1 Introduction

The Junior Certificate and Leaving Certificate mathematics course committees of the National Council for Curriculum and Assessment (NCCA) accepted the recommendation contained in the paper [4] to base the logical structure of post-primary school geometry on the level 1 account in Professor Barry's book [1].

To quote from [4]: We distinguish three levels:

Level 1: The fully-rigorous level, likely to be intelligible only to professional mathematicians and advanced third- and fourth-level students.

Level 2: The semiformal level, suitable for digestion by many students from (roughly) the age of 14 and upwards.

Level 3: The informal level, suitable for younger children.

This document sets out the agreed geometry for post-primary schools. It has been updated to reflect the changes to the study of geometry introduced in the 2018 Junior Cycle Mathematics specification. Readers should refer to the Junior Cycle Mathematics specification and the Leaving Certificate Mathematics syllabus document for the expected learning outcomes at each level. A summary of the corresponding content is given in sections 9–13 of this document.

2 The system of geometry used for the purposes of formal proofs

In the following, Geometry refers to plane geometry.

There are many formal presentations of geometry in existence, each with its own set of axioms and primitive concepts. What constitutes a valid proof in the context of one system might therefore not be valid in the context of another. Given that students will be expected to present formal proofs in the examinations, it is therefore necessary to specify the system of geometry that is to form the context for such proofs.

The formal underpinning for the system of geometry on the Junior and Leaving Certificate courses is that described by Prof. Patrick D. Barry in [1]. A properly formal presentation of such a system has the serious disadvantage that it is not readily accessible to students at this level. Accordingly, what is presented below is a necessarily simplified version that treats many concepts far more loosely than a truly formal presentation would demand. Any readers who wish to rectify this deficiency are referred to [1] for a proper scholarly treatment of the material.

Barry's system has the primitive undefined terms **plane, point, line, $<_l$ (precedes on a line), (open) half-plane, distance, and degree-measure**, and seven axioms: A_1: about incidence, A_2: about order on lines, A_3: about how lines separate the plane, A_4: about distance, A_5: about degree measure, A_6: about congruence of triangles, A_7: about parallels.

3 Guiding Principles

In constructing a level 2 account, we respect the principles about the relationship between the levels laid down in [4, Section 2].

The choice of material to study should be guided by applications (inside and outside Mathematics proper).

The most important reason to study synthetic geometry is to prepare the ground logically for the development of trigonometry, coordinate geometry, and vectors, which in turn have myriad applications.

We aim to keep the account as simple as possible.

We also take it as desirable that the official Irish syllabus should avoid imposing terminology that is nonstandard in international practice, or is used in a nonstandard way.

No proof should be allowed at level 2 that cannot be expanded to a complete rigorous proof at level 1, or that uses axioms or theorems that come later in the logical sequence. We aim to supply adequate proofs for all the theorems, but do not propose that only those proofs will be acceptable. It should be open to teachers and students to think about other ways to prove the results, provided they are correct and fit within the logical framework. Indeed, such activity is to be encouraged. Naturally, teachers and students will need some assurance that such variant proofs will be acceptable if presented in examination. We suggest that the discoverer of a new proof should discuss it with students and colleagues, and (if in any doubt) should refer it to the National Council for Curriculum and Assessment and/or the State Examinations Commission.

It may be helpful to note the following non-exhaustive list of salient differences between Barry's treatment and our less formal presentation.

- Whereas we may use set notation and we expect students to understand the conceptualisation of geometry in terms of sets, we more often use the language that is common when discussing geometry informally, such as "the point is/lies on the line", "the line passes through the point", etc.

- We accept and use a much lesser degree of precision in language and notation (as is apparent from some of the other items on this list).

- We state five explicit axioms, employing more informal language than Barry's, and we do not explicitly state axioms corresponding to Axioms A2 and A3 – instead we make statements without fanfare in the text.

- We accept a much looser understanding of what constitutes an **angle**, making no reference to angle-supports. We do not define the term angle. We mention reflex angles from the beginning (but make no use of them until we come to angles in circles), and quietly assume (when the time comes) that axioms that are presented by Barry in the context of wedge-angles apply also in the naturally corresponding way to reflex angles.

- When naming an angle, it is always assumed that the non-reflex angle is being referred to, unless the word "reflex" precedes or follows.

- We make no reference to results such as Pasch's property and the "crossbar theorem". (That is, we do not expect students to consider the necessity to prove such results or to have them given as axioms.)

- We refer to "the number of degrees" in an angle, whereas Barry treats this more correctly as "the degree-measure" of an angle.

- We take it that the definitions of parallelism, perpendicularity and "sidedness" are readily extended from lines to half-lines and line segments. (Hence, for example, we may refer to the opposite sides of a particular quadrilateral as being parallel, meaning that the lines of which they are subsets are parallel).

- We do not refer explicitly to triangles being **congruent** "under the correspondence $(A, B, C) \to (D, E, F)$", taking it instead that the correspondence is the one implied by the order in which the vertices are listed. That is, when we say "ΔABC is congruent to ΔDEF" we mean, using Barry's terminology, "Triangle [A,B,C] is congruent to triangle [D,E,F] under the correspondence $(A, B, C) \to (D, E, F)$".

- We do not always retain the distinction in language between an angle and its measure, relying frequently instead on the context to make the meaning clear. However, we continue the practice of distinguishing notationally between the angle $\angle ABC$ and the number $|\angle ABC|$ of degrees in the angle[1]. In the same spirit, we may refer to two angles being equal, or one being equal to the sum of two others, (when we should more precisely say that the two are equal in measure, or that the measure of one is equal to the sum of the measures of the other two). Similarly, with length, we may loosely say, for example: "opposite sides of a parallelogram are equal", or refer to "a circle of radius r". Where ambiguity does not arise, we may refer to angles using a single letter. That is, for example, if a diagram includes only two rays or segments from the point A, then the angle concerned may be referred to as $\angle A$.

Having pointed out these differences, it is perhaps worth mentioning some significant structural aspects of Barry's geometry that are retained in our less formal version:

[1] In practice, the examiners do not penalise students who leave out the bars.

- The primitive terms are almost the same, subject to the fact that their properties are conceived less formally. We treat **angle** as an extra undefined term.

- We assume that results are established in the same order as in Barry [1], up to minor local rearrangement. The exception to this is that we state all the axioms as soon as they are useful, and we bring the theorem on the angle-sum in a triangle forward to the earliest possible point (short of making it an axiom). This simplifies the proofs of a few theorems, at the expense of making it easy to see which results are theorems of so-called Neutral Geometry[2].

- **Area** is not taken to be a primitive term or a given property of regions. Rather, it is defined for triangles following the establishment of the requisite result that the products of the lengths of the sides of a triangle with their corresponding altitudes are equal, and then extended to convex quadrilaterals.

- **Isometries or other transformations** are not taken as primitive. Indeed, in our case, the treatment does not extend as far as defining them. Thus they can play no role in our proofs.

4 Outline of the Level 2 Account

We present the account by outlining:

1. A list (Section 5), of the terminology for the geometrical concepts. Each term in a theory is either undefined or defined, or at least definable. There have to be some undefined terms. (In textbooks, the undefined terms will be introduced by descriptions, and some of the defined terms will be given explicit definitions, in language appropriate to the level. We assume that previous level 3 work will have laid a foundation that will allow students to understand the undefined terms. We do not give the explicit definitions of all the definable terms. Instead we rely on the student's ordinary language, supplemented sometimes by informal remarks. For instance, we do not write out in cold blood the definition of the **side opposite** a given angle in a triangle, or the

[2] Geometry without the axiom of parallels. This is not a concern in secondary school.

definition (in terms of set membership) of what it means to say that a line **passes through** a given point. The reason why some terms **must** be given explicit definitions is that there are alternatives, and the definition specifies the starting point; the alternative descriptions of the term are then obtained as theorems.

2. A logical account (Section 6) of the synthetic geometry theory. All the material through to LC higher is presented. The individual syllabuses will identify the relevant content by referencing it by number (e.g. Theorems 1,2, 9).

3. The geometrical constructions (Section 7) that will be studied. Again, the individual syllabuses will refer to the items on this list by number when specifying what is to be studied.

4. Some guidance on teaching (Section 8).

5. Syllabus content summaries for each of JC-OL, JC-HL, LC-FL, LC-OL, LC-HL.

5 Terms

Undefined Terms: angle, degree, length, line, plane, point, ray, real number, set.

Most important Defined Terms: area, parallel lines, parallelogram, right angle, triangle, congruent triangles, similar triangles, tangent to a circle, area.

Other Defined terms: acute angle, alternate angles, angle bisector, arc, area of a disc, base and corresponding apex and height of triangle or parallelogram, chord, circle, circumcentre, circumcircle, circumference of a circle, circumradius, collinear points, concurrent lines, convex quadrilateral, corresponding angles, diameter, disc, distance, equilateral triangle, exterior angles of a triangle, full angle, hypotenuse, incentre, incircle, inradius, interior opposite angles, isosceles triangle, median lines, midpoint of a segment, null angle, obtuse angle, perpendicular bisector of a segment, perpendicular lines, point of contact of a tangent, polygon, quadrilateral, radius, ratio, rectangle, reflex angle ordinary angle, rhombus, right-angled triangle, scalene triangle, sector, segment, square, straight angle, subset, supplementary angles, transversal line, vertically-opposite angles.

