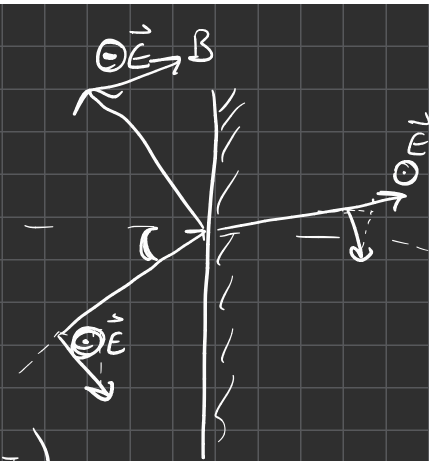
9.17




For $\perp$ polarized wave in an arbitrary direction,

$$\tilde{\vec{E}}_I = \tilde{E}_{0,I} e^{i(k_I\cdot r - \omega t)} \hat{y}$$

$$\tilde{\vec{B}}_I = \frac{1}{v_1} \tilde{E}_{0,I} e^{i(k_I\cdot r - \omega t)} (-\cos\theta_I \hat{x} + \sin\theta_I \hat{z})$$

$$\tilde{\vec{E}}_R = \tilde{E}_{0,R} e^{i(k_R\cdot r - \omega t)} \hat{y}$$

$$\tilde{\vec{B}}_R = \frac{1}{v_1} \tilde{E}_{0,R} e^{i(k_R\cdot r - \omega t)} (\hat{x}\cdot\cos\theta_I + \hat{z}\cdot\sin\theta_I)$$

$$\tilde{\vec{E}}_T = \tilde{E}_{0,T} e^{i(k_T\cdot r - \omega t)} \hat{y}$$

$$\tilde{\vec{B}}_T = \frac{1}{v_T} \tilde{E}_{0,T} e^{i(k_T\cdot r - \omega t)} (-\cos\theta_T \hat{x} + \sin\theta_T \hat{z})$$

We can use boundary condition

$B_1^{\perp} = B_2^{\perp}$ to get $B_{z1} = B_{z2}$

so $\frac{1}{v_1}\sin\theta_1 = \frac{1}{v_2}\sin\theta_2$

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

We can use $E_1^{\parallel} = E_2^{\parallel}$ to get $E_{y1} = E_{y2}$, so

$$\tilde{E}_{0,I} + \tilde{E}_{0,R} = \tilde{E}_{0,T}$$

And then using $\frac{1}{\mu_1} B_1^{\parallel} = \frac{1}{\mu_2} B_2^{\parallel}$, we can say

$$\frac{1}{\mu_1 v_1}\left[\tilde{E}_{0,I}(-\cos\theta_I) + \tilde{E}_{0,R}(\cos\theta_I)\right] = \frac{1}{\mu_2 v_2} \tilde{E}_{0,T}\cos\theta_T$$

$$\tilde{E}_{0I} - \tilde{E}_{0R} = \frac{\mu_1 v_1 \cos\theta_2}{\mu_2 v_2 \cos\theta_1} \tilde{E}_{0,T}$$ which is

$$\tilde{E}_{0I} - \tilde{E}_{0R} = \alpha\beta \tilde{E}_{0T}$$

so,

$$\frac{\tilde{E}_{0I} - \tilde{E}_{0R}}{\alpha\beta} = \tilde{E}_{0I} + \tilde{E}_{0R}$$

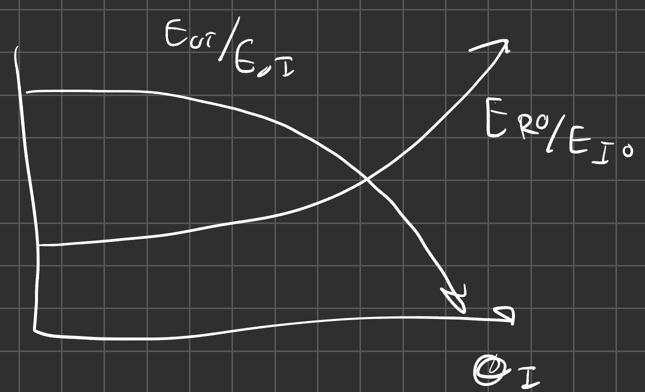
$$\tilde{E}_{0I}(1 - \alpha\beta) = \tilde{E}_{0,R}(1 + \alpha\beta)$$

