rightarrow \quad (a-1)^2 \geq 0,$ true where $(a+2) > 0$

20. If $\dfrac{a}{b} > \dfrac{c}{d}$,

$\Rightarrow \quad \dfrac{a}{b}(bd) > \dfrac{c}{d}(bd)$ if a, b, c, d are positive numbers

$\Rightarrow \quad ad > cd$

Hence, $\dfrac{a+c}{b+d} > \dfrac{c}{d}$

$\Rightarrow \quad \dfrac{(a+c)}{(b+d)} \cdot (d)(b+d) > \dfrac{c}{d}(d)(b+d)$ if a, b, c, d are positive numbers

$\Rightarrow \quad ad + cd > cb + cd$

$\Rightarrow \quad ad > cb,$ true

21. $(a^3 - b^3)(a-b) = (a-b)(a^2+ab+b^2)(a-b)$

$= (a-b)^2(a^2+ab+b^2)$

If $a > b \Rightarrow (a > b) > 0$.

Hence, $(a-b)^2$ is positive,

and $a^2 + ab + b^2 > a^2 - 2ab + b^2 = (a-b)^2 > 0$

Hence, $a^2 + ab + b^2$ is a positive, true if a, b, c, d are positive numbers.

Hence, $a^4 + b^4 \geq a^3b + ab^3$

$\Rightarrow \quad a^4 - a^3b - ab^3 + b^4 \geq 0$

$\Rightarrow \quad a^3(a-b) - b^3(a-b) \geq 0$

$\Rightarrow \quad (a-b)(a^3-b^3) \geq 0$

$\Rightarrow \quad (a-b)(a-b)(a^2+ab+b^2) \geq 0$

$\Rightarrow \quad (a-b)^2(a^2+ab+b^2) \geq 0,$ true for $a, b \in R$ and $a > b$

Exercise 12.6

1. (i) $a^2 \times a^3 = a^{2+3} = a^5$

(ii) $x \,.\, x \,.\, x^2 = x^{1+1+2} = x^4$

(iii) $2x^3 \times 3x^3 = 6x^{3+3} = 6x^6$

(iv) $\dfrac{x^5}{x^2} = x^{5-2} = x^3$

(v) $\dfrac{x^4}{x^5} = x^{4-5} = x^{-1}$

(vi) $a^0 = 1$

(vii) $\sqrt[3]{27} = 3$

(viii) $(a^3)^2 = a^6$

(ix) $\dfrac{(x^3)^2}{x^3} = \dfrac{x^6}{x^3} = x^{6-3} = x^3$

(x) $(3ab)^2 = 9a^2b^2$

2. (i) $\sqrt[3]{64} = 4$

(ii) $3^{-2} = \dfrac{1}{3^2} = \dfrac{1}{9}$

(iii) $\dfrac{1}{2^{-3}} = 2^3 = 8$

(iv) $\dfrac{2^{-2}}{3^{-2}} = \dfrac{3^2}{2^2} = \dfrac{9}{4}$

(v) $\dfrac{1}{4^{-\frac{1}{2}}} = 4^{\frac{1}{2}} = 2$

3. (i) $8^{\frac{2}{3}} = (2^3)^{\frac{2}{3}} = 2^2 = 4$

(ii) $16^{\frac{3}{4}} = (2^4)^{\frac{3}{4}} = 2^3 = 8$

(iii) $27^{\frac{2}{3}} = (3^3)^{\frac{2}{3}} = 3^2 = 9$

(iv) $81^{\frac{3}{4}} = (3^4)^{\frac{3}{4}} = 3^3 = 27$

(v) $125^{\frac{2}{3}} = (5^3)^{\frac{2}{3}} = 5^2 = 25$

4. (i) $\left(\dfrac{2}{3}\right)^{-2} = \dfrac{1}{\left(\frac{2}{3}\right)^2} = \dfrac{1}{\frac{4}{9}} = \dfrac{9}{4}$

(ii) $\left(\dfrac{4}{9}\right)^{-\frac{1}{2}} = \dfrac{1}{\left(\frac{4}{9}\right)^{\frac{1}{2}}} = \dfrac{1}{\frac{2}{3}} = \dfrac{3}{2}$

(iii) $\left(\dfrac{9}{25}\right)^{-\frac{3}{2}} = \dfrac{1}{\left(\frac{9}{25}\right)^{\frac{3}{2}}} = \dfrac{1}{\left(\left(\frac{3}{5}\right)^2\right)^{\frac{3}{2}}} = \dfrac{1}{\left(\frac{3}{5}\right)^3} = \dfrac{1}{\frac{27}{125}} = \dfrac{125}{27}$

(iv) $\left(\dfrac{27}{125}\right)^{-\frac{2}{3}} = \left(\left(\frac{3}{5}\right)^3\right)^{\frac{-2}{3}} = \left(\dfrac{3}{5}\right)^{-2} = \dfrac{1}{\left(\frac{3}{5}\right)^2} = \dfrac{1}{\frac{9}{25}} = \dfrac{25}{9}$

(v) $\left(3\dfrac{3}{8}\right)^{\frac{1}{3}} = \left(\dfrac{27}{8}\right)^{\frac{1}{3}} = \dfrac{(27)^{\frac{1}{3}}}{(8)^{\frac{1}{3}}} = \dfrac{3}{2}$

5. $\dfrac{4^2 \times 16^{\frac{1}{2}}}{64^{\frac{2}{3}} \times 4^3} = \dfrac{4^2 \times 4}{(4^3)^{\frac{2}{3}} \times 4^3} = \dfrac{4^3}{4^2 \times 4^3} = \dfrac{4^3}{4^5} = 4^{3-5} = 4^{-2}$

6. $$\frac{3^{\frac{1}{4}} \times 3 \times 3^{\frac{1}{6}}}{\sqrt{3}} = \frac{3^{\frac{1}{4}+1+\frac{1}{6}}}{3^{\frac{1}{2}}} = \frac{3^{1\frac{5}{12}}}{3^{\frac{1}{2}}} = 3^{1\frac{5}{12}-\frac{1}{2}} = 3^{\frac{11}{12}}$$

$$3^{P} = 3^{\frac{11}{12}} \Rightarrow p = \frac{11}{12}$$

7. (i) $$\frac{(xy^2)^3 \times (x^2y)^{-2}}{xy} = \frac{x^3y^6 \,.\, x^{-4}y^{-2}}{xy} = \frac{x^{-1}y^4}{x^1y^1}$$

$$= \frac{y^{4-1}}{x^{1+1}} = \frac{y^3}{x^2}$$

(ii) $$\left(\frac{p^2q}{p^{-1}q^3}\right)^4 = \frac{p^8\,q^4}{p^{-4}\,q^{12}} = \frac{p^{8+4}}{q^{12-4}} = \frac{p^{12}}{q^8}$$

(iii) $$a^{\frac{1}{4}} \times a^{\frac{-5}{4}} = a^{\frac{1}{4}-\frac{5}{4}} = a^{-1} = \frac{1}{a}$$

(iv) $$\left(\frac{y^{-2}}{y^{-3}}\right)^{\frac{2}{3}} = (y^{-2+3})^{\frac{2}{3}} = (y^1)^{\frac{2}{3}} = y^{\frac{2}{3}}$$

(v) $$\frac{(a\sqrt{b})^{-3}}{\sqrt{a^3b}} = \frac{(a^1b^{\frac{1}{2}})^{-3}}{(a^3b^1)^{\frac{1}{2}}} = \frac{a^{-3}b^{-\frac{3}{2}}}{a^{\frac{3}{2}}b^{\frac{1}{2}}} = \frac{1}{a^{3+\frac{3}{2}}b^{\frac{1}{2}+\frac{3}{2}}} = \frac{1}{a^{\frac{9}{2}}b^2}$$

(vi) $$\frac{\sqrt[4]{x^7}}{\sqrt{x^3}} = \frac{x^{\frac{7}{4}}}{x^{\frac{3}{2}}} = x^{\frac{7}{4}-\frac{3}{2}} = x^{\frac{1}{4}}$$

8. (i) $$\frac{x^{\frac{1}{2}} + x^{-\frac{1}{2}}}{x^{\frac{1}{2}}} = \frac{x^{-\frac{1}{2}}(x+1)}{x^{\frac{1}{2}}} = \frac{x+1}{x^{\frac{1}{2}+\frac{1}{2}}} = \frac{x+1}{x}$$

(ii) $$(x + x^{\frac{1}{2}})(x - x^{\frac{1}{2}}) = x^2 - x\,.\,x^{\frac{1}{2}} + x\,.\,x^{\frac{1}{2}} - x^{\frac{1}{2}+\frac{1}{2}}$$
$$= x^2 - x$$

(iii) $$\frac{\sqrt{x} + \sqrt{x^3}}{\sqrt{x}} = \frac{x^{\frac{1}{2}} + x^{\frac{3}{2}}}{x^{\frac{1}{2}}} = \frac{x^{\frac{1}{2}}(1+x)}{x^{\frac{1}{2}}} = 1 + x$$

9. $$\frac{(x-1)^{\frac{1}{2}} + (x-1)^{-\frac{1}{2}}}{(x-1)^{\frac{1}{2}}} = \frac{(x-1)^{\frac{1}{2}}(x-1)^{\frac{1}{2}} + (x-1)^{\frac{1}{2}}(x-1)^{-\frac{1}{2}}}{(x-1)^{\frac{1}{2}}(x-1)^{\frac{1}{2}}}$$

$$= \frac{(x-1)^1 + (x-1)^0}{(x-1)^1} = \frac{x-1+1}{x-1} = \frac{x}{x-1}$$

10. $$\sqrt{3^{2n+1}} \times \sqrt[3]{3^{-3n}} = (3^{2n+1})^{\frac{1}{2}} \times (3^{-3n})^{\frac{1}{3}}$$
$$= 3^{n+\frac{1}{2}} \times 3^{-n} = 3^{n+\frac{1}{2}-n} = 3^{\frac{1}{2}}$$
$$\Rightarrow 3^k = 3^{\frac{1}{2}} \Rightarrow k = \frac{1}{2}$$

11. $$220\,.\,2^{\frac{1}{2}}\,.\,2^{\frac{1}{12}}\,.\,2^{\frac{1}{12}} = 220\,.\left(2^{\frac{1}{4}}\right)$$
$$= 261\,.626$$
$$= 262 \text{ Hz}$$

12. $$\frac{A_1}{A_2} = \frac{4\pi R_1^2}{4\pi R_2^2} = \frac{R_1^2}{R_2^2}$$

$$\left(\frac{V_1}{V_2}\right)^{\frac{2}{3}} = \left(\frac{\frac{4}{3}\pi R_1^3}{\frac{4}{3}\pi R_2^3}\right)^{\frac{2}{3}} = \frac{\left(\frac{4}{3}\right)^{\frac{2}{3}}(\pi)^{\frac{2}{3}}R_1^{3\left(\frac{2}{3}\right)}}{\left(\frac{4}{3}\right)^{\frac{2}{3}}(\pi)^{\frac{2}{3}}R_2^{3\left(\frac{2}{3}\right)}} = \frac{R_1^2}{R_2^2}$$

Hence, $$\frac{A_1}{A_2} = \left(\frac{V_1}{V_2}\right)^{\frac{2}{3}}$$

Hence, $$\frac{A_1}{A_2} = \left(\frac{162}{384}\right)^{\frac{2}{3}} = \left(\frac{27}{64}\right)^{\frac{2}{3}} = \frac{9}{16}$$

13. $f(n) = 3^n \quad \Rightarrow \quad$ (i) $f(n+3) = 3^{n+3}$

and (ii) $f(n+1) = 3^{n+1}$

$$\Rightarrow f(n+3) - f(n+1) = 3^{n+3} - 3^{n+1}$$
$$= 3^n \,.\, 3^3 - 3^n \,.\, 3^1$$
$$= 3^n(27 - 3) = 24(3^n)$$
$$\Rightarrow k\,f(n) = k(3^n) = 24(3^n)$$
$$\Rightarrow \quad k = 24$$

14. $f(n) = 3^{n-1} \Rightarrow f(n+3) + f(n) = k\,f(n)$

$$\Rightarrow 3^{n+3-1} + 3^{n-1} = k(3^{n-1})$$
$$\Rightarrow 3^{n-1} \,.\, 3^3 + 3^{n-1} = k(3^{n-1})$$
$$\Rightarrow 3^{n-1}(27+1) = k(3^{n-1})$$
$$\Rightarrow \quad 28 = k$$

Exercise 12.7

1. (i) $2^x = 32$

$$\Rightarrow 2^x = 2^5 \quad \Rightarrow x = 5$$

(ii) $16^x = 64$

$$\Rightarrow (4^2)^x = 4^3$$
$$\Rightarrow 4^{2x} = 4^3 \quad \Rightarrow 2x = 3 \quad \Rightarrow x = \frac{3}{2}$$

(iii) $25^x = 125$

$$\Rightarrow (5^2)^x = 5^3$$
$$\Rightarrow \quad 5^{2x} = 5^3 \quad \Rightarrow 2x = 3 \quad \Rightarrow x = \frac{3}{2}$$

(iv) $3^x = \frac{1}{27}$

$$\Rightarrow 3^x = \frac{1}{3^3}$$
$$\Rightarrow 3^x = 3^{-3} \quad \Rightarrow x = -3$$

2. (i) $9^x = \frac{1}{27}$

$$\Rightarrow (3^2)^x = \frac{1}{3^3}$$
$$\Rightarrow \quad 3^{2x} = 3^{-3} \quad \Rightarrow 2x = -3 \quad \Rightarrow x = -\frac{3}{2}$$

(ii) $4^x = \frac{1}{32}$

$$\Rightarrow \quad (2^2)^x = \frac{1}{2^5}$$
$$\Rightarrow \quad 2^{2x} = 2^{-5} \quad \Rightarrow 2x = -5 \quad \Rightarrow x = -\frac{5}{2}$$

(iii) $4^{x-1} = 2^{x+1}$

$$\Rightarrow (2^2)^{x-1} = 2^{x+1}$$
$$\Rightarrow \quad 2^{2x-2} = 2^{x+1} \Rightarrow 2x - 2 = x + 1$$
$$\Rightarrow x = 3$$

(iv) $\frac{1}{9^x} = 27$

$$\Rightarrow \frac{1}{(3^2)^x} = 3^3$$
$$\Rightarrow \quad \frac{1}{3^{2x}} = 3^3$$
$$\Rightarrow 3^{-2x} = 3^3 \Rightarrow -2x = 3$$
$$\Rightarrow \quad x = -\frac{3}{2}$$

3. (i) $2^x = \frac{\sqrt{2}}{2}$

$$\Rightarrow 2^x = \frac{2^{\frac{1}{2}}}{2^1}$$
$$\Rightarrow 2^x = 2^{\frac{1}{2}-1} = 2^{-\frac{1}{2}}$$
$$\Rightarrow \quad x = -\frac{1}{2}$$

(ii) $25^x = \frac{125}{\sqrt{5}}$

$$\Rightarrow (5^2)^x = \frac{5^3}{5^{\frac{1}{2}}}$$
$$\Rightarrow \quad 5^{2x} = 5^{3-\frac{1}{2}} = 5^{\frac{5}{2}}$$
$$\Rightarrow \quad 2x = \frac{5}{2}$$
$$\Rightarrow \quad x = \frac{5}{4}$$

(iii) $\frac{1}{8^x} = \sqrt{2}$

$$\Rightarrow \frac{1}{(2^3)^x} = 2^{\frac{1}{2}}$$
$$\Rightarrow \quad \frac{1}{2^{3x}} = 2^{\frac{1}{2}}$$
$$\Rightarrow \quad 2^{-3x} = 2^{\frac{1}{2}}$$
$$\Rightarrow -3x = \frac{1}{2}$$
$$\Rightarrow \quad x = -\frac{1}{6}$$

(iv) $7^x = \frac{1}{\sqrt[3]{7}}$

$$\Rightarrow 7^x = \frac{1}{7^{\frac{1}{3}}}$$
$$\Rightarrow 7^x = 7^{-\frac{1}{3}}$$
$$\Rightarrow \quad x = -\frac{1}{3}$$

4. $\sqrt{32} = (2^5)^{\frac{1}{2}} = 2^{\frac{5}{2}}$

$\Rightarrow \quad 16^{x-1} = 2\sqrt{32}$

$\Rightarrow \quad (2^4)^{x-1} = 2^1 \,.\, 2^{\frac{5}{2}}$

$\Rightarrow \quad 2^{4x-4} = 2^{1+\frac{5}{2}} = 2^{\frac{7}{2}}$

$\Rightarrow \quad 4x - 4 = \frac{7}{2}$

$\Rightarrow \quad 4x = \frac{7}{2} + 4 = \frac{15}{2}$

$\Rightarrow \quad x = \frac{15}{8}$

5. $27^x = 9$ and $2^{x-y} = 64$

$\Rightarrow (3^3)^x = 3^2 \qquad \Rightarrow \quad 2^{x-y} = 2^6$

$\Rightarrow \quad 3^{3x} = 3^2 \qquad \Rightarrow \quad x - y = 6$

$\Rightarrow \quad 3x = 2$

$\Rightarrow \quad x = \frac{2}{3} \qquad \Rightarrow \quad \frac{2}{3} - y = 6$

$-y = 6 - \frac{2}{3} = \frac{16}{3}$

$\Rightarrow \quad y = -\frac{16}{3}$

6. (i) $2^{x+2} = 2^x \,.\, 2^2 = 4 \,.\, 2^x$

(ii) $2^x + 2^x = 2 \,.\, 2^x$

Hence, $2^x + 2^x = 2^{x+2}(c - 2)$

$\Rightarrow \quad 2 \,.\, 2^x = 4 \,.\, 2^x(c - 2)$

$\Rightarrow \quad 2 = 4c - 8$

$\Rightarrow \quad 4c = 2 + 8 = 10$

$\Rightarrow \quad c = \frac{10}{4} = \frac{5}{2}$

7. $3^x = y \quad \Rightarrow \quad 3^{2x} = (3^x)^2 = y^2$

Solve $3^{2x} - 12(3^x) + 27 = 0$

Let $y = 3^x \quad \Rightarrow \quad y^2 - 12y + 27 = 0$

$\Rightarrow \quad (y - 3)(y - 9) = 0$

$\Rightarrow \quad y = 3, \; y = 9$

Hence, $3^x = 3, \; 3^x = 9$

$\Rightarrow \quad 3^x = 3^1, \; 3^x = 3^2$

$\Rightarrow \quad x = 1, \; x = 2$

8. Solve $2^{2x} - 3(2^x) - 4 = 0$

Let $y = 2^x \Rightarrow y^2 - 3y - 4 = 0$

$\Rightarrow (y - 4)(y + 1) = 0$

$\Rightarrow \quad y = 4, \; y = -1$

Hence, $2^x = 4 = 2^2, \; 2^x = -1$ (Not valid)