Definable terms used without explicit definition: angles, adjacent sides, arms or sides of an angle, centre of a circle, endpoints of segment, equal angles, equal segments, line passes through point, opposite sides or angles of a quadrilateral, or vertices of triangles or quadrilaterals, point lies on line, side of a line, side of a polygon, the side opposite an angle of a triangle, vertex, vertices (of angle, triangle, polygon).

6 The Theory

Line[3] is short for straight line. Take a fixed **plane**[4], once and for all, and consider just lines that lie in it. The plane and the lines are **sets**[5] of **points**[6]. Each line is a **subset** of the plane, i.e. each element of a line is a point of the plane. Each line is endless, extending forever in both directions. Each line has infinitely-many points. The points on a line can be taken to be ordered along the line in a natural way. As a consequence, given any three distinct points on a line, exactly one of them lies **between** the other two. Points that are not on a given line can be said to be on one or other **side** of the line. The sides of a line are sometimes referred to as **half-planes**.

Notation 1. We denote points by roman capital letters A, B, C, etc., and lines by lower-case roman letters l, m, n, etc.

Axioms are statements we will accept as true[7].

Axiom 1 (Two Points Axiom)**.** *There is exactly one line through any two given points. (We denote the line through A and B by AB.)*

Definition 1. The line **segment** $[AB]$ is the part of the line AB between A and B (including the endpoints). The point A divides the line AB into two

[3] Line is undefined.
[4] Undefined term
[5] Undefined term
[6] Undefined term
[7] An **axiom** is a statement accepted without proof, as a basis for argument. A **theorem** is a statement deduced from the axioms by logical argument.

pieces, called **rays**. The point A lies between all points of one ray and all points of the other. We denote the ray that starts at A and passes through B by $[AB$. Rays are sometimes referred to as **half-lines**.

Three points usually determine three different lines.

Definition 2. If three or more points lie on a single line, we say they are **collinear**.

Definition 3. Let A, B and C be points that are not collinear. The **triangle** ΔABC is the piece of the plane enclosed by the three line segments $[AB]$, $[BC]$ and $[CA]$. The segments are called its **sides**, and the points are called its **vertices** (singular **vertex**).

6.1 Length and Distance

We denote the set of all **real numbers**[8] by $\mathbb{R}$.

Definition 4. We denote the **distance**[9] between the points A and B by $|AB|$. We define the **length** of the segment $[AB]$ to be $|AB|$.

We often denote the lengths of the three sides of a triangle by a, b, and c. The usual thing for a triangle ΔABC is to take $a = |BC|$, i.e. the length of the side opposite the vertex A, and similarly $b = |CA|$ and $c = |AB|$.

Axiom 2 (Ruler Axiom[10]). *The distance between points has the following properties:*

1. *the distance $|AB|$ is never negative;*
2. *$|AB| = |BA|$;*
3. *if C lies on AB, between A and B, then $|AB| = |AC| + |CB|$;*
4. *(marking off a distance) given any ray from A, and given any real number $k \geq 0$, there is a unique point B on the ray whose distance from A is k.*

[8] Undefined term
[9] Undefined term
[10] Teachers used to traditional treatments that follow Euclid closely should note that this axiom (and the later Protractor Axiom) guarantees the existence of various points (and lines) without appeal to postulates about constructions using straight-edge and compass. They are powerful axioms.

Definition 5. The **midpoint** of the segment $[AB]$ is the point M of the segment with [11]

$$|AM| = |MB| = \frac{|AB|}{2}.$$

6.2 Angles

Definition 6. A subset of the plane is **convex** if it contains the whole segment that connects any two of its points.

For example, one side of any line is a convex set, and triangles are convex sets.

We do not define the term angle formally. Instead we say: There are things called **angles**. To each angle is associated:

1. a unique point A, called its **vertex**;
2. two rays $[AB$ and $[AC$, both starting at the vertex, and called the **arms** of the angle;
3. a piece of the plane called the **inside** of the angle.

An angle is either a null angle, an ordinary angle, a straight angle, a reflex angle or a full angle, Unless otherwise specified, you may take it that any angle we talk about is an ordinary angle.

Definition 7. An angle is a **null angle** if its arms coincide with one another and its inside is the empty set.

Definition 8. An angle is an **ordinary angle** if its arms are not on one line, and its inside is a convex set.

Definition 9. An angle is a **straight angle** if its arms are the two halves of one line, and its inside is one of the sides of that line.

Definition 10. An angle is a **reflex angle** if its arms are not on one line, and its inside is not a convex set.

Definition 11. An angle is a **full angle** if its arms coincide with one another and its inside is the rest of the plane.

[11] Students may notice that the first equality implies the second.

Definition 12. Suppose that A, B, and C are three noncollinear points. We denote the (ordinary) angle with arms $[AB$ and $[AC$ by $\angle BAC$ (and also by $\angle CAB$). We shall also use the notation $\angle BAC$ to refer to straight angles, where A, B, C are collinear, and A lies between B and C (either side could be the inside of this angle).

Sometimes we want to refer to an angle without naming points, and in that case we use lower-case Greek letters, α,β,γ, etc.

6.3 Degrees

Notation 2. We denote the number of **degrees** in an angle $\angle BAC$ or α by the symbol $|\angle BAC|$, or $|\angle\alpha|$, as the case may be.

Axiom 3 (Protractor Axiom)**.** *The number of degrees in an angle (also known as its degree-measure) is always a number between* $0°$ *and* $360°$*. The number of degrees of an ordinary angle is less than* $180°$*. It has these properties:*

1. *A straight angle has* $180°$*.*
2. *Given a ray* $[AB$*, and a number* d *between* 0 *and* 180*, there is exactly one ray from* A *on each side of the line* AB *that makes an (ordinary) angle having* d *degrees with the ray* $[AB$*.*
3. *If* D *is a point inside an angle* $\angle BAC$*, then*

$$|\angle BAC| = |\angle BAD| + |\angle DAC|.$$

Null angles are assigned $0°$, full angles $360°$, and reflex angles have more than $180°$. To be more exact, if A, B, and C are noncollinear points, then the reflex angle "outside" the angle $\angle BAC$ measures $360° - |\angle BAC|$, in degrees.

Definition 13. The ray $[AD$ is the **bisector** of the angle $\angle BAC$ if

$$|\angle BAD| = |\angle DAC| = \frac{|\angle BAC|}{2}.$$

We say that an angle is 'an angle of' (for instance) $45°$, if it has 45 degrees in it.

Definition 14. A **right angle** is an angle of exactly $90°$.

Definition 15. An angle is **acute** if it has less than $90°$, and **obtuse** if it has more than $90°$.

Definition 16. If $\angle BAC$ is a straight angle, and D is off the line BC, then $\angle BAD$ and $\angle DAC$ are called **supplementary angles**. They add to $180°$.

Definition 17. When two lines AB and AC cross at a point A, they are **perpendicular** if $\angle BAC$ is a right angle.

Definition 18. Let A lie between B and C on the line BC, and also between D and E on the line DE. Then $\angle BAD$ and $\angle CAE$ are called **vertically-opposite angles**.

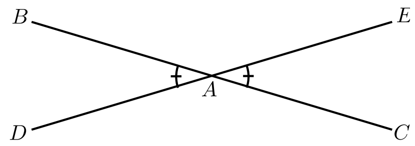

Figure 1.

Theorem 1 (Vertically-opposite Angles)**.**
Vertically opposite angles are equal in measure.

Proof. See Figure 1. The idea is to add the same supplementary angles to both, getting $180°$. In detail,

$$\begin{aligned} |\angle BAD| + |\angle BAE| &= 180°, \\ |\angle CAE| + |\angle BAE| &= 180°, \end{aligned}$$

so subtracting gives:

$$\begin{aligned} |\angle BAD| - |\angle CAE| &= 0°, \\ |\angle BAD| &= |\angle CAE|. \end{aligned}$$

□

6.4 Congruent Triangles

Definition 19. Let A, B, C and A', B', C' be triples of non-collinear points. We say that the triangles ΔABC and $\Delta A'B'C'$ are **congruent** if all the sides and angles of one are equal to the corresponding sides and angles of the other, i.e. $|AB| = |A'B'|$, $|BC| = |B'C'|$, $|CA| = |C'A'|$, $|\angle ABC| = |\angle A'B'C'|$, $|\angle BCA| = |\angle B'C'A'|$, and $|\angle CAB| = |\angle C'A'B'|$. See Figure 2.