$$\tilde{E}_{0,R} = \frac{(1-\alpha\beta)}{(1+\alpha\beta)} \tilde{E}_{0I}$$

$$\hat{E}_{0I} - \alpha\beta\tilde{E}_{0T} = \tilde{E}_{0T} - \tilde{E}_{0I}$$

$$2\tilde{E}_{0I} = \tilde{E}_{0T}(1 + \alpha\beta)$$

$$E_T = \frac{2}{(1+\alpha\beta)} \tilde{E}_{0I}$$



$E_{0,R} = 0$ results in

$$\frac{(1-\alpha\beta)}{(1+\alpha\beta)} = 0$$

$$\alpha\beta = 1$$

$$\alpha = \frac{\sqrt{1-(v_2/v_1)^2\sin^2\theta}}{\cos\theta} = \frac{1}{\beta} = \frac{\mu_2 v_2}{\mu_1 v_1}$$

$$1-\left(\frac{v_2}{v_1}\right)^2\sin^2\theta = \left(\frac{\mu_2 v_2}{\mu_1 v_1}\right)^2\cos^2\theta$$

$$1 = \left(\frac{v_2}{v_1}\right)^2\left[\sin^2\theta + \underbrace{(\mu_2/\mu_1)^2}\cos^2\theta\right)$$

this should be around 1, so

$1 \approx \left(\frac{v_2}{v_1}\right)^2$, so Brewsters angle only exists with 2 very similar materials

@ normal incidence, $\alpha = 1$, so

$$E_{0T} = \left(\frac{2}{1+\beta}\right)E_{0I} \quad \& \quad E_{0R} = \left(\frac{1-\beta}{1+\beta}\right)E_{0I} \quad \checkmark$$

$$R = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2$$

$$T = \alpha\beta\left(\frac{2}{1+\alpha\beta}\right)^2$$

$$R+T = \frac{(1+\alpha\beta)^2}{(1+\alpha\beta)^2} = 1$$

9.18

(a)

$$\frac{E_R}{E_I} = \frac{\alpha-\beta}{\alpha+\beta} = \frac{1-2.42}{1+2.42} = -0.4$$

$$\frac{E_T}{E_I} = \frac{2}{1+2.42} = 0.585$$

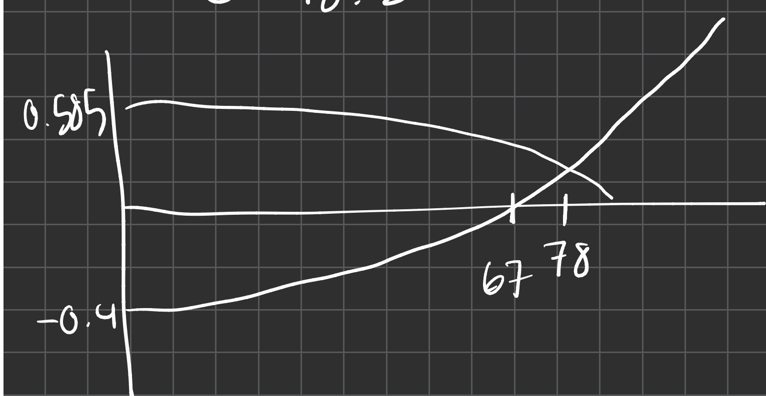
(b) $\theta_B = \arctan(2.42) = 67.5°$

(c) $$\frac{\alpha-\beta}{\alpha+\beta} = \frac{2}{\alpha+\beta} \Rightarrow \alpha-\beta = 2$$

$\alpha = 4.4$

$$4.4 = \frac{\sqrt{1-\left(\sin\theta/2.42\right)^2}}{\cos\theta}$$

$\theta = 78.3$



9.39

a) $\tilde{\vec{E}}_T = \tilde{E}_{0T} e^{i(\vec{k}\cdot r - \omega t)}$

$$\vec{k}\cdot r = k_T \sin\theta_T x \hat{x} + k_T \cos\theta_T z \hat{z}$$

$$= \frac{\omega n_2}{c} \cdot \frac{n_1}{n_2} \sin\theta_I x \hat{x} + \frac{\omega n_2 i}{c} \sqrt{\sin^2\theta_T - 1}\, z \hat{z}$$

$$= \underbrace{\frac{\omega n_1}{c} \sin\theta_I}_{k \text{ as defined in book}} x \hat{x} + \frac{\omega n_2 i}{c} \sqrt{\frac{n_1^2 \sin^2\theta_I}{n_2^2} - 1}\, z \hat{z}$$

$$= k x \hat{x} + \underbrace{\frac{\omega i}{c} \sqrt{n_1^2 \sin^2\theta_I - n_2^2}}_{K i \text{ as defined in book, so}} z \hat{z}$$