$\Rightarrow x = 2$

9. (i) Solve $2^{2x} - 9(2^x) + 8 = 0$

Let $y = 2^x \Rightarrow \quad (y^2 - 9y + 8 = 0$

$\Rightarrow \quad (y - 1)(y - 8) = 0$

$\Rightarrow \quad y = 1, \; y = 8$

Hence, $2^x = 1 = 2^0, \; 2^x = 8 = 2^3$

$\Rightarrow x = 0, \qquad x = 3$

(ii) Solve $3^{2x} - 10(3^x) + 9 = 0$

Let $y = 3^x \Rightarrow y^2 - 10y + 9 = 0$

$\Rightarrow (y-1)(y-9) = 0$

$\Rightarrow y = 1, \; y = 9$

Hence, $3^x = 1 = 3^0, \; 3^x = 9 = 3^2$

$\Rightarrow x = 0, \qquad x = 2$

10. $y = 2^x \Rightarrow$ (i) $2^{2x} = (2^x)^2 = y^2$

(ii) $2^{2x+1} = (2^x)^2 . 2^1 = 2y^2$

(iii) $2^{2x+3} = 2^x . 2^3 = 8y$

Hence, solve $2^{2x+1} - 2^{x+3} - 2^x + 4 = 0$

Let $y = 2^x \Rightarrow 2y^2 - 8y - y + 4 = 0$

$\Rightarrow 2y^2 - 9y + 4 = 0$

$\Rightarrow (2y-1)(y-4) = 0$

$\Rightarrow 2y = 1, \; y = 4$

$\Rightarrow y = \frac{1}{2}$

Hence $2^x = \frac{1}{2} = 2^{-1}, \; 2^x = 4 = 2^2$

$\Rightarrow x = -1, \; x = 2$

11. Solve $3 . 3^x + 3^{-x} = 4$

Let $y = 3^x \Rightarrow 3y + \frac{1}{y} = 4$

$\Rightarrow 3y^2 + 1 = 4y$

$\Rightarrow 3y^2 - 4y + 1 = 0$

$\Rightarrow (3y-1)(y-1) = 0$

$\Rightarrow y = \frac{1}{3}, \; y = 1$

Hence, $3^x = \frac{1}{3} = 3^{-1}, \; 3^x = 1 = 3^0$

$\Rightarrow x = -1, \; x = 0$

12. Solve $2(4^x) + 4^{-x} = 3$

Let $y = 4^x \Rightarrow 2y + \frac{1}{y} = 3$

$\Rightarrow 2y^2 + 1 = 3y$

$\Rightarrow 2y^2 - 3y + 1 = 0$

$\Rightarrow (2y-1)(y-1) = 0$

$\Rightarrow y = \frac{1}{2}, \; y = 1$

Hence, $4^x = \frac{1}{2}, \quad 4^x = 1$

$\Rightarrow (2^2)^x = 2^{-1}, \qquad (2^2)^x = 1 = 2^0$

$\Rightarrow 2^{2x} = 2^{-1} \qquad 2^{2x} = 2^0$

$\Rightarrow 2x = -1 \qquad \Rightarrow 2x = 0$

$\Rightarrow x = -\frac{1}{2} \qquad \Rightarrow x = 0$

13. Solve $3^x - 28 + 27(3^{-x}) = 0$

Let $y = 3^x \Rightarrow y - 28 + 27\left(\frac{1}{y}\right) = 0$

$\Rightarrow y^2 - 28y + 27 = 0$

$\Rightarrow (y-1)(y-27) = 0$

$\Rightarrow y = 1, \; y = 27$

Hence, $3^x = 1 = 3^0, \; 3^x = 27 = 3^3$

$\Rightarrow x = 0, \qquad x = 3$

14. Solve $2^{x+1} + 2(2^{-x}) - 5 = 0$

Let $2^x = y \Rightarrow 2y + 2\left(\frac{1}{y}\right) - 5 = 0$

$\Rightarrow 2y^2 + 2 - 5y = 0$

$\Rightarrow 2y^2 - 5y + 2 = 0$

$\Rightarrow (2y - 1)(y - 2) = 0$

$\Rightarrow y = \frac{1}{2}, \quad y = 2$

Hence, $2^x = \frac{1}{2} = 2^{-1}, \qquad 2^x = 2^1$

$\Rightarrow x = -1, \quad x = 1$

15. Solve $3^x + 81(3^{-x}) - 30 = 0$

Let $y = 3^x \Rightarrow y + 81\left(\frac{1}{y}\right) - 30 = 0$

$\Rightarrow y^2 + 81 - 30y = 0$

$\Rightarrow y^2 - 30y + 81 = 0$

$\Rightarrow (y - 3)(y - 27) = 0$

$\Rightarrow y = 3, \quad y = 27$

Hence, $3^x = 3^1, \qquad 3^x = 27 = 3^3$

$\Rightarrow x = 1, \qquad x = 3$

Exercise 12.8

1. (i) B (ii) A (iii) D (iv) C

2. $A(n) = 1000 \times 2^{0.2n}$

(i) $A(0) = 1000 \times 2^{0.2(0)}$
$= 1000 \times 2^0$
$= 1000 \times 1 = 1000$ ha

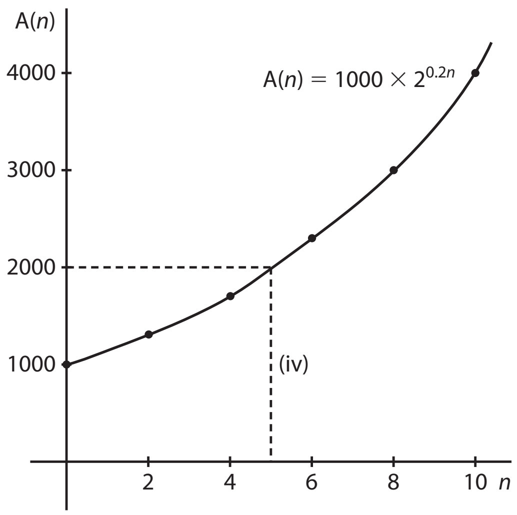
(ii) (a) $A(10) = 1000 \times 2^{0.2(10)}$
$= 1000 \times 2^2$
$= 1000 \times 4 = 4000$ ha

(b) $A(12) = 1000 \times 2^{0.2(12)}$
$= 1000 \times 2^{2.4}$
$= 1000(5.278) = 5278$ ha

(iii)

n **=**	0	2	4	6	8	10
A(*n*) =	1000	1320	1741	2297	3031	4000

(iv) 5 weeks

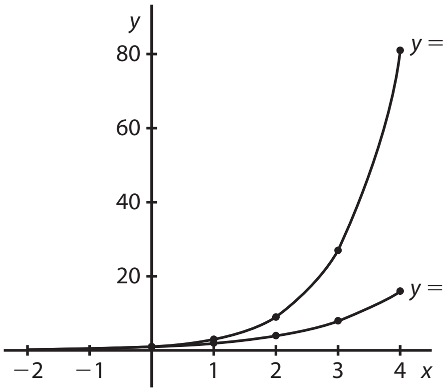


3. (i) Decreasing
(ii) Decreasing
(iii) Increasing
(iv) Decreasing

4. (i) $x = 0 \quad \Rightarrow \quad y = (0.6)2^0 = (0.6)(1) = 0.6$
(ii) $x = 0 \quad \Rightarrow \quad y = 3(2^{-0}) = 3(1) = 3$
(iii) $x = 0 \quad \Rightarrow \quad y = 8(2^0) = 8(1) = 8$
(iv) $x = 0 \quad \Rightarrow \quad y = 6(4^{-0}) = 6(1) = 6$

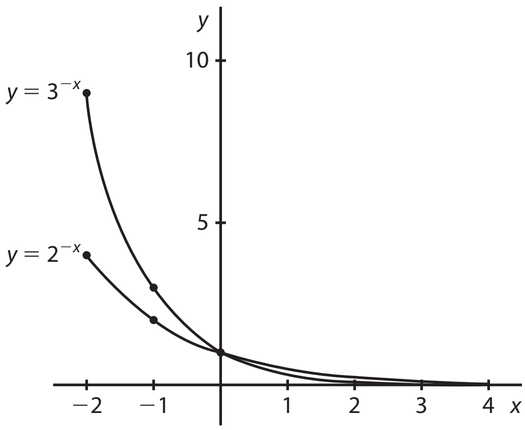
5. (i)

$x =$	-2	-1	0	1	2	3	4
$2^x =$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16
$3^x =$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27	81



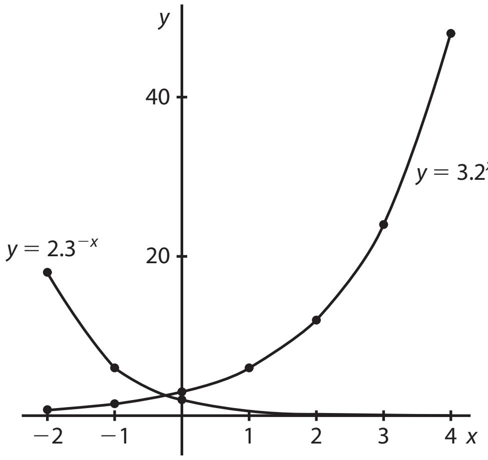
(ii)

$x =$	-2	-1	0	1	2	3	4
$2^{-x} =$	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
$3^{-x} =$	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$	$\frac{1}{81}$



(iii)

$x =$	-2	-1	0	1	2	3	4
$3.2^x =$	$\frac{3}{4}$	$1\frac{1}{2}$	3	6	12	24	48
$2.3^{-x} =$	18	6	2	$\frac{2}{3}$	$\frac{2}{9}$	$\frac{2}{27}$	$\frac{2}{81}$



(iv) $-2 \leqslant x < 0$

(v) $0 < x \leqslant 4$

(vi) $x = 0$

(vii) $0 < x \leqslant 4$

6. $D = 18(0.72)^T$

(i) Decay: Graph is decreasing

(ii) (a) $T = 5°\text{C} \Rightarrow D = 18(0.72)^5 = 3.48 = 3$ days

(b) $T = 2°\text{C} \Rightarrow D = 18(0.72)^2 = 9.33 = 9$ days

(c) $T = 0°\text{C} \Rightarrow D = 18(0.72)^0 = 18$ days

(iii) $18(0.72)^T \geqslant 5$

$$\Rightarrow (0.72)^T \geqslant \frac{5}{18}$$

$$\Rightarrow \log(0.72)^T \geqslant \log\frac{5}{18}$$

$$\Rightarrow T(\log 0.72) \geqslant \log\frac{5}{18}$$

$$\Rightarrow T \geqslant \frac{\log\frac{5}{18}}{\log(0.72)}$$

$$\Rightarrow T \geqslant 3.899$$

$$\Rightarrow T = 3.9°\text{C}$$

7. $P = 100(0.99988)^n$

(a) (i) $n = 200 \Rightarrow P = 100(0.99988)^{200} = 97.628 = 97.6\%$

(ii) $n = 500 \Rightarrow P = 100(0.99988)^{500} = 94.176 = 94.2\%$

(b) $n = 5000 \Rightarrow P = 100(0.99988)^{5000} = 54.879 = 54.9\%$

$n = 6000 \Rightarrow P = 100(0.99988)^{6000} = 48.673 = 48.7\%$

Hence, $100(0.99988)^n = 50$

$$\Rightarrow (0.99988)^n = 0.5$$

$$\Rightarrow \log(0.99988)^n = \log(0.5)$$

$$\Rightarrow n\log(0.99988) = \log(0.5)$$

$$\Rightarrow n = \frac{\log(0.5)}{\log(0.99988)} = 5775.88 = 5780 \text{ years}$$

(c) $100(0.99988)^n = 79$

$\Rightarrow \quad (0.99988)^n = 0.79$

$\Rightarrow \quad \log(0.99988)^n = \log(0.79)$

$\Rightarrow \quad n\log(0.99988) = \log(0.79)$

$\Rightarrow \quad n = \dfrac{\log(0.79)}{\log(0.99988)} = 1964.2 = 1964$ years

8. $A(n) = 1000 \times 2^{0.2n}$

(i) $A(0) = 1000 \times 2^{0.2(0)}$

$= 1000 \times 2^0$

$= 1000 \times 1 = 1000$ ha

(ii) (a) $A(5) = 1000 \times 2^{0.2(5)}$

$= 1000 \times 2^1 = 2000$ ha

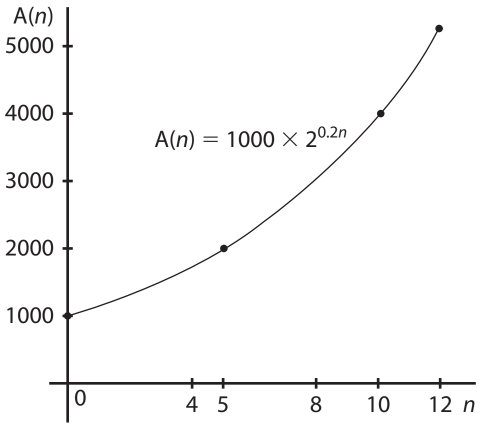
(b) $A(10) = 1000 \times 2^{0.2(10)}$

$= 1000 \times 2^2 = 4000$ ha

(c) $A(12) = 1000 \times 2^{0.2(12)}$

$= 1000 \times 2^{2.4}$

$= 1000(5.278) = 5278$ ha



9. $P(t) = 90 \times 3^{-0.25t} + 50$

(i)

$t =$	0	2	4	6	8	10
$90 \times 3^{-0.25t} + 50$	140	102	80	67	60	56

(ii) $P(0) = 90 \times 3^{-0.25(0)} + 50$

$= 90 \times 3^0 + 50$

$= 90 + 50 = 140$ beats/min

(iii) (a) $90 \times 3^{-0.25t} + 50 = 70$

$\Rightarrow \quad 90 \times 3^{-0.25t} = 20$

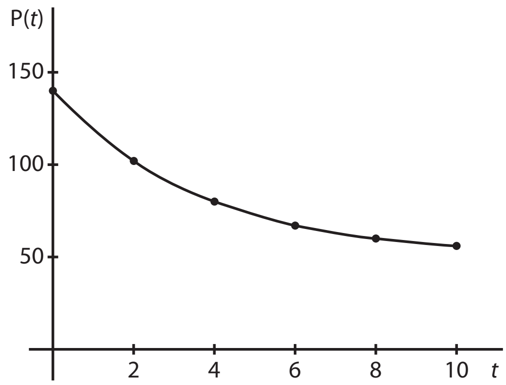
$\Rightarrow \quad 3^{-0.25t} = \dfrac{20}{90} = \dfrac{2}{9}$

$\Rightarrow \quad \log 3^{-0.25t} = \log \dfrac{2}{9}$

$\Rightarrow \quad -0.25t(\log 3) = \log \dfrac{2}{9}$

$\Rightarrow \quad -0.25t = \dfrac{\log \frac{2}{9}}{\log 3} = -1.369$

$\Rightarrow \quad t = \dfrac{-1.369}{-0.25} = 5.476 = 5.5$ minutes



(b) $90 \times 3^{-0.25t} + 50 = 55$

$\Rightarrow \quad 90 \times 3^{-0.25t} = 5$

$\Rightarrow \quad 3^{-0.25t} = \dfrac{5}{90} = \dfrac{1}{18}$

$\Rightarrow \quad \log 3^{-0.25t} = \log \dfrac{1}{18}$

$\Rightarrow \quad -0.25t(\log 3) = \log \dfrac{1}{18}$

$\Rightarrow \quad -0.25t = \dfrac{\log \frac{1}{18}}{\log 3} = -2.63$

$\Rightarrow \quad t = \dfrac{-2.63}{-0.25} = 10.52 = 10.5$ minutes

(iv) 50 beats/min because $P(t) = 50$ as t gets larger using graph of $P(t)$.