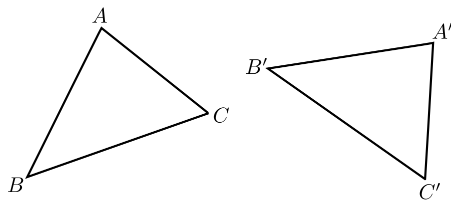

Figure 2.

Notation 3. Usually, we abbreviate the names of the angles in a triangle, by labelling them by the names of the vertices. For instance, we write $\angle A$ for $\angle CAB$.

Axiom 4 (SAS+ASA+SSS[12])**.**
If (1) $|AB| = |A'B'|$, $|AC| = |A'C'|$ *and* $|\angle A| = |\angle A'|$,
or
(2) $|BC| = |B'C'|$, $|\angle B| = |\angle B'|$, *and* $|\angle C| = |\angle C'|$,
or
(3) $|AB| = |A'B'|$, $|BC| = |B'C'|$, *and* $|CA| = |C'A'|$
then the triangles ΔABC *and* $\Delta A'B'C'$ *are congruent.*

Definition 20. A triangle is called **right-angled** if one of its angles is a right angle. The other two angles then add to $90°$, by Theorem 4, so are both acute angles. The side opposite the right angle is called the **hypotenuse**.

Definition 21. A triangle is called **isosceles** if two sides are equal[13]. It is **equilateral** if all three sides are equal. It is **scalene** if no two sides are equal.

Theorem 2 (Isosceles Triangles)**.**
(1) In an isosceles triangle the angles opposite the equal sides are equal.
(2) Conversely, If two angles are equal, then the triangle is isosceles.

Proof. (1) Suppose the triangle ΔABC has $AB = AC$ (as in Figure 3). Then ΔABC is congruent to ΔACB [SAS]
$\therefore \angle B = \angle C$.

[12]It would be possible to prove all the theorems using a weaker axiom (just SAS). We use this stronger version to shorten the course.
[13] The simple "equal" is preferred to "of equal length"

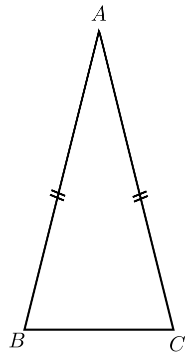

Figure 3.

(2) Suppose now that $\angle B = \angle C$. Then
ΔABC is congruent to ΔACB [ASA]
$\therefore |AB| = |AC|, \Delta ABC$ is isosceles. $\square$

Acceptable Alternative Proof of (1). Let D be the midpoint of $[BC]$, and use SSS to show that the triangles ΔABD and ΔACD are congruent. (This proof is more complicated, but has the advantage that it yields the extra information that the angles $\angle ADB$ and $\angle ADC$ are equal, and hence both are right angles (since they add to a straight angle)). $\square$

6.5 Parallels

Definition 22. Two lines l and m are **parallel** if they are either identical, or have no common point.

Notation 4. We write $l||m$ for "l is parallel to m".

Axiom 5 (Axiom of Parallels)**.** *Given any line l and a point P, there is exactly one line through P that is parallel to l.*

Definition 23. If l and m are lines, then a line n is called a **transversal** of l and m if it meets them both.

Definition 24. Given two lines AB and CD and a transversal BC of them, as in Figure 4, the angles $\angle ABC$ and $\angle BCD$ are called **alternate** angles.

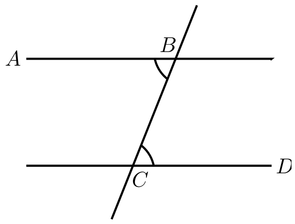

Figure 4.

Theorem 3 (Alternate Angles)**.** *Suppose that A and D are on opposite sides of the line BC.*
(1) If $|\angle ABC| = |\angle BCD|$, then $AB||CD$. In other words, if a transversal makes equal alternate angles on two lines, then the lines are parallel.
(2) Conversely, if $AB||CD$, then $|\angle ABC| = |\angle BCD|$. In other words, if two lines are parallel, then any transversal will make equal alternate angles with them.

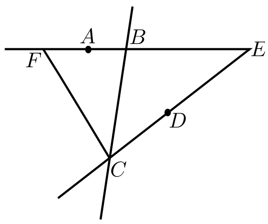

Figure 5.

Proof. (1) Suppose $|\angle ABC| = |\angle BCD|$. If the lines AB and CD do not meet, then they are parallel, by definition, and we are done. Otherwise, they meet at some point, say E. Let us assume that E is on the same side of BC as D.[14] Take F on EB, on the same side of BC as A, with $|BF| = |CE|$ (see Figure 5). [Ruler Axiom]

[14]Fuller detail: There are three cases:
1°: E lies on BC. Then (using Axiom 1) we must have $E = B = C$, and $AB = CD$.
2°: E lies on the same side of BC as D. In that case, take F on EB, on the same side of BC as A, with $|BF| = |CE|$. [Ruler Axiom]
Then ΔBCE is congruent to ΔCBF. [SAS]
Thus

$$|\angle BCF| = |\angle CBE| = 180^\circ - |\angle ABC| = 180^\circ - |\angle BCD|,$$

Then ΔBCE is congruent to ΔCBF. [SAS]
Thus

$$|\angle BCF| = |\angle CBE| = 180^\circ - |\angle ABC| = 180^\circ - |\angle BCD|,$$

so that F lies on DC. [Ruler Axiom]
Thus AB and CD both pass through E and F, and hence coincide, [Axiom 1]
Hence AB and CD are parallel. [Definition of parallel]

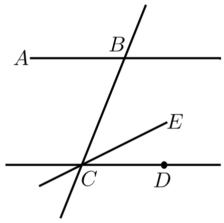

Figure 6.

(2) To prove the converse, suppose $AB||CD$. Pick a point E on the same side of BC as D with $|\angle BCE| = |\angle ABC|$. (See Figure 6.) By Part (1), the line CE is parallel to AB. By Axiom 5, there is only one line through C parallel to AB, so $CE = CD$. Thus $|\angle BCD| = |\angle BCE| = |\angle ABC|$. □

Theorem 4 (Angle Sum 180)**.** *The angles in any triangle add to* 180°*.*

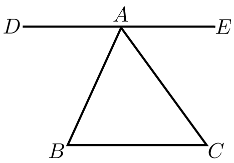

Figure 7.

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so that F lies on DC. [Protractor Axiom]
Thus AB and CD both pass through E and F, and hence coincide. [Axiom 1]
3°: E lies on the same side of BC as A. Similar to the previous case.
Thus, in all three cases, $AB = CD$, so the lines are parallel.

Proof. Let ΔABC be given. Take a segment $[DE]$ passing through A, parallel to BC, with D on the opposite side of AB from C, and E on the opposite side of AC from B (as in Figure 7). [Axiom of Parallels]
Then AB is a transversal of DE and BC, so by the Alternate Angles Theorem,

$$|\angle ABC| = |\angle DAB|.$$

Similarly, AC is a transversal of DE and BC, so

$$|\angle ACB| = |\angle CAE|.$$

Thus, using the Protractor Axiom to add the angles,

$$\begin{aligned} & |\angle ABC| + |\angle ACB| + |\angle BAC| \\ = & |\angle DAB| + |\angle CAE| + |\angle BAC| \\ = & |\angle DAE| = 180^\circ, \end{aligned}$$

since $\angle DAE$ is a straight angle. □

Definition 25. Given two lines AB and CD, and a transversal AE of them, as in Figure 8(a), the angles $\angle EAB$ and $\angle ACD$ are called **corresponding** angles[15].

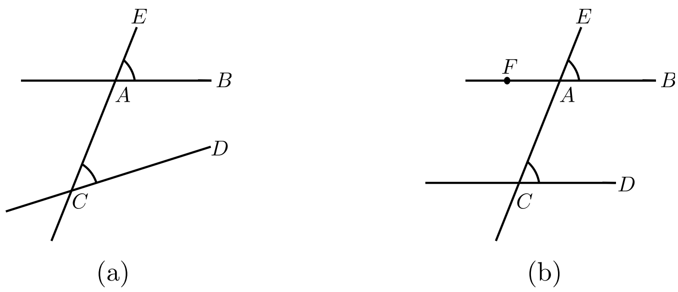

Figure 8.

Theorem 5 (Corresponding Angles)**.** *Two lines are parallel if and only if for any transversal, corresponding angles are equal.*

[15]with respect to the two lines and the given transversal.

Proof. See Figure 8(b). We first assume that the corresponding angles $\angle EAB$ and $\angle ACD$ are equal. Let F be a point on AB such that F and B are on opposite sides of AE. Then we have

$|\angle EAB| = |\angle FAC|$ [Vertically opposite angles]

Hence the alternate angles $\angle FAC$ and $\angle ACD$ are equal and therefore the lines $FA = AB$ and CD are parallel.