$$\tilde{\vec{E}}_T = \tilde{E}_{0T} e^{i(kx + Kiz - \omega t)}$$

$$= \tilde{E}_{0T} e^{-Kz} e^{i(kx - \omega t)} \checkmark$$

b) $R = \left| \frac{\alpha - \beta}{\alpha + \beta} \right|^2$ $\alpha$ is imaginary so to square multiply by complex conjugate

$$= \left| \frac{(\alpha - \beta)(-\alpha - \beta)}{(\alpha + \beta)(-\alpha + \beta)} \right| = \left| \frac{(\alpha - \beta)(-1)(\alpha + \beta)}{(\alpha + \beta)(-1)(\alpha - \beta)} \right| = 1 \quad \text{100\% Reflection}$$

c) $R = \left| \frac{1 + \alpha\beta}{1 - \alpha\beta} \right|^2$ again $\alpha$ is imaginary, so

$$\left| \frac{(1 + \alpha\beta)(1 - \alpha\beta)}{(1 - \alpha\beta)(1 + \alpha\beta)} \right| = 1 \checkmark \text{ again, 100\% reflection}$$

a) Result of 9.17 is

$$\tilde{\vec{E}}_T = \tilde{E}_{0T} e^{i(k_T \cdot r - \omega t)} \hat{y} \quad \tilde{B}_T = \frac{1}{V_2} \tilde{E}_{0T} e^{i(k_T \cdot r - \omega t)} (-\cos\theta_T \hat{x} + \sin\theta_T \hat{z})$$

and in a, $k \cdot r = kx + iKz$

$$\sin\theta_T = \frac{ck}{\omega n_2}, \quad \cos\theta_T = i\frac{cK}{\omega n_2}$$

so,

$$\tilde{\vec{E}} = \tilde{E}_0 e^{-Kz} e^{i(kx - \omega t)} \hat{y}$$

$$\tilde{\vec{B}} = \frac{1}{V_2} \tilde{E}_0 e^{-Kz} e^{i(kx - \omega t)} \left(-\frac{icK}{\omega n_2} \hat{x} + \frac{ck}{\omega n_2} \hat{z}\right)$$

E is easy to show the real part,

$$\vec{E} = E_0 e^{-Kz} \cos(kx - \omega t) \hat{y}$$

B is harder, real multiplied with real + im · im is real, so

$$\frac{1}{V_2} E_0 e^{-Kz} \left[ (\cos(kx - \omega t) + i\sin(kx - \omega t)) \left( -\frac{icK}{\omega n_2} \hat{x} + \frac{ck}{\omega n_2} \hat{z} \right) \right]$$

$$\vec{B} = \frac{1}{V_2} E_0 e^{-Kz} \left[ \sin(kx - \omega t) \frac{cK}{\omega n_2} \hat{x} + \cos(kx - \omega t) \frac{ck}{\omega n_2} \hat{z} \right]$$

$V_2 = \frac{c}{n_2}$, so

$$\vec{B} = \frac{1}{\omega} E_0 e^{-Kz} \left[ K \sin(kx - \omega t) \hat{x} + k \cos(kx - \omega t) \hat{z} \right] \checkmark$$

e) $\nabla \cdot E = \frac{\rho}{\varepsilon_0} = 0$ w/ no extra charge

$$\nabla \cdot E = \frac{\partial}{\partial y}\left[E_0 e^{-kz}\cos(hx-\omega t)\right] = 0 \checkmark$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E & 0 \end{vmatrix} = -\frac{\partial}{\partial z}E\hat{x} + \frac{\partial}{\partial x}E\hat{z}$$

$$= -kE_0 e^{-kz}\cos(hx-\omega t)\hat{x} - hE_0 e^{-kz}\sin(hx-\omega t)\hat{z}$$

$$\frac{\partial \vec{B}}{\partial t} = \frac{E_0 e^{-kz}}{\omega}\left[-\omega k\cos(hx-\omega t)\hat{x} - \omega h\sin(hx-\omega t)\hat{z}\right]$$

cancel the omegas & $\hat{x} + \hat{z}$ components are equal ✓ ^^

$$\nabla \cdot \vec{B} = 0$$

$$= \frac{E_0}{\omega} e^{-kz}\left[kh\cos(hx-\omega t)\right] + \frac{E_0}{\omega}(-k)e^{-kz}\left[h\cos(hx-\omega t)\right] = 0 \quad ^^$$