10. $E \cong 10^{1.5M+4.8}$

$M = 7 \quad \Rightarrow \quad E = 10^{1.5(7)+4.8} = 10^{15.3}$

$M = 5 \quad \Rightarrow \quad E = 10^{1.5(5)+4.8} = 10^{12.3}$

$\Rightarrow \quad \text{No. of times} = \dfrac{10^{15.3}}{10^{12.3}} = 10^{15.3-12.3} = 10^3 = 1000$

11. $P(t) = 40b^t$

(i) $t = 0 \quad \Rightarrow \quad P(0) = 40b^0 = 40$

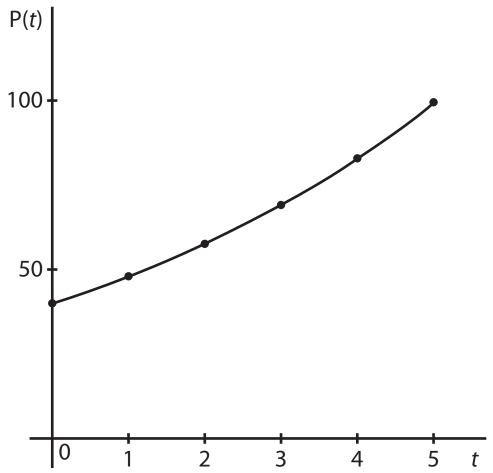
(ii) $t = 1 \quad \Rightarrow \quad P(1) = 40b^1 = 48$

$\Rightarrow \quad b = \dfrac{48}{40} = 1.2 > 1$

$\Rightarrow$ No. of flies is increasing

(iii)

$t =$	0	1	2	3	4	5
$P(t) = 40(1.2)^t$	40	48	57.6	69.1	82.9	99.5



Exercise 12.9

1. (i) $\log_2 4 = 2$

(ii) $\log_3 81 = 4$

(iii) $\log_{10} 1000 = 3$

(iv) $\log_2 64 = 6$

2. (i) $\log_8 16 = x \quad \Rightarrow \quad 8^x = 16$

$\Rightarrow \quad (2^3)^x = 2^4$

$\Rightarrow \quad 2^{3x} = 2^4$

$\Rightarrow \quad 3x = 4 \quad \Rightarrow \quad x = \dfrac{4}{3}$

(ii) $\log_9 27 = x \quad \Rightarrow \quad 9^x = 27$

$\Rightarrow \quad (3^2)^x = 3^3$

$\Rightarrow \quad 3^{2x} = 3^3$

$\Rightarrow \quad 2x = 3 \quad \Rightarrow \quad x = \dfrac{3}{2}$

(iii) $\log_{16} 32 = x \quad \Rightarrow \quad 16^x = 32$

$\Rightarrow \quad (2^4)^x = 2^5$

$\Rightarrow \quad 2^{4x} = 2^5$

$\Rightarrow \quad 4x = 5 \quad \Rightarrow \quad x = \dfrac{5}{4}$

(iv) $\log_{\frac{1}{2}} 8 = x \Rightarrow \left(\frac{1}{2}\right)^x = 8$

$\Rightarrow (2^{-1})^x = 2^3$

$\Rightarrow 2^{-x} = 2^3$

$\Rightarrow -x = 3 \Rightarrow x = -3$

(v) $\log_{\frac{1}{3}} 81 = x \Rightarrow \left(\frac{1}{3}\right)^x = 81$

$\Rightarrow (3^{-1})^x = 3^4$

$\Rightarrow 3^{-x} = 3^4$

$\Rightarrow -x = 4 \Rightarrow x = -4$

3. (i) $\log_{\frac{1}{3}} 27 = x \Rightarrow \left(\frac{1}{3}\right)^x = 27$

$\Rightarrow (3^{-1})^x = 3^3$

$\Rightarrow 3^{-x} = 3^3$

$\Rightarrow -x = 3 \Rightarrow x = -3$

(ii) $\log_{\sqrt{2}} 4 = x \Rightarrow (\sqrt{2})^x = 4$

$\Rightarrow (2^{\frac{1}{2}})^x = 2^2$

$\Rightarrow 2^{\frac{1}{2}x} = 2^2$

$\Rightarrow \frac{1}{2}x = 2$

$\Rightarrow x = 4$

(iii) $\log_8 x = 2$

$\Rightarrow 8^2 = x$

$\Rightarrow x = 64$

(iv) $\log_{64} x = \frac{1}{2}$

$\Rightarrow 64^{\frac{1}{2}} = x$

$\Rightarrow x = 8$

4. (i) $\log_2 x = -1$

$\Rightarrow 2^{-1} = x$

$\Rightarrow x = \frac{1}{2}$

(ii) $\log_3 \sqrt{27} = x$

$\Rightarrow 3^x = \sqrt{27} = (3^3)^{\frac{1}{2}} = 3^{\frac{3}{2}}$

$\Rightarrow x = \frac{3}{2}$

(iii) $\log_x 2 = 2 \Rightarrow x^2 = 2$

$\Rightarrow x = 2^{\frac{1}{2}} = \sqrt{2}$

(iv) $\log_2 (0.5) = x \Rightarrow 2^x = \frac{1}{2} = 2^{-1}$

$\Rightarrow x = -1$

5. (i) $\log_4 2 + \log_4 32 = \log_4 (2).(32)$

$= \log_4 64 = x$

$\Rightarrow 4^x = 64 = 4^3$

$\Rightarrow x = 3$

(ii) $\log_6 9 + \log_6 8 - \log_6 2 = \log_6 \frac{(9)(8)}{(2)}$

$= \log_6 36 = x$

$\Rightarrow 6^x = 36 = 6^2$

$\Rightarrow x = 2$

(iii) $\log_6 4 + 2\log_6 3 = \log_6 4 + \log_6 3^2$
$= \log_6 4 + \log_6 9 = \log_6 36 = 2$

6. (i) $\log_3 2 + 2\log_3 3 - \log_3 18$
$= \log_3 2 + \log_3 3^2 - \log_3 18$
$= \log_3 2 + \log_3 9 - \log_3 18$
$= \log_3 \frac{(2)(9)}{18} = \log_3 1 = 0$

(ii) $\log_8 72 - \log_8 \left(\frac{9}{8}\right)$
$= \log_8 72 - (\log_8 9 - \log_8 8)$
$= \log_8 72 - \log_8 9 + \log_8 8$
$= \log_8 \frac{(72)(8)}{(9)} = \log_8 64 = 2$

7. $\log_3 5 = a$

(i) $\log_3 15 = \log_3 5 \,.\, 3 = \log_3 5 + \log_3 3 = a + 1$

(ii) $\log_3 \left(\frac{5}{3}\right) = \log_3 5 - \log_3 3 = a - 1$

(iii) $\log_3 \left(8\frac{1}{3}\right) = \log_3 \left(\frac{25}{3}\right) = \log_3 25 - \log_3 3$
$= \log_3 5^2 - 1$
$= 2\log_3 5 - 1 = 2a - 1$

(iv) $\log_3 \left(\frac{25}{27}\right) = \log_3 25 - \log_3 27$
$= \log_3 (5)^2 - \log_3 (3)^3$
$= 2\log_3 5 - 3\log_3 3 = 2a - 3$

(v) $\log_3 75 = \log_3 (25)\,.\,(3) = \log_3 25 + \log_3 3$
$= \log_3 5^2 + 1$
$= 2\log_3 5 + 1$
$= 2a + 1$

8. (i) $2^x = 200$
$\Rightarrow \log 2^x = \log 200$
$\Rightarrow x\log 2 = \log 200$
$\Rightarrow x = \frac{\log 200}{\log 2} = 7.643 = 7.64$

(ii) $5^x = 500$
$\Rightarrow \log 5^x = \log 500$
$\Rightarrow x\log 5 = \log 500$
$\Rightarrow x = \frac{\log 500}{\log 5} = 3.861 = 3.86$

(iii) $3^{x+1} = 25$
$\Rightarrow \log 3^{(x+1)} = \log 25$
$\Rightarrow (x+1)\log 3 = \log 25$
$\Rightarrow x + 1 = \frac{\log 25}{\log 3} = 2.929$
$\Rightarrow x = 1.929 = 1.93$

(iv) $5^{2x+3} = 51$
$\Rightarrow \log 5^{(2x+3)} = \log 51$
$\Rightarrow (2x+3)\log 5 = \log 51$
$\Rightarrow 2x + 3 = \frac{\log 51}{\log 5} = 2.4429$
$\Rightarrow 2x = -0.5571$
$\Rightarrow x = -0.2785 = -0.279$

9. (i) $y = 2^{x+1} + 3$

$$\Rightarrow 2^{x-1} = y - 3$$
$$\Rightarrow \log(2^{x-1}) = \log(y-3)$$
$$\Rightarrow (x-1)\log 2 = \log(y-3)$$
$$\Rightarrow x - 1 = \frac{\log(y-3)}{\log 2}$$
$$\Rightarrow x = \frac{\log(y-3)}{\log 2} + 1$$

(ii) $y = 8 \Rightarrow x = \dfrac{\log(8-3)}{\log 2} + 1 = \dfrac{\log 5}{\log 2} + 1 = 2.321928 + 1$

$= 3.321928$

$= 3.3219$

10. $\log_{10} x = 1 + a \Rightarrow 10^{1+a} = x$

$\log_{10} y = 1 - a \Rightarrow 10^{1-a} = y$

$\Rightarrow xy = 10^{1+a} . 10^{1-a} = 10^{1+a+1-a}$

$= 10^2 = 100$

11. $p = \log_a \dfrac{21}{4} = \log_a 21 - \log_a 4$

$= \log_a 7.3 - \log_a 4$

$= \log_a 7 + \log_a 3 - \log_a 4$

$q = \log_a \dfrac{7}{3} = \log_a 7 - \log_a 3$

$\Rightarrow p + q = \log_a 7 + \log_a 3 - \log_a 4 + \log_a 7 - \log_a 3$

$= 2\log_a 7 - \log_a 4$

$2r = 2\log_a \dfrac{7}{2} = 2[\log_a 7 - \log_a 2]$

$= 2\log_a 7 - 2\log_a 2$

$= 2\log_a 7 - \log_a 2^2 = 2\log_a 7 - \log_a 4$

Hence, $p + q = 2r$

12. If $\log_a x = 4$ and $\log_a y = 5$

(i) $\log_a x^2 y = \log_a x^2 + \log_a y$

$= 2\log_a x + \log_a y = 2(4) + 5 = 13$

(ii) $\log_a axy = \log_a a + \log_a y$

$= 1 + 4 + 5 = 10$

(iii) $\log_a \dfrac{\sqrt{x}}{y} = \log_a \sqrt{x} - \log_a y$

$= \log_a x^{\frac{1}{2}} - \log_a y$

$= \dfrac{1}{2}\log_a x - \log_a y$

$= \dfrac{1}{2}(4) - 5 = -3$

13. Prove $\log_{25} x = \dfrac{1}{2}\log_5 x$

Proof $\log_{25} x = \dfrac{\log_5 x}{\log_5 25}$

$= \dfrac{\log_5 x}{2} = \dfrac{1}{2}\log_5 x$

14. (i) $\log_{10} 4 = 0.60205999 = 0.602$

(ii) $\log_{10} 27 = 1.43136 = 1.43$

(iii) $\log_{10} 356 = 2.5514 = 2.55$

(iv) $\log_{10} 5600 = 3.748 = 3.75$

(v) $\log_{10} 29\,000 = 4.462 = 4.46$
(vi) $\log_{10} 35\,0000 = 5.544 = 5.54$
(vii) $\log_{10} 38\,70\,000 = 6.5877 = 6.59$

15. Minimum $= 10^3 = 1000$
Maximum $= 10^4 = 10\,000$

16. $\log_3 15 = \dfrac{\log_{10} 15}{\log_{10} 3} = \dfrac{1.17609}{0.47712} = 2.464977$

$\log_2 5 = \dfrac{\log_{10} 5}{\log_{10} 2} = \dfrac{0.69897}{0.30103} = 2.3219$

Hence, $\log_3 15 - \log_2 5 = 2.464977 - 2.3219$
$= 0.143077 = 0.143$

17. (i) $\log_{27} 81 = \dfrac{\log_3 81}{\log_3 27} = \dfrac{4}{3}$

(ii) $\log_{32} 8 = \dfrac{\log_2 8}{\log_2 32} = \dfrac{3}{5}$

18. Show: $\log_b a = \dfrac{1}{\log_a b}$

Proof: $\log_b a = \dfrac{\log_a a}{\log_a b} = \dfrac{1}{\log_a b}$

19. $\dfrac{1}{\log_2 x} + \dfrac{1}{\log_3 x} + \dfrac{1}{\log_5 x} = \log_x 2 + \log_x 3 + \log_x 5$
$= \log_x (2)(3)(5)$
$= \log_x 30$
$= \dfrac{1}{\log_{30} x}$

20. $\log_r p = \log_r 2 + 3 \log_r q$
$\Rightarrow \log_r p = \log_r 2 + \log_r q^3$
$\Rightarrow \log_r p = \log_r 2q^3$
$\Rightarrow p = 2q^3$

21. $\log_3 a + \log_9 a = \dfrac{3}{4}$
$\Rightarrow \log_3 a + \dfrac{\log_3 a}{\log_3 9} = \dfrac{3}{4}$
$\Rightarrow \log_3 a + \dfrac{\log_3 a}{2} = \dfrac{3}{4}$
$\Rightarrow 2 \log_3 a + \log_3 a = \dfrac{3}{2}$
$\Rightarrow 3\log_3 a = \dfrac{3}{2}$
$\Rightarrow \log_3 a = \dfrac{1}{2}$
$\Rightarrow 3^{\frac{1}{2}} = a \Rightarrow a = \sqrt{3}$

22. $3 \ln 41.5 - \ln 250 = 3(3.7256934) - 5.52146$
$= 11.17708 - 5.52146$
$= 5.6556$
$= 5.66$

23. Solve $\log_2 (x - 2) + \log_2 x = 3$

$\Rightarrow \quad \log_2 (x - 2)(x) = 3$

$\Rightarrow \quad x^2 - 2x = 2^3 - 8$

$\Rightarrow \quad x^2 - 2x - 8 = 0$

$\Rightarrow \quad (x - 4)(x + 2) = 0$

$\Rightarrow \quad x = 4, \qquad x = -2$ (Not valid)

24. $\log_{10} (x^2 + 6) - \log_{10} (x^2 - 1) = 1$

$\Rightarrow \quad \log_{10} \dfrac{x^2 + 6}{x^2 - 1} = 1$

$\Rightarrow \quad \dfrac{x^2 + 6}{x^2 - 1} = 10^1 = 10$

$\Rightarrow \quad 10x^2 - 10 = x^2 + 6$

$\Rightarrow \quad 9x^2 - 16 = 0$

$\Rightarrow \quad (3x + 4)(3x - 4) = 0$

$\Rightarrow \quad x = -\dfrac{4}{3}, x = \dfrac{4}{3} \quad \Rightarrow \quad x = \pm \dfrac{4}{3}$

25. $\log 2x - \log (x - 7) = \log 3$

$\Rightarrow \quad \log \dfrac{2x}{x - 7} = \log 3$

$\Rightarrow \quad \dfrac{2x}{x - 7} = 3$

$\Rightarrow \quad 3x - 21 = 2x$

$\Rightarrow \quad x = 21$

26. $\log (2x + 3) + \log(x - 2) = 2 \log x$

$\Rightarrow \quad \log (2x + 3)(x - 2) = \log x^2$

$\Rightarrow \quad 2x^2 - x - 6 = x^2$

$\Rightarrow \quad x^2 - x - 6 = 0$

$\Rightarrow \quad (x - 3)(x + 2) = 0$

$\Rightarrow \quad x = 3, x = -2$ (Not valid)

27. $\log_{10} (17 - 3x) + \log_{10} x = 1$

$\Rightarrow \quad \log_{10} (17 - 3x)(x) = 1$

$\Rightarrow \quad 10^1 = 17x - 3x^2$

$\Rightarrow \quad 3x^2 - 17x + 10 = 0$

$\Rightarrow \quad (3x - 2)(x - 5) = 0 \quad \Rightarrow x = \dfrac{2}{3}, 5$

28. $\log_{10} (x^2 - 4x - 11) = 0$

$\Rightarrow \quad 10^0 = x^2 - 4x - 11$

$\Rightarrow \quad x^2 - 4x - 11 = 1$

$\Rightarrow \quad x^2 - 4x - 12 = 0$

$\Rightarrow \quad (x - 6)(x + 2) = 0$

$\Rightarrow \quad x = 6, -2$

29. $2 \log_2 x = y$ and $\log_2 (2x) = y + 4$

$\Rightarrow \quad \log_2 x^2 = y \quad \Rightarrow \quad 2^{y+4} = 2x$

$\Rightarrow \quad 2^y = x^2 \quad \Rightarrow \quad 2^y . 2^4 = 2x$

$\Rightarrow \quad x^2 . 16 = 2x$

$\Rightarrow \quad 16x^2 - 2x = 0$

$\Rightarrow \quad 8x^2 - x = 0$

$\Rightarrow \quad x(8x - 1) = 0$

$\Rightarrow \quad x = \dfrac{1}{8}, x = 0$ (Not valid)

30. $\log_6 x + \log_6 y = 1$

$\Rightarrow \quad \log_6 x \, . \, y = 1 \qquad \Rightarrow \quad 6 = xy$

$\Rightarrow \quad x = \frac{6}{y}$

Solve $\log_6 x + \log_6 y = 1 \quad \cap \quad 5x + y = 17$

$\Rightarrow x = \frac{6}{y} \qquad \Rightarrow \quad 5\left(\frac{6}{y}\right) + y = 17$