For the converse, let us assume that the lines AB and CD are parallel. Then the alternate angles $\angle FAC$ and $\angle ACD$ are equal. Since

$|\angle EAB| = |\angle FAC|$ [Vertically opposite angles]

we have that the corresponding angles $\angle EAB$ and $\angle ACD$ are equal. □

Definition 26. In Figure 9, the angle α is called an **exterior angle** of the triangle, and the angles β and γ are called (corresponding) **interior opposite angles**.[16]

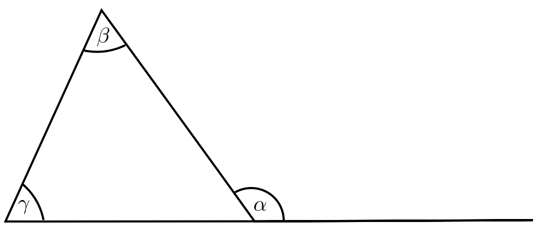

Figure 9.

Theorem 6 (Exterior Angle)**.** *Each exterior angle of a triangle is equal to the sum of the interior opposite angles.*

Proof. See Figure 10. In the triangle ΔABC let α be an exterior angle at A. Then

$|\alpha| + |\angle A| = 180°$ [Supplementary angles]

and

$|\angle B| + |\angle C| + |\angle A| = 180°$. [Angle sum 180°]

Subtracting the two equations yields $|\alpha| = |\angle B| + |\angle C|$. □

[16] The phrase **interior remote angles** is sometimes used instead of **interior opposite angles**.

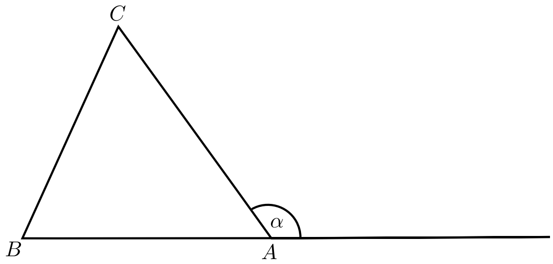

Figure 10.

Theorem 7.
(1) In ΔABC, suppose that $|AC| > |AB|$. Then $|\angle ABC| > |\angle ACB|$. In other words, the angle opposite the greater of two sides is greater than the angle opposite the lesser side.
(2) Conversely, if $|\angle ABC| > |\angle ACB|$, then $|AC| > |AB|$. In other words, the side opposite the greater of two angles is greater than the side opposite the lesser angle.

Proof.
(1) Suppose that $|AC| > |AB|$. Then take the point D on the segment $[AC]$ with
$|AD| = |AB|$. [Ruler Axiom]

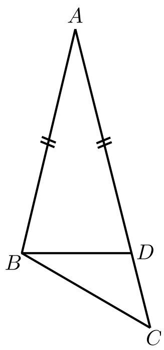

Figure 11.

See Figure 11. Then ΔABD is isosceles, so

$$\begin{array}{rcll} |\angle ACB| & < & |\angle ADB| & \text{[Exterior Angle]} \\ & = & |\angle ABD| & \text{[Isosceles Triangle]} \\ & < & |\angle ABC|. & \end{array}$$

Thus $|\angle ACB| < |\angle ABC|$, as required.

(2)(This is a Proof by Contradiction!)
Suppose that $|\angle ABC| > |\angle ACB|$. See Figure 12.

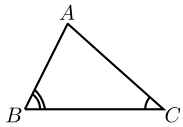

Figure 12.

If it could happen that $|AC| \leq |AB|$, then
either Case 1°: $|AC| = |AB|$, in which case ΔABC is isosceles, and then $|\angle ABC| = |\angle ACB|$, which contradicts our assumption,
or Case 2°: $|AC| < |AB|$, in which case Part (1) tells us that $|\angle ABC| < |\angle ACB|$, which also contradicts our assumption. Thus it cannot happen, and we conclude that $|AC| > |AB|$. □

Theorem 8 (Triangle Inequality)**.**
Two sides of a triangle are together greater than the third.

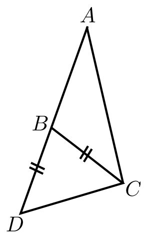

Figure 13.

Proof. Let ΔABC be an arbitrary triangle. We choose the point D on AB such that B lies in $[AD]$ and $|BD| = |BC|$ (as in Figure 13). In particular

$$|AD| = |AB| + |BD| = |AB| + |BC|.$$

Since B lies in the angle $\angle ACD$[17] we have

$$|\angle BCD| < |\angle ACD|.$$

[17] B lies in a segment whose endpoints are on the arms of $\angle ACD$. Since this angle is $< 180°$ its inside is convex.

Because of $|BD| = |BC|$ and the Theorem about Isosceles Triangles we have $|\angle BCD| = |\angle BDC|$, hence $|\angle ADC| = |\angle BDC| < |\angle ACD|$. By the previous theorem applied to ΔADC we have

$$|AC| < |AD| = |AB| + |BC|.$$

□

6.6 Perpendicular Lines

Proposition 1. [18] *Two lines perpendicular to the same line are parallel to one another.*

Proof. This is a special case of the Alternate Angles Theorem. □

Proposition 2. *There is a unique line perpendicular to a given line and passing though a given point. This applies to a point on or off the line.*

Definition 27. The **perpendicular bisector** of a segment $[AB]$ is the line through the midpoint of $[AB]$, perpendicular to AB.

6.7 Quadrilaterals and Parallelograms

Definition 28. A closed chain of line segments laid end-to-end, not crossing anywhere, and not making a straight angle at any endpoint encloses a piece of the plane called a **polygon**. The segments are called the **sides** or edges of the polygon, and the endpoints where they meet are called its **vertices**. Sides that meet are called **adjacent sides**, and the ends of a side are called **adjacent vertices**. The angles at adjacent vertices are called **adjacent angles**. A polygon is called **convex** if it contains the whole segment connecting any two of its points.

Definition 29. A **quadrilateral** is a polygon with four vertices.

Two sides of a quadrilateral that are not adjacent are called **opposite sides**. Similarly, two angles of a quadrilateral that are not adjacent are called **opposite angles**.

[18] In this document, a proposition is a useful or interesting statement that could be proved at this point, but whose proof is not stipulated as an essential part of the programme. Teachers are free to deal with them as they see fit. For instance, they might be just mentioned, or discussed without formal proof, or used to give practice in reasoning for HLC students. It is desirable that they be mentioned, at least.

Definition 30. A **rectangle** is a quadrilateral having right angles at all four vertices.

Definition 31. A **rhombus** is a quadrilateral having all four sides equal.

Definition 32. A **square** is a rectangular rhombus.

Definition 33. A polygon is **equilateral** if all its sides are equal, and **regular** if all its sides and angles are equal.

Definition 34. A **parallelogram** is a quadrilateral for which both pairs of opposite sides are parallel.

Proposition 3. *Each rectangle is a parallelogram.*

Theorem 9. *In a parallelogram, opposite sides are equal, and opposite angles are equal.*

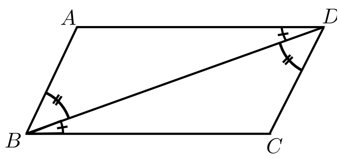

Figure 14.

Proof. See Figure 14. Idea: Use Alternate Angle Theorem, then ASA to show that a diagonal divides the parallelogram into two congruent triangles. This gives opposite sides and (one pair of) opposite angles equal.

In more detail, let $ABCD$ be a given parallelogram, $AB||CD$ and $AD||BC$. Then

$|\angle ABD| = |\angle BDC|$ [Alternate Angle Theorem]
$|\angle ADB| = |\angle DBC|$ [Alternate Angle Theorem]
ΔDAB is congruent to ΔBCD. [ASA]

$$\therefore |AB| = |CD|, \; |AD| = |CB|, \text{ and } |\angle DAB| = |\angle BCD|.$$

□

Remark 1. Sometimes it happens that the converse of a true statement is false. For example, it is true that if a quadrilateral is a rhombus, then its diagonals are perpendicular. But it is not true that a quadrilateral whose diagonals are perpendicular is always a rhombus.

It may also happen that a statement admits several valid converses. Theorem 9 has two:

Converse 1 to Theorem 9: *If the opposite angles of a convex quadrilateral are equal, then it is a parallelogram.*

Proof. First, one deduces from Theorem 4 that the angle sum in the quadrilateral is 360°. It follows that adjacent angles add to 180°. Theorem 3 then yields the result. □

Converse 2 to Theorem 9: *If the opposite sides of a convex quadrilateral are equal, then it is a parallelogram.*

Proof. Drawing a diagonal, and using SSS, one sees that opposite angles are equal. □

Corollary 1. *A diagonal divides a parallelogram into two congruent triangles.*

Remark 2. The converse is false: It may happen that a diagonal divides a convex quadrilateral into two congruent triangles, even though the quadrilateral is not a parallelogram.

Proposition 4. *A quadrilateral in which one pair of opposite sides is equal and parallel, is a parallelogram.*

Proposition 5. *Each rhombus is a parallelogram.*

Theorem 10. *The diagonals of a parallelogram bisect one another.*

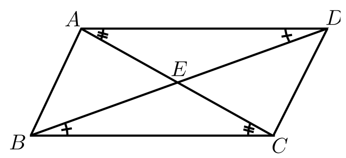

Figure 15.