$\nabla \times \vec{B} = \mu_2 \varepsilon_2 \frac{\partial E}{\partial t}$ w/ no extra current

$$\nabla \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & 0 & B_z \end{vmatrix} = 0\hat{x} + \left[\frac{\partial}{\partial z}B_x - \frac{\partial}{\partial x}B_z\right]\hat{y} + 0\hat{z}$$

$$= -\frac{kE_0}{\omega}e^{-kz}k\sin(hx-\omega t) + \frac{E_0}{\omega}e^{-kz}h^2\sin(hx-\omega t)$$

$$= (h^2 - k^2)\frac{E_0}{\omega}e^{-kz}\sin(hx-\omega t)$$

$(k^2 - K^2) =$

$$\frac{\omega^2 n_1^2}{c^2}\sin^2\theta_I - \frac{\omega^2}{c^2}\left[n_1^2\sin^2\theta_I - n_2^2\right] = \left(\frac{\omega n_2}{c}\right)^2$$

reduces to

$\omega^2 \varepsilon_2 \mu_2$

$$\nabla \times B = \omega \varepsilon_2 \mu_2 E_0 e^{-kz}\sin(kx - \omega t)\hat{y}$$

$$\mu_2 \varepsilon_2 \frac{\partial E}{\partial t} = \mu_2 \varepsilon_2\left[E_0 e^{-kz}\omega \sin(kx - \omega t)\right]\hat{y}$$ which is equal :)

f) $$\vec{S} = \frac{1}{\mu_2}\left[\vec{E} \times \vec{B}\right] = \frac{1}{\mu_2}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & E & 0 \\ B_x & 0 & B_z \end{vmatrix}$$

$$= \frac{1}{\mu_0}\left[\frac{E_0^2}{\omega} e^{-2kz} k\cos^2(kx - \omega t)\hat{x} + -\frac{E_0^2}{\omega} e^{-2kz} K\cos(kx-\omega t)\sin(kx-\omega t)\hat{z}\right]$$

average of $\cos^2 = 1/2$ + average of $\sin\cos = 0$

$\langle S \rangle = \frac{E_0^2 k}{2\mu_2\omega} e^{-2kz}\hat{x}$, so energy is only transmitted in x direction

I should not have waited to the end of break to do this :( it was so long

Exercise 1

We were given

$$B_{0z} = B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

So for $m=1, n=0$

$$B_{0z} = B_0 \cos\left(\frac{\pi x}{a}\right)$$

So $B_z = B_0 \cos\left(\frac{\pi x}{a}\right)\cos(kz-\omega t)$

In TE mode, $E_z = 0$, so

$$B_{0x} = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\left(k\frac{\partial B_{0z}}{\partial x}\right)$$

$$= \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\left(k \cdot \frac{\pi}{a} B_0 \cdot -\sin\left(\frac{\pi x}{a}\right)\right)$$

$$= \frac{-\pi k B_0 i}{a\left(\left(\frac{\omega}{c}\right)^2 - k^2\right)} \sin\left(\frac{\pi x}{a}\right)$$

$$\left(\frac{\omega}{c}\right)^2 - k^2 = \left(\frac{\omega}{c}\right)^2 - \left(\frac{\omega}{c}\right)^2 + \pi^2\left[\left(\frac{1}{a^2}\right)\right]$$

$$= \frac{\pi^2}{a^2}$$

$$B_{0x} = -\frac{kaB_0 i \sin(\pi x/a)}{\pi}$$

$$B_x = \frac{ka}{\pi} B_0 \sin\left(\frac{\pi x}{a}\right)\sin(kz-\omega t)$$

$$E_{oy} = \frac{i}{\pi^2/a^2}\left(-\omega \frac{\partial B_{oz}}{\partial x}\right)$$

$$= \frac{a^2 i}{\pi^2}\left(\frac{\omega \pi B_o}{a} \sin\left(\frac{\pi x}{a}\right)\right)$$

$$E_y = -\frac{\omega a}{\pi} B_o \sin\left(\frac{\pi x}{a}\right) \sin(kz - \omega t)$$

$E_z$ must be zero b/c of how TE waves are defined

$$E_{ox} = \frac{ia^2}{\pi^2}\left[k \cdot 0 + \omega \cdot 0\right] = 0$$

$$B_{oy} = \frac{ia^2}{\pi^2}\left(k \cdot 0 + \frac{\omega}{c^2} \cdot 0\right) = 0$$