$\Rightarrow \quad \frac{30}{y} + y = 17$

$\Rightarrow \quad 30 + y^2 = 17y$

$\Rightarrow \quad y^2 - 17y + 30 = 0$

$(y - 2)(y - 15) = 0$

$\Rightarrow \quad y = 2, \quad y = 15$

$\Rightarrow \quad x = \frac{6}{2} = 3, x = \frac{6}{15} = \frac{2}{5}$

31. (i) Solve $4 \log_x 2 - \log_2 x - 3 = 0$

$\Rightarrow \quad 4 \log_x 2 - \frac{\log_x x}{\log_x 2} - 3 = 0$

$\Rightarrow \quad 4 \log_x 2 - \frac{1}{\log_x 2} - 3 = 0$

Let $y = \log_x 2 \quad \Rightarrow \quad 4y - \frac{1}{y} - 3 = 0$

$\Rightarrow \quad 4y^2 - 1 - 3y = 0$

$\Rightarrow \quad 4y^2 - 3y - 1 = 0$

$\Rightarrow \quad (4y + 1)(y - 1) = 0$

$\Rightarrow \quad y = -\frac{1}{4}, y = 1$

Hence, $\quad \log_x 2 = -\frac{1}{4}, \log_x 2 = 1$

$\Rightarrow \quad x^{-\frac{1}{4}} = 2, x^1 = 2$

$\Rightarrow \quad x = 2^{-4} = \frac{1}{16}, x = 2$

(ii) Solve $2 \log_4 x + 1 = \log_x 4$

$\Rightarrow \quad 2 \log_4 x + 1 = \frac{\log_4 4}{\log_4 x}$

$\Rightarrow \quad 2 \log_4 x + 1 = \frac{1}{\log_4 x}$

Let $y = \log_4 x \quad \Rightarrow \quad 2y + 1 = \frac{1}{y}$

$\Rightarrow \quad 2y^2 + y = 1$

$\Rightarrow \quad 2y^2 + y - 1 = 0$

$\Rightarrow \quad (2y - 1)(y + 1) = 0$

$\Rightarrow \quad y = \frac{1}{2}, y = -1$

Hence, $\log_4 x = \frac{1}{2}, \qquad \log_4 x = -1$

$\Rightarrow \quad 4^{\frac{1}{2}} = x, \qquad 4^{-1} = x$

$\Rightarrow \quad x = \sqrt{4} = 2, \qquad x = \frac{1}{4}$

Exercise 12.10

1. Property 1 $\Rightarrow$ $\log_a 1 = 0$ $\Rightarrow a^0 = 1$, true

Property 2 $\Rightarrow$ $\log_2 2 = 1$ $\Rightarrow 2^1 = 2$, true

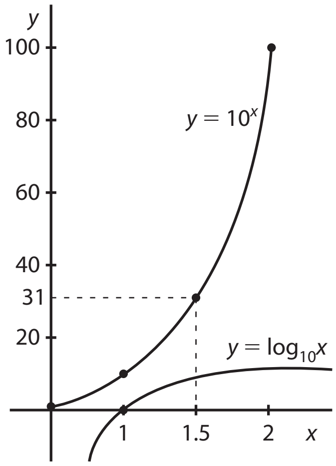
$\Rightarrow$ $\log_e e = 1$ $\Rightarrow e^1 = e$, true

$\Rightarrow$ $\log_{10} 10 = 1$ $\Rightarrow 10^1 = 10$, true

Property 5 $\Rightarrow$ $\log_a 0 = y$ $\Rightarrow a^y = 0 \Rightarrow$ No solution here

2.

$x =$	0	1	2
$y = 10^x =$	1	10	100
$y = \log_{10} x =$	undef.	0	0.3



3. (i)

$y =$

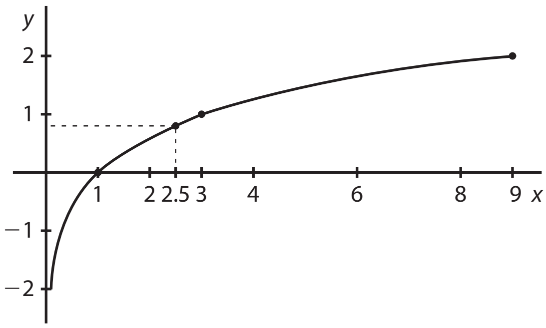
$x =$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$y = \log_3 x =$	-2	-1	0	1	2

(ii) Graph $y = \log_3 x$

(iii) $\log_3 2.5 = 0.8$, using graph

(iv) $\log_3 2.5 = \dfrac{\log_{10} 2.5}{\log_{10} 3}$

$= \dfrac{0.39794}{0.47712} = 0.834$



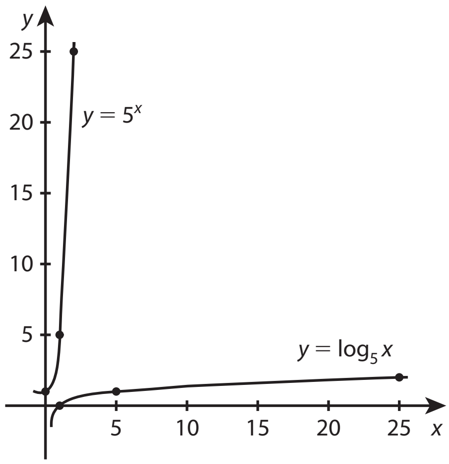
4. (i)

$x =$	0	1	2
$y = 5^x =$	1	5	25

(ii)

$x =$	0	1	5	10	25
$y = \log_5 x =$	undef.	0	1	1.4	2

(iii) One graph is the inverse of the other.

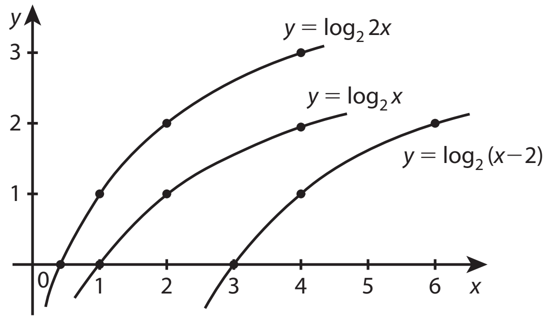


5.

$x =$	0	1	2	4
$y = \log_2 x =$	undef.	0	1	2

$x =$	0	1	2	4
$2x =$	0	2	4	8
$y = \log_2 2x =$	undef.	1	2	3

$x =$	2	3	4	6
$x - 2 =$	0	1	2	4
$y = \log_2 (x - 2) =$	undef.	0	1	2

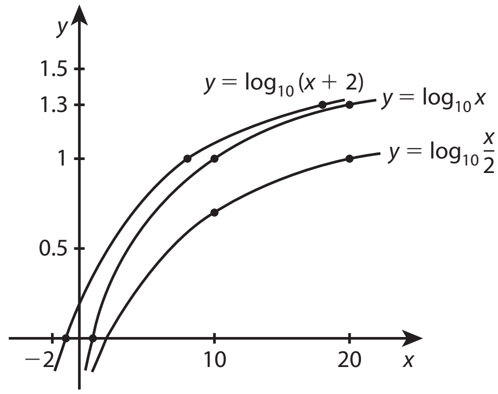


6.

$x =$	0	1	10	20
$y = \log_{10} x =$	undef.	0	1	1.3

$x =$	0	2	10	20
$\frac{x}{2} =$	0	1	5	10
$y = \log_{10} \frac{x}{2} =$	undef.	0	0.7	1

$x =$	-2	-1	8	18
$x + 2 =$	0	1	10	20
$y = \log_{10} (x + 2) =$	u.	0	1	1.3



7. (i)

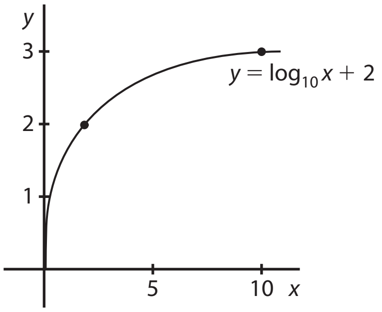
$$3^{x+2} = y + 5$$
$$\Rightarrow \log 3^{x+2} = \log (y + 5)$$
$$\Rightarrow (x + 2) \log 3 = \log (y + 5)$$
$$\Rightarrow x + 2 = \frac{\log (y + 5)}{\log 3}$$
$$\Rightarrow x = \frac{\log (y + 5)}{\log 3} - 2 \quad \text{or} \quad \log_3 (y + 5) - 2$$

(ii) $x = \dfrac{\log (y + 5)}{\log 3} - 2$

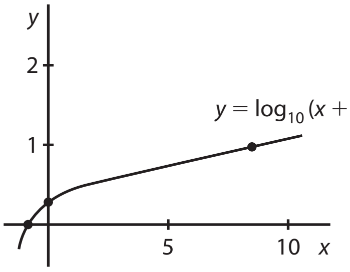
when $y = 30 \Rightarrow x = \dfrac{\log (30 + 5)}{\log 3} - 2 = \dfrac{\log 35}{\log 3} - 2$

$$= \frac{1.544068}{0.47712} - 2$$
$$= 3.2362 - 2$$
$$x = 1.2362 = 1.236$$

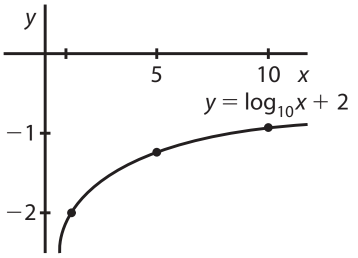
8. (i) Graph of $y = \log_{10} x + 2$



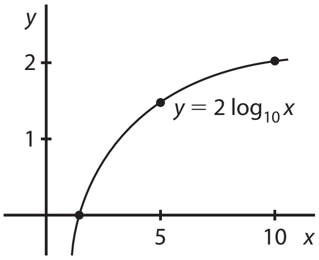
(ii) Graph of $y = \log_{10}(x + 2)$



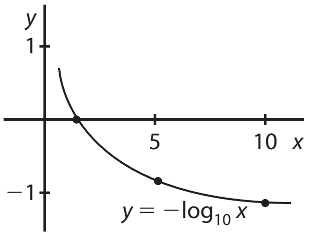
(iii) Graph of $y = \log_{10} x - 2$



(iv) Graph of $y = 2\log_{10} x$

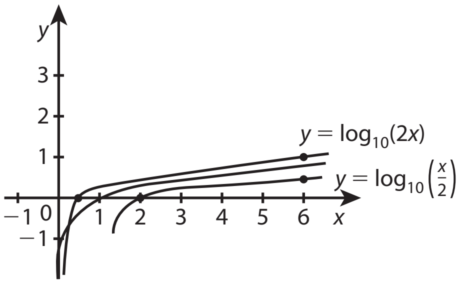


(v) $y = -\log_{10} x$



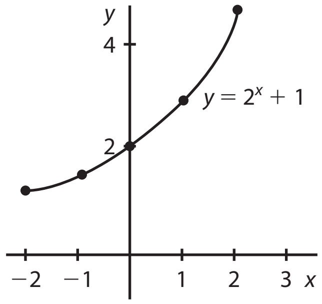
9. Graphs of (i) $y = \log_{10}(2x)$

(ii) $y = \log_{10}\left(\frac{x}{2}\right)$

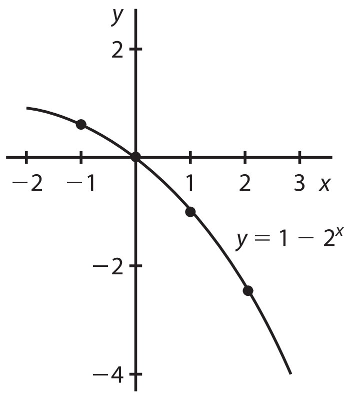


10. Graphs for:

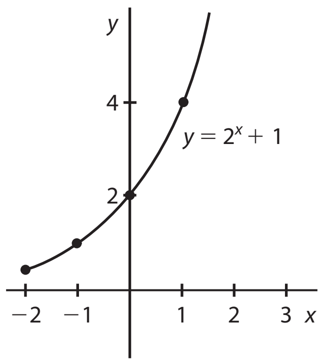
(i) $y = 2^x + 1$



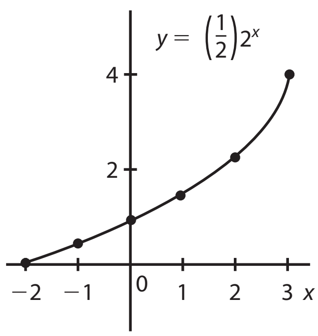
(ii) $y = 1 - 2^x$



(iii) $y = 2^{x+1}$



(iv) $y = \left(\frac{1}{2}\right) . 2^x$



Exercise 12.11

1. (i) (a) $A = 5000 + 5000\left(\frac{0.6}{100}\right)^1 = 5000(1 + 0.006)$

$= 5000(1.006) = €5030$

(b) $A = 5000 + 5000\left(\frac{0.6}{100}\right)^2 = 5000(1.006)^2 = €5060.18$

(c) $A = 5000 + 5000\left(\frac{0.6}{100}\right)^3 = 5000(1.006)^3 = €5090.54$

(ii) $5000(1.006)^t$

(iii) $5000(1.006)^t = 10\,000$

$\Rightarrow \quad (1.006)^t = \frac{10\,000}{5000} = 2$

$\Rightarrow \quad \ln(1.006)^t = \ln 2$

$\Rightarrow \quad t\ln(1.006) = \ln 2$

$\Rightarrow t = \frac{\ln 2}{\ln 1.006} = 115.87 = 116$ months

2. $y = Ae^{bt}$

$t = 0 \Rightarrow 100 = Ae^{b(0)} = Ae^0 = A\,.\,1$

$\Rightarrow \quad A = 100$

Hence, $y = 100e^{bt}$

$t = 6 \quad \Rightarrow 100e^{b(6)} = 450$

$\Rightarrow \quad e^{6b} = \frac{450}{100} = 4.5$

$\Rightarrow \quad \ln e^{6b} = \ln 4.5$

$\Rightarrow \quad 6b\ln e = 6b(1) = \ln 4.5$

$\Rightarrow \quad b = \frac{\ln 4.5}{6} = 0.2507 = 0.25$

3. $T = 15 + 30 \times 10^{-0.02t}$

(i) $t = 0 \Rightarrow T = 15 + 30 \times 10^{-0.02(0)}$

$= 15 + 30 \times 10^0 = 15 + 30(1) = 45°\text{C}$

(ii) $T = 35°\text{C} \Rightarrow 15 + 30 \times 10^{-0.02t} = 35$

$\Rightarrow \quad 30 \times 10^{-0.02t} = 35 - 15 = 20$

$\Rightarrow \quad 10^{-0.02t} = \frac{20}{30} = \frac{2}{3}$

$\Rightarrow \quad \log_{10} 10^{-0.02t} = \log_{10}\frac{2}{3}$

$\Rightarrow \quad -0.02t(\log_{10} 10) = -0.02t(1) = \log\frac{2}{3}$

$\Rightarrow \quad t = \frac{\log\frac{2}{3}}{-0.02}$

$= \frac{-0.17609}{-0.02}$

$= 8.8$ minutes

(iii) As t gets larger, $30 \times 10^{-0.02t}$ gets smaller.

Hence, smallest value for $30 \times 10^{-0.02t} = 0$.

$\Rightarrow$ Room temperature $= 15°\text{C}$

4. $L = 10\log_{10}\left(\frac{I}{I_o}\right) = 10\log_{10}\left(\frac{I}{1\times 10^{-12}}\right)$

(i) $L = 100 \quad \Rightarrow 100 = 10[\log_{10} I - \log_{10}(1\times 10^{-12})]$

$\Rightarrow 10 = \log_{10} I + 12$

$\Rightarrow \log_{10} I = -2$

$I = 10^{-2} = 0.01\ \text{Wm}^{-2}$

$L = 110 \quad \Rightarrow 110 = 10[\log_{10} I - \log_{10}(1\times 10^{-12})]$

$11 = \log_{10} I + 12$

$\Rightarrow \log_{10} I = -1 \Rightarrow I = 10^{-1} = 0.1\ \text{Wm}^{-2}$

$\Rightarrow$ Range is between 0.01 Wm^{-2} and 0.1 $\text{Wm}{-}^{2}$

(ii) $I = 10\ \text{Wm}^{-2} \quad \Rightarrow \quad L = 10\log_{10}\left(\frac{10}{1\times 10^{-12}}\right)$

$= 10\log_{10}10^{13}$

$= 10\,.\,13\log_{10}10$

$= 10\,.\,13(1) = 130$ dB

5. $A = 10^M$ and $E \cong 10^{1.5M+4.8}$

$= 10^{1.5M}\cdot 10^{4.8}$

$= (10^M)^{1.5}\cdot 10^{4.8}$

$= A^{1.5}\cdot 10^{4.8} = 10^a A^b$

Hence, $a = 4.8$ and $b = 1.5$

6. Value = €100 in 2000

(i) t years later $\Rightarrow$ Value $= 100\left(1 + \frac{4.5}{100}\right)^t$

$= 100(1.045)^t$

(ii) 10 years later $\Rightarrow t = 10 \Rightarrow$ Value $= 100(1.045)^{10}$

$= 155.2969 =$ €155.30

(iii) 5 years earlier $\Rightarrow t = -5 \Rightarrow$ Value $= 100(1.045)^{-5}$

$= 80.2451 =$ €80.25

7. $W = 0.6\times 1.15^t$

(i) $t = 0 \quad \Rightarrow \quad W = 0.6\times 1.15^0$

$= 0.6\times 1 = 0.6$ kg

(ii) $1.15 = 1 + 0.15 = 1 + \frac{15}{100}$

$\Rightarrow$ growth constant = 15%

(iii) $W = 2(0.6) = 1.2 \Rightarrow = 0.6\times 1.15^t = 1.2$

$\Rightarrow 1.15^t = \frac{1.2}{0.6} = 2$

$\Rightarrow \ln 1.15^t = \ln 2$

$\Rightarrow t\ln 1.15 = \ln 2$

$\Rightarrow t = \frac{\ln 2}{\ln 1.15} = 4.959 = 5$ months

8. $M = M_0e^{-kt}$

(i) $M = 10$ when $t = 0 \quad \Rightarrow \quad M_0e^{-k(0)} = 10$

$\Rightarrow \quad M_0e^0 = 10$

$\Rightarrow \quad M_0(1) = M_0 = 10$

Hence, $M = 10e^{-kt}$

$M = 5$ when $t = 140 \quad \Rightarrow \quad 10 . e^{-k(140)} = 5$

$\Rightarrow \quad e^{-140k} = \frac{5}{10} = 0.5$

$\Rightarrow \quad \ln e^{-140k} = \ln 0.5$

$\Rightarrow \quad -140k(\ln e) = \ln 0.5$

$\Rightarrow \quad -140k(1) = -0.693147$

$\Rightarrow \quad k = \frac{-0.693147}{-140}$

$= 0.00495$

(ii) $M = 10e^{-0.00495t}$

$t = 70 \quad \Rightarrow \quad M = 10\, e^{-0.00495(70)}$

$= 10\, e^{-0.3465}$

$= 10(0.7071)$

$= 7.071 = 7g$

(iii) $M = 2g \quad \Rightarrow \quad 10e^{-0.00495t} = 2$

$\Rightarrow \quad e^{-0.00495t} = \frac{2}{10} = 0.2$

$\Rightarrow \quad \ln e^{-0.00495t} = \ln 0.2$

$\Rightarrow \quad -0.00495t(\ln e) = \ln 0.2$

$-0.00495t(1) = -1.609438$

$\Rightarrow \quad t = \frac{-1.609438}{-0.00495} = 325.1 = 325$ days

Exercise 12.12(A)

1. Proof:

(i) $n = 1? \quad \Rightarrow 2(1) = 1(1 + 1) \Rightarrow 2 = 2$, true $n = 1$

(ii) Assume true for $n = k$.