Proof. See Figure 15. Idea: Use Alternate Angles and ASA to establish congruence of ΔADE and ΔCBE.

In detail: Let AC cut BD in E. Then

$$\begin{array}{rcll} |\angle EAD| & = & |\angle ECB| \text{ and} & \\ |\angle EDA| & = & |\angle EBC| & \text{[Alternate Angle Theorem]} \\ |AD| & = & |BC|. & \text{[Theorem 9]} \end{array}$$

$\therefore \Delta ADE$ is congruent to ΔCBE. [ASA] □

Proposition 6 (Converse)**.** *If the diagonals of a quadrilateral bisect one another, then the quadrilateral is a parallelogram.*

Proof. Use SAS and Vertically Opposite Angles to establish congruence of ΔABE and ΔCDE. Then use Alternate Angles. □

6.8 Ratios and Similarity

Definition 35. If the three angles of one triangle are equal, respectively, to those of another, then the two triangles are said to be **similar**.

Remark 3. Obviously, two right-angled triangles are similar if they have a common angle other than the right angle.

(The angles sum to 180°, so the third angles must agree as well.)

Theorem 11. *If three parallel lines cut off equal segments on some transversal line, then they will cut off equal segments on any other transversal.*

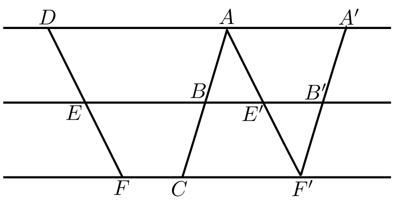

Figure 16.

Proof. Uses opposite sides of a parallelogram, AAS, Axiom of Parallels.

In more detail, suppose $AD||BE||CF$ and $|AB| = |BC|$. We wish to show that $|DE| = |EF|$.

Draw $AE'||DE$, cutting EB at E' and CF at F'.
Draw $F'B'||AB$, cutting EB at B'. See Figure 16.

Then

$\|B'F'\|$	$=$	$\|BC\|$	[Theorem 9]
	$=$	$\|AB\|$.	[by Assumption]
$\|\angle BAE'\|$	$=$	$\|\angle E'F'B'\|$.	[Alternate Angle Theorem]
$\|\angle AE'B\|$	$=$	$\|\angle F'E'B'\|$.	[Vertically Opposite Angles]
$\therefore \Delta ABE'$	is congruent to	$\Delta F'B'E'$.	[ASA]
$\therefore \|AE'\|$	$=$	$\|F'E'\|$.	

But
$|AE'| = |DE|$ and $|F'E'| = |FE|$. [Theorem 9]
$\therefore |DE| = |EF|$. □

Definition 36. Let s and t be positive real numbers. We say that a point C **divides the segment** $[AB]$ **in the ratio** $s : t$ if C lies on the line AB, and is between A and B, and

$$\frac{|AC|}{|CB|} = \frac{s}{t}.$$

We say that a line l **cuts** $[AB]$ **in the ratio** $s : t$ if it meets AB at a point C that divides $[AB]$ in the ratio $s : t$.

Remark 4. It follows from the Ruler Axiom that given two points A and B, and a ratio $s : t$, there is exactly one point that divides the segment $[AB]$ in that exact ratio.

Theorem 12. *Let ΔABC be a triangle. If a line l is parallel to BC and cuts $[AB]$ in the ratio $s : t$, then it also cuts $[AC]$ in the same ratio.*

Proof. We prove only the commensurable case.

Let l cut $[AB]$ in D in the ratio $m : n$ with natural numbers m, n. Thus there are points (Figure 17)

$$D_0 = A, D_1, D_2, \ldots, D_{m-1}, D_m = D, D_{m+1}, \ldots, D_{m+n-1}, D_{m+n} = B,$$

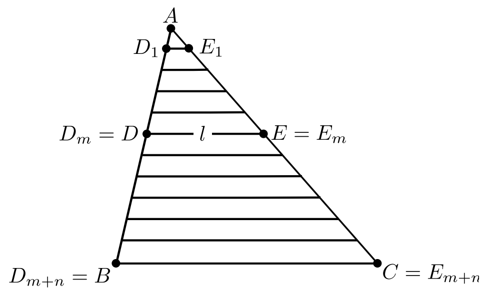

Figure 17.

equally spaced along $[AB]$, i.e. the segments

$$[D_0D_1], [D_1D_2], \ldots [D_iD_{i+1}], \ldots [D_{m+n-1}D_{m+n}]$$

have equal length.

Draw lines $D_1E_1, D_2E_2, \ldots$ parallel to BC with $E_1, E_2, \ldots$ on $[AC]$.

Then all the segments

$$[AE_1], [E_1E_2], [E_2E_3], \ldots, [E_{m+n-1}C]$$

have the same length, [Theorem 11]
and $E_m = E$ is the point where l cuts $[AC]$. [Axiom of Parallels]
Hence E divides $[AC]$ in the ratio $m : n$. □

Proposition 7. *If two triangles ΔABC and $\Delta A'B'C'$ have*

$$|\angle A| = |\angle A'|, \ \textit{and} \ \frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|},$$

then they are similar.

Proof. Suppose $|A'B'| \leq |AB|$. If equal, use SAS. Otherwise, note that then $|A'B'| < |AB|$ and $|A'C'| < |AC|$. Pick B'' on $[AB$ and C'' on $[AC$ with $|A'B'| = |AB''|$ and $|A'C'| = |AC''|$. [Ruler Axiom] Then by SAS, $\Delta A'B'C'$ is congruent to $\Delta AB''C''$.

Draw $[B''D$ parallel to BC [Axiom of Parallels], and let it cut AC at D. Now the last theorem and the hypothesis tell us that D and C'' divide $[AC]$ in the same ratio, and hence $D = C''$.

Thus

$$\begin{aligned} |\angle B| &= |\angle AB''C''| \text{ [Corresponding Angles]} \\ &= |\angle B'|, \end{aligned}$$

and
$$|\angle C| = |\angle AC''B''| = |\angle C'|,$$
so ΔABC is similar to $\Delta A'B'C'$. [Definition of similar]

□

Remark 5. The **Converse to Theorem 12** is true:
Let ΔABC be a triangle. If a line l cuts the sides AB and AC in the same ratio, then it is parallel to BC.

Proof. This is immediate from Proposition 7 and Theorem 5. □

Theorem 13. *If two triangles ΔABC and $\Delta A'B'C'$ are similar, then their sides are proportional, in order:*
$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|}.$$

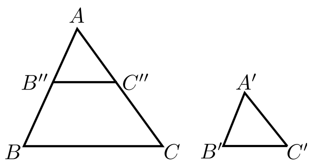

Figure 18.

Proof. We may suppose $|A'B'| \leq |AB|$. Pick B'' on $[AB]$ with $|AB''| = |A'B'|$, and C'' on $[AC]$ with $|AC''| = |A'C'|$. Refer to Figure 18. Then

$\Delta AB''C''$	is congruent to	$\Delta A'B'C'$	[SAS]
$\therefore \|\angle AB''C''\|$	$=$	$\|\angle ABC\|$	
$\therefore B''C''$	$\|\|$	BC	[Corresponding Angles]
$\therefore \frac{\|A'B'\|}{\|A'C'\|}$	$=$	$\frac{\|AB''\|}{\|AC''\|}$	[Choice of B'', C'']
	$=$	$\frac{\|AB\|}{\|AC\|}$	[Theorem 12]
$\frac{\|AC\|}{\|A'C'\|}$	$=$	$\frac{\|AB\|}{\|A'B'\|}$	[Re-arrange]

Similarly, $\dfrac{|BC|}{|B'C'|} = \dfrac{|AB|}{|A'B'|}$ □

Proposition 8 (Converse)**.** *If*

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|},$$

then the two triangles ΔABC and $\Delta A'B'C'$ are similar.

Proof. Refer to Figure 18. If $|A'B'| = |AB|$, then by SSS the two triangles are congruent, and therefore similar. Otherwise, assuming $|A'B'| < |AB|$, choose B'' on AB and C'' on AC with $|AB''| = |A'B'|$ and $|AC''| = |A'C'|$. Then by Proposition 7, $\Delta AB''C''$ is similar to ΔABC, so

$$|B''C''| = |AB''| \cdot \frac{|BC|}{|AB|} = |A'B'| \cdot \frac{|BC|}{|AB|} = |B'C'|.$$

Thus by SSS, $\Delta A'B'C'$ is congruent to $\Delta AB''C''$, and hence similar to ΔABC. □

6.9 Pythagoras

Theorem 14 (Pythagoras)**.** *In a right-angle triangle the square of the hypotenuse is the sum of the squares of the other two sides.*

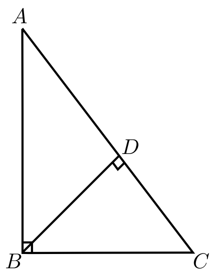

Figure 19.