Exercise 2

Given, $B_{0z} = B_0 \cos(\frac{m\pi x}{a})\cos(n\pi y/b)$

$$\tilde{B}_z = B_0 \cos(\frac{m\pi x}{a})\cos(\frac{n\pi y}{b})\, e^{i(kz-\omega t)}$$

only real part will be the cosine, so

$$B_z = B_0 \cos(\frac{m\pi x}{a})\cos(\frac{n\pi y}{b})\cos(kz-\omega t)$$

$$E_{x0} = \frac{i}{(\omega/c)^2 - k^2}\left(k \cdot 0 + \omega \frac{\partial B_{0z}}{\partial y}\right)$$

$$(\omega/c)^2 - k^2 = \left(\frac{\omega}{c}\right)^2 - \left(\frac{\omega^2 - \omega_{mn}^2}{c^2}\right) = \frac{\omega_{mn}^2}{c^2}$$

$$E_{x0} = \frac{i}{(\frac{\omega_{mn}}{c})^2}\left(\omega B_0 \cos(\frac{m\pi x}{a}) - \sin(\frac{n\pi y}{b})\right)\left(\frac{n\pi}{b}\right)$$

$$E_{x0} = \frac{-i\omega B_0}{\frac{\omega_{mn}^2}{c^2}}\left(\frac{n\pi}{b}\right)\cos(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})$$

Since this is imaginary only sin part will survive when we take the real part, so

$$E_x = \frac{\omega}{\omega_{mn}^2/c^2}\left(\frac{n\pi}{b}\right) B_0 \cos(\frac{m\pi x}{a})\sin\left(\frac{n\pi y}{b}\right)\sin(kz-\omega t)$$

$$E_{y0} = \frac{i}{(\omega_{mn}/c)^2}\left(-\omega \frac{\partial B_{0z}}{\partial x}\right) = \frac{\omega i}{(\omega_{mn}/c^2)^2} B_0 \sin\left(\frac{m\pi x}{a}\right)\left(\frac{m\pi}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

So similarly,

$$E_y = -\frac{\omega}{\omega_{mn}^2/c^2}\left(\frac{m\pi}{a}\right) B_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\sin(kz-\omega t)$$

$$B_{0x} = \frac{i}{\omega_{mn}^2/c^2}\left[k\frac{\partial B_{0z}}{\partial x}\right] = \frac{-ki}{\omega_{mn}^2/c^2} B_0 \sin\left(\frac{m\pi x}{a}\right)\left(\frac{m\pi}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

So

$$B_x = \frac{k}{\omega_{mn}^2/c^2}\left(\frac{m\pi}{a}\right) B_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\sin(kz-\omega t)$$

$$B_{0y} = \frac{i}{\omega_{mn}^2/c^2}\left(k\frac{B_{0z}}{\partial y}\right) = \frac{-ik}{\omega_{mn}^2/c^2} B_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\left(\frac{n\pi}{b}\right)$$

So

$$B_y = \frac{k}{\omega_{mn}^2/c^2}\left(\frac{n\pi}{b}\right) B_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin(kz-\omega t)$$

$E_z = 0$ by definition

## Exercise 4

For TM waves, $B_z = 0$, so

$$E_z(x,y) = X(x)Y(y)$$

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} + [(\omega/c)^2 - k^2]XY = 0$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k_x^2 \qquad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_y^2$$

$$-k_x^2 - k_y^2 + (\omega/c)^2 - k^2 = 0$$

$$X(x) = A\sin(k_x x) + B\cos(k_x x)$$

Boundary conditions require $B_z$ to be zero at edges, so only sine term stays, and

$k_x = \frac{m\pi x}{a}$ $k_y = \frac{n\pi y}{b}$, just like before

$\therefore$,

$$B_{0z} = B_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

So

$$B_z = B_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cos(kz - \omega t)$$

Cutoff frequency is still

$$\omega_{mn} = c\pi\sqrt{(m/a)^2 + (n/b)^2}$$

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}$$ ← this doesn't change from TE

wave velocity $v = \frac{\omega}{k} = \frac{c}{\sqrt{1-(\omega_{mn}/\omega)^2}}$

group velocity

$$v_g = c\sqrt{1-(\omega_{mn}/\omega)^2}$$

The major difference is that for TM waves, the lowest mode is the $m=1, n=1$ mode. Because $E_z$ has two sine factors rather than cosine factors, if m or n is zero, the whole function goes to zero.

$$\omega_{11} = c\pi\sqrt{\frac{1}{a^2}+\frac{1}{b^2}} \quad \text{for TM}$$

$$\omega_{10} = c\pi\sqrt{\frac{1}{a^2}} = \frac{c\pi}{a}$$

$$\frac{\omega_{11}}{\omega_{10}} = \frac{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}{\sqrt{\frac{1}{a^2}}} = 1 + \sqrt{\frac{a^2}{b^2}} = 1 + \frac{a}{b}$$