$\Rightarrow 2 + 4 + 6 + 8 + \ldots 2k = k(k + 1)$

(iii) Also true for $n = k + 1$?

$\Rightarrow \quad 2 + 4 + 6 + 8 + \ldots + 2k + 2(k + 1) = k(k + 1) + 2(k + 1)$

$\Rightarrow \quad 2 + 4 + 6 + 8 + \ldots + 2k + 2(k + 1) = k(k + 1)(k + 2)$

$\Rightarrow \quad 2 + 4 + 6 + 8 + \ldots + 2k + 2(k + 1) = k(k + 1)[(k + 1) + 1]$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.

And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3 \ldots$ etc.

(v) Therefore, it is true for all positive integer values of n.

2. Proof:

(i) $n = 1? \quad \Rightarrow 1 = \frac{1}{2}[3(1) - 1] \quad \Rightarrow 1 = \frac{1}{2}(2) = 1$, true $n = 1$,

(ii) Assume true for $n = k$.

$\Rightarrow 1 + 4 + 7 + 10 + \ldots(3k - 2) = \frac{k}{2}(3k - 1)$

(iii) Also true for $n = k + 1$?

$\Rightarrow 1 + 4 + 7 + 10 + \ldots (3k - 2) + (3k + 1) = \frac{k}{2}(3k - 1) + (3k + 1)$

$\Rightarrow 1 + 4 + 7 + 10 + \ldots (3k - 2) + (3k + 1) = \frac{k(3k - 1) + 2(3k + 1)}{2}$

$\Rightarrow 1 + 4 + 7 + 10 + \ldots (3k - 2) + (3k + 1) = \dfrac{3k^2 - k + 6k + 2}{2}$

$\Rightarrow 1 + 4 + 7 + 10 + \ldots (3k - 2) + (3k + 1) = \dfrac{3k^2 + 5k + 2}{2}$

$\Rightarrow 1 + 4 + 7 + 10 + \ldots (3k - 2) + (3k + 1) = \dfrac{(k + 1)(3k + 2)}{2}$

$\Rightarrow 1 + 4 + 7 + 10 + \ldots (3k - 2) + (3k + 1) = \dfrac{(k + 1)[3(k + 1) - 1]}{2}$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3 \ldots$ etc.

(v) Therefore, it is true for all positive integer values of n.

3. Proof:

(i) $n = 1?$ $\quad \Rightarrow 1\,.\,2 = \frac{1}{3}(1 + 1)(1 + 2)$ $\quad \Rightarrow 2 = \frac{1}{3}(2)(3)$ $\quad \Rightarrow 2 = 2$, true $n = 1$.

(ii) Assume true for $n = k$.

$\Rightarrow 1\,.\,2 + 2\,.\,3 + 3\,.\,4 + 4\,.\,5 + \ldots k(k + 2) = \frac{k}{3}(k + 1)\,(k + 2)$

(iii) Also true for $n = k + 1$?

$\Rightarrow 1\,.\,2 + 2\,.\,3 + 3\,.\,4 + 4\,.\,5 + \ldots\, k(k + 1) + (k + 1)(k + 2) = \frac{k}{3}(k + 1)(k + 2) + (k + 1)(k + 2)$

$\Rightarrow 1\,.\,2 + 2\,.\,3 + 3\,.\,4 + 4\,.\,5 + \ldots\, k(k + 1) + (k + 1)(k + 2) = \dfrac{k(k + 1)(k + 2) + 3(k + 1)(k + 2)}{3}$

$\Rightarrow 1\,.\,2 + 2\,.\,3 + 3\,.\,4 + 4\,.\,5 + \ldots\, k(k + 1) + (k + 1)(k + 2) = \dfrac{(k + 1)(k + 2)(k + 3)}{3}$

$\Rightarrow 1\,.\,2 + 2\,.\,3 + 3\,.\,4 + 4\,.\,5 + \ldots\, k(k + 1) + (k + 1)(k + 2) = \dfrac{(k + 1)[(k + 1) + 1][(k + 1) + 2]}{3}$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3 \ldots$ etc.

(v) Therefore, it is true for all positive integer values of n.

4. Proof:

(i) $n = 1?$ $\quad \Rightarrow \dfrac{1}{2\,.\,3} = \dfrac{1}{2(1 + 2)}$ $\quad \Rightarrow \dfrac{1}{2\,.\,3} = \dfrac{1}{2\,.\,3}$ $\quad \Rightarrow$ true $n = 1$,

(ii) Assume true for $n = k$.

$\Rightarrow \dfrac{1}{2\,.\,3} + \dfrac{1}{3\,.\,4} + \dfrac{1}{4\,.\,5} + \dfrac{1}{5\,.\,6} + \ldots \dfrac{1}{(k + 1)(k + 2)} + \dfrac{k}{2(k + 2)}$

(iii) Also true for $n = k + 1$?

$$\Rightarrow \frac{1}{2\,.\,3} + \frac{1}{3\,.\,4} + \frac{1}{4\,.\,5} + \frac{1}{5\,.\,6} + \ldots \frac{1}{(k + 1)(k + 2)} + \frac{1}{(k + 2)(k + 3)} = \frac{k}{2(k + 2)} + \frac{1}{(k + 2)(k + 3)}$$

$$= \frac{k(k + 3) + 2(1)}{2(k + 2)(k + 3)}$$

$$= \frac{k^2 + 3k + 2}{2(k + 2)(k + 3)}$$

$$= \frac{(k + 1)(k + 2)}{2(k + 2)(k + 3)}$$

$$= \frac{k + 1}{2(k + 2)} = \frac{k + 1}{2[(k + 1) + 2]}$$

$\therefore$ It is true for $n = k + 1$

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ … etc.

(v) Therefore, it is true for all positive integer values of n.

5. Proof:

(i) $n = 1$? $\Rightarrow \frac{1}{4\,.\,5} = \frac{1}{4(1+4)} \quad \Rightarrow \frac{1}{4\,.\,5} = \frac{1}{4\,.\,5}$, true $n = 1$

(ii) Assume true for $n = k$.

$$\Rightarrow \frac{1}{4\,.\,5} + \frac{1}{5\,.\,6} + \frac{1}{6\,.\,7} + \dots \frac{1}{(k+3)(k+4)} = \frac{k}{4(k+4)}$$

(iii) Also true for $n = k + 1$?

$$\begin{aligned}\Rightarrow \quad \frac{1}{4\,.\,5} + \frac{1}{5\,.\,6} + \frac{1}{6\,.\,7} + \dots \frac{1}{(k+3)(k+4)} + \frac{1}{(k+4)(k+5)} &= \frac{k}{4(k+4)} + \frac{1}{(k+4)(k+5)} \\ &= \frac{k(k+5) + 4(1)}{4(k+4)(k+5)} \\ &= \frac{k^2 + 5k + 4}{4(k+4)(k+5)} \\ &= \frac{(k+1)(k+4)}{4(k+4)(k+5)} \\ &= \frac{k+1}{4(k+5)} = \frac{k+1}{4[(k+1)+4]}\end{aligned}$$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ … etc.

(v) Therefore, it is true for all positive integer values of n.

6. Proof:

(i) $n = 1$? $\Rightarrow 1^3 = \frac{(1)^2}{4}(1+1)^2 \quad \Rightarrow 1 = \frac{1}{4} \quad \Rightarrow 1 = 1$, true $n = 1$.

(ii) Assume true for $n = k \quad \Rightarrow 1^3 + 2^3 + 3^3 + \dots k^3 = \frac{k^2}{4}(k+1)^2$.

(iii) Also true for $n = k + 1$?

$$\begin{aligned}\Rightarrow \quad 1^3 + 2^3 + 3^3 + \dots k^3 + (k+1)^3 &= \frac{k^2}{4}(k+1)^2 + (k+1)^3 \\ &= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\ &= \frac{(k+1)^2[k^2 + 4(k+1)]}{4} \\ &= \frac{(k+1)^2(k+2)(k+2)}{4} \\ &= \frac{(k+1)^2(k+2)^2}{4} = \frac{(k+1)^2[(k+1)+1]^2}{4}\end{aligned}$$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ … etc.

(v) Therefore, it is true for all positive integer values of n.

7. $\displaystyle\sum_{n=1}^{n} n(n+2) = 1\,.\,3 + 2\,.\,4 + 3\,.\,5 + \dots n(n+2) = \frac{n(n+1)(2n+7)}{6}$

Proof:

(i) $n = 1$? $\Rightarrow 1\,.\,3 = \frac{1(1+1)(2+7)}{6} \quad \Rightarrow 3 = \frac{2\,.\,9}{6} \quad \Rightarrow 3 = 3$, true $n = 1$.

(ii) Assume true for $n = k \Rightarrow 1.3 + 2.4 + 3.5 + \ldots k(k+2) = \frac{k(k+1)(2k+7)}{6}$

(iii) Also true for $n = k + 1$?

$$\Rightarrow \quad 1.3 + 2.4 + 3.5 + \ldots k(k+2) + (k+1)(k+3) = \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3)$$
$$= \frac{k(k+1)(2k+7) + 6(k+1)(k+3)}{6}$$
$$= \frac{(k+1)[2k^2 + 7k + 6k + 18]}{6}$$
$$= \frac{(k+1)[2k^2 + 13k + 18]}{6}$$
$$= \frac{(k+1)(k+2)(2k+9)}{6} = \frac{(k+1)[(k+1)+1][2(k+1)+7]}{6}$$

$\therefore$ It is true for $n = k + 1$

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3 \ldots$ etc.

(v) Therefore, it is true for all positive integer values of n.

8. Proof:

(i) $n = 1$? $\quad \Rightarrow x = \frac{x(x^1 - 1)}{x - 1} \quad \Rightarrow x = x,\quad$ true $n = 1$.

(ii) Assume true for $n = k \Rightarrow x + x^2 + x^3 + x^4 + \ldots + x^k = \frac{x(x^k - 1)}{x - 1}$

(iii) Also true for $n = k + 1$?

$$\Rightarrow x + x^2 + x^3 + \ldots + x^k + x^{k+1} = \frac{x(x^k - 1)}{x - 1} + x^{k+1}$$
$$= \frac{x(x^k - 1) + x^{k+1}(x - 1)}{x - 1}$$
$$= \frac{x.x^k - x + x.x^{k+1} - x^{k+1}}{x - 1}$$
$$= \frac{x^{k+1} + x.x^{k+1} - x - x^{k+1}}{x - 1}$$
$$= \frac{x(x^{k+1} - 1)}{x - 1}$$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3 \ldots$ etc.

(v) Therefore, it is true for all positive integer values of n.

Exercise 12.12(B)

1. Proof:

(i) True for $n = 1$? $\quad \Rightarrow 5$ is a factor of $6^1 - 1 = 5,\quad$ true $n = 1$.

(ii) Assume true for $n = k \quad \Rightarrow 5$ is a factor of $6^k - 1,\quad k \in N$.

(iii) Also true for $n = k + 1$?

$$\Rightarrow 6^{k+1} - 1 = 6^k.6^1 - 1$$
$$= 6^k(5 + 1) - 1$$
$$= 5.6^k + (6^k - 1)$$

Since 5.6^k is divisible by 5 and $(6^k - 1)$ is assumed divisible by 5,

$\therefore 5.6^k + (6^k - 1)$ is divisible by 5.

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3 \ldots$ etc.

(v) Therefore, it is true for all positive integer values of n.

2. Proof:

(i) True for $n = 1$? $\Rightarrow$ 4 is a factor of $5^1 - 1 = 4$, true $n = 1$.

(ii) Assume true for $n = k$ $\Rightarrow$ 4 is a factor of $5^k - 1$.

(iii) Also true for $n = k + 1$?

$$\begin{aligned}\Rightarrow 5^{k+1} - 1 &= 5^k . 5^1 - 1\\ &= 5^k(4 + 1) - 1\\ &= 4 . 5^k + (5^k - 1)\end{aligned}$$

Since $4 . 5^k$ is divisible by 4 and $(5^k - 1)$ is assumed divisible by 4,

$\therefore 4 . 5^k + (5^k - 1)$ is divisible by 4.

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.

And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ … etc.

(v) Therefore, it is true for all positive integer values of n.

3. Proof:

(i) True for $n = 1$? $\Rightarrow$ 4 is a factor of $9^1 - 5^1 = 4$, true $n = 1$.

(ii) Assume true for $n = k$ $\Rightarrow$ 4 is a factor of $9^k - 5^k$.

(iii) Also true for $n = k + 1$?

$$\begin{aligned}\Rightarrow 9^{k+1} - 5^{k+1} &= 9^k . 9^1 - 5^k . 5^1\\ &= 9^k(8 + 1) - 5^k(4 + 1)\\ &= 8 . 9^k + 9^k - 4 . 5^k - 5^k\\ &= 8 . 9^k - 4 . 5^k + (9^k - 5^k)\end{aligned}$$

Since $8 . 9^k - 4 . 5^k$ is divisible by 4 and $9^k - 5^k$ is assumed divisible by 4,

$\therefore 8 . 9^k - 4 . 5^k + (9^k - 5^k)$ is divisible by 4.

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.

And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ … etc.

(v) Therefore, it is true for all positive integer values of n.

4. Proof:

(i) True for $n = 1$? $\Rightarrow$ 8 is a factor of $3^{2(1)} - 1 = 9 - 1 = 8$, true $n = 1$.

(ii) Assume true for $n = k$ $\Rightarrow$ 8 is a factor of $3^{2k} - 1$.

(iii) Also true for $n = k + 1$?

$$\begin{aligned}\Rightarrow 3^{2(k+1)} - 1 &= 3^{2k+2} - 1\\ &= 3^{2k} . 3^2 - 1\\ &= 3^{2k} . 9 - 1\\ &= 3^{2k}(8 + 1) - 1\\ &= 8 . 3^{2k} + (3^{2k} - 1)\end{aligned}$$

Since $8 . 3^{2k}$ is divisible by 8 and $(3^{2k} - 1)$ is assumed divisible by 8,

$\therefore 8 . 3^{2k} + (3^{2k} - 1)$ is divisible by 8.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.

And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ … etc.

(v) Therefore, it is true for all positive integer values of n.

5. Proof:

(i) True for $n = 1$? $\Rightarrow$ 5 is a factor of $7^1 - 2^1 = 5$, true $n = 1$

(ii) Assume true for $n = k$ $\Rightarrow$ 5 is a factor of $7^k - 2^k$.

(iii) Also true for $n = k + 1$?

$$\begin{aligned}\Rightarrow 7^{k+1} - 2^{k+1} &= 7^k . 7^1 - 2^k . 2^1\\ &= 7^k(5 + 2) - 2 . 2^k\\ &= 5 . 7^k + 2 . 7^k - 2 . 2^k\\ &= 5 . 7^k + 2(7^k - 2^k)\end{aligned}$$

Since 5.7^k is divisible by 5 and $(7^k - 2^k)$ is assumed divisible by 5,

$\therefore 5 \,.\, 7^k + 2(7^k - 2^k)$ is divisible by 5.

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3 \ldots$ etc.

(v) Therefore, it is true for all positive integer values of n.

6. Proof:

(i) True for $n = 1$? $\Rightarrow$ 8 is a factor of $7^{2(1)+1} + 1 = 7^3 + 1 = 344 = 8 \,.\, 43$ $\Rightarrow$ true $n = 1$.

(ii) Assume true for $n = k$ $\Rightarrow 7^{2k+1} + 1$ is divisible by 8.

(iii) Also true for $n = k + 1$?

$$\begin{aligned}\Rightarrow 7^{2(k+1)+1} + 1 &= 7^{2k+2+1} + 1\\ &= 7^{2k+1} \,.\, 7^2 + 1\\ &= 7^{2k+1} \,.\, 49 + 1\\ &= 7^{2k+1}(48 + 1) + 1\\ &= 48 \,.\, 7^{2k+1} + (7^{2k+1} + 1)\end{aligned}$$

Since $48 \,.\, 7^{2k+1}$ is divisible by 8 and $(7^{2k+1} + 1)$ is assumed divisible by 8,

$\therefore 48 \,.\, 7^{2k+1} + (7^{2k+1} + 1)$ is divisible by 8.