Proof. Let ΔABC have a right angle at B. Draw the perpendicular BD from the vertex B to the hypotenuse AC (shown in Figure 19).

The right-angle triangles ΔABC and ΔADB have a common angle at A.
$\therefore$ ΔABC is similar to ΔADB.

$$\therefore \frac{|AC|}{|AB|} = \frac{|AB|}{|AD|},$$

so

$$|AB|^2 = |AC| \cdot |AD|.$$

Similarly, ΔABC is similar to ΔBDC.

$$\therefore \frac{|AC|}{|BC|} = \frac{|BC|}{|DC|},$$

so

$$|BC|^2 = |AC| \cdot |DC|.$$

Thus

$$\begin{aligned} |AB|^2 + |BC|^2 &= |AC| \cdot |AD| + |AC| \cdot |DC| \\ &= |AC|\left(|AD| + |DC|\right) \\ &= |AC| \cdot |AC| \\ &= |AC|^2. \end{aligned}$$

□

Theorem 15 (Converse to Pythagoras)**.** *If the square of one side of a triangle is the sum of the squares of the other two, then the angle opposite the first side is a right angle.*

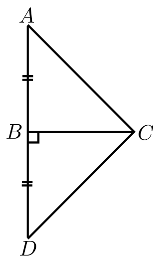

Figure 20.

Proof. (Idea: Construct a second triangle on the other side of $[BC]$, and use Pythagoras and SSS to show it congruent to the original.)
In detail: We wish to show that $|\angle ABC| = 90°$.

Draw $BD \perp BC$ and make $|BD| = |AB|$ (as shown in Figure 20).

Then

$$\begin{aligned} |DC| &= \sqrt{|DC|^2} \\ &= \sqrt{|BD|^2 + |BC|^2} && \text{[Pythagoras]} \\ &= \sqrt{|AB|^2 + |BC|^2} && [\ |AB| = |BD|\ ] \\ &= \sqrt{|AC|^2} && \text{[Hypothesis]} \\ &= |AC|. \end{aligned}$$

$\therefore$ ΔABC is congruent to ΔDBC. [SSS]
$\therefore$ $|\angle ABC| = |\angle DBC| = 90°$. □

Proposition 9 (RHS)**.** *If two right angled triangles have hypotenuse and another side equal in length, respectively, then they are congruent.*

Proof. Suppose ΔABC and $\Delta A'B'C'$ are right-angle triangles, with the right angles at B and B', and have hypotenuses of the same length, $|AC| = |A'C'|$, and also have $|AB| = |A'B'|$. Then by using Pythagoras' Theorem, we obtain $|BC| = |B'C'|$, so by SSS, the triangles are congruent. □

Proposition 10. *Each point on the perpendicular bisector of a segment* $[AB]$ *is equidistant from the ends.*

Proposition 11. *The perpendiculars from a point on an angle bisector to the arms of the angle have equal length.*

6.10 Area

Definition 37. If one side of a triangle is chosen as the base, then the opposite vertex is the **apex** corresponding to that base. The corresponding **height** is the length of the perpendicular from the apex to the base. This perpendicular segment is called an **altitude** of the triangle.

Theorem 16. *For a triangle, base times height does not depend on the choice of base.*

Proof. Let AD and BE be altitudes (shown in Figure 21). Then ΔBCE and ΔACD are right-angled triangles that share the angle C, hence they are similar. Thus

$$\frac{|AD|}{|BE|} = \frac{|AC|}{|BC|}.$$

Re-arrange to yield the result. □

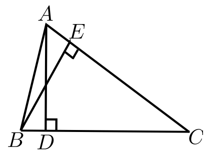

Figure 21.

Definition 38. The **area** of a triangle is half the base by the height.

Notation 5. We denote the area by "area of ΔABC"[19].

Proposition 12. *Congruent triangles have equal areas.*

Remark 6. This is another example of a proposition whose converse is false. It may happen that two triangles have equal area, but are not congruent.

Proposition 13. *If a triangle ΔABC is cut into two by a line AD from A to a point D on the segment $[BC]$, then the areas add up properly:*

$$\text{area of } \Delta ABC = \text{area of } \Delta ABD + \text{area of } \Delta ADC.$$

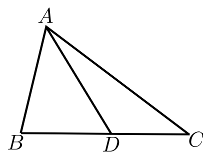

Figure 22.

Proof. See Figure 22. All three triangles have the same height, say h, so it comes down to

$$\frac{|BC| \times h}{2} = \frac{|BD| \times h}{2} + \frac{|DC| \times h}{2},$$

which is obvious, since

$$|BC| = |BD| + |DC|.$$

□

[19] $|\Delta ABC|$ will also be accepted.

If a figure can be cut up into nonoverlapping triangles (i.e. triangles that either don't meet, or meet only along an edge), then its area is taken to be the sum of the area of the triangles[20].

If figures of equal areas are added to (or subtracted from) figures of equal areas, then the resulting figures also have equal areas[21].

Proposition 14. *The area of a rectangle having sides of length a and b is ab.*

Proof. Cut it into two triangles by a diagonal. Each has area $\frac{1}{2}ab$. □

Theorem 17. *A diagonal of a parallelogram bisects the area.*

Proof. A diagonal cuts the parallelogram into two congruent triangles, by Corollary 1. □

Definition 39. Let the side AB of a parallelogram $ABCD$ be chosen as a base (Figure 23). Then the **height** of the parallelogram **corresponding to that base** is the height of the triangle ΔABC.

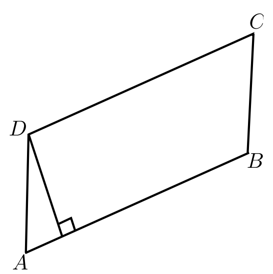

Figure 23.

Proposition 15. *This height is the same as the height of the triangle ΔABD, and as the length of the perpendicular segment from D onto AB.*

[20] If students ask, this does not lead to any ambiguity. In the case of a convex quadrilateral, $ABCD$, one can show that

$$\text{area of } \Delta ABC + \text{area of } \Delta CDA = \text{area of } \Delta ABD + \text{area of } \Delta BCD.$$

In the general case, one proves the result by showing that there is a common refinement of any two given triangulations.

[21] Follows from the previous footnote.

Theorem 18. *The area of a parallelogram is the base by the height.*

Proof. Let the parallelogram be $ABCD$. The diagonal BD divides it into two triangles, ΔABD and ΔCDB. These have equal area, [Theorem 17] and the first triangle shares a base and the corresponding height with the parallelogram. So the areas of the two triangles add to $2 \times \frac{1}{2} \times \text{base} \times \text{height}$, which gives the result. □

6.11 Circles

Definition 40. A **circle** is the set of points at a given distance (its **radius**) from a fixed point (its **centre**). Each line segment joining the centre to a point of the circle is also called **a** radius. The plural of radius is radii. A **chord** is the segment joining two points of the circle. A **diameter** is a chord through the centre. All diameters have length twice the radius. This number is also called **the** diameter of the circle.

Two points A, B on a circle cut it into two pieces, called **arcs**. You can specify an arc uniquely by giving its endpoints A and B, and one other point C that lies on it. A **sector** of a circle is the piece of the plane enclosed by an arc and the two radii to its endpoints.

The length of the whole circle is called its **circumference**. For every circle, the circumference divided by the diameter is the same. This ratio is called π.

A **semicircle** is an arc of a circle whose ends are the ends of a diameter.

Each circle divides the plane into two pieces, the inside and the outside. The piece inside is called a **disc**.

If B and C are the ends of an arc of a circle, and A is another point, not on the arc, then we say that the angle $\angle BAC$ is the angle at A **standing on the arc**. We also say that it **stands on the chord** $[BC]$.

Theorem 19. *The angle at the centre of a circle standing on a given arc is twice the angle at any point of the circle standing on the same arc.*

Proof. There are several cases for the diagram. It will be sufficient for students to examine one of these. The idea, in all cases, is to draw the line through the centre and the point on the circumference, and use the Isosceles Triangle Theorem, and then the Protractor Axiom (to add or subtract angles, as the case may be).

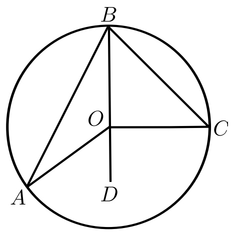

Figure 24.

In detail, for the given figure, Figure 24, we wish to show that $|\angle AOC| = 2|\angle ABC|$.