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3 \ldots$ etc.

(v) Therefore, it is true for all positive integer values of n.

7. Proof:

(i) True for $n = 1$? $\Rightarrow$ 7 is a factor of $2^{3(1)-1} + 3 = 2^2 + 3 = 7$, true $n = 1$.

(ii) Assume true for $n = k$ $\Rightarrow$ 7 is a factor of $2^{3k-1} + 3$, $k \in N$.

(iii) Also true for $n = k + 1$?

$$\begin{aligned}\Rightarrow 2^{3(k+1)-1} + 3 &= 2^{3k+3-1} + 3\\ &= 2^{3k-1} \,.\, 2^3 + 3\\ &= 2^{3k-1} \,.\, 8 + 3\\ &= 2^{3k-1}(7 + 1) + 3\\ &= 7 \,.\, 2^{3k-1} + (2^{3k-1} + 3)\end{aligned}$$

Since $7 \,.\, 2^{3k-1}$ is divisible by 7 and $2^{3k-1} + 3$ is assumed divisible by 7,

$\therefore 7 \,.\, 2^{3k-1} + (2^{3k-1} + 3)$ is divisible by 7.

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it must now be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3 \ldots$ etc.

(v) Therefore, it is true for all positive integer values of n.

8. Proof:

(i) True for $n = 1$? $\Rightarrow$ 4 is a factor of $5^1 - 4(1) + 3 = 4$, true $n = 1$.

(ii) Assume true for $n = k$ $\Rightarrow$ 4 is a factor of $5^k - 4k + 3$, $k \in N$.

(iii) Also true for $n = k + 1$?

$$\begin{aligned}\Rightarrow 5^{k+1} - 4(k + 1) + 3 &= 5^k \,.\, 5^1 - 4k - 4 + 3\\ &= 5^k(4 + 1) - 4k - 4 + 3\\ &= 4 \,.\, 5^k + 5^k - 4k - 4 + 3\\ &= 4 \,.\, 5^k - 4 + (5^k - 4k + 3)\end{aligned}$$

Since $4 \,.\, 5^k - 4$ is divisible by 4 and $(5^k - 4k + 3)$ is assumed divisible by 4,

$\therefore 4 \,.\, 5^k - 4 + (5^k - 4k + 3)$ is divisible by 4.

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it must now be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ … etc.
(v) Therefore, it is true for all positive integer values of n.

9. Proof:
(i) True for $n = 1$? $\Rightarrow$ 6 is a factor of $7^1 + 4^1 + 1 = 12$ $\Rightarrow$ true $n = 1$.
(ii) Assume true for $n = k$ $\Rightarrow$ 6 is a factor of $7^k + 4^k + 1$, $k \in N$.
(iii) Also true for $n = k + 1$?

$$\begin{aligned}\Rightarrow 7^{k+1} + 4^{k+1} + 1 &= 7^k . 7^1 + 4^k . 4^1 + 1\\ &= 7^k(6 + 1) + 4^k(3 + 1) + 1\\ &= 6 . 7^k + 7^k + 3 . 4^k + 4^k + 1\\ &= 6 . 7^k + 3 . 4^k + (7^k + 4^k + 1)\end{aligned}$$

4^k is an even number $\Rightarrow 3 . 4^k$ is divisible by 6 and
$7^k + 4^k + 1$ is assumed divisible by 6;
$\therefore 6 . 7^k + 3 . 4^k + (7^k + 4^k + 1)$ is divisible by 6.
$\therefore$ It is true for $n = k + 1$.
(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ … etc.
(v) Therefore, it is true for all positive integer values of n.

10. Proof:
(i) True for $n = 1$? $\Rightarrow 1(1+1)[2(1) + 1] = (1)(2)(3) = 6$ $\Rightarrow$ true $n = 1$.
(ii) Assume true for $n = k$ $\Rightarrow$ 3 is a factor of $k(k + 1)(2k + 1) = 2k^3 + 3k^2 + k$, $k \in N$.
(iii) Also true for $n = k + 1$?

$$\begin{aligned}\Rightarrow (k + 1)[(k + 1) + 1][2(k + 1) + 1] &= (k + 1)(k + 2)(2k + 3)\\ &= (k + 1)(2k^2 + 7k + 6)\\ &= 2k^3 + 9k^2 + 13k + 6\\ &= 2k^3 + 6k^2 + 3k^2 + 12k + k + 6\\ &= 6k^2 + 12k + 6 + (2k^3 + 3k^2 + k)\end{aligned}$$

Since $6k^2 + 12k + 6$ is divisible by 3, and $(2k^3 + 3k^2 + k)$ is assumed divisible by 3,
$\therefore 6k^2 + 12k + 6 + (2k^3 + 3k^2 + k)$ is divisible by 3.
(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ … etc.
(v) Therefore, it is true for all positive integer values of n.

11. Prove $n^3 - n$ is divisible by 3 for $n \in N$.
Proof:
(i) True for $n = 1$? $\Rightarrow$ 3 is a factor of $1^3 - 1 = 1 - 1 = 0$ $\Rightarrow$ true $n = 1$.
(ii) Assume true for $n = k$ $\Rightarrow k^3 - k$ is divisible by 3.
(iii) Also true for $n = k + 1$?

$$\begin{aligned}\Rightarrow (k + 1)^3 - (k + 1) &= k^3 + 3k^2 + 3k + 1 - k - 1\\ &= 3k^2 + 3k + (k^3 - k)\end{aligned}$$

Since $3k^2 + 3k$ is divisible by 3, and $(k^3 - k)$ is assumed divisible by 3,
$\therefore 3k^2 + 3k + (k^3 - k)$ is divisible by 3.
$\therefore$ It is true for $n = k + 1$.
(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ … etc.
(v) Therefore, it is true for all positive integer values of n.

12. Prove $13^n - 6^{n-2}$ is divisible by 7 for $n \in N$.

Proof:

(i) True for $n = 2$? $\Rightarrow$ 7 is a factor of $13^2 - 6^{2-2} = 169 - 1 = 168 = 7 \,.\, 24$ $\Rightarrow$ true $n = 2$.

(ii) Assume true for $n = k$ $\Rightarrow 13^k - 6^{k-2}$ is divisible by 7.

(iii) Also true for $n = k + 1$?

$$\begin{aligned} \Rightarrow 13^{k+1} - 6^{(k+1)-2} &= 13^k \,.\, 13^1 - 6^{k-2} \,.\, 6^1 \\ &= 13^k(7 + 6) - 6^{k-2} \,.\, 6 \\ &= 7 \,.\, 13^k + 6 \,.\, 13^k - 6 \,.\, 6^{k-2} \\ &= 7 \,.\, 13^k + 6(13^k - 6^{k-2}) \end{aligned}$$

Since $7 \,.\, 13^k$ is divisible by 7, and $(13^k - 6^{k-2})$ is assumed divisible by 7,

$\therefore 7 \,.\, 13^k + 6(13^k - 6^{k-2})$ is divisible by 7.

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.

And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ … etc.

(v) Therefore, it is true for all positive integer values of n excluding 1.

Exercise 12.12(C)

1. Proof:

(i) True for $n = 3$? $\Rightarrow 2^3 > 2(3) + 1$ $\Rightarrow 8 > 7$ $\Rightarrow$ true $n = 3$.

(ii) Assume true for $n = k$ $\Rightarrow 2^k > 2k + 1,\ k \in N,\ k \geqslant 3$.

(iii) Also true for $n = k + 1$?

$$\begin{aligned} \Rightarrow 2^{k+1} &= 2 \,.\, 2^k \\ &> 2(2k + 1) && \text{since } 2^k > 2k + 1 \text{ (assumed)} \\ &= 4k + 2 \\ &= 2k + 2k + 2 \\ &> 2k + 3 && \text{since } 2k + 2 > 3, \text{ for } k \geqslant 3 \\ &= 2(k + 1) + 1 \end{aligned}$$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 3$, it now must be true for $n = 3 + 1 = 4$.

And if it is true for $n = 4$, it is true for $n = 4 + 1 = 5$ … etc.

(v) Therefore, it is true for all values of n, $n \geqslant 3$, $n \in N$.

2. Proof:

(i) True for $n = 2$? $\Rightarrow 3^2 > 2^2$ $\Rightarrow 9 > 4$ $\Rightarrow$ true $n = 2$.

(ii) Assume true for $n = k$ $\Rightarrow 3^k > k^2,\ k \in N$, and $k \geqslant 2$.

(iii) Also true for $n = k + 1$?

$$\begin{aligned} \Rightarrow 3^{k+1} &= 3^1 \,.\, 3^k \\ &> 3 \,.\, k^2 && \text{since } 3^k > k^2 \text{ (assumed)} \\ &= k^2 + k^2 + k^2 \\ &> k^2 + 2k + 1 && \text{since } 2k^2 > 2k + 1, \text{ for } k \geqslant 2 \\ &= (k + 1)(k + 1) \\ &= (k + 1)^2 \end{aligned}$$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 2$, it now must be true for $n = 2 + 1 = 3$.

And if it is true for $n = 3$, it is true for $n = 3 + 1 = 4$ … etc.

(v) Therefore, it is true for all values of n, $n \geqslant 2$, $n \in N$.

3. Proof:

(i) True for $n = 2$? $\Rightarrow 3^2 > 2(2) + 2$ $\Rightarrow 9 > 6$ $\Rightarrow$ true $n = 2$.

(ii) Assume true for $n = k$ $\Rightarrow 3^k > 2k + 2, k \in N, k \geqslant 2$.

(iii) Also true for $n = k + 1$?

$$\begin{aligned} \Rightarrow 3^{k+1} &= 3 \,.\, 3^k \\ &> 3(2k + 2) && \text{since } 3^k > 2k + 2 \text{ (assumed)} \\ &= 6k + 6 \\ &= 2k + 4k + 2 + 4 \\ &= 2k + 4 + 4k + 2 \\ &> 2k + 4 && \text{since } 4k + 2 > 0, \text{ for } k \geqslant 2 \\ &= 2(k + 1) + 2 \end{aligned}$$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 2$, it now must be true for $n = 2 + 1 = 3$.
And if it is true for $n = 3$, it is true for $n = 3 + 1 = 4$ … etc.

(v) Therefore, it is true for all values of n, $n \geqslant 2$, $n \in N$.

4. Proof:

(i) True for $n = 3$? $\Rightarrow 3! > 2^{3-1}$ $\Rightarrow 6 > 4$ $\Rightarrow$ true $n = 3$.

(ii) Assume true for $n = k$ $\Rightarrow k! > 2^{k-1}$, $k \in N$, $k \geqslant 3$.

(iii) Also true for $n = k + 1$?

$$\begin{aligned} \Rightarrow (k + 1)! &= (k + 1)k! \\ &> (k + 1) \,.\, 2^{k-1} && \text{since } k! > 2^{k-1} \text{ (assumed)} \\ &> (1 + 1) \,.\, 2^{k-1} && \text{since } k + 1 > 2, \text{ for } k \geqslant 3 \\ &= 2 \,.\, 2^{k-1} \\ &= 2^k \\ &= 2^{(k+1)-1} \end{aligned}$$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 3$, it now must be true for $n = 3 + 1 = 4$.
And if it is true for $n = 4$, it is true for $n = 4 + 1 = 5$ … etc.

(v) Therefore, it is true for all values of n, $n \geqslant 3$, $n \in N$.

5. Proof:

(i) True for $n = 2$? $\Rightarrow (2 + 1)! > 2^2$ $\Rightarrow 6 > 4$ $\Rightarrow$ true $n = 2$.

(ii) Assume true for $n = k$ $\Rightarrow (k + 1)! > 2^k$.

(iii) Also true for $n = k + 1$?

$$\begin{aligned} \Rightarrow (k + 1 + 1)! &= (k + 2)! \\ &= (k + 2)(k + 1)! \\ &> (k + 2) \,.\, 2^k && \text{since } (k + 1)! > 2^k \text{ (assumed)} \\ &> 2 \,.\, 2^k && \text{since } (k + 2) > 2, \text{ for } k \geqslant 2 \\ &= 2 \,.\, 2^{k+1} \end{aligned}$$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 2$, it now must be true for $n = 2 + 1 = 3$.
And if it is true for $n = 3$, it is true for $n = 3 + 1 = 4$ … etc.

(v) Therefore, it is true for all values of n, $n \geqslant 2$, $n \in N$.

6. Proof:

(i) True for $n = 1$? $\Rightarrow (1 + 2x)^1 = 1 + 2(1)x$ $\Rightarrow 1 + 2x = 1 + 2x$ $\Rightarrow$ true $n = 1$.

(ii) Assume true for $n = k$ $\Rightarrow (1 + 2x)^k \geqslant 1 + 2kx$ for $x > 0$, $k \in N$.

(iii) Also true for $n = k + 1$?

$\Rightarrow (1 + 2x)^{k+1} = (1 + 2x)(1 + 2x)^k$

$\geqslant (1 + 2x)(1 + 2kx)$ since $(1 + 2x)^k \geqslant 1 + 2kx$ (assumed)

$= 1 + 2kx + 2x + 4kx^2$

$\geqslant 1 + 2kx + 2x$ since $4kx^2 \geqslant 0$, for $x > 0, k \geqslant 1$

$= 1 + 2(k+1)x$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.

And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ … etc.

(v) Therefore, it is true for all values of n, $n \geqslant 1$, $n \in N$.

7. Prove $(1 + ax)^n \geqslant 1 + anx$ for $a > 0$, $x > 0$, $n \in N$.

Proof:

(i) True for $n = 1$? $\Rightarrow (1 + ax)^1 \geqslant 1 + a(1)x$ $\Rightarrow 1 + ax \geqslant 1 + ax$, true $n = 1$.

(ii) Assume true for $n = k$ $\Rightarrow (1 + ax)^k \geqslant 1 + akx$.

(iii) Also true for $n = k + 1$?

$\Rightarrow (1 + ax)^{k+1} = (1 + ax)(1 + ax)^k$

$\geqslant (1 + ax)(1 + akx)$ since $(1 + ax)^k \geqslant 1 + akx$ (assumed)

$= 1 + akx + ax + a^2kx^2$

$\geqslant 1 + akx + ax$ since $a^2kx^2 \geqslant 0$, for $a > 0, x > 0$

$= 1 + a(k + 1)x$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.

And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ … etc.

(v) Therefore, it is true for all values of n, $n \geqslant 1$, $n \in N$.

Revision Exercise12 (Core)

1. $-1 \leqslant \dfrac{2x + 4}{3} \leqslant 2, x \in R$

$\Rightarrow -3 \leqslant 2x + 4 \leqslant 6$

$\Rightarrow -7 \leqslant 2x \leqslant 2$

$\Rightarrow -3.5 \leqslant x \leqslant 1$

2. (a) (i) $10^{3.5} = 3162.278 = 3162$

(ii) $\log_{10} 4.5 = 0.6532 = 0.65$

(iii) $t = 0.04 \Rightarrow 10^{3t} = 10^{3(0.04)} = 10^{0.12} = 1.318 = 1.32$

(iv) $n = 100 \Rightarrow \log 5n = \log 5(100) = \log 500 = 2.69897 = 2.7$

(b) (i) $e^{3.4} = 29.964 = 30$

(ii) $\ln 589 = 6.378 = 6.38$

(iii) $t = 40 \Rightarrow e^{-0.02t-4} = e^{-0.02(40)-4} = e^{-0.8-4} = e^{-4.8}$

$= 0.0082297 = 0.00823$

(iv) $k = 3.7 \Rightarrow \ln\left(\dfrac{10}{k}\right) = \ln\left(\dfrac{10}{3.7}\right) = 0.994 = 0.99$

3. (i) $f(x) = 3 \times 4^x$

Point $(a, 6)$ $\Rightarrow$ $f(a) = 3 \times 4^a = 6$

$\Rightarrow 4^a = 2$

$\Rightarrow (2^2)^a = 2$

$\Rightarrow 2^{2a} = 2^1$

$\Rightarrow 2a = 1 \Rightarrow a = \dfrac{1}{2}$

(ii) Point $\left(-\frac{1}{2}, b\right) \Rightarrow f\left(-\frac{1}{2}\right) = 3 \times 4^{-\frac{1}{2}} = b$

$$\Rightarrow 3 \times \frac{1}{2} = b$$

$$\Rightarrow b = \frac{3}{2}$$

4. $x - 8 = -3$ or $x - 8 = 3$

$\Rightarrow x = 5$ or $x = 11$

5. (i) $5^{2n} \times 25^{2n-1} = 625$

$\Rightarrow 5^{2n} \times (5^2)^{2n-1} = 5^4$

$\Rightarrow 5^{2n} \times 5^{4n-2} = 5^4$

$\Rightarrow 5^{2n+4n-2} = 5^{6n-2} = 5^4$

$\Rightarrow 6n - 2 = 4$

$\Rightarrow 6n = 6 \Rightarrow n = 1$

(ii) $27^{n-2} = 9^{3n+2}$

$\Rightarrow (3^3)^{n-2} = (3^2)^{3n+2}$

$\Rightarrow 3^{3n-6} = 3^{6n+4}$

$\Rightarrow 3n - 6 = 6n + 4$

$\Rightarrow -3n = 10 \Rightarrow n = -\frac{10}{3}$

6. $y = a2^x + b$

(i) Point (0, 2.5) $\Rightarrow$ $2.5 = a\,.\,2^0 + b = a\,.\,1 + b \Rightarrow$ $a + b = 2.5$

Point (2, 4) $\Rightarrow$ $4 = a\,.\,2^2 + b = 4a + b$ $\Rightarrow$ $4a + b = 4$

(ii) $4a + b = 4$

$a + b = 2.5$

$\Rightarrow 3a = 1.5$ $\Rightarrow a = \frac{1.5}{3} = 0.5$

and $0.5 + b = 2.5$ $\Rightarrow b = 2.5 - 0.5 = 2$

7. (i) Curve C $= \ln(x) \Rightarrow$ Point (1, 0) $\Rightarrow \ln 1 = 0$, true

(ii) Curve $B = \ln(x + 1) \Rightarrow$ Point (0, 0) $\Rightarrow \ln(0 + 1) = \ln 1 = 0$, true

(iii) Curve A $= \ln(x) + 1 \Rightarrow$ Point (1, 1) $\Rightarrow \ln 1 + 1 = 0 + 1 = 1$, true

8. Solve $\ln(x - 1) + \ln(x + 2) = \ln(6x - 8)$

$\Rightarrow \ln(x - 1)(x + 2) = \ln(6x - 8)$

$\Rightarrow (x - 1)(x + 2) = 6x - 8$

$\Rightarrow x^2 + x - 2 = 6x - 8$

$\Rightarrow x^2 - 5x + 6 = 0$

$\Rightarrow (x - 2)(x - 3) = 0$

$\Rightarrow x = 2, x = 3$

9. $y = Ae^{bt}$

$y = 6$ when $t = 1$ $\Rightarrow 6 = Ae^{b(1)}$ $\Rightarrow Ae^b = 6 \Rightarrow e^b = \frac{6}{A}$

$y = 8$ when $t = 2$ $\Rightarrow 8 = Ae^{b(2)}$ $\Rightarrow Ae^{2b} = 8$

$\Rightarrow A(e^b)^2 = 8$

$\Rightarrow A\left(\frac{6}{A}\right)^2 = 8$

$\Rightarrow A \cdot \frac{36}{A^2} = \frac{36}{A} = 8$

$\Rightarrow 8A = 36 \Rightarrow A = \frac{9}{2}$

$$e^b = \frac{6}{\frac{9}{2}} = \frac{4}{3}$$

$\Rightarrow \ln e^b = \ln \frac{4}{3}$

$\Rightarrow b(\ln e) = b = \ln \frac{4}{3}$

10. $y = a \log_2 (x - b)$

$(5, 2) \Rightarrow 2 = a \log_2 (5 - b)$

$\Rightarrow 2 = \log_2 (5 - b)^a \Rightarrow (5 - b)^a = 2^2 = 4$

$\Rightarrow 5 - b = 4^{\frac{1}{a}}$

$\Rightarrow b = 5 - 4^{\frac{1}{a}}$

$(7, 4) \Rightarrow 4 = a \log_2(7 - b)$

$\Rightarrow 4 = \log_2(7 - b)^a \Rightarrow (7 - b)^a = 2^4 = 16$

$\Rightarrow 7 - b = 16^{\frac{1}{a}} = (4^2)^{\frac{1}{a}} = 4^{\frac{2}{a}}$

$\Rightarrow b = 7 - 4^{\frac{2}{a}}$

Hence, $5 - 4^{\frac{1}{a}} = 7 - 4^{\frac{2}{a}}$

$\Rightarrow 4^{\frac{2}{a}} - 4^{\frac{1}{a}} = 2$

$\Rightarrow (4^{\frac{1}{a}})^2 - 4^{\frac{1}{a}} - 2 = 0$

Let $y = 4^{\frac{1}{a}} \Rightarrow \quad y^2 - y - 2 = 0$

$\Rightarrow (y - 2)(y + 1) = 0$

$\Rightarrow y = 2, y = -1$

$\Rightarrow 4^{\frac{1}{a}} = 2$ or $4^{\frac{1}{a}} = -1$ (Not valid)

$\Rightarrow 4 = 2^a$

$\Rightarrow 2^2 = 2^a \Rightarrow a = 2$

$\Rightarrow b = 5 - 4^{\frac{1}{2}} = 5 - 2 = 3$

11. Solve $32^{x-1} = 28$

$\Rightarrow \ln 32^{x-1} = \ln 28$

$\Rightarrow (x - 1) \ln 32 = \ln 28$

$\Rightarrow x - 1 = \dfrac{\ln 28}{\ln 32} = 0.96147$

$\Rightarrow x = 1.96147$

$\Rightarrow x = 1.96$

12. Proof:

(i) $n = 1? \quad \Rightarrow 3 = \frac{3(1)}{2}(1 + 1) \quad \Rightarrow 3 = 3$, true $n = 1$.

(ii) Assume true for $n = k \quad \Rightarrow 3 + 6 + 9 + ... + 3k = \frac{3k}{2}(k + 1)$.

(iii) Also true for $n = k + 1$?

$\Rightarrow 3 + 6 + 9 + ... + 3k + (3k + 3) = \frac{3k}{2}(k + 1) + 3(k + 1)$

$\Rightarrow 3 + 6 + 9 + ... + 3k + 3(k + 1) = \dfrac{3k(k + 1) + 2\,.\,3(k + 1)}{2}$

$\Rightarrow 3 + 6 + 9 + ... + 3k + 3(k + 1) = \frac{3}{2}(k + 1)(k + 2)$

$\Rightarrow 3 + 6 + 9 + ... + 3k + 3(k + 1) = \frac{3}{2}(k + 1)[(k + 1) + 1]$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ ... etc.

(v) Therefore, it is true for all positive integer values of n.

13. Proof:

(i) True for $n = 1$? $\Rightarrow$ 7 is a factor of $8^1 + 6 = 14$ $\Rightarrow$ true $n = 1$.

(ii) Assume true for $n = k$ $\Rightarrow 8^k + 6$ is divisible by 7.

(iii) Also true for $n = k + 1$?

$$\Rightarrow 8^{k+1} + 6 = 8^k . 8^1 + 6$$
$$= 8^k(7 + 1) + 6$$
$$= 7 . 8^k + (8^k + 6)$$

Since $7 . 8^k$ is divisible by 7, and $(8^k + 6)$ is assumed divisible by 7,

$\therefore 7 . 8^k + (8^k + 6)$ is divisible by 7.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.

And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$ ... etc.

(v) Therefore, it is true for all positive integer values of n.

14. Prove by induction that $n^2 > 4n + 3$ for $n \geqslant 5, n \in N$.

Proof:

(i) True for $n = 5$? $\Rightarrow 5^2 > 4(5) + 3$ $\Rightarrow 25 > 23$ $\Rightarrow$ true $n = 5$.

(ii) Assume true for $n = k$ $\Rightarrow k^2 > 4k + 3$ for $k \geqslant 5, k \in N$.

(iii) Also true for $n = k + 1$?

$$\Rightarrow (k + 1)^2 = k^2 + 2k + 1$$
$$> 4k + 3 + 2k + 1 \qquad \text{since } k^2 > 4k + 3 \text{ (assumed)}$$
$$= 4k + 4 + 2k$$
$$> 4(k + 1) + 3 \qquad \text{since } 2k > 3, \text{ for } k \geq 5$$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 5$, it now must be true for $n = 5 + 1 = 6$.

And if it is true for $n = 6$, it is true for $n = 6 + 1 = 7$ ... etc.

(v) Therefore, it is true for all values of $n, n \geqslant 5, n \in N$.

Revision Exercise12 (Advanced)

1. (i) $3x + 4 < x^2 - 6$

$x^2 - 3x - 10 > 0$

Let $x^2 - 3x - 10 = 0$

$(x - 5)(x + 2) = 0$

$x = 5$ or $x = -2$

Solution: $x < -2$ or $x < 5$

(ii) $x^2 - 6 < 9 - 2x$

$x^2 + 2x - 15 < 0$

Let $x^2 + 2x - 15 = 0$

$(x + 5)(x - 3) = 0$

$x = -5$ or $x = 3$

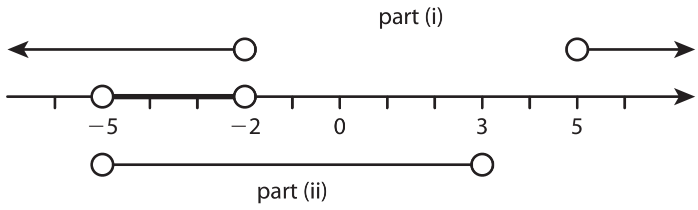
Solution: $-5 < x < 3$

(iii) The solution of

$$3x + 4 < x^2 - 6 < 9 - 2x$$

is then the intersection of the above solution sets, i.e.

$$-5 < x < -2.$$



2. $M = 30 \times 2^{-0.001t}$

(i) $t = 0 \Rightarrow M = 30 \times 2^{-0.001(0)} = 30 \times 2^0 = 30 \times 1 = 30\,\text{g}$

(ii) $M = 10$

$$\Rightarrow 30 \times 2^{-0.001t} = 10$$
$$\Rightarrow 2^{-0.001t} = \frac{10}{30} = \frac{1}{3}$$
$$\Rightarrow \log 2^{-0.001t} = \log \frac{1}{3}$$
$$\Rightarrow -0.001t\,.\log 2 = \log \frac{1}{3}$$
$$\Rightarrow -0.001t = \frac{\log \frac{1}{3}}{\log 2} = -1.5849625$$
$$\Rightarrow t = \frac{-1.5849625}{-0.001} = 1584.9625$$
$$\Rightarrow t = 1585 \text{ years}$$

(iii) 1% of 30 = 0.3

$$\Rightarrow 30 \times 2^{-0.001t} = 0.3$$
$$\Rightarrow 2^{-0.001t} = \frac{0.3}{30} = 0.01$$
$$\Rightarrow \log 2^{-0.001t} = \log 0.01$$
$$\Rightarrow -0.001t\,.\log 2 = \log 0.01$$
$$\Rightarrow -0.001t = \frac{\log 0.01}{\log 2} = -6.643856$$
$$\Rightarrow t = \frac{-6.643856}{-0.001} = 6643.856 = 6644 \text{ years}$$

3. $I = I_0 \times 10^{0.1S}$

(i) $S = 30 \Rightarrow I = I_0 \times 10^{0.1(30)}$

$= I_0 \times 10^3 = 1000\,I_0 \Rightarrow$ Answer $= 1000$

(ii) $S = 28 \Rightarrow I = I_0 \times 10^{0.1(28)} = I_0 \times 10^{2.8}$

$S = 15 \Rightarrow I = I_0 \times 10^{0.1(15)} = I_0 \times 10^{1.5}$

$$\Rightarrow \text{No. of times} = \frac{I_0 \times 10^{2.8}}{I_0 \times 10^{1.5}} = 10^{1.3} = 19.95 = 20 \text{ Times}$$

4. Solve $\log_5 x - 1 = 6 \log_x 5$.

$$\Rightarrow \log_5 x - 1 = 6\frac{\log_5 5}{\log_5 x} = 6\,.\frac{1}{\log_5 x} = \frac{6}{\log_5 x}$$

Let $y = \log_5 x$

$$\Rightarrow y - 1 = \frac{6}{y}$$
$$\Rightarrow y^2 - y = 6$$
$$\Rightarrow y^2 - y - 6 = 0$$
$$\Rightarrow (y-3)(y+2) = 0$$
$$\Rightarrow y = 3, y = -2$$

Hence, $\log_5 x = 3$ or $\log_5 x = -2$

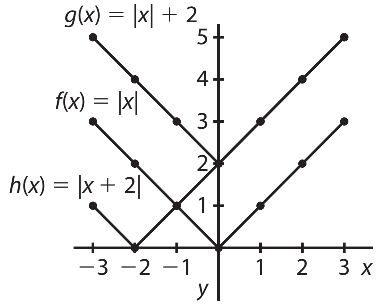
$\Rightarrow x = 5^3 = 125$ or $x = 5^{-2} = \frac{1}{25}$

5. Solve $(0.7)^x \geqslant 0.3$

$$\Rightarrow \log(0.7)^x \geqslant \log(0.3)$$
$$\Rightarrow x\,.\log(0.7) \geqslant \log(0.3)$$
$$\Rightarrow x(-0.1549) \geqslant -0.5228787$$
$$\Rightarrow x(0.1549) \leqslant 0.5228787$$
$$\Rightarrow x \leqslant \frac{0.5228787}{0.1549} = 3.3755$$
$$\Rightarrow x \leqslant 3.38$$

6. (i) (ii) (iii)

$x =$	-3	-2	-1	0	1	2	3
$f(x) = \lvert x \rvert =$	3	2	1	0	1	2	3
$g(x) = \lvert x \rvert + 2$	5	4	3	2	3	4	5
$x + 2 =$	-1	0	1	2	3	4	5
$h(x) = \lvert x + 2 \rvert$	1	0	1	2	3	4	5



(iv) $f(x) \cap h(x) \quad \Rightarrow \quad x = -1$

(v) $g(x) > h(x) \quad \Rightarrow \quad -3 \leq x < 0$

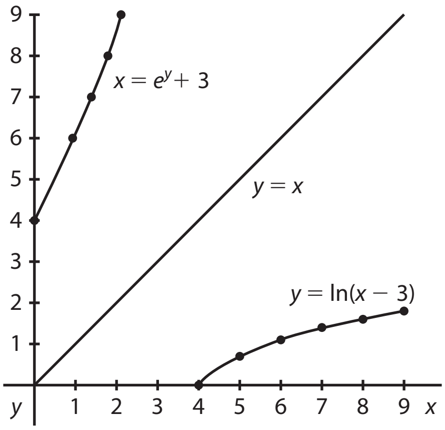
7.

$x =$	4	5	6	7	8	9
$x - 3 =$	1	2	3	4	5	6
$\ln(x - 3) =$	0	0.7	1.1	1.4	1.6	1.8

$\ln(x - 3) = \log_e(x - 3) = y$

$\Rightarrow e^y = x - 3$

$\Rightarrow x = e^y + 3$

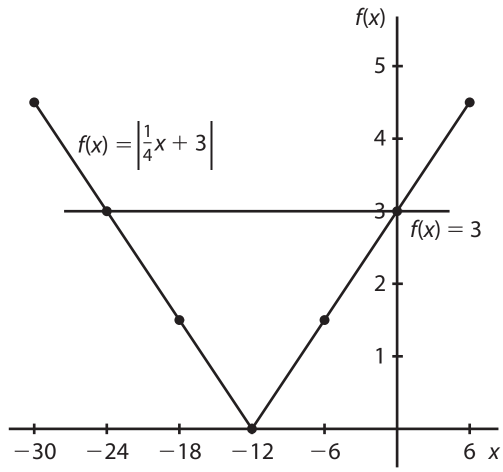


8.

$x = 1$	-30	-24	-18	-12	-6	0	6
$\frac{1}{4}x =$	-7.5	-6	-4.5	-3	-1.5	0	1.5
$\frac{1}{4}x + 3 =$	-4.5	-3	-1.5	0	1.5	3	4.5
$f(x) = \left\lvert \frac{1}{4}x + 3 \right\rvert =$	4.5	3	1.5	0	1.5	3	4.5

Solve $\left\lvert \frac{1}{4}x + 3 \right\rvert \geqslant 3$

$\Rightarrow -24 \geqslant x \geqslant 0$



9. $\dfrac{x^{\frac{3}{2}} - x^{-\frac{1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}$

$$= \frac{x^{-\frac{1}{2}}(x^2 - 1)}{x^{-\frac{1}{2}}(x - 1)} = \frac{(x + 1)(x - 1)}{x - 1} = x + 1$$

10. Proof:

(i) True for $n = 1$? $\Rightarrow \dfrac{1}{(1 + r)^1} \leqslant \dfrac{1}{1 + (1)r} \Rightarrow \dfrac{1}{1 + r} \leqslant \dfrac{1}{1 + r} \Rightarrow$ true $n = 1$.

(ii) Assume true for $n = k \Rightarrow \dfrac{1}{(1 + r)^k} \leqslant \dfrac{1}{1 + kr}$ for $k \geqslant 1,\ r > 0,\ k \in N$.

(iii) Also true for $n = k + 1$?

$$\Rightarrow \frac{1}{(1 + r)^{k+1}} = \frac{1}{(1 + r)^k(1 + r)} \leqslant \frac{1}{(1 + kr)(1 + r)} \qquad \text{since } \frac{1}{(1 + r)^k} \leqslant \frac{1}{1 + kr} \text{ (assumed)}$$

$$= \frac{1}{1 + kr + r + kr^2}$$

$$\leqslant \frac{1}{1+kr+r} \qquad \text{since } kr^2 > 0 \text{ for } k \geqslant 1,\ r > 0$$

$$= \frac{1}{1+(k+1)r}$$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3 \ldots$ etc.

(v) Therefore, it is true for all positive integer values of n.

11. Prove $\frac{4x}{(x+1)^2} \leqslant 1$ for all $x \in R,\ x \neq -1$.

Proof: $\frac{4x}{(x+1)^2} \cdot (x+1)^2 \leqslant 1(x+1)^2$

$\Rightarrow\quad 4x \leqslant x^2 + 2x + 1$

$\Rightarrow\quad -x^2 + 2x - 1 \leqslant 0$

$\Rightarrow\quad x^2 - 2x + 1 \leqslant 0$

$\Rightarrow\quad (x-1)^2 \leqslant 0$ true for all $x \in R,\ x \neq -1$

Hence, $\frac{4x}{(x+1)^2} \leqslant 1$.