Join B to O and continue the line to D. Then

$$\begin{array}{rcll} |OA| &=& |OB|. & \text{[Definition of circle]} \\ \therefore |\angle BAO| &=& |\angle ABO|. & \text{[Isosceles triangle]} \\ \therefore |\angle AOD| &=& |\angle BAO| + |\angle ABO| & \text{[Exterior Angle]} \\ &=& 2 \cdot |\angle ABO|. & \end{array}$$

Similarly,

$$|\angle COD| = 2 \cdot |\angle CBO|.$$

Thus

$$\begin{array}{rcl} |\angle AOC| &=& |\angle AOD| + |\angle COD| \\ &=& 2 \cdot |\angle ABO| + 2 \cdot |\angle CBO| \\ &=& 2 \cdot |\angle ABC|. \end{array}$$

□

Corollary 2. *All angles at points of the circle, standing on the same arc, are equal. In symbols, if A, A', B and C lie on a circle, and both A and A' are on the same side of the line BC, then $\angle BAC = \angle BA'C$.*

Proof. Each is half the angle subtended at the centre. □

Remark 7. The converse is true, but one has to careful about sides of BC:

Converse to Corollary 2: *If points A and A' lie on the same side of the line BC, and if $|\angle BAC| = |\angle BA'C|$, then the four points A, A', B and C lie on a circle.*

Proof. Consider the circle s through A, B and C. If A' lies outside the circle, then take A'' to be the point where the segment $[A'B]$ meets s. We then have

$$|\angle BA'C| = |\angle BAC| = |\angle BA''C|,$$

by Corollary 2. This contradicts Theorem 6.

A similar contradiction arises if A' lies inside the circle. So it lies on the circle. □

Corollary 3. *Each angle in a semicircle is a right angle. In symbols, if BC is a diameter of a circle, and A is any other point of the circle, then $\angle BAC = 90°$.*

Proof. The angle at the centre is a straight angle, measuring 180°, and half of that is 90°. □

Corollary 4. *If the angle standing on a chord $[BC]$ at some point of the circle is a right angle, then $[BC]$ is a diameter.*

Proof. The angle at the centre is 180°, so is straight, and so the line BC passes through the centre. □

Definition 41. A **cyclic** quadrilateral is one whose vertices lie on some circle.

Corollary 5. *If $ABCD$ is a cyclic quadrilateral, then opposite angles sum to 180°.*

Proof. The two angles at the centre standing on the same arcs add to 360°, so the two halves add to 180°. □

Remark 8. The converse also holds: *If $ABCD$ is a convex quadrilateral, and opposite angles sum to 180°, then it is cyclic.*

Proof. This follows directly from Corollary 5 and the converse to Corollary 2. □

It is possible to approximate a disc by larger and smaller equilateral polygons, whose area is as close as you like to πr^2, where r is its radius. For this reason, we say that the area of the disc is πr^2.

Proposition 16. *If l is a line and s a circle, then l meets s in zero, one, or two points.*

Proof. We classify by comparing the length p of the perpendicular from the centre to the line, and the radius r of the circle. If $p > r$, there are no points. If $p = r$, there is exactly one, and if $p < r$ there are two. □

Definition 42. The line l is called a **tangent** to the circle s when $l \cap s$ has exactly one point. The point is called the **point of contact** of the tangent.

Theorem 20.
(1) Each tangent is perpendicular to the radius that goes to the point of contact.
(2) If P lies on the circle s, and a line l through P is perpendicular to the radius to P, then l is tangent to s.

Proof. (1) This proof is a proof by contradiction.

Suppose the point of contact is P and the tangent l is not perpendicular to OP.

Let the perpendicular to the tangent from the centre O meet it at Q. Pick R on PQ, on the other side of Q from P, with $|QR| = |PQ|$ (as in Figure 25).

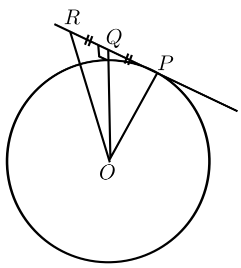

Figure 25.

Then ΔOQR is congruent to ΔOQP. [SAS]

$$\therefore |OR| = |OP|,$$

so R is a second point where l meets the circle. This contradicts the given fact that l is a tangent.

Thus l must be perpendicular to OP, as required.

(2) (Idea: Use Pythagoras. This shows directly that each other point on l is further from O than P, and hence is not on the circle.)

In detail: Let Q be any point on l, other than P. See Figure 26. Then

$$\begin{aligned} |OQ|^2 &= |OP|^2 + |PQ|^2 && \text{[Pythagoras]} \\ &> |OP|^2. \\ \therefore |OQ| &> |OP|. \end{aligned}$$

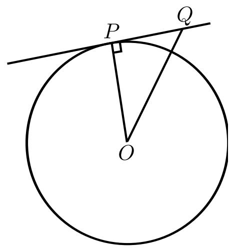

Figure 26.

$\therefore Q$ is not on the circle. [Definition of circle]
$\therefore P$ is the only point of l on the circle.
$\therefore l$ is a tangent. [Definition of tangent]
□

Corollary 6. *If two circles share a common tangent line at one point, then the two centres and that point are collinear.*

Proof. By part (1) of the theorem, both centres lie on the line passing through the point and perpendicular to the common tangent. □

The circles described in Corollary 6 are shown in Figure 27.

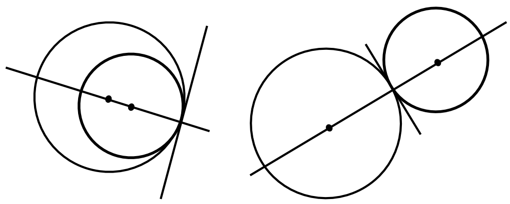

Figure 27.

Remark 9. Any two distinct circles will intersect in 0, 1, or 2 points.

If they have two points in common, then the common chord joining those two points is perpendicular to the line joining the centres.

If they have just one point of intersection, then they are said to be *touching* and this point is referred to as their *point of contact*. The centres and the point of contact are collinear, and the circles have a common tangent at that point.

Theorem 21.
(1) The perpendicular from the centre to a chord bisects the chord.
(2) The perpendicular bisector of a chord passes through the centre.

Proof. (1) (Idea: Two right-angled triangles with two pairs of sides equal.) See Figure 28.

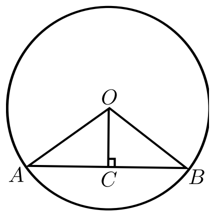

Figure 28.

In detail:

$$\begin{aligned} |OA| &= |OB| && \text{[Definition of circle]} \\ |OC| &= |OC| && \\ |AC| &= \sqrt{|OA|^2 - |OC|^2} && \text{[Pythagoras]} \\ &= \sqrt{|OB|^2 - |OC|^2} && \\ &= |CB|. && \text{[Pythagoras]} \end{aligned}$$

$\therefore \Delta OAC$ is congruent to ΔOBC. [SSS]
$\therefore |AC| = |CB|$.

(2) This uses the Ruler Axiom, which has the consequence that a segment has exactly one midpoint.
Let C be the foot of the perpendicular from O on AB.
By Part (1), $|AC| = |CB|$, so C is the midpoint of $[AB]$.
Thus CO is the perpendicular bisector of AB.
Hence the perpendicular bisector of AB passes through O. □

6.12 Special Triangle Points

Proposition 17. *If a circle passes through three non-collinear points A, B, and C, then its centre lies on the perpendicular bisector of each side of the triangle ΔABC.*

Definition 43. The **circumcircle** of a triangle ΔABC is the circle that passes through its vertices (see Figure 29). Its centre is the **circumcentre** of the triangle, and its radius is the **circumradius**.

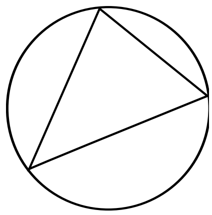

Figure 29.

Proposition 18. *If a circle lies inside the triangle ΔABC and is tangent to each of its sides, then its centre lies on the bisector of each of the angles $\angle A$, $\angle B$, and $\angle C$.*

Definition 44. The **incircle** of a triangle is the circle that lies inside the triangle and is tangent to each side (see Figure 30). Its centre is the **incentre**, and its radius is the **inradius**.

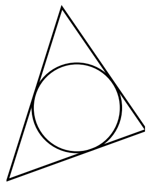

Figure 30.

Proposition 19. *The lines joining the vertices of a triangle to the centre of the opposite sides meet in one point.*

Definition 45. A line joining a vertex of a triangle to the midpoint of the opposite side is called a **median** of the triangle. The point where the three medians meet is called the **centroid**.

Proposition 20. *The perpendiculars from the vertices of a triangle to the opposite sides meet in one point.*

Definition 46. The point where the perpendiculars from the vertices to the opposite sides meet is called the **orthocentre** (see Figure 31).

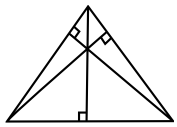

Figure 31.

7 Constructions to Study

The instruments that may be used are:

straight-edge: This may be used (together with a pencil) to draw a straight line passing through two marked points.

compass: This instrument allows you to draw a circle with a given centre, passing through a given point. It also allows you to take a given segment $[AB]$, and draw a circle centred at a given point C having radius $|AB|$.

ruler: This is a straight-edge marked with numbers. It allows you measure the length of segments, and to mark a point B on a given ray with vertex A, such that the length $|AB|$ is a given positive number. It can also be employed by sliding it along a set square, or by other methods of sliding, while keeping one or two points on one or two curves.

protractor: This allows you to measure angles, and mark points C such that the angle $\angle BAC$ made with a given ray $[AB$ has a given number of degrees. It can also be employed by sliding it along a line until some line on the protractor lies over a given point.

set-squares: You may use these to draw right angles, and angles of $30°$, $60°$, and $45°$. It can also be used by sliding it along a ruler until some coincidence occurs.