12. $(1+2k)x^2 - 10x + (k-2) = 0$

(i) Real Roots

$$\begin{aligned}
&\Rightarrow & b^2 - 4ac &\geqslant 0\\
&\Rightarrow & (-10)^2 - 4(1+2k)(k-2) &\geqslant 0\\
&\Rightarrow & 100 - 4(2k^2 - 3k - 2) &\geqslant 0\\
&\Rightarrow & 100 - 8k^2 + 12k + 8 &\geqslant 0\\
&\Rightarrow & -8k^2 + 12k + 108 &\geqslant 0\\
&\Rightarrow & 8k^2 - 12k - 108 &\leqslant 0\\
&\Rightarrow & 2k^2 - 3k - 27 &\leqslant 0
\end{aligned}$$

Factors: $(k+3)(2k-9) = 0$

Roots: $k = -3,\ k = 4\frac{1}{2}$

Hence, $-3 \leqslant k \leqslant 4\frac{1}{2}$.

(ii) Sum of roots > 5

$$\begin{aligned}
&\Rightarrow & \frac{10}{1+2k} &> 5\\
&\Rightarrow & \frac{10}{1+2k}(1+2k)^2 &> 5(1+2k)^2\\
&\Rightarrow & 10(1+2k) &> 5(1+2k)^2\\
&\Rightarrow & 2(1+2k) &> (1+2k)^2\\
&\Rightarrow & 2 + 4k &> 1 + 4k + 4k^2\\
&\Rightarrow & -4k^2 + 1 &> 0\\
&\Rightarrow & 4k^2 - 1 &< 0
\end{aligned}$$

Factors: $(2k+1)(2k-1) = 0$

Roots: $k = -\frac{1}{2},\ k = \frac{1}{2}$

Hence, $-\frac{1}{2} < k < \frac{1}{2}$.

13. Proof:

(i) True for $n = 1$? $\Rightarrow 1 = (1-1)2^1 + 1 \quad \Rightarrow 1 = 0 + 1 \quad \Rightarrow 1 = 1 \quad \Rightarrow$ true $n = 1$.

(ii) Assume true for $n = k \quad \Rightarrow 1 + 2\,.\,2 + 3\,.\,2^2 + \cdots\cdots k\,.\,2^{k-1} = (k-1)\,.\,2^k + 1$.

(iii) Also true for $n = k + 1$?

$\Rightarrow\ 1 + 2.2 + 3.2^2 + \cdots + k.2^{k-1} + (k+1).2^{(k+1)-1}$

$= (k-1)2^k + 1 + (k+1)2^{(k+1)-1}$

$= (k-1)2^k + 1 + (k+1)2^k$

$= k.2^k - 2^k + 1 + k.2^k + 2^k$

$= 2k.2^k + 1$

$= k.2^{k+1} + 1 = [(k+1) - 1].2^{k+1} + 1$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it is now must be true for $n = 1 + 1 = 2$.

And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3 \cdots$ etc.

(v) Therefore, it is true for all positive integer values of n.

14. $u_n = (n - 20).2^n$

$\Rightarrow\ u_{n+1} = (n + 1 - 20).2^{n+1} = (n - 19).2.2^n$

$\Rightarrow\ u_{n+2} = (n + 2 - 20).2^{n+2} = (n - 18).2^2.2^n = (n - 18).4.2^n$

Hence, $u_{n+2} - 4u_{n+1} + 4u_n$

$= (n - 18).4.2^n - 4(n - 19).2.2^n + 4.(n - 20)2^n$

$= 2^n[4n - 72 - 8n + 152 + 4n - 80]$

$= 2^n[8n - 8n + 152 - 152] = 0$

15. $2\log y = \log 2 + \log x$ and $2^y = 4^x$ for $y > 0$

$\Rightarrow\ \log y^2 = \log 2x$ $\quad\Rightarrow\ 2^y = (2^2)^x$

$\Rightarrow\ y^2 = 2x$ $\quad\Rightarrow\ 2^y = 2^{2x}$

$\Rightarrow\ y^2 = y$ $\quad\Rightarrow\ y = 2x$

$\Rightarrow\ y^2 - y = 0$

$\Rightarrow\ y(y - 1) = 0$

$\Rightarrow y = 0$ (Not Valid) or $y = 1 \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}$

16. (i) An exponential function : $P = 40\,000(1.03)^n$

(ii) $n = 12 \Rightarrow P = 40\,000(1.03)^{12}$

$= 40\,000(1.42576)$

$= 57\,030.43 = 57\,030$

(iii) $n = 0 \Rightarrow P = 40\,000(1.03)^0$

$= 40\,000(1) = 40\,000$

(iv) $P = 80\,000 \Rightarrow 40\,000(1.03)^n = 80\,000$

$$\Rightarrow\ (1.03)^n = \frac{80\,000}{40\,000} = 2$$

$$\Rightarrow\ \log(1.03)^n = \log 2$$

$$\Rightarrow\ n\log(1.03) = \log 2$$

$$\Rightarrow\ n = \frac{\log 2}{\log 1.03} = 23.449 = 23.5 \text{ years}$$

17. (i) $P = Ae^{kt}$, where $A = 8000$, $P = 15\,000$, $t = 8$; find k.

$\Rightarrow\ 15\,000 = 8000e^{k(8)}$

$$\Rightarrow\ e^{8k} = \frac{15\,000}{8000} = 1.875$$

$\Rightarrow\ \ln e^{8k} = \ln 1.875$

$\Rightarrow\ 8k(\ln e) = 8k(1) = 8k = 0.62860865$

$$\Rightarrow k = \frac{0.62860865}{8} = 0.078576$$

Hence, $P = Ae^{kt}$, where $k = 0.078576$, t = number of years and $A = 8000$.

(ii) $t = 10 \Rightarrow P = 8000\,e^{0.078576(10)}$

$= 8000\,e^{0.78576} = 8000(2.1940738)$

$= 17\,552.59 = 17\,553$

(iii) $P = 30\,000 \Rightarrow 15\,000\,e^{0.078576t} = 30\,000$

$\Rightarrow \quad e^{0.078576t} = \dfrac{30\,000}{15\,000} = 2$

$\Rightarrow \quad \ln e^{0.078576t} = \ln 2$

$\Rightarrow \quad 0.078576t \ln e = 0.69314718$

$\Rightarrow \quad 0.078576t(1) = 0.69314718$

$\Rightarrow \quad t = \dfrac{0.69314718}{0.078576} = 8.82 = 9 \text{ years}$

Ans = 2016

Revision Exercise 12 (Extended-Response Questions)

1. $N = 5000e^{-0.15t}$

(i) $t = 0 \quad \Rightarrow N = 5000e^{-0.15(0)} = 5000e^{0} = 5000(1) = 5000$

$t = 5 \quad \Rightarrow N = 5000e^{-0.15(5)} = 5000e^{-0.75}$

$= 5000(0.47236655)$

$= 2361.83 = 2362$

$\Rightarrow$ Claim is justified.

(ii) $t = 10 \Rightarrow N = 5000e^{-0.15(10)}$

$= 5000e^{-1.5}$

$= 5000(0.23313)$

$= 1115.65$

(iii) $t = 0 \quad \Rightarrow N = 5000e^{-0.15(0)} = 5000\,.\,e^{0} = 5000\,.\,1 = 5000$

(iv) $N = 100 \Rightarrow \quad 5000e^{-0.15t} = 100$

$\Rightarrow \quad e^{-0.15t} = \dfrac{100}{5000} = 0.02$

$\Rightarrow \quad \ln e^{-0.15t} = \ln(0.02)$

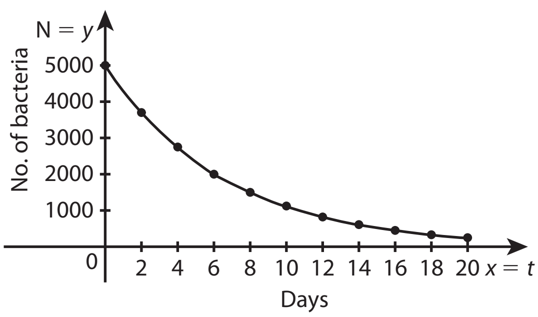
$\Rightarrow \quad -0.15t(\ln e) = \ln(0.02)$

$\Rightarrow \quad -0.15t(1) = -0.15t = -3.912$

$\Rightarrow \quad t = \dfrac{-3.912}{-0.15} = 26.08 = 26.1 \text{ days}$

(v)

$t =$	0	2	4	6	8	10	12	14	16	18	20
$N = 5000^{-0.15t}$	5000	3704	2744	2033	1506	1116	826	612	454	336	249



2. (i) $A = 0.02(0.92)^{\frac{x}{10}}$

(ii) Length $= \dfrac{5}{3} = 1.66667$

$\Rightarrow \quad A = 0.02(0.92)^{\frac{1.66667}{10}} = 0.02(0.92)^{0.166667}$

$= 0.01972 = 0.0197$

(iii) $S = (0.92)^{10-3x}$

$$W = S \times A$$
$$= 0.02(0.92)^{\frac{x}{10}}.(0.92)^{10-3x}$$
$$= 0.02(0.92)^{0.1x+10-3x}$$
$$= 0.02(0.92)^{10-2.9x}$$

(iv) $W < (0.02)(0.92)^{2.5}$

$$\Rightarrow 0.02(0.92)^{10-2.9x} < (0.02)(0.92)^{2.5}$$
$$\Rightarrow \quad 10 - 2.9x < 2.5$$
$$\Rightarrow \quad -2.9x < 2.5 - 10 = -7.5$$
$$\Rightarrow \quad 2.9x > 7.5$$
$$\Rightarrow \quad x > \frac{7.5}{2.9} = 2.586$$
$$\Rightarrow \quad x > 2.59$$

3. (i) $A = (0.83)^n\, I$; $\quad B = (0.66)(0.89)^n\, I$

(ii) $(0.83)^n\, I = (0.66)(0.89)^n\, I$

$$\Rightarrow \quad \log(0.83)^n = \log(0.66)(0.89)^n$$
$$\Rightarrow \quad n\log(0.83) = \log(0.66) + \log(0.89)^n$$
$$\Rightarrow \quad n\log(0.83) = \log(0.66) + n\log(0.89)$$
$$\Rightarrow \quad n[\log(0.83) - \log(0.89)] = \log(0.66)$$
$$\Rightarrow \quad n[-0.0809219 + 0.05061] = -0.180456$$
$$\Rightarrow \quad n(-0.0303119) = -0.180456$$
$$\Rightarrow \quad n = \frac{-0.180456}{-0.0303119} = 5.9533$$
$$\Rightarrow \quad n = 6 \text{ stations}$$

4. (i) $P_g = A\left(1 + \frac{11}{100}\right)^t = A(1.11)^t$

(ii) $P_r = 10A\left(1 - \frac{5}{100}\right)^t = 10A(0.95)^t$

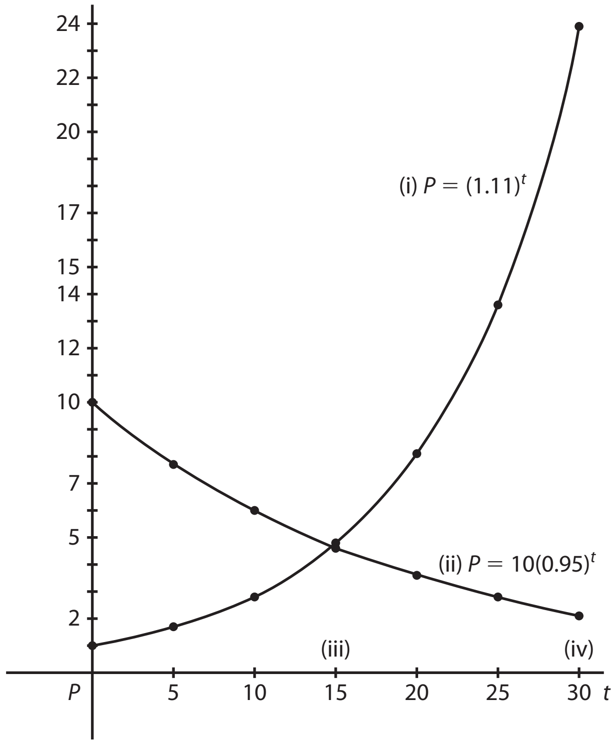
(iii)
$$A(1.11)^t = 10A(0.95)^t$$
$$\Rightarrow \quad \frac{(1.11)^t}{(0.95)^t} = 10$$
$$\Rightarrow \quad \left(\frac{1.11}{0.95}\right)^t = 10$$
$$\Rightarrow \quad (1.168421)^t = 10$$
$$\Rightarrow \quad \log(1.168421)^t = \log 10$$
$$\Rightarrow \quad t\log(1.168421) = 1$$
$$\Rightarrow \quad t = \frac{1}{\log(1.168421)} = 14.793 = 14.8 \text{ years}$$

(iv) $A(1.11)^t = 100A(0.95)^t$ .... [Note: $P_g = 10P_r$ when proportions are reversed.]

$$\Rightarrow \quad \left(\frac{1.11}{0.95}\right)^t = 100$$
$$\Rightarrow \quad (1.168421)^t = 100$$
$$\Rightarrow \quad \log(1.168421)^t = \log 100$$
$$\Rightarrow \quad t\log(1.168421) = 2$$
$$\Rightarrow \quad t = \frac{2}{\log(1.168421)} = 29.586$$
$$\Rightarrow \quad t = 29.6 \text{ years}$$

(v)

$t =$	0	5	10	15	20	25	30
$P = 10(0.95)^t$	10	7.7	6.0	4.6	3.6	2.8	2.1
$P = (1.11)^t$	1	1.7	2.8	4.8	8.1	13.6	23.9



5. $n = A(1 - e^{-bt})$

(i) Growth; as t increases, n also increases.

(ii) $t = 2$ when $n = 10\,000 \Rightarrow 10\,000 = A(1 - e^{-2b})$

$$\Rightarrow \frac{10{,}000}{A} = 1 - e^{-2b}$$

$$\Rightarrow \quad e^{-2b} = 1 - \frac{10\,000}{A}$$

$t = 4$ when $n = 15\,000 \Rightarrow 15\,000 = A(1 - e^{-4b})$

$$\Rightarrow \frac{15\,000}{A} = 1 - e^{-4b}$$

$$\Rightarrow \quad e^{-4b} = 1 - \frac{15\,000}{A}$$

Hence, $2e^{-4b} - 3e^{-2b} + 1$

$$= 2\left(1 - \frac{15\,000}{A}\right) - 3\left(1 - \frac{10\,000}{A}\right) + 1$$

$$= 2 - \frac{30\,000}{A} - 3 + \frac{30\,000}{A} + 1 = 0$$

(iii) $a = e^{-2b} \Rightarrow e^{-4b} = (e^{-2b})^2 = a^2$

Hence, $\quad 2e^{-4b} - 3e^{-2b} + 1 = 0$

becomes $\quad 2a^2 - 3a + 1 = 0$

(iv) Factors: $\quad (a - 1)(2a - 1) = 0$

Roots: $\quad a = 1, a = \frac{1}{2}$

(v) $e^{-2b} = 1$ or $e^{-2b} = \frac{1}{2}$

$\Rightarrow e^{-2b} = e^0 \qquad \Rightarrow \qquad \ln e^{-2b} = \ln \frac{1}{2}$

$\Rightarrow -2b = 0 \qquad \Rightarrow \qquad -2b(\ln e) = \ln 1 - \ln 2$

$\Rightarrow \quad b = 0$ (Not valid) $\qquad \Rightarrow \qquad -2b = -\ln 2$

$\Rightarrow \qquad b = \frac{1}{2} \ln 2$

(vi) $n = A(1 - e^{-t \cdot \frac{1}{2}\ln 2})$

$t = 2$ when $n = 10\,000 \Rightarrow 10\,000 = A(1 - e^{-2 \cdot \frac{1}{2}\ln 2})$

$\Rightarrow 10\,000 = A(1 - e^{-\ln 2})$

$\Rightarrow 10\,000 = A(1 - 0.5)$

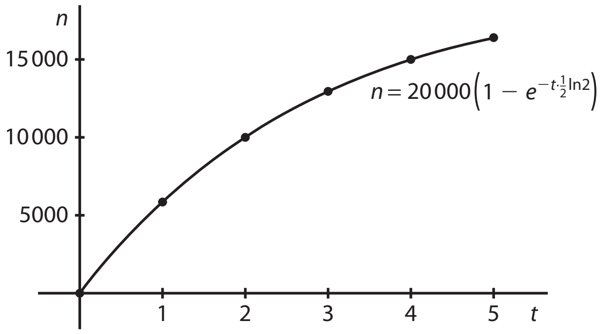
$\Rightarrow 10\,000 = A(0.5)$

$\Rightarrow 20\,000 = A(1)$

$\Rightarrow \quad A = 20\,000$

(vii) $n = 20\,000(1 - e^{-t \cdot \frac{1}{2}\ln 2})$

$t =$	0	1	2	3	4	5
$n =$	0	5858	10 000	12 929	15 000	16 464



(viii) $n = 18\,000 \Rightarrow 20\,000(1 - e^{-t \cdot \frac{1}{2}\ln 2}) = 18\,000$

$\Rightarrow \quad (1 - e^{-t \cdot \frac{1}{2}\ln 2}) = \frac{18\,000}{20\,000} = 0.9$

$\Rightarrow \quad -e^{-t \cdot \frac{1}{2}\ln 2} = 0.9 - 1 = -0.1$

$\Rightarrow \quad e^{-t \cdot \frac{1}{2}\ln 2} = 0.1$

$\Rightarrow \quad \ln e^{-t \cdot \frac{1}{2}\ln 2} = \ln 0.1$

$\Rightarrow \quad -t \cdot \frac{1}{2}\ln 2(\ln e) = -2.3$

$\Rightarrow \quad t = \frac{2.3}{\frac{1}{2}\ln 2}$

$\Rightarrow \quad t = \frac{2.3}{0.34657} = 6.636 = 6.64$ hours