The prescribed constructions are:

1. Bisector of a given angle, using only compass and straight edge.
2. Perpendicular bisector of a segment, using only compass and straight edge.
3. Line perpendicular to a given line l, passing through a given point not on l.

4. Line perpendicular to a given line l, passing through a given point on l.

5. Line parallel to given line, through given point.

6. Division of a segment into 2, 3 equal segments, without measuring it.

7. Division of a segment into any number of equal segments, without measuring it.

8. Line segment of given length on a given ray.

9. Angle of given number of degrees with a given ray as one arm.

10. Triangle, given lengths of three sides.

11. Triangle, given SAS data.

12. Triangle, given ASA data.

13. Right-angled triangle, given the length of the hypotenuse and one other side.

14. Right-angled triangle, given one side and one of the acute angles (several cases).

15. Rectangle, given side lengths.

16. Circumcentre and circumcircle of a given triangle, using only straight-edge and compass.

17. Incentre and incircle of a given triangle, using only straight-edge and compass.

18. Angle of $60°$, without using a protractor or set square.

19. Tangent to a given circle at a given point on it.

20. Parallelogram, given the length of the sides and the measure of the angles.

21. Centroid of a triangle.

22. Orthocentre of a triangle.

8 Teaching Approaches

8.1 Practical Work

Practical exercises and experiments should be undertaken before the study of theory. These should include:

1. Lessons along the lines suggested in the Guidelines for Teachers [2]. We refer especially to Section 4.6 (7 lessons on Applied Arithmetic and Measure), Section 4.9 (14 lessons on Geometry), and Section 4.10 (4 lessons on Trigonometry).

2. Ideas from Technical Drawing.

3. Material in [3].

8.2 From Discovery to Proof

It is intended that all of the geometrical results on the course would first be encountered by students through investigation and discovery. As a result of various activities undertaken, students should come to appreciate that certain features of certain shapes or diagrams appear to be independent of the particular examples chosen. These apparently constant features therefore seem to be general results that we have reason to believe might always be true. At this stage in the work, we ask students to accept them as true for the purpose of applying them to various contextualised and abstract problems, but we also agree to come back later to revisit this question of their truth. Nonetheless, even at this stage, students should be asked to consider whether investigating a number of examples in this way is sufficient to be convinced that a particular result always holds, or whether a more convincing argument is required. Is a person who refuses to believe that the asserted result will always be true being unreasonable? An investigation of a statement that appears at first to be always true, but in fact is not, may be helpful, (e.g. the assertion that $n^2 + n + 41$ is prime for all $n \in \mathbb{N}$). Reference might be made to other examples of conjectures that were historically believed to be true until counterexamples were found.

Informally, the ideas involved in a mathematical proof can be developed even at this investigative stage. When students engage in activities that lead to closely related results, they may readily come to appreciate the manner in which these results are connected to each other. That is, they may see for themselves or be led to see that the result they discovered today is an inevitable logical consequence of the one they discovered yesterday. Also, it should be noted that working on problems or "cuts" involves logical deduction from general results.

Later, students at the relevant levels need to proceed beyond accepting a result on the basis of examples towards the idea of a more convincing logical argument. Informal justifications, such as a dissection-based proof of Pythagoras' theorem, have a role to play here. Such justifications develop an argument more strongly than a set of examples. It is worth discussing what the word "prove" means in various contexts, such as in a criminal trial, or in a civil court, or in everyday language. What mathematicians regard as a "proof" is quite different from these other contexts. The logic involved in the various steps must be unassailable. One might present one or more of the readily available dissection-based "proofs" of fallacies and then probe a dissection-based proof of Pythagoras' theorem to see what possible gaps might need to be bridged.

As these concepts of argument and proof are developed, students should be led to appreciate the need to formalise our idea of a mathematical proof to lay out the ground rules that we can all agree on. Since a formal proof only allows us to progress logically from existing results to new ones, the need for axioms is readily identified, and the students can be introduced to formal proofs.

9 JCOL Content

9.1 Concepts

Set, subset, plane, point, line, ray, angle, real number, length,

degree, triangle, right-angle, congruent triangles, similar triangles, parallel lines, parallelogram, area,

segment, collinear points, distance, midpoint of a segment,

reflex angle, ordinary angle, straight angle, null angle, full angle, supplementary angles, vertically-opposite angles, acute angle, obtuse angle,

angle bisector, perpendicular lines, perpendicular bisector of a segment, ratio,

isosceles triangle, equilateral triangle, scalene triangle, right-angled triangle, exterior angles of a triangle, interior opposite angles, hypotenuse, alternate angles, corresponding angles,

polygon, quadrilateral, convex quadrilateral, rectangle, square, rhombus, base and corresponding apex and height of triangle or parallelogram, transversal line,

circle, centre of a circle, radius, diameter, chord, arc, sector, circumference of a circle, disc, area of a disc, circumcircle, tangent to a circle, point of contact of a tangent,

vertex, vertices (of angle, triangle, polygon), endpoints of segment, arms of an angle, equal segments, equal angles, adjacent sides, angles, or vertices of triangles or quadrilaterals, the side opposite an angle of a triangle, opposite sides or angles of a quadrilateral.

9.2 Constructions

Students will study constructions 1, 2, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15.

9.3 Axioms and Proofs

Students should be exposed to situations where they are required to use the terms theorem proof, axiom, corollary, converse, and implies. The students should be exposed to some formal proofs. They will not be examined on these. They will see Axioms 1,2,3,4,5, and study the proofs of Theorems 1, 2, 3, 4, 5, 6, 9, 10, 14, 15; and direct proofs of Corollaries 3, 4. They will study the statement and use of Theorem 13, but need not study its formal proof.

10 Additional Content for JCHL

10.1 Concepts

Concurrent lines.

10.2 Constructions

Constructions 3 and 7.

10.3 Logic, Axioms and Theorems

Students should be exposed to situations where they are required to use and explain the terms Theorem, proof, axiom, corollary, converse, implies.

They will study Axioms 1, 2, 3, 4, 5. They will study the proofs of Theorems 13, 19, Corollaries 1, 2, 3, 4, 5, and their converses.

They will make use of Theorems 11 and 12, but need not study their formal proofs. The formal material on area will not be studied at this level. Students will deal with area only as part of the material on arithmetic and mensuration. They will study geometrical problems.

11 Syllabus for LCFL

Students are expected to build on their mathematical experiences to date.

11.1 Constructions

Students revisit constructions 4, 5, 10, 13, 15, and learn how to apply these in real-life contexts.

12 Syllabus for LCOL

12.1 Constructions

A knowledge of the constructions prescribed for JC-OL will be assumed, and may be examined. In addition, students will study constructions 16–21.

12.2 Theorems and Proofs

Students will be expected to understand the meaning of the following terms related to logic and deductive reasoning: **Theorem, proof, axiom, corollary, converse, implies.**

A knowledge of the Axioms, concepts, Theorems and Corollaries prescribed for JC-OL will be assumed.

Students will study proofs of Theorems 7, 8, 11, 12, 13, 16, 17, 18, 20, 21, and Corollary 6.

No proofs are examinable. Students will be examined using problems that can be attacked using the theory.

13 Syllabus for LCHL

13.1 Constructions

A knowledge of the constructions prescribed for JC-HL will be assumed, and may be examined. In addition, students will study the constructions prescribed for LC-OL, and construction 22.

13.2 Theorems and Proofs

Students will be expected to understand the meaning of the following terms related to logic and deductive reasoning: **Theorem, proof, axiom, corollary, converse, implies, is equivalent to, if and only if, proof by contradiction.**

A knowledge of the Axioms, concepts, Theorems and Corollaries prescribed for JC-HL will be assumed.

Students will study all the theorems and corollaries prescribed for LC-OL, but will not, in general, be asked to reproduce their proofs in examination.

However, they may be asked to give proofs of the Theorems 11, 12, 13, concerning ratios, which lay the proper foundation for the proof of Pythagoras studied at JC, and for trigonometry.

They will be asked to solve geometrical problems (so-called "cuts") and write reasoned accounts of the solutions. These problems will be such that they can be attacked using the given theory. The study of the propositions may be a useful way to prepare for such examination questions.

References

[1] Patrick D. Barry. *Geometry with Trigonometry.* Horwood. Chichester. 2001. ISBN 1-898563-69-1.

[2] Junior Cycle Course Committee, NCCA. *Mathematics: Junior Certificate Guidelines for Teachers.* Stationary Office, Dublin. 2002. ISBN 0-7557-1193-9.

[3] Fiacre O'Cairbre, John McKeon, and Richard O. Watson. A Resource for Transition Year Mathematics Teachers. DES. Dublin. 2006.

[4] Anthony G. O'Farrell. *School Geometry.* IMTA Newsletter 109 (2009) 21-